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DISCRETE GROUPS AND THIN SETS

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Abstract. Let Γ be a discrete group of Möbius transformations acting on and preserving the unit ball in $\mathbf{R}^d$ (i.e. Fuchsian groups in the planar case). We will put a hyperbolic ball around each orbit point of the origin and refer to their union as the *archipelago of* Γ.

The main topic of this paper is the question: "How big is the archipelago of Γ?" We will study different ways to answer various meanings of that question using concepts from potential theory such as *minimal thinness* and *rarefiedness* in order to give connections between the theory of discrete groups and small sets in potential theory.

One of the answers that will be given says that the critical exponent of Γ equals the Hausdorff dimension of the set on the unit sphere where the archipelago of Γ is not minimally thin. Another answer tells us that the limit set of a geometrically finite Fuchsian group Γ is the set on the boundary where the archipelago of Γ is not rarefied.

1. Plan of the paper

We start with some basic definitions and results about discrete groups in Section 2. Section 3 deals with limit sets of discrete groups. We are especially interested in two subsets of the limit set, the classical non-tangential limit set and the "non-osculating limit set" which we define in an analogous way to the non-tangential.

In Section 4 we study the connection between the theory of discrete groups and potential theory in the sense of the question: "Is the archipelago of Γ thin at the boundary?" giving both local and global answers to that question.

In Section 5, we show that the set where the archipelago is not minimally thin is close to the non-tangential limit set. That is, the sets have the same Hausdorff dimension and they coincide when Γ is of geometrical finite type, see Theorem 5.4 and Corollary 6.1.

We conclude by studying the case when Γ is a Fuchsian group of geometrical finite type, i.e. finitely generated. We show in Theorem 6.2 that the set on the boundary where the archipelago is not rarefied coincides with the limit set of Γ. We will also show that this result in not valid in higher dimensions.

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2. Discrete groups

We will recall some basic facts about discrete groups and their limit sets. (A good first introduction to this area is given in Beardon's "An introduction to hyperbolic geometry" in [8, pp. 1–33].)

2.1. Basic properties of discrete groups. Denote by $\mathcal{M}$ the group of Möbius transformations in $\mathbf{R}^d$ that keep the unit ball B invariant. In other words, $\mathcal{M}$ is taken to be the group of all finite compositions of reflections (in spheres or planes) that preserve the orientation.

We recall that a Möbius mapping of the unit disk onto itself is a conformal mapping of the form

$$\gamma(z) = \frac{az+\bar{c}}{cz+\bar{a}}, \qquad \text{where } a,c \in \mathbf{C} \text{ and } |a|^2-|c|^2=1.$$

The elements in $\mathcal{M}$ are for our purpose too "dense". We thus need to select a sparser subgroup. This idea translates to a discreteness or discontinuity condition.

Definition 2.1. Let us view $\mathcal{M}$ as a topological group and Γ as a subgroup of $\mathcal{M}$. We say that Γ is *discrete* if each point is isolated. That is, if Γ is discrete and γ_n tends to the identity mapping I, then there is an N such that $\gamma_n = I$ for all $n \geq N$. Here $\{\gamma_n\}$ denotes the members of Γ.

Remark 2.2. One sometimes uses a different kind of subgroups of $\mathcal{M}$, the *discontinuously acting groups*. In our case discrete and discontinuously acting groups are equivalent. This follows from the fact that the elements in Γ preserve the unit ball and from Theorem 5.3.14(i) in [7].

Remark 2.3. We will distinguish between the planar case and the general higher dimensional case by calling discrete subgroups of $\mathcal{M}$ *Fuchsian* groups if $\mathrm{d}=2$.

The natural metric when dealing with the Möbius group is the hyperbolic metric, since the members of $\mathcal{M}$ act as isometry mappings with respect to this metric.

Definition 2.4. We define the *hyperbolic distance*, $d(\,\cdot\,,\cdot\,)$, between x and y in B by

$$d(x,y) = \inf_{\nu} \int_{\nu} \frac{2\,|dz|}{1-|z|^2}, \qquad \text{where } \nu \text{ is a smooth arc joining } x \text{ and } y.$$

We will also need a measure of the density of the orbit of Γ with respect to the origin.

Definition 2.5. Let $n(r)$ be the *orbital counting function*, i.e. the number of elements γ_n in Γ such that $d\big(0,\gamma_n(x_0)\big) < r$. The *critical exponent* is defined as

$$\delta = \limsup_{r\to\infty} \frac{1}{r}\log\big(n(r)\big).$$

(δ is independent of x_0, see for example [19, pp. 260]).

Let us also recall the following fundamental series.

Definition 2.6. The *Poincaré series* of Γ is defined as

$$h_s(x,y) = \sum_{\gamma_n\in\Gamma} e^{-sd(x,\gamma_n(y))}.$$

The convergence of this series depends on the parameter s, but is independent of x and y. More precisely,

$$h_s(x,y)\begin{cases}\text{converges} & \text{if } s>\delta,\\ \text{diverges} & \text{if } s<\delta.\end{cases} \tag{1}$$

(See for example [19, pp. 259–260] for the proof.) Let us therefore denote $h_s(0,0)$ simply by h_s.

The subgroup Γ of $\mathscr{M}$ is said to be of *convergence type* if the Poincaré series converges with $s = \mathrm{d}-1$, the dimension of the boundary. Note that many authors use the term convergence type when the Poincaré series converges at the critical exponent δ, rather than at $\mathrm{d}-1$ as here.

Remark 2.7. By using the above Definition 2.4 we see that

$$d(0,z) = \log\left(\frac{1+|z|}{1-|z|}\right).$$

Thus the series $\sum_{\gamma_n\in\Gamma}\big(1-|\gamma_n(0)|\big)^s$ converges if and only if the Poincaré series h_s converges.

2.2. The non-tangential limit set. Let us study the orbit of a discrete group Γ which is a set $\Gamma(x_0) = \{\gamma(x_0) : \gamma\in\Gamma\}$ of points in the unit ball; see Figure 7 for an example of an orbit. We will usually choose the reference point x_0 to be 0.

Since Γ is discrete, the points cannot cluster inside the unit ball, but since it is infinite (unless Γ is trivial) the orbit points must cluster at the unit sphere ∂B. We call this cluster set the (total) limit set and denote it by Λ.

There is a special subset of Λ called the non-tangential limit set Λ_c containing the limit points that are the cluster points on ∂B of orbit points "clustering in a non-tangential way". Λ_c is also called the conical limit set. See Remark 3.5 for a technical definition of Λ_c.

If a discrete group is of convergence type then the non-tangential limit set has Lebesgue measure zero. We denote by $|\cdot|$ the Lebesgue measure on the surface ∂B.

Theorem A. *If* Γ *is of convergence type then* $|\Lambda_c| = 0$.

The proof can be found in [14, Theorem 4, p. 29], or in [1, Lemma 3, p. 93]. In 1978 D. Sullivan presented the converse relation.

Theorem B. *If* $|\Lambda_c| = 0$ *then* Γ *is of convergence type.*

See for example [1, p. 97] for the proof.

We will now follow Garnett's presentation in [14] but we will study *horocycles* instead of cones and give an analogue to Theorem A in Proposition 3.10 below.

3. The non-osculating limit set

Let B be the d-dimensional unit sphere. The *horocycles* are defined as:

Definition 3.1. A *horocycle* is the truncated sphere in B which is tangent to ∂B at $z \in \partial B$ with radius $= M/(M+1)$, or in other words:

$$\widetilde{H}(z, M) = \{x \in B : |x| > \tfrac{1}{2};\ |x-z|^2 < M(1-|x|^2)\}.$$

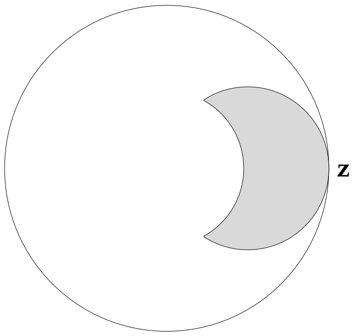

Figure 1. Horocycle, $\widetilde{H}(z, M)$, with $M = 1$ and $z = 1$.

