PERCOLATION DIFFUSION

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ABSTRACT. Let a Brownian motion in the unit ball be absorbed if it hits a set generated by a radially symmetric Poisson point process. The point set is "fattened" by putting a ball with a constant hyperbolic radius on each point. When is the probability non zero that the Brownian motion hits the boundary of the unit ball? That is, manage to avoid all the Poisson balls and "percolate diffusively" all the way to the boundary. We will show that if the bounded Poisson intensity at a point z is $\nu(d(0,z))$, where $d(\cdot,\cdot)$ is the hyperbolic metric, then the Brownian motion percolates diffusively if and only if $\nu \in L^1$.

Keywords: Percolation, Brownian motion, Poisson process, hyperbolic geometry, minimal thinness.

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1. INTRODUCTION

A *percolation model* was mathematically first set up in 1957 by S. Broadbent and J. Hammersley in [2]. In their introduction they write:

> "There are many physical phenomena in which a *fluid* spreads randomly through a *medium*."
>
> "According to the nature of the problem, it may be natural to ascribe the random mechanism either to the fluid or to the medium. Most mathematical analyses are confined to the former alternative, for which we retain the usual name of *diffusion process*: in contrast, there is (as far as we know) little published work on the latter alternative, which we shall call a *percolation process*."

We want in the present paper to address a problem that takes account both environment and particle stochastics, i.e. both percolation and diffusion. A physical motivation could be to understand when a certain type of idealized randomized gas mask works, e.g. Example 4 in [2], where we add the absorption of gas molecules at the surface of the solid.

Furthermore, instead of working in a discrete lattice, we will be using a continuous setting. This is now a standard approach using a Poisson process, see for example [9].

The diffusion part is generated by a Brownian motion, and the percolation part is created by a Poisson point process, with variable intensity, in the hyperbolic unit ball. Let us here mention two authors that have done very interesting work in a similar set up, but with constant intensities: R. Lyons calculated in [8] the critical intensity for almost sure blocking of all rays—not Brownian paths—from the origin, for an appropriately chosen radius. A.-S. Sznitman in used in [11] and [12], a constant Poisson intensity together with increasing domains in the hyperbolic ball.

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2. The set up

Let $d(\cdot,\cdot)$ denote the hyperbolic metric in the unit n–dimensional ball $B = B^n$. That is, if x and y are in B, then $d(x,y) = \inf_\gamma \int \frac{2\|dz|}{1-|z|^2}$, where the infimum is taken over all curves γ from x to y.

Let $\mathcal{S}$ be a point sequence in the open unit ball B given by a Poisson point process in the hyperbolic space with intensity that is a radially symmetric continuous function, $\beta(z)$. Due to the symmetry, we can write

$$\beta(z) = \nu((d(0,z)), \quad \text{for a } \nu : [0,\infty)\to[0,\infty). \tag{1}$$

Remark 2.1. Let us separate the intensity function in the following way. Let M be the supremum of β in B. By assumption $M < \infty$. Now let ν_1 be the function from $[0,\infty)$ to $[0,1]$ defined by $\beta(z) = M\nu_1((d(0,z))$.

We can think of our non–homogeneous Poisson process as a *thinning* of a stationary process. We get a realization of the process by taking a realization of a stationary Poisson process with (constant) intensity M. Then, if z_k is a point in the realization, we remove it by probability $1-\nu_1(d(0,z_k))$ to obtain our point sequence. By [9, Proposition 1.3] this process is the same as the non–homogeneous Poisson process with intensity β.

Let us now "fatten" the point sequence $\mathcal{S}$ by putting balls centered on each point in $\mathcal{S}$ and having a given constant radius ϱ. That is, for every $z_i \in \mathcal{S}$ let

$$B_i = \{z \in B; d(z,z_i) \le \varrho\}. \tag{2}$$

(These balls are called *clouds* in [8].) Let us denote the union of the balls, i.e. the *random archipelago*, by $\mathcal{A}$.

$$\mathcal{A} = \bigcup B_i. \tag{3}$$

Let us now consider a Brownian motion started at the origin. What is the probability that it reaches the unit sphere without hitting the archipelago $\mathcal{A}$? See figure 1.

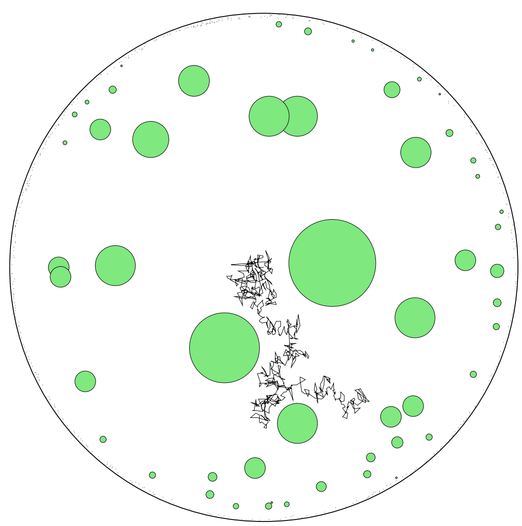

FIGURE 1. A schematic picture of the situation. A Brownian motion in the unit disk. What is the probability that it reaches the unit sphere without hitting the random archipelago $\mathcal{A}$?

If we instead of using an Euclidean Brownian motion, take a process in the hyperbolic geometry, we reformulate the question above by: What is the probability that a (hyperbolic) Brownian motion started at the origin never hits the random archipelago $\mathcal{A}$? Let us use the following notation for that "escape" probability.

$$\pi_e = \mathrm{P}[BM \text{ hits } \partial B \text{ before hitting } \mathcal{A}]. \tag{4}$$

Note that we have two different random processes involved, one for the generation of $\mathcal{A}$ and one for the Brownian motion in the complement of $\mathcal{A}$. Let us combine the two processes in the following definition.

Definition 2.2. *We say that we have* **percolation diffusion** *if there is a non-zero probability that $\mathcal{A}$ is created in such a way that a Brownian motion, BM, avoids hitting $\mathcal{A}$ with non-zero probability, i.e.*

$$\mathrm{P}_{\mathcal{A}}[\mathrm{P}_{BM}[BM \text{ hits } \partial B \text{ before hitting } \mathcal{A}] > 0] > 0,$$

or in other words: $\mathrm{P}[\pi_e > 0] > 0$.

3. An integral criterion for percolation diffusion

Theorem 3.1. *Let $\mathcal{S}$ be a realized set of point in the hyperbolic unit ball from a Poisson point process with intensity $\beta(z) = \nu(d(0, z))$. Let $\mathcal{A}$ be a random archipelago based on $\mathcal{S}$ as in (3). We have percolation diffusion if and only if $\nu \in L^1$.*

Remark 3.2. Note that $\nu \in L^1$ if and only if the expected number of Poisson points are finite.

The proof of the theorem will be based on a result concerning the concept of minimal thinness given in the following section.

4. Minimal thinness

Let us denote the class of non-negative superharmonic functions in the unit ball by $SH(B)$, and the Poisson kernel at $\tau \in \partial B$, $\frac{1-|z|^2}{|z-\tau|^n}$, by P_τ.

Definition 4.1. *The* **reduced function** *of h with respect to the a subset E of B is defined as*

$$R_h^E(w) = \inf\{u(w) : u \in SH(B) \text{ and } u \geq h \text{ on } E\}.$$

We can make this function lower semi continuous by regularizing it to the **regularized reduced function** $\hat{R}_h^E(z) = \liminf_{w\to z} R_h^E(w)$.