Definition 3.2. A *horocap* is the part of ∂B reached by paths totally inside horocycles from a point x in B, where every path lies in a horocycle containing x. If $M > 0$, the horocap $C_h(x, M)$ is given by

$$C_h(x, M) = \{z \in \partial B : |x-z|^2 < M(1-|x|^2)\}.$$

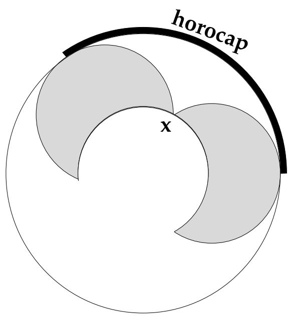

Figure 2. Horocap, $C_h(x, M)$, with $M = 1$.

Remark 3.3. The horocycle and the horocap are related by:

$$x \in \widetilde{H}(z,M) \quad \Longleftrightarrow \quad z \in C_h(x,M).$$

By taking the union of the horocaps as $|\gamma_n x_0|$ tends to 1 (which is equivalent to $n \to \infty$) we get a pre-version of the desired general limit set in the following definition.

Definition 3.4. The *non-osculating limit set of* x_0 *and* M is defined as

$$\Lambda_h(x_0\ ,M) = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} C_h\big(\gamma_n x_0, M\big), \qquad \gamma_n \in \Gamma.$$

Remark 3.5. The non-tangential limit set $\Lambda_c(x_0,M)$ used by Garnett is defined in the following way. Let

$$C_c(x,M) := \{z \in \partial B : |x-z| < M(1-|x|)\}.$$

Then

$$\Lambda_c(x_0,M) := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} C_c\big(\gamma_n(x_0), M\big).$$

For details, we refer to [14, p. 26] where the notation Λ is used for the non-tangential limit set, which we have reserved here to denote the whole limit set.

We can now give the following

Definition 3.6. The *non-osculating limit set* is defined as

$$\Lambda_h = \bigcup_{M>0} \Lambda_h(x_0,M).$$

Remark 3.7. It is not hard to see that Λ_h is independent of x_0.

Remark 3.8. Nicholls defines, on p. 37 in [18], the *horospherical limit set*, H. It can be shown that H is a subset of the non-osculating limit set and that the set difference, $\Lambda_h \setminus H$, is the set of *Garnett points.* Theorem 2.6.6 in [18] tells us that the Lebesgue measure of the set of Garnett points is always zero.

Remark 3.9. The author was kindly informed by L. Ward that in [17], K. Matsuzaki defines *the weak horocyclic limit set* which is equivalent to Λ_h.

We aim at the following analogue to Theorem A.

Proposition 3.10. *If* $h_{(\mathrm{d}-1)/2} < \infty$ *then* $|\Lambda_h| = 0$.

We can give a proof by applying Garnett's technique in [14] to our horocaps. However, we shall take a short-cut and use a powerful result in [18] instead. To be able to do that we need some more definitions and a lemma.

The following two notations are from [18], the first on p. 5 and the second on p. 23.

Definition 3.11. For $a \in B$ and $k, \alpha > 0$,

$$I(a : k, \alpha) = \left\{ x \in \partial B : \left| x - \frac{a}{|a|} \right| < k(1 - |a|)^\alpha \right\}.$$

Definition 3.12.

$$L(0 : k, \alpha) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} I\big(\gamma_n(0) : k, \alpha\big).$$

From the definitions of the non-osculating limit set of 0 and M in Definition 3.4 and the non-tangential limit set of 0 and M in Remark 3.5, we obtain the following results.

Lemma 3.13. *For the non-tangential limit set*

$$L(0 : k, 1) = \Lambda_c\Big(0, \sqrt{1 + k^2}\,\Big),$$

and for the non-osculating limit set

$$L\big(0 : k, \tfrac{1}{2}\big) = \Lambda_h\big(0, \tfrac{1}{2}k^2\big).$$

Proof. Using a new parameter l, we write

$$\begin{aligned} I(a : k, \alpha) &= \left\{ x \in \partial B : \left| x - \frac{a}{|a|} \right| < k(1 - |a|)^\alpha \right\} \\ &= \{ x \in \partial B : |x - a| < l(1 - |a|)^\alpha \}. \end{aligned}$$

and denote $1 - |a|$ by t. Definition 3.12 tells us that to obtain the limit sets the "lim sup process", i.e. $\bigcap_{m=1}^{\infty} \bigcup_{n>m}^{\infty}$, is required. In other words, we are only interested in the limit case as $t \to 0$. We will then obtain the following asymptotic relation $t^2 + k^2 t^{2\alpha} = l^2 t^{2\alpha}$, i.e.

$$l = \sqrt{t^{2(1-\alpha)} + k^2}\,. \tag{2}$$

Thus we immediately obtain the first result for $\alpha = 1$ by the definition of the non-tangential limit set of 0 and M.

In order to consider the latter statement in the lemma, let us now put $\alpha = \frac{1}{2}$.

$$\big\{ x \in \partial B : |x - a|^2 < \tfrac{1}{2} k^2 (1 - |a|^2) \big\} = \left\{ x \in \partial B : |x - a| < k \sqrt{\tfrac{1}{2}(1 + |a|)}\, \sqrt{1 - |a|}\, \right\}.$$

Since $\alpha < 1$ we have by (2) that $l \to k$ as $t \to 0$. Thus

$$\left\{ x \in \partial B : |x - a| < k \sqrt{\tfrac{1}{2}(1 + |a|)}\, \sqrt{1 - |a|}\, \right\} \to \left\{ x \in \partial B : \left| \frac{x - a}{|a|} \right| < k(1 - |a|)^{1/2} \right\}$$

as $|a| \to 1$, which is what happens when we construct the limit set. We obtain the last statement in the lemma. □

To get an opening angle independent non-osculating limit set, the union was taken over all opening angles in Definition 3.6. The same thing can be done in our generalized situation.

Definition 3.14. Let us denote the α-limit set by

$$\mathscr{L}(\alpha) = \bigcup_{k>0} L(0:k,\alpha).$$

We cite Theorem 2.1.1 in [18]:

Theorem C. *If* Γ *is a discrete group acting in* B *such that the series* $\sum_{\gamma\in\Gamma}(1-|\gamma(0)|)^{(\mathrm{d}-1)\alpha}$ *converges, then* $|\mathscr{L}(\alpha)| = 0$.

From Lemma 3.13 we see that the non-osculating limit set (Definition 3.6) is in fact $\mathscr{L}(\frac{1}{2})$. We can therefore obtain the result in Proposition 3.10 by applying Theorem C for $\alpha = \frac{1}{2}$.

4. Thinness of the archipelago

We will "fatten" the orbit points to get a non discrete set called the archipelago of Γ, whose "thinness" at the boundary will then be studied.

We will use four different kinds of thinness at the boundary, two local and two global ones.

4.1. Two kinds of thin sets at a boundary point. Let us denote the class of non-negative superharmonic functions in the unit ball by $SH(B)$, and let P_τ denote the Poisson kernel $(1-|z|^2)/(|z-\tau|^{\mathrm{d}})$ at $\tau \in \partial B$.

The Poisson kernel is a harmonic function and it is minimal in the sense that if h is a positive harmonic function such that $h(z) \le P_s(z)$ for all $z \in B$, then $h(z) \equiv 0$ or $h(z) = cP_s(z)$ for a constant c.

Let us now make a variant of this. Let $u \in SH(B)$ such that $u(z) \ge P_s(z)$ holds on a subset E of the unit ball. How strong is this condition? Can there be such a function u and a point z in $B \setminus E$ such that $u(z) < P_s(z)$? The answer depends on how "big" E is close to the basepoint s. The concept of *minimal thinness* was introduced for the study of similar questions in [16]. Let us now turn to the definition.

The *reduced function* of h with respect to a subset E of B is defined as

$$R_h^E(w) = \inf\{u(w) : u \in SH(B) \text{ and } u \ge h \text{ on } E\}.$$

We can make this function lower semicontinuous by regularizing it, obtaining the *regularized reduced function* $\widehat{R}_h^E(z) = \liminf_{w\to z} R_h^E(w)$.