Definition 4.2. *A set E is* **minimally thin** *at $\tau \in \partial B$ if there is a z_0 in the unit ball such that $\hat{R}_{P_\tau}^E(z_0) < P_\tau(z_0)$.*

Let us now go back to our random archipelago and study the following set.

$$\mathfrak{M} = \{\tau \in \partial B \text{ such that the random archipelago } \mathcal{A} \text{ is minimally thin at } \tau\}.$$

Let us use the notation $|\cdot|$ for the surface area on the unit ball (in $\mathbb{R}^n$), and let ω_{n-1} be the full area, i.e. $\omega_{n-1} = |\partial B|$. We will use the following zero-one law for the above defined set $\mathfrak{M}$.

Lemma 4.3. *With probability one, we have that $\mathcal{A}$ is such that*

$$\frac{1}{\omega_{n-1}}|\mathfrak{M}| = \begin{cases} 0, & \nu \notin L^1 \\ 1, & \nu \in L^1. \end{cases}$$

Proof. We will use a Wiener series criterion for minimal thinness developed in [5] and in [1]. Let $\{Q_k\}$ be a Whitney decomposition of B, and let q_k be the Euclidean distance from the center of the Whitney cube Q_k to the boundary ∂B and let $\rho_k(\tau)$ be the distance from the center of Q_k to the boundary point τ. See Figure 2. By cap we denote the logarithmic capacity when $n = 2$, and the Newtonian capacity when $n \geq 3$; see for example [6].

We will now use the following series.

$$W(\tau) = W(\tau, \mathcal{A}) = \begin{cases} \displaystyle\sum_k \frac{q_k^2}{\rho_k(\tau)^2}\left(\log\frac{4q_k}{\operatorname{cap}(\mathcal{A}\cap Q_k)}\right)^{-1} & \text{if } n = 2, \\ \displaystyle\sum_k \frac{q_k^2}{\rho_k(\tau)^n}\operatorname{cap}(\mathcal{A}\cap Q_k) & \text{if } n \geq 3. \end{cases} \tag{5}$$

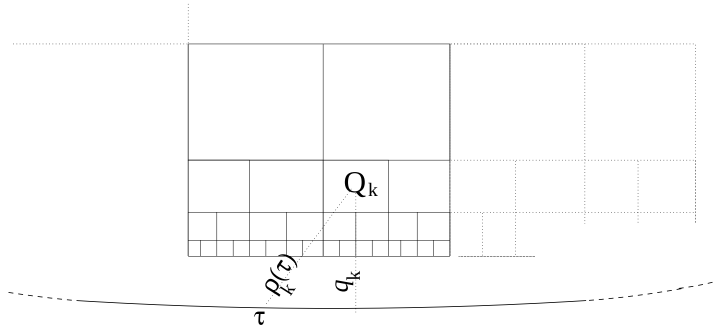

FIGURE 2. Part of a Whitney decomposition of the unit disk. The side length of the squares (or cubes in higher dimensions) are comparable to its distance to the boundary, ∂B. For more background and details see for example [10] on page 16.

A result of Essén in [5] (for $n = 2$) and Aikawa in [1] (for $n \geq 3$), gives us that $\mathcal{A}$ is minimally thin at a boundary point τ if and only if $W(\tau, \mathcal{A})$ converges.

Let $\widetilde{Q_k}$ be the extended cube obtained by adding all points with hyperbolic distance to Q_k at most ϱ, the radius of the islands. We can now estimate the probability that, for a given cube Q_k, $\mathcal{A} \cap Q_k$ is empty by the probability that there is no Poisson point in $\widetilde{Q_k}$, which is $\exp(-\int_{\widetilde{Q_k}} \beta(z)\,dz)$ see for example [9, p. 12]. This can be approximated, since β is continuous, by

$$\exp(-\beta(\text{center}(Q_k)\text{Volume}(\widetilde{Q_k})),$$

where Volume is the hyperbolic area (or volume in higher dimensions). We have also that $\text{Volume}(\widetilde{Q_k}) \approx \text{Volume}(Q_k)$ since the diameter of the cubes are approximately constant in the hyperbolic metric. (We use the standard notion that two positive functions u and v are comparable, i.e. $u \approx v$, if there is a constant $C \geq 1$ such that $C^{-1}u \leq v \leq Cu$ holds.) Thus, the probability that $\mathcal{A} \cap Q_k$ is *not* empty is comparable to

$$1 - \exp(-\beta(\text{center}(Q_k)\text{Volume}(Q_k)).$$

We will obtain an estimating series of W by changing Q_k to $\widetilde{Q_k}$ in Equation (5), that series, denoted by $\widetilde{W}$ will converge together with W.