Definition 4.1. A set E is *minimally thin* at $\tau \in \partial B$ if there is a z in the unit ball such that $\widehat{R}_{P_\tau}^E(z) < P_\tau(z)$.

In [13] and [4] a metric condition for a set to be minimally thin is stated. Let $\{Q_k\}$ be a Whitney decomposition of B. We set q_k to be the (Euclidean) distance from the Whitney cube Q_k to the boundary ∂B and $\varrho_k(\tau)$ to be the distance from Q_k to the boundary point τ. By cap we denote the logarithmic capacity when $\mathrm{d}=2$, and the Newtonian capacity when $\mathrm{d}\ge 3$ (see for example [15]).

Definition 4.2. We put

$$W(\xi)=W(\xi,E)=\begin{cases}\displaystyle\sum_k \frac{q_k^2}{\varrho_k(\xi)^2}\left(\log\frac{4q_k}{\operatorname{cap}(E\cap Q_k)}\right)^{-1} & \text{if } \mathrm{d}=2,\\ \displaystyle\sum_k \frac{q_k^2}{\varrho_k(\xi)^{\mathrm{d}}}\operatorname{cap}(E\cap Q_k) & \text{if } \mathrm{d}\ge 3.\end{cases}$$

Theorem D (Essén [13]). *E is minimally thin at a boundary point ξ if and only if $W(\xi,E)$ converges.*

The other local thinness measure we will use is *rarefiedness*. This is a concept tailored to fit in the upper half space, but it can, by a Möbius transformation, be expressed for subsets in the unit ball. For the definition, we will need to recall that the Riesz mass of a positive superharmonic function u is a measure μ such that, by the Riesz representation theorem, $u(x)=G\mu+h$, where $G\mu$ is the Green potential of the measure μ and h is a harmonic function.

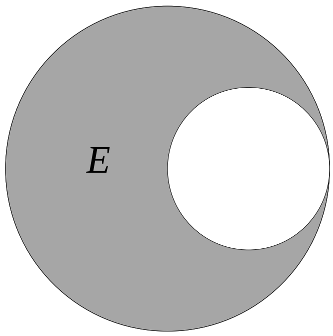

Figure 3. E is the complement, in the unit disc, to a horo-disc tangent to the unit circle at 1. E is minimally thin at the boundary point 1 as can be concluded by the following argument. First note that the level set of the Poisson kernel consists of horocycles. Let u be the truncated Poisson kernel that coincides with P_1 on E and is constant off E. It is easy to see that $u\in SH(B)$ and that $u\le P_1$ everywhere and $u<P_1$ off E. Therefore $\widehat{R}^E_{P_1}(z)\le u(z)<P_1(z)$ for $z\notin E$.

Definition 4.3 (cf. Definition 12.4 at p. 74 in [5]). A subset E of the unit ball is *rarefied* at $\tau\in\partial B$ if there exists a positive function u in the upper half space $\mathbf{H}=\{x=(x_1,\dots,x_d):0<x_d\}$ with no Riesz mass at infinity such that

$$u(x)\ge |x|,\qquad x\in E',$$

where E' is the image of E under the Möbius transformation that maps B to $\mathbf{H}$ and τ to ∞.

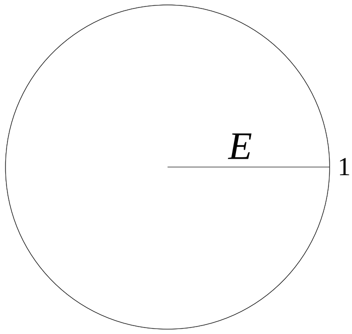

Figure 4. The line segment E is not minimally thin at 1. This can be seen from Theorem D, since a line segment of length l has logarithmic capacity $\frac{1}{4}l$; see p. 172 in [15].

There is also a Wiener type criterion for rarefiedness. The following theorem is cited from [4, Theorem 3.2]; see also [12].

Theorem E (Aikawa [4]). *Let* $\mathbf{H}$ *be the upper half space and let* $X \in \partial\mathbf{H}$. *Suppose that* E *is a bounded subset of* $\mathbf{H}$. *Then* E *is rarefied at* X *if and only if* E *has a decomposition* $E' \cup E''$ *such that*

$$(3)\qquad \begin{cases} \displaystyle\sum_k \frac{q_k}{\varrho_k(X)}\left(\log\frac{4q_k}{\operatorname{cap}(E'\cap Q_k)}\right)^{-1} < \infty & \textit{for } \mathrm{d}=2, \\ \displaystyle\sum_k \frac{q_k}{\varrho_k(X)^{\mathrm{d}-1}}\operatorname{cap}(E'\cap Q_k) < \infty & \textit{for } \mathrm{d}\geq 3, \end{cases}$$

where E'' *has a covering* $\bigcup_i B(X_i, r_i)$ *with* $X_i \in \partial\mathbf{H}$, $0 < 2r_i < |X - X_i|$ *and*

$$(4)\qquad \sum_i \left(\frac{r_i}{|X_i - X|}\right)^{\mathrm{d}-1} < \infty.$$

Remark 4.4. Note that rarefiedness implies minimal thinness; see for example [2, p. 425]. The opposite is not true. The set E in Figure 3 is minimally thin at 1 but not rarefied there.

4.2. Two kinds of globally thin sets at the boundary. In [4], Aikawa introduces two global characterizations of exceptional sets:

Definition 4.5. A set E in B is *thin with respect to capacity* if

$$(5)\qquad \begin{cases} \displaystyle\sum_k q_k\left(\log\frac{4q_k}{\operatorname{cap}(E\cap Q_k)}\right)^{-1} < \infty & \text{for } \mathrm{d}=2, \\ \displaystyle\sum_k q_k\operatorname{cap}(E\cap Q_k) < \infty & \text{for } \mathrm{d}\geq 3. \end{cases}$$

The other global measure is more easily stated in the upper half space.

Definition 4.6. A set E in the upper-half space $\mathbf{H}$ is *thin with respect to measure* if

$$H(E \cap D_\varepsilon) \to 0 \qquad \text{as } \varepsilon \to 0,$$

where the Hausdorff-type outer measure H is defined by

$$H(A) := \inf\Big\{ \sum_i r_i^{\mathrm{d}-1} : A \subset \bigcup_i \overline{B}(X_i, r_i),\ X_i \in \partial\mathbf{H} \Big\}.$$

and where D_ε is the hyper-strip $\{x \in \mathbf{H} : 0 < x_d < \varepsilon,\ x = (x_1, \dots, x_d)\}$.

Remark 4.7. Theorem 4.2 in [4] tells us that E is minimally thin almost everywhere at the boundary if and only if E is a union of sets thin with respect to capacity and thin with respect to measure.

4.3. The archipelago of Γ. In order to investigate connections between discrete groups and thin sets at the boundary we have to "put on some flesh" on the set of orbit points to make the point set visible by our potential-theoretic eyes. We do that by the following construction.

Let Γ be a discrete group; see Remark 2.3. Since Γ is discrete it is possible to find an $r_\Gamma > 0$ such that the balls B_j do not intersect each other, where $B_j := \{z \in B : d(z, \gamma_i(0)) < r_\Gamma,\ \gamma_i \in \Gamma \setminus \{I\}\}$. Let us fix such an r_Γ. Let $E := \bigcup_j B_j$. That is, E is the "fattened" orbit of Γ. We call it *the archipelago of* Γ.

By construction, the archipelago E covers the orbit of the origin by Euclidean balls with radii comparable to the distance from the boundary ∂B; see (8) below. Using the similarity between these balls (or disks) and the Whitney cubes (or squares) in some Whitney decomposition, we can obtain relations between discrete groups and thin sets, which was our main goal.

4.4. Global properties.

Proposition 4.8. *If E is the archipelago of a discrete group Γ, the following statements are equivalent:*

- *Γ is of convergence type.*
- *E is thin with respect to capacity.*
- *E is thin with respect to measure.*

Proof. Let us denote $t_i = 1 - |\gamma_i(0)|$. We see from Remark 2.7 that Γ is of convergence type if and only if $\sum_i t_i^{\mathrm{d}-1} < \infty$. On the other hand, by Definition 4.5, E is thin with respect to capacity if and only if (5) holds.

By Lemma 4.11 below we have the following comparison.

$$(5) \iff \sum_{E \cap Q_k \neq \emptyset} q_k^{\mathrm{d}-1} < \infty \iff \sum_i t_i^{\mathrm{d}-1} < \infty,$$

since

$$(c_4' b_3)^{\mathrm{d}-1} \sum_{i:\gamma_i \in \Gamma} t_i^{\mathrm{d}-1} \leq \sum_{E \cap Q_k \neq \emptyset} q_k^{\mathrm{d}-1} \leq (c_4 b_4)^{\mathrm{d}-1} \sum_{i:\gamma_i \in \Gamma} t_i^{\mathrm{d}-1},$$

with the constants taken from the proof of Lemma 4.11. We have obtained the first equivalence:

$$\Gamma \text{ is of convergence type} \quad \Longleftrightarrow \quad E \text{ is thin with respect to capacity.}$$

By Definition 4.6, E is thin with respect to measure if $H(E \cap D_\varepsilon) \to 0$ when $\varepsilon \to 0$. Suppose now that Γ is of convergence type and consider the upper half plane case

$$H(E \cap D_\varepsilon) \leq \sum_{i > I(\varepsilon)} (2t_i)^{\mathrm{d}-1},$$

where $I(\varepsilon)$ tends to ∞ as ε tends to 0. The sum on the right tends to zero as t tends to zero.

Thus,

$$\Gamma \text{ is of convergence type} \quad \Longrightarrow \quad E \text{ is thin with respect to measure.}$$

To prove the converse implication, let us now assume that E is thin with respect to measure. The *essential projection* E^* of E is defined as

$$E^* = \bigl\{ X \in \mathbf{R}^{\mathrm{d}-1} : \forall \varepsilon > 0 \ \exists\, y \text{ such that } 0 < y < \varepsilon \text{ and } (X, y) \in E \bigr\}.$$

We now choose an M, $1 < M < M_{r_\Gamma}$, where M_{r_Γ} is a constant depending only on our hyperbolic radius constant r_Γ, which forces us to choose an M close enough to 1. Let us then construct a non-tangential limit set $\Lambda_c(0, M)$ with respect to Γ and the parameter M (cf. Remark 3.5). The construction was carried out in the unit ball in [14], but it can immediately be applied to the upper half plane as well. For every B_i in $E \cap D_\varepsilon$, the cap $C\bigl(\gamma_i(0), M\bigr) = \bigl\{ X \in \partial B : |X - \gamma_i(0)| < M(1 - |\gamma_i(0)|) \bigr\}$ lies in the projection of the disk B_i by the choice of M. (See Figure 5.) We have that

$$\Lambda_c(0, M) \subset E^*. \tag{6}$$

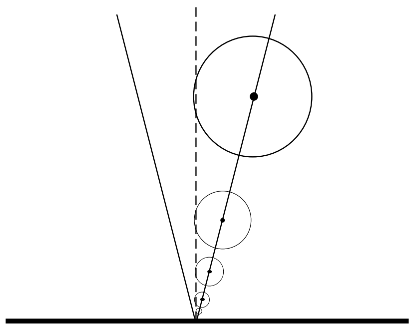

Figure 5. If the opening-angle is small enough (i.e. M small enough), the vertex of the cone will be contained in the essential projection of E.

The non-tangential limit set is defined as

$$\Lambda_c = \bigcup_{M>1} \Lambda_c(0,M).$$

It is only "slightly dependent" on the parameter M. In fact,

$$|\Lambda_c| = |\Lambda_c(0,M)| \qquad \text{for all } M > 1;$$

see [14, p. 29]. We then have from equation (6)

$$|E^*| \geq |\Lambda_c(0,M)| = |\Lambda_c|.$$

Since we assumed that E was thin with respect to measure, we can use Lemma 6.35 in [4] and deduce that $|E^*| = 0$. Hence $|\Lambda_c| = 0$, and by applying Theorem B, we conclude that Γ is of convergence type. That is

$$E \text{ is thin with respect to measure} \quad \Longrightarrow \quad G \text{ is of convergence type.}$$

We have proved the proposition. □

Let us use a variant of the Wiener type series in Definition 4.2.

Definition 4.9. Suppose that $\{Q_k\}$ is a Whitney decomposition of B. We define the following series.

$$W_0(\tau) := \sum{}' \big(q_k/\varrho_k(\tau)\big)^{\mathrm{d}},$$

where $q_k = \operatorname{dist}(\partial B, Q_k)$ and $\varrho_k(\tau) = \operatorname{dist}(Q_k, \tau)$. The notation $\sum'$ means that the summation is over all indices k such that $Q_k \cap E \neq \emptyset$ (cf. [13, p. 88]).

Remark 4.10. We say that two positive functions u and v are comparable, i.e. $u \approx v$ if there is a constant $C \geq 1$ such that $C^{-1}u \leq v \leq Cu$ holds.

Lemma 4.11. *Let E, $t_i = 1 - |\gamma_i(0)|$, $R_i(\tau) = |\gamma_i(0) - \tau|$, q_k and $\varrho_k(\tau)$ be as above. Then*
(1) *$t_i \approx q_k$ if $Q_k \cap B_i \neq \emptyset$,*
(2) *$R_i(\tau) \approx \varrho_k(\tau)$ if $Q_k \cap B_i \neq \emptyset$,*
(3) *$W(\tau) \approx W_0(\tau)$,*
(4) *(5) $\iff \sum_{E\cap Q_k \neq \emptyset} q_k^{\mathrm{d}-1} < \infty$.*

Proof. Suppose that $\{Q_k\}$ is a Whitney decomposition such that

$$c_1 q_k < \operatorname{diameter}(Q_k) < c_2 q_k \qquad \text{where } c_1 > 0,\ c_2 < 1. \tag{7}$$

The balls B_i are controlled by the choice of the hyperbolic radius r_Γ in a similar way,

$$b_1 t_i < \operatorname{radius}(B_i) < b_2 t_i \qquad \text{where } b_1 > 0 \text{ and } b_2 < 1, \tag{8}$$

where radius$(\,\cdot\,)$ stands for the Euclidean radius. We can now get the first two estimates in the following way. Let $\{Q_{i_k}\}$ be the Whitney cubes (or squares) that intersect the ball (or disk) B_i. Then

$$q_{i_k} \geq t_i - \mathrm{radius}\,(B_i) - \mathrm{diam}\,(Q_{i_k}) \geq t_i - b_2 t_i - c_2 q_{i_k}.$$

Thus, by putting $b_3 := (1-b_2)/(1+c_2)$, we have the estimate $q_{i_k} \geq b_3 t_i$. In a similar way we have by setting $b_4 := 1 + 2b_2$ that $q_{i_k} \leq b_4 t_i$, where the factor 2 is to compensate for the fact that the hyperbolic center of the ball, $\gamma_i(0)$, is closer to the boundary than the Euclidean center of B_i. Thus $t_i \approx q_{i_k}$. The argument holds without any change when we compare ϱ_{i_k} and R_i. Hence

$$b_3 R_i(\tau) \leq \varrho_{i_k}(\tau) \leq b_4 R_i(\tau).$$

The first two statements in the lemma are proved.

We also have a size relation between intersecting balls and cubes. If $Q_k \cap B_i \neq \emptyset$ holds, then

$$\frac{\mathrm{diam}\,(Q_k)}{\mathrm{diam}\,(B_i)} \geq \frac{c_1 q_k}{2b_2 t_i} \geq \frac{c_1 b_3 t_i}{2b_2 t_i}.$$

So, by letting $c_3 := c_1 b_3 / 2b_2$ we have that

$$\min_k \bigl(\mathrm{diam}\,(Q_k) : Q_k \cap B_i \neq \emptyset\bigr) \geq c_3 \mathrm{diam}\,(B_i). \tag{9}$$

The number of Whitney cubes that intersect a hyperbolic ball is then estimated above by $c_4 := \bigl((1/c_3)+1\bigr)^{\mathrm{d}}$. Analogously, we can get an upper estimate of the number of balls B_{k_i} intersecting a Whitney cube Q_k by $c_4' := \bigl((1/c_3')+1\bigr)^{\mathrm{d}}$, where $c_3' := 2b_1/c_2 b_4$.