Suppose now that Q_k and B_i intersect. Then they have to be comparable in size. Furthermore $B_i \cap \widetilde{Q_k} = B_i$. Hence

$$B_i \cap Q_k \neq \emptyset \text{ implies } \text{cap}(\widetilde{Q_k} \cap B_i) = \text{cap}(B_i) \approx \text{cap}(Q_k).$$

Now using calculated values for capacities. See for example [6, p. 165, 172], we can get simplified series.

$$\text{cap}(Q_k) \approx \begin{cases} \text{diam}(Q_k) \approx q_k & \text{if } n = 2, \\ \text{diam}(Q_k)^{n-2} \approx q_k^{n-2} & \text{if } n \geq 3. \end{cases}$$

$$W(\tau) \approx \widetilde{W}(\tau) \approx \sum_{k: B_i \cap Q_k} \frac{q_k^n}{\rho_k(\tau)^n} = \sum_k \frac{q_k^n}{\rho_k(\tau)^n} \chi_{\mathcal{A} \cap Q_k} \tag{6}$$

$$\text{E}[W(\tau)] \approx \sum_k \frac{q_k^n}{\rho_k(\tau)^n} \text{E}[\chi_{\mathcal{A} \cap Q_k}] = \sum_k \frac{q_k^n}{\rho_k(\tau)^n} (1 - e^{-\beta(\text{center}(Q_k))\text{Volume}(Q_k)}) \approx$$

$$\approx \sum_k \frac{q_k^n}{\rho_k(\tau)^n} (\beta(\text{center}(Q_k))\text{Volume}(Q_k)).$$

We now use that the hyperbolic area of the Whitney cubes are about constant and that $\beta(z) = \nu(d(z,0))$, giving us

$$\mathrm{E}[W(\tau)] \approx \sum_k \frac{q_k^n}{\rho_k(\tau)^n}\nu(d(\mathrm{center}(Q_k),0)).$$

Let $z = \mathrm{center}(Q_k)$. From the definition of the hyperbolic metric it follows that

$$d(z,0) = \log\left(\frac{1+|z|}{1-|z|}\right).$$

Hence we have that

$$d(\mathrm{center}(Q_k),0) \approx \log\frac{2}{q_k},$$

close to the boundary, which is what we need since minimally thinness is a local concept.

Thus,

$$\mathrm{E}[W(\tau)] \approx \sum_k \frac{q_k^n}{\rho_k(\tau)^n}\nu(\log\frac{2}{q_k}).$$

We have therefore that $\mathrm{E}[W(\tau)]$ converges if and only if the integral I in (7) below converges, where we transformed unit ball to the upper half space such that τ goes to 0, and used cylindrical coordinates. We also truncated for the "height" coordinate, here denoted by y, since we can again use the fact that it is sufficient only to consider the domain close to the boundary to decide if we have convergence or not.

$$I = \int_{y=0}^{2} \omega_{n-2}\int_{r=0}^{\infty}\frac{y^n}{(r^2+y^2)^{n/2}}\nu(\log\frac{2}{y})\frac{r^{n-2}dr\,dy}{y^n} = \tag{7}$$

$$= \omega_{n-2}\int_{y=0}^{2}\nu(\log\frac{2}{y})\int_{r=0}^{\infty}\frac{r^{n-2}}{(r^2+y^2)^{n/2}}\,dr\,dy.$$

Let us study the inner integral, using the substitution $s = r/y$,

$$\int_{r=0}^{\infty}\frac{r^{n-2}}{(r^2+y^2)^{n/2}}\,dr = \frac{1}{y}\int_0^{\infty}\frac{s^{n-2}}{(s^2+1)^{\frac{n}{2}}}\,ds = \frac{1}{y}C_n.$$

where $C_n \leq \frac{\pi}{2}$.