For simplicity, let us now treat the planar situation separately. That is, let for a while $d = 2$. The logarithmic capacity of a square of side-length a is bounded above by $0.6a$, see [15, p. 172]. Therefore we have

$$\mathrm{cap}(E_k) = \mathrm{cap}(E \cap Q_k) \leq \mathrm{cap}(Q_k) \leq 0.6 \frac{1}{\sqrt{2}} \mathrm{diam}\,(Q_k) \leq \frac{0.6}{\sqrt{2}} c_2 q_k < q_k,$$

where we use the notation $E_k := E \cap Q_k$. Hence

$$\left(\log\left(\frac{4q_k}{\mathrm{cap}(E_k)}\right)\right)^{-1} \leq \bigl(\log(4)\bigr)^{-1} \leq 1. \tag{10}$$

It follows that

$$W(\tau) \leq W_0(\tau). \tag{11}$$

We will now obtain an opposite inequality.

$$W(\tau) = \sum_{E\cap Q_k\neq\emptyset} \big(q_k/\varrho_k(\tau)\big)^2 \left(\log\left(\frac{4q_k}{\operatorname{cap}(E_k)}\right)\right)^{-1}$$
$$\geq \sum_{i:\gamma_i\in\Gamma} (q_{i_k}/\varrho_{i_k}(\tau))^2 \left(\log\left(\frac{4q_{i_k}}{\operatorname{cap}(E_{i_k})}\right)\right)^{-1},$$

where the E_{i_k} is chosen such that $\operatorname{cap}(E_{i_k}) = \max_k \operatorname{cap}(E_k\cap B_i)$. Let us estimate the logarithmic capacity of E_{i_k}. If we shrink the radius of the disk B_i by the factor c_3 we have from (9) that a copy of E_{i_k} could cover the shrunken disk. Since the logarithmic capacity of a ball of radius a is a, see [15, p. 172], we obtain the following estimate.

$$\operatorname{cap}(E_{i_k}) \geq c_3 \operatorname{cap}(B_i) \geq c_3 b_1 t_i \geq c_3 b_1 \frac{1}{b_4} q_{i_k}.$$

Hence

$$\frac{4q_i}{\operatorname{cap}(E_{i_k})} \leq \frac{4b_4}{c_3 b_1}.$$

Let

$$c_5 := \left(\log\left(\frac{4b_4}{c_3 b_1}\right)\right)^{-1}.$$

Then

$$W(\tau) \geq \sum_{i:\gamma_i\in\Gamma} (q_{i_k}/\varrho_{i_k}(\tau))^2 c_5 \geq \frac{1}{c_4} c_5 \sum_{E\cap Q_k\neq\emptyset} \big(q_k/\varrho_k(\tau)\big)^2.$$

We conclude that

$$W(\tau) \leq W_0(\tau) \leq c_W W(\tau).$$

We are done with the two-dimensional case.

For the higher dimensions, we argue completely analogously noting that from p. 165 in [15] $\operatorname{cap}(Q_i) \approx q_i^{\mathrm{d}-2}$ and $\operatorname{cap}(B_k) \approx t_k^{\mathrm{d}-2}$. Hence from Definition 4.2

$$W(\tau) = \sum_k \frac{q_k^2}{\varrho_k(\tau)^{\mathrm{d}}} \operatorname{cap}(E\cap Q_k) \approx \sum_k \frac{q_k^{\mathrm{d}}}{\varrho_k(\tau)^{\mathrm{d}}} \approx W_0(\tau).$$

We see that the last statement, (5) $\Leftrightarrow \sum_{E\cap Q_k\neq\emptyset} q_k^{\mathrm{d}-1} < \infty$, can be proved exactly the same way as we proved that $W(\tau)\approx W_0(\tau)$ above.

The lemma is proved. □

We are now ready to state a result concerning the relation between the non-osculating limit set and rarefiedness. Let $\complement\Lambda_h$ denote the set $\partial B\setminus\Lambda_h$.

Proposition 4.12. *If $h_{(\mathrm{d}-1)/2} < \infty$ and $\tau \in \complement\Lambda_h$ then the archipelago is rarefied at τ.*

Proof. Let us denote as above $t_i := 1-|\gamma_i(0)|$ and $R_i = R_i(\tau) := |1-\overline{\gamma_i(0)}\tau| = |\gamma_i(0) - \tau|$. Let us also recall the notion of the non-osculating limit set Λ_h, given in Definitions 3.4 and 3.6.

Let $\tau \in \complement\Lambda_h$. Then $\tau \notin \bigcup_{M>0} \Lambda_h(0, M)$, or in other words, $\tau \in \complement\Lambda_h(0, M)$ for all $M > 0$. With $M > 0$ fixed,

$$\tau \in \complement\Lambda_h(0, M) = \complement \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} C_h\big(\gamma_i(0), M\big) = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} \complement C_h\big(\gamma_i(0), M\big).$$

This is a "lim inf"-construction telling us that there exists a natural number $I = I(M)$ such that if $i > I$ then $\tau \notin C_h\big(\gamma_i(0), M\big)$.

Theorem E gives a necessary and sufficient condition for a set to be rarefied at τ. Let us define an auxiliary series in the same spirit as W_0 in Definition 4.9 above,

$$W_0^r(\tau) = {\textstyle\sum_k'} \big(q_k/\varrho_k(\tau)\big)^{\mathrm{d}-1}. \tag{12}$$

(Recall that the prime sign indicates that we only sum over those k for which $Q_k \cap E \neq \emptyset$.) We can show that $W_0^r(\tau) \approx W^r(\tau)$ completely analogously to the proof indicating that $W_0(\tau) \approx W(\tau)$ in Lemma 4.11. We have also from Lemma 4.11 that $t_i \approx q_k$ and $R_i(\tau) \approx \varrho_k(\tau)$ if $Q_k \cap E \neq \emptyset$, and that the number of intersections are controlled (by c_4, c_4'). Thus we have that

$${\textstyle\sum_k'} \big(q_k/\varrho_k(\tau)\big)^{\mathrm{d}-1} \approx \sum_i \big(t_i/R_i(\tau)\big)^{\mathrm{d}-1}.$$

Let us divide the series into two parts.

$$\sum_i \big(t_i/R_i(\tau)\big)^{\mathrm{d}-1} = \sum_{i \le I} (t_i/R_i)^{\mathrm{d}-1} + \sum_{i > I} (t_i/R_i)^{\mathrm{d}-1}.$$

Denote the finite summation $c_0 = \sum_{i \le I} (t_i/R_i)^{\mathrm{d}-1}$. We note that $c_0 \le I$. For the other series we have $i > I$ which implies $\tau \notin C_h\big(\gamma_i(0), M\big)$, which in turn implies

$$|\gamma_i(0) - \tau| \ge \sqrt{M(1 - |\gamma_i(0)|^2)} \qquad \text{and} \qquad R_i \ge \sqrt{Mt_i}\,,$$

by the construction of the horocap; see Definition 3.2. Therefore,

$$\sum_i \big(t_i/R_i(\tau)\big)^{\mathrm{d}-1} \le c_0 + \sum_{i>I} \left(\frac{t_i}{\sqrt{Mt_i}}\right)^{\mathrm{d}-1} = c_0 + \frac{1}{M^{(\mathrm{d}-1)/2}} \sum_{i>I} t_i^{(\mathrm{d}-1)/2}.$$

Since $h_{(\mathrm{d}-1)/2} < \infty$, the series $\sum_{i>I} t_i^{(\mathrm{d}-1)/2}$ converges. Hence we have that $W_0^r(\tau) < \infty$, and it follows that $W^r(\tau) < \infty$ by the proof of Lemma 4.11. Hence E is rarefied at τ. □

If $h_{(\mathrm{d}-1)/2} < \infty$ we can use Proposition 3.10 which tells us that the non-osculating limit set Λ_h has measure zero. Hence we have the following corollary.

Corollary 4.13. *If $h_{(\mathrm{d}-1)/2} < \infty$ then the archipelago is rarefied a.e.*

4.5. A local property. Except Proposition 4.12, all the above propositions are global. Let us now turn to questions of local behavior. What can we say about a given point on the boundary? To answer this question, we will again consider the non-tangential limit set Λ_c of the discrete group Γ.

Proposition 4.14. *If $\tau \in \Lambda_c$ then the archipelago is* not *minimally thin at τ.*

Remark 4.15. This holds independently of the value of δ.

Proof. Since τ is in $\Lambda_c = \bigcup_{M>0} \Lambda_c(0,M)$, there exists $M > 0$ such that $\tau \in \Lambda_c(0,M) = \bigcap_{j=1}^{\infty} \bigcup_{i>j}^{\infty} C\bigl(\gamma_i(0), M\bigr)$. This is a "lim sup"-construction and we conclude that $\tau \in C\bigl(\gamma_i(0), M\bigr)$ for infinitely many i, say, for all i in the index set $I(M)$.

We will now estimate the series

$$W_0(\tau) \approx \sum_i \left(\frac{t_i}{\varrho_i(\tau)}\right)^{\mathrm{d}} \geq \sum_{i \in I(M)} \left(\frac{1 - |\gamma_i(0)|}{|\gamma_i(0) - \tau|}\right)^{\mathrm{d}}.$$

Since $\tau \in C\bigl(\gamma_i(0), M\bigr)$, we have

$$|\gamma_i(0) - \tau| < M\bigl(1 - |\gamma_i(0)|\bigr).$$

Hence,

$$W_0(\tau) \gtrapprox \sum_{i \in I(M)} \frac{1}{M^{\mathrm{d}}} = \infty.$$

This implies that $W(\tau) = \infty$ by Lemma 4.11, and we conclude that E is not minimally thin at τ. We also note that we only use the fact that the cardinality of the index set $I(M)$ is infinite. We do not use any convergence properties. That is, the result is independent of δ. □

We will give another local result in Theorem 5.3 below supplying a sufficient condition for E to be minimally thin at τ.

4.6. A global result. The following result concerning the global size of the archipelago E follows now easily from the Propositions 4.8 and 4.14 above.

Theorem 4.16. *Let E be the archipelago of Γ. Γ is of convergence type if and only if E is minimally thin a.e. on the boundary.*

Proof. Proposition 4.8 gives the necessary part, since we know by [4, Theorem 1.2] that either *thin with respect to capacity* or *thin with respect to measure* gives minimal thinness a.e.

The sufficient part is obtained by the following reasoning. If Γ is not of convergence type we know that the conical limit set has full Lebesgue measure (see for example [14, Theorem 5, p. 29]). That is, almost every point on the boundary is in the conical limit set. If τ is such a point, we know by Proposition 4.14 that E is not minimally thin at τ. We conclude that E is not minimally thin at almost every point on the boundary. □

5. Generalized limit sets

In the previous sections we have considered two different limit sets, the non-tangential and the non-osculating limit set. In this section we will study the more general family of limit sets introduced in Section 3.

5.1. The non-minimally thin set $\mathfrak{N}$.

Definition 5.1. We define the set $\mathfrak{N}$ to be

$$\mathfrak{N} = \{x \in \partial B : \text{ the archipelago is not minimally thin at } x\}.$$

In this section we will show that $\mathfrak{N}$ is close to the non-tangential limit set Λ_c.

We can introduce a strong type of the limit set $\mathscr{L}(\alpha)$ by taking the intersection instead of the union in the following manner.

Definition 5.2. We define the *strong* α*-limit set* to be

$$\mathscr{L}_s(\alpha) = \bigcap_{k>0} L(0 : k, \alpha).$$

Thus we have that

$$\partial B \supset \mathscr{L}(\alpha) \supset \mathscr{L}_s(\alpha) \supset \mathscr{L}(\alpha + \varepsilon) \qquad \text{for all } \varepsilon > 0.$$

Proposition 4.14 above says that if $\tau \in \mathscr{L}(1)$ then the archipelago is not minimally thin at τ. We will in this section study the opposite relation.

We will show the following result.

Theorem 5.3. *If $\alpha < 1$ and $\tau \notin \mathscr{L}_s(\alpha)$ then the archipelago of Γ is minimally thin at τ.*

In the proof we will need the following result which we state in the form given in [4, p. 357].

Theorem F (e.g. [10] for the planar case, [2, p. 440], [11, p. 98]). *Let us consider the upper half space*

$$\mathbf{H} = \{z = (X, y) \in \mathbf{R}^{\mathrm{d}} : X = (x_1, x_2, \dots, x_{\mathrm{d}-1}) \in \mathbf{R}^{\mathrm{d}-1} \text{ and } y > 0\}$$

and the subset

$$E_f = \{z = (X, y) \in \mathbf{R}^{\mathrm{d}} : 0 < y < f(|X|)\},$$

where f is a positive non-decreasing function on $(0, \infty)$. Then E_f is minimally thin at 0 if and only if there is an $\varepsilon > 0$ such that

$$\int_0^{\varepsilon} \frac{f(x)}{x^2}\, dx < \infty.$$

Proof of Theorem 5.3. Without loss of generality, we can assume that $\tau = 0$. If $0 \notin \mathscr{L}_s(\alpha)$ then there exists a $k > 0$ such that $0 \notin L(0 : k, \alpha)$, i.e. there are only finitely many orbit points in the truncated "α-cone", $\mathscr{C}_\alpha(k, 0)$, which we define as $(\mathbf{H} \setminus E_f) \cap B(0, 1)$, where

$$f(x) = \left(\frac{1}{k} x\right)^{1/\alpha}.$$

To obtain the archipelago E, we fatten the point sequence. We will have to take care of the extra intersections—which may be infinitely many. See Figure 6 where the point sequence lies outside the "undashed" α-cone, but every hyperbolic ball intersects it.

We will now show that it is possible to get a slightly smaller cone by changing k to $\frac{1}{2}k$ in $\mathscr{C}_\alpha$ so that the number of balls B_i in E that intersect $\mathscr{C}_\alpha\left(\frac{1}{2}k, 0\right)$ is finite.

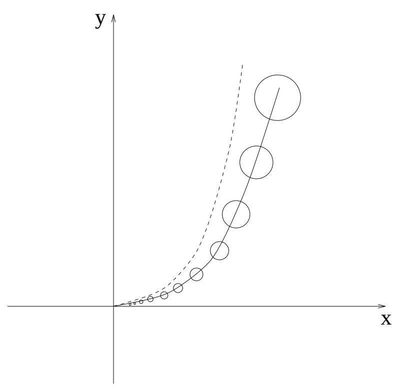

Figure 6. The undashed curve represents $x = ky^\alpha$ (i.e. $x = f^{-1}(y)$) and the dashed curve $x = ky^\alpha - py$, with $p = p(r_\Gamma)$.

We see that only finitely many balls B_i in E can reach inside the α-cone $\mathscr{C}_\alpha(k,0)$ more than a hyperbolic distance r_Γ. In the $\mathbf{H}$ model, the hyperbolic distance is approximately the Euclidean divided by the distance to the boundary (i.e. y) for small horizontal hyperbolic distances. Due to the fact that the hyperbolic distances we are interested in are bounded by the constant r_Γ we can find a p, only depending on r_Γ, such that a hyperbolic ball with radius r_Γ with its (hyperbolic) center on the curve $x = ky^\alpha$ does not intersect the upper "dashed curve" $x = ky^\alpha - py$; see Figure 6.

We see that

$$\frac{k}{2}y^\alpha < ky^\alpha - py \text{ holds for every } y < \left(\frac{k}{2p}\right)^{1/(1-\alpha)}.$$

Since Γ is discrete we know that the orbit cannot have a cluster-point inside $\mathbf{H}$. Hence there are only finitely many $\gamma_i \in \Gamma$ such that

$$\operatorname{Im}\bigl(\gamma_i(0)\bigr) \geq \left(\frac{k}{2p}\right)^{1/(1-\alpha)}.$$

We conclude that the number of balls B_i in E that intersect $\mathscr{C}_\alpha\bigl(\frac{1}{2}k,0\bigr)$ is finite.