Thus going back to the original integral I we have that

$$I \approx \int_0^2 \frac{\nu(\log\frac{2}{y})}{y}\,dy = \int_0^{\infty}\nu(t)\,dt.$$

Thus we see that

$$\mathrm{E}[W(\tau)] \approx I = \begin{cases} = \infty & \text{if } \nu \notin L^1, \\ < \infty & \text{if } \nu \in L^1. \end{cases} \tag{8}$$

Suppose now that we pick a Whitney decomposition where the side lengths, measured in the hyperbolic metric, of the cubes is greater than 2ϱ, i.e. the diameter of the islands in $\mathcal{A}$. It is then possible, by partition each dimension with the help of two different "layers", to split such a W series into 2^n independent series, such that one island can not intersect two different cubes in the same sub-series. See Figure 3 for a schematic depiction in the planar case, where

$$W(\tau) \approx \sum_1 \frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k} + \sum_2 \frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k} + \sum_3 \frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k} + \sum_4 \frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k}.$$

Thanks to the independence in each sub–series, we can use Kolmogorov's three series theorem, see for example [3] on p. 118 (where we let X_k be the random variable $\frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k}$; and noting that, $0 \leq X_k \leq 1$, we can pick $A = 1$.)

Hence we have that

$$\sum_N \frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k} < \infty \text{ a.s.}$$

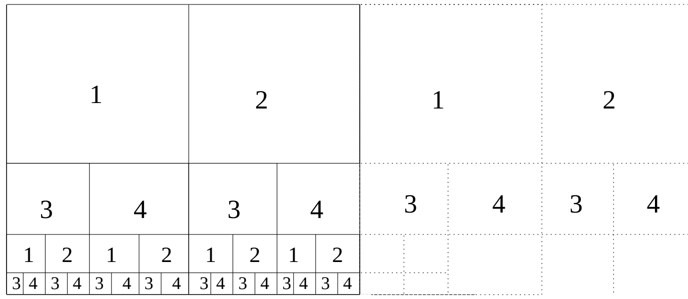

FIGURE 3. In each sub–sum, e.g. $\sum_N$, the event $\mathcal{A}\cap Q_k$ and $\mathcal{A}\cap Q_l$ are independent for different cubes Q_k and Q_l.

if and only if both

$$\sum_N \mathrm{E}[\frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k}] < \infty \text{ and } \sum_N \mathrm{V}[\frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k}] < \infty,$$

where V is the variance.

As we noted above, $0 \leq \frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k} \leq 1$, giving us that

$$\mathrm{V}[\frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k}] \leq \mathrm{E}[\frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k}].$$

Therefore $W(\tau)$ converges if and only if all sub–series

$$\sum_N \mathrm{E}[\frac{q_k^n}{\rho_k(\tau)^n}\chi_{\mathcal{A}\cap Q_k}] \text{ converges.}$$

Hence we see from (8) that $W(\tau) < \infty$ a.s. if and only if $\nu \in L^1$. That is, $\mathcal{A}$ is minimally thin with probability one at τ if and only if $\nu \in L^1$.

Remark 4.4. Note that since we assumed that ν was continuous, we have that the convergence only depends on the tail, which corresponds to the fact that minimal thinness is a local property at the boundary.