Now, let us split E into two parts

$$E_1 := E \setminus \mathscr{C}_\alpha\bigl(\tfrac{1}{2}k,0\bigr) \qquad \text{and} \qquad E_2 := E \cap \mathscr{C}_\alpha\bigl(\tfrac{1}{2}k,0\bigr).$$

From Theorem F above with $f(x) = \bigl((2/k)x\bigr)^{1/\alpha}$ and the fact that minimal thinness is a local property, we know that $A = \mathbf{H} \setminus \mathscr{C}_\alpha\bigl(\frac{1}{2}k,0\bigr)$ is minimally thin at 0. Since $A \supset E_1$ it follows that E_1 is minimally thin at the origin.

For the "inner set" E_2, we consider a slightly bigger set

$$\widetilde{E}_2 := \bigcup_{B_i \cap \mathscr{C}_\alpha\left(\frac{1}{2}k,0\right) \neq \emptyset} B_i.$$

From Lemma 4.11 and the fact that B_i intersects $\mathscr{C}_\alpha\bigl(\frac{1}{2}k,0\bigr)$ at most finitely many times we conclude that

$$W(0,\widetilde{E}_2) \leq W_0(0,\widetilde{E}_2) < \infty.$$

In other words, E_2 is minimally thin at 0; cf. Theorem D.

Hence both $W(0,E_1)$ and $W(0,E_2)$ are finite. Thus

$$W(0,E) \leq W_0(0,E) \leq W_0(0,E_1) + W_0(0,E_2) \leq c_W\bigl(W(0,E_1) + W(0,E_2)\bigr) < \infty,$$

where c_W is the constant from the proof of Lemma 4.11. We have shown that the archipelago E is minimally thin at 0. □

5.2. The Hausdorff dimension of $\mathfrak{N}$. From Proposition 4.14 we learn that Λ_c is a subset of $\mathfrak{N}$, but Theorem 5.3 tells us that for all $\alpha < 1$, $\mathscr{L}_s(\alpha) \supset \mathfrak{N}$. We also have that $\mathscr{L}_s(1) \subseteq \mathscr{L}(1) = \Lambda_c$. This implies that the sets $\mathfrak{N}$ and Λ_c cannot differ very much. In fact, they are of the same dimension.

Theorem 5.4. *Let Γ be a non-elementary discrete group. The Hausdorff dimension of the non-minimal thin set $\mathfrak{N}$ equals the critical exponent of Γ.*

Proof. Let us view the situation in the upper half space $\mathbf{H}$. As usual, let us denote by δ the critical exponent of Γ. That implies

$$\sum_{\gamma_i \in \Gamma} \big(1 - |\gamma_i(0)|\big)^{\delta+\varepsilon} < \infty \qquad \text{for all } \varepsilon > 0. \tag{13}$$

In estimating the one-dimensional Hausdorff measure H_1 of the limit set $\mathscr{L}_s(\alpha)$ we use the the α-caps; see Figure 2 where a $\frac{1}{2}$-cap is drawn. The radius of a ball in $\mathbf{R}^{\mathrm{d}}$, centered at $\gamma_i(0)/|\gamma_i(0)|$, that exactly covers the α-cap at $\gamma_i(0)$ is $k\big(1 - |\gamma_i(0)|\big)^{\alpha}$; see Definitions 3.11, 3.12 and 5.2. Hence we get the estimate

$$H_1(\mathscr{L}_s(\alpha)) \le \sum k\big(1 - |\gamma_i(0)|\big)^{\alpha}.$$

By equation (13) we have that

$$H_{(\delta+\varepsilon)/\alpha}(\mathscr{L}_s(\alpha)) \le \sum k^{(\delta+\varepsilon)/\alpha}\big(1 - |\gamma_i(0)|\big)^{\alpha(\delta+\varepsilon)/\alpha} = \sum k_1\big(1 - |\gamma_i(0)|\big)^{\delta+\varepsilon} < \infty.$$

Thus we see that the Hausdorff dimension of $\mathscr{L}_s(\alpha)$ is less than or equal to $(\delta+\varepsilon)/\alpha$ for any $\varepsilon > 0$. Since $\mathscr{L}_s(\alpha)$ is independent of ε, we have $\dim(\mathscr{L}_s(\alpha)) \le \delta/\alpha$.

By Theorem 5.3, $\mathfrak{N} \subset \mathscr{L}_s(\alpha)$ for every $\alpha < 1$. Thus the Hausdorff dimension of $\mathfrak{N}$ is less than or equal to δ/α, and since $\mathfrak{N}$ is independent of α we obtain $\dim(\mathfrak{N}) \le \delta$. From Proposition 4.14 it follows that Λ_c is a subset of $\mathfrak{N}$. Since Γ is non-elementary, Theorem 1.1 in [9] gives us that $\delta = \dim(\Lambda_c)$. Hence, $\dim(\mathfrak{N}) = \delta$. □

The question is now: Is in fact $\mathfrak{N} = \Lambda_c$? If we limit ourselves to groups of geometrically finite type, we will have an affirmative answer in Corollary 6.1 below.

6. The geometrically finite situation

A discrete group Γ is geometrically finite if some convex fundamental polyhedron has finitely many faces. For the planar case, Γ is geometrically finite if and only if it is finitely generated; see [7, p. 254].

Corollary 6.1. *If Γ is a geometrically finite discrete group then $\mathfrak{N} = \Lambda_c$.*

Proof. First, we note that $\mathfrak{N}$ is a subset of the limit set Λ since if τ is not in Λ then there exists a neighborhood in the unit ball of τ such that the archipelago E of Γ does not intersect that neighborhood.

We also have that for a geometrically finite group the limit set Λ is the union of non-tangential limit points (i.e. Λ_c) and parabolic fixed points; see [6]. Let now τ be a parabolic fixed point. We have then that $\tau \notin \mathscr{L}_s(\alpha)$ for all $\alpha > \frac{1}{2}$. (We have in fact $\tau \notin \mathscr{L}_s(\frac{1}{2})$ but $\tau \in \mathscr{L}(\frac{1}{2})$.) By choosing an $\alpha \in (\frac{1}{2}, 1)$, we have from Theorem 5.3 that E is minimally thin at τ. We conclude that $\mathfrak{N} \subset \Lambda_c$. Since $\mathfrak{N} \supset \Lambda_c$ due to Proposition 4.14, we are done. □

In this situation we can also be precise about rarefiedness.

Theorem 6.2. *If Γ is a geometrically finite (i.e. finitely generated) Fuchsian group and E the archipelago of Γ then*

$$\{\tau \in \partial B : E \text{ not rarefied at } \tau\} = \Lambda.$$

We need the following lemma to prove this and to motivate why we have to restrict ourselves to the plane.

Lemma 6.3. *Suppose Γ is a discrete group generated by a single parabolic transformation such that the fixed point τ is of rank 1. Then the archipelago E is not rarefied at τ if and only if $d = 2$.*

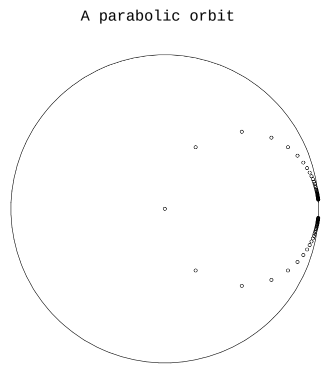

Figure 7. Here is an example of a single parabolic Fuchsian group. It is generated by the parabolic mapping $z \mapsto z + 1$ in the upper half plane which is then mapped onto the unit disc by a Möbius transformation. The critical exponent δ is $\frac{1}{2}$ and the archipelago is minimally thin but not rarefied at 1.

Before we go to the proofs, we cite a theorem from [5, p. 74].

Theorem G. *A subset E in the upper half space* **H** *is* rarefied *at ∞ if*

$$\sum_{1}^{\infty} \lambda'(E^{(n)}) 2^{n(1-\mathrm{d})} < \infty,$$

where d *is the dimension,* $E^{(n)}$ *the intersection of* E *with the half annulus* $\{x \in \mathbf{H} \cup \partial\mathbf{H} : 2^n \leq |x| < 2^{n+1}\}$ *and* $\lambda'(E)$ *is the* Green mass *of* E *(see the definition below).*

Definition 6.4. Let E be a bounded set in $\mathbf{H}$ and let μ_1 and μ_2 be measures on $\partial\mathbf{H}$ and $\mathbf{H}$ such that

$$\widehat{R}_1^E(x) = \int_{\partial\mathbf{H}} P_y(x)\, d\mu_1(y) + \int_{\mathbf{H}} G(x,y)\, d\mu_2(y),$$

where $G(x,y)$ is the Green function, $P_y(x)$ the Poisson kernel and $\widehat{R}_1^E(x)$ the regularized reduced function of 1 with respect to E in $\mathbf{H}$. Then we define the *Green mass* of E to be

$$\lambda'(E) = \mu_1(\partial\mathbf{H}) + \int_{\mathbf{H}} y_{\mathrm{d}}\, d\mu_2(y).$$

Proof of Lemma 6.3. We start by estimating the Poincaré series.

Let us do the following geometric construction: Let us map the unit ball by a Möbius transformation to the upper half space $\mathbf{H}$ such that τ is mapped on the origin and the origin on $(0,\dots,0,1)$. If we project orbit points from the origin onto the horizontal line $(x_1,\dots,x_{d-1},1)$, we will get something like Figure 8. Note that the intersections of the projection lines with the horizontal line are at a constant distance c from each other, since we can think of the projected points as being orbit points transformed by the reflection in the unit ball (i.e. $z \mapsto 1/\bar{z}$ if $d=2$) which is a Möbius map from the upper half space onto itself that takes the origin to infinity and fixes $(0,\dots,0,1)$.

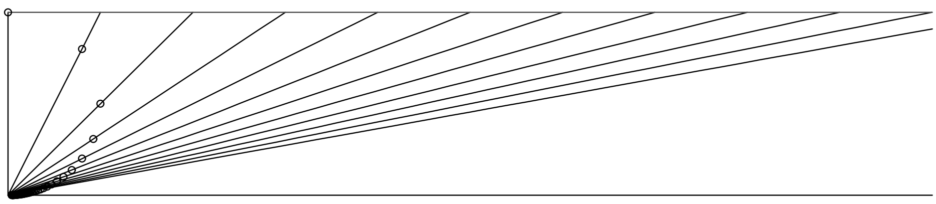

Figure 8. The projected orbit—or transforming the parabolic fixed point to infinity. (In this example $c = \frac{1}{2}$.)

We now use the fact that the orbit points approach the fixed point parabolically, i.e. approaching the graph of a quadratic function when the index n is getting large. By similarity of triangles, we get for large n that $t_n/\sqrt{t_n}$ is approximately $1/nc$, where t_n is the distance from the nth orbit point to the boundary. This asymptotic estimate still holds when we transform back to the unit ball, i.e.

$$\frac{1-|\gamma_n(0)|}{\sqrt{1-|\gamma_n(0)|}} \approx \frac{1}{n}.$$

Hence,

$$e^{-d(0,\gamma_n(0))} = \frac{1-|\gamma_n(0)|}{1+|\gamma_n(0)|} \approx 1-|\gamma_n(0)| \approx \frac{1}{n^2}.$$

We can now estimate the Poincaré series.

$$h_s = \sum_{\gamma_n \in \Gamma} e^{-sd(0,\gamma_n(0))} \approx \sum_{\gamma_n \in \Gamma} \frac{1}{n^{2s}} \begin{cases} = \infty & \text{if } s \le \frac{1}{2}, \\ < \infty & \text{if } s > \frac{1}{2}. \end{cases}$$

From this and equation (1) we see that $\delta = \frac{1}{2}$.

To see that the Fuchsian archipelago E is not rarefied at τ, we use the Poincaré series in the following way. By the fact that the orbit points lie outside any given cone with vertex at τ if n is large enough, we can estimate $|\tau - \gamma_n(0)| \approx \sqrt{1-|\gamma_n(0)|}$. Hence,

$$W_0^r(\tau) \approx \sum{}' \left(\frac{1-|\gamma_n(0)|}{\sqrt{1-|\gamma_n(0)|}} \right)^{\mathrm{d}-1} \approx h_{(\mathrm{d}-1)/2} \begin{cases} = \infty & \text{if } d = 2, \\ < \infty & \text{if } d > 2. \end{cases}$$

We see now by Theorem E that E is rarefied at τ if $\mathrm{d} > 2$. However, the situation is a little bit more complicated for the planar case since the convergence of $W_0^r(\tau)$ is not a necessary condition for rarefiedness. We have to examine the situation more carefully.

Let us as above transform the situation to the upper half plane such that τ is mapped on ∞. All the "islands" will then have the same distance to the boundary and the same distance to their closest neighbors. Since we are going to use Theorem G, we need an estimate of the Green mass $\lambda'(E^{(n)})$.

Let us first study the case when $E^{(n)}$ is just one island. We have then that $\mu_1 = 0$ and the support of μ_2 is on the boundary of the island; see the graph of $\widehat{R}_1^{E^{(n)}}$ in Figure 9.

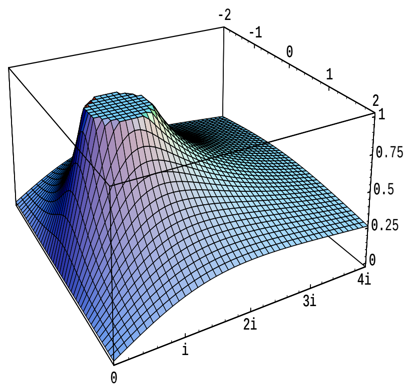

Figure 9. The reduced function $\widehat{R}_1^{E^{(n)}}(x)$ where $E^{(n)}$ is a single "island" centered at i in the upper half plane. (Note that r_Γ is rather big here.)

Now, since the islands line up horizontal and equidistant, we can estimate $\lambda'(E^{(n)})$ by the number of islands inside $E^{(n)}$ which is proportional to the width of the nth half annulus, i.e. 2^{n+1}. Hence,

$$\lambda'(E^{(n)}) \approx 2^n.$$

Thus we have in Theorem G

$$\sum_1^\infty \lambda'(E^{(n)}) 2^{-n} = \infty.$$

Hence we see that the archipelago E is not rarefied at τ if Γ is a Fuchsian group with a single parabolic generator whose fixed point is at τ. □

Proof of Theorem 6.2. Suppose first that $\tau \notin \Lambda$. Since $\partial B \setminus \Lambda$ is an open set there is an open ball centered at τ such that the ball does not intersect E. We can then use Theorem G to see that E is rarefied at τ, since the test-series becomes a finite summation of bounded terms.

Let us now suppose that $\tau \notin \Lambda$. As in the proof of Corollary 6.1 above, we take advantage of the following two facts: Γ is geometrically finite if and only if it is finitely generated; and the limit set Λ is the union of non-tangential limit points (i.e. Λ_c) and parabolic fixed points, see [7, p. 254] and [6].

From Lemma 6.3 we have that if τ is a parabolic fixed point then E is not rarefied at τ. Hence we are done if we can show that $\tau \in \Lambda_c$ implies that E is not rarefied at τ.

For completeness we give two arguments to show why this is true. By Proposition 4.14, $\tau \in \Lambda_c$ implies that E is not minimally thin at τ. We also have that rarefiedness implies minimal thinness; see for example [2, p. 425]. Thus $\tau \in \Lambda_c$ implies that E is not rarefied at τ.

We can also use Theorem G to get this result. We have then to show that we cannot divide E into two parts satisfying both (3) and (4). If (3) holds then E'' has to contain points from infinitely many "islands" inside a cone with vertex at τ by arguments similar to those given in the proof of Proposition 4.14. Then due to the condition $2r_i < |\tau - X_i|$ one needs infinitely many balls to cover points from E'' in the cone. For those balls, thanks to the cone-assumption, $r_i \approx |\tau - X_i|$. Hence (4) does not hold and we are done. □

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