Since we picked an arbitrary point τ on the boundary ∂B, we know that if $\nu \in L^1$

$$\forall \tau \in \partial B,\ \mathrm{P}[\mathcal{A} \text{ is minmally thin at } \tau] = 1.$$

By Fubini's theorem,

$$\mathrm{E}[\int_{\partial B} \chi_{\mathcal{A} \text{ minimally thin at } \tau}\, d\tau] = \int_{\partial B} \mathrm{P}[\mathcal{A} \text{ is minimally thin at } \tau]\, d\tau =$$

$$= \int_{\partial B} 1\, d\tau = |\partial B| = \omega_{n-1}.$$

Therefore the random variable

$$\frac{1}{\omega_{n-1}} \int_{\partial B} \chi_{\mathcal{A} \text{ minimally thin at } \tau}\, d\tau$$

which is positive, and is at most 1, has 1 as expected value and hence is 1 a.s. Hence

$$\frac{1}{\omega_{n-1}}|\mathfrak{M}| = 1 \text{ a.s.}$$

Using the same argument for the case $\nu \notin L^1$ gives us the result of the lemma. $\square$

5. The proof of the Theorem

Proof of Theorem 3.1 . Let BM be an ordinary Euclidean Brownian motion started at the origin. We can find a very nice potential theoretic interpretation of π_e given in [4, p. 653], which can be stated in the following manner.

$$\hat{R}_1^{\mathcal{A}}(0) = \mathrm{P}[BM \text{ started in } 0 \text{ hits } \mathcal{A} \text{ before hitting the boundary } \partial B].$$

That is,

$$\pi_e = 1 - \hat{R}_1^{\mathcal{A}}(0). \tag{9}$$

Let us study the following function in the unit ball.

$$J(z) = \frac{1}{|\partial B|} \int_{\partial B} \hat{R}_{P_\tau}^{\mathcal{A}}(z)\, d\tau.$$

It is then not hard to check that:

- J is positive in B.
- $J \leq 1$ in general, since $\hat{R}_{P_\tau}^{\mathcal{A}}(z) \leq P_\tau(z)$.
- $J = 1$ quasi everywhere on $\mathcal{A}$.
- J is harmonic in $B \setminus \mathcal{A}$ and in $\mathcal{A}^o$, since $\hat{R}_{P_\tau}^{\mathcal{A}}$ is harmonic there.
- J is superharmonic in B, since it can be viewed as the minimum of two harmonic functions.

Definition 4.1 tells us then that J is the regularized reduced function on $\mathcal{A}$ with respect to 1, i.e.

$$J(z) = \hat{R}_1^{\mathcal{A}}(z). \tag{10}$$

We then have, thanks to equations (9) and (10), that

$$\pi_e > 0 \Leftrightarrow \hat{R}_1^{\mathcal{A}}(0) < 1 \Leftrightarrow \frac{1}{|\partial B|} \int_{\partial B} \hat{R}_{P_\tau}^{\mathcal{A}}(0)\, d\tau < 1 \Leftrightarrow |\{\tau \in \partial B, \hat{R}_{P_\tau}^{\mathcal{A}}(0) < 1\}| > 0.$$

Let us denote by $\mathfrak{M}_0$ the set $\{\tau \in \partial B, \hat{R}_{P_\tau}^{\mathcal{A}}(0) < 1\}$. Hence,

$$\pi_e > 0 \Leftrightarrow |\mathfrak{M}_0| > 0. \tag{11}$$

Trivially we have that $\mathfrak{M}_0 \subset \mathfrak{M}$ from where we deduce, using Lemma 4.3, that

$$\nu \notin L^1 \Rightarrow \mathrm{P}[|\mathfrak{M}| = 0] = 1 \Rightarrow \mathrm{P}[|\mathfrak{M}_0| > 0] = 0 \Rightarrow \mathrm{P}[\pi_e > 0] = 0. \tag{12}$$

We expand the definition of $\mathfrak{M}_0$ in the following way. Suppose for a moment that $\mathcal{A}$ is a given fixed archipelago. For any $\tau \in \mathfrak{M}$, define $\Sigma(\tau)$ to be the subset of $B \setminus \mathcal{A}$ such that

$$\hat{R}_{P_\tau}^{\mathcal{A}}(z) < P_\tau(z) \text{ if } z \in \Sigma(\tau).$$

Furthermore, let

$$\delta(\tau) = \inf_{z \in \Sigma(\tau)} d(0, z).$$

In view of Definition 4.2, such a $\delta(\tau)$ exists for all $\tau \in \mathfrak{M}$.

We define the following subset $\mathfrak{M}_\Delta$ of $\mathfrak{M}$. Let

$$\mathfrak{M}_\Delta = \{\tau \in \mathfrak{M} \text{ such that } \delta(\tau) \leq \Delta\}.$$

Note that $\mathfrak{M}_0$ agrees with our earlier definition of $\mathfrak{M}_0$ above, and that $\mathfrak{M}_\Delta \nearrow \mathfrak{M}$ as Δ goes to ∞. By the monotone convergence theorem, we then have that

$$\lim_{\Delta\to\infty} |\mathfrak{M}_\Delta| = \lim_{\Delta\to\infty} \int \chi_{\mathfrak{M}_\Delta}\, d\lambda = \int \lim_{\Delta\to\infty} \chi_{\mathfrak{M}_\Delta}\, d\lambda = |\lim_{\Delta\to\infty} \mathfrak{M}_\Delta| = |\mathfrak{M}|.$$

Hence if $|\mathfrak{M}| = \omega_{n-1}$ we can always pick a Δ such that $|\mathfrak{M}_\Delta| > 0$.

Let us now look at a general archipelago $\mathcal{A}$ and suppose that $\mathrm{P}[|\mathfrak{M}| = \omega_{n-1}] = 1$. Note that the subsets $\{|\mathfrak{M}_\Delta| > 0\}$ in the sample space Ω converges to $\{|\mathfrak{M}| > 0\}$ as Δ goes to ∞. Then again by monotonicity we have that

$$\lim_{\Delta\to\infty} \mathrm{P}[|\mathfrak{M}_\Delta| > 0] = \mathrm{P}[|\mathfrak{M}| > 0] = 1.$$

Hence there is a Δ such that

$$\mathrm{P}[|\mathfrak{M}_\Delta| > 0] > 0. \tag{13}$$

We will now use this to obtain the opposite implication of (12). Suppose that $\nu \in L^1$, then we have from Lemma 4.3 that $\mathrm{P}[|\mathfrak{M}| = \omega_{n-1}] = 1$, and there is a Δ such that $\mathrm{P}[|\mathfrak{M}_\Delta| > 0] > 0$. Let us by D_s denote the ball centered at the origin with hyperbolic radius s. We have that $\mathcal{A} \cap D_{2\Delta}$ is empty with probability

$$\exp(-\int_{D_{2\Delta}} \beta(z)\,dz) > 0 \text{ since } \nu \in L^1.$$

Thus with positive probability, and for any $\tau \in \mathfrak{M}_\Delta$, there is a z' such that z' and 0 are in the same open component of $B \setminus \mathcal{A}$, and $P_\tau(z') - \hat{R}^{\mathcal{A}}_{P_\tau}(z') > 0$. Thanks to the maximum principle, which we can use since $\hat{R}^{\mathcal{A}}_{P_\tau}$ is harmonic in each component of $B \setminus \mathcal{A}$ (see for example p. 39 in [4]), we have that $\hat{R}^{\mathcal{A}}_{P_\tau}(0) < P_\tau(0) = 1$. That is, $\tau \in \mathfrak{M}_0$.

Hence we have that $\mathrm{P}[|\mathfrak{M}_0| > 0] > 0$ which by (11) implies that $\mathrm{P}[\pi_e > 0] > 0$. Thanks to (12) we have that

$$\nu \in L^1 \Leftrightarrow \mathrm{P}[\pi_e > 0] > 0.$$

That is, we have percolation diffusion if and only if $\nu \in L^1$. □

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