GEODESICS ON QUOTIENT–MANIFOLDS AND THEIR CORRESPONDING LIMIT POINTS

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Abstract. The motivation of this paper is twofold.

We address the following question left open by the author in [6]. *Is the set on the boundary where the so called archipelago of* Γ *is not minimally thin equal to the conical limit set?* We will show that this is not true in general by constructing a counterexample in Section 9.

We are also considering a problem, suggested to the author by Chris Bishop, about generalizing the well known result which gives a correspondence between returning geodesics on Riemann manifolds and conical limit points.

Keywords: Discrete group, Fuchsian group, Kleinian group, horocycle, limit set, non-tangential limit set, minimal thinness.

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1. The set up

Let B be the unit ball in $\mathbb{R}^n$, or the unit disk if $n = 2$, and let Γ be a discrete group that preserves B. We will denote the elements in Γ by γ_i. Let $S = B/\Gamma$ be the (Riemann) quotient–manifold obtained from B by identification of Γ–equivalent points. (If n ≤ 3 S is a manifold, for higher dimensions it may not be a manifold, but we will adopt the notion from [1] p. 79 and call it a quotient–manifold nevertheless.)

Furthermore, let x_0 be the base point on S corresponding to the origin in B; and let $g(t)$ be a parameterized geodesic on S such that $g(0) = x_0$ and such that the arc length of $g(t)$ for t from 0 to τ is τ. Let $\varphi(t)$ be the distance $d(g(t), x_0)$ on the manifold. Thus we have that $\varphi(t) \leq t$. The geodesic from x_0 is viewed in B as a straight line from the origin to a boundary point ξ which thus corresponds to a limit point, $\lim_{t\to\infty} g(t)$, on S.

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It is a well known fact that if $\varphi(t)$ is bounded then the corresponding limit point ξ, for the discrete group Γ, is a non-tangential limit point, or in other words, a conical limit point. *What can be said in general about the limit point ξ if $\varphi(t)$ is known?*

We can think of S as the result of taking the Dirichlet domain in B around 0 and gluing together corresponding sides according to the generators of Γ. The "seams" on S will then correspond to the set on S where the graph of φ "has a corner", i.e. there are at least two different geodesics from x_0 to a seam-point; see Figure 2.

We will now give some definitions taken from [6] and [7]. The following is cited from [7, p. 5].

Definition 1.1. *Let $a \in B$ and $k, \alpha > 0$. We define*

$$I(a : k, \alpha) = \{x \in \partial B : \left| x - \frac{a}{|a|} \right| < k(1 - |a|)^{\alpha}\}.$$

Let us also cite page 23 in [7] for the following definition.

Definition 1.2. *Let γ_i be the elements of the discrete group Γ and let z be the base point of the orbit. Then*

$$L(z : k, \alpha) = \bigcap_{m=1}^{\infty} \bigcup_{i>m}^{\infty} I(\gamma_i(z) : k, \alpha).$$

Let us cite Definition 3.14 in [6] where we take the base point to be the origin.

Definition 1.3. *Let us denote the α-limit set by*

$$\mathcal{L}(\alpha) = \bigcup_{k>0} L(0 : k, \alpha)$$

Remark 1.4. The special case when $\alpha = 1$ give us the conical limit set, also called the non–tangential limit set, i.e. $\mathcal{L}(1) = \Lambda_c$, see for example Lemma 3.13 in [6] for a more detailed comparasion.

In Definition 5.2 in [6], a subset of the limit set $\mathcal{L}(\alpha)$ was introduced by taking the intersection instead of the union in the following manner.

Definition 1.5. *We define the* **strong α-limit set** *to be*

$$\mathcal{L}_s(\alpha) = \bigcap_{k>0} L(0 : k, \alpha).$$

We have that for any strictly positive α

$$\partial B \supset \mathcal{L}(\alpha) \supset \mathcal{L}_s(\alpha) \supset \mathcal{L}(\alpha + \varepsilon) \quad \text{for all } \varepsilon > 0.$$

It is well known that the conical limit set, i.e. $\mathcal{L}(1)$, is independent of the choise of base point; see for example p. 29 in [5]. We will show that the same holds when $\alpha \in (0,1)$, telling us that our restriction in Definitions 1.3 and 1.5 to fix the base point to the origin is not that essential.

Lemma 1.6. *For any point z in B,*

$$\bigcup_{k>0} L(z : k, \alpha) = \bigcup_{k>0} L(0 : k, \alpha)$$

if $\alpha \le 1$. That is, $\mathcal{L}(\alpha)$ is independent of the base point of the orbit if $\alpha \le 1$. Similarly, $\mathcal{L}_s(\alpha)$ is independent of the base point of the orbit if $\alpha < 1$.

Proof. Given an $\alpha \le 1$ and a $z \in B$. Suppose $x \in \bigcup_{k>0} L(z : k, \alpha)$, then $x \in L(z : k, \alpha)$ for some $k > 0$. We want to show that there is a K such that $x \in L(0 : K, \alpha)$, where K is dependent on α, z and k.

Let us by δ denote the hyperbolic distance from 0 to z, i.e. $\delta = d(0, z)$. Since a Möbius mapping acts as an isometry , we then have $\delta = d(\gamma_i(0), \gamma_i(z))$.

Since $x \in L(z : k, \alpha)$, we have that $x \in I(\gamma_i(z), k, \alpha)$ for infinitely many indices i. Call that set of indices J. Let $\varepsilon < \frac{1}{2e^{\delta}}$ and define J_ε to be the (infinite) subset of J such that

$$J_\varepsilon = \{i \in J; 1 - |\gamma_i(z)| < \varepsilon\}.$$

Thus if $i \in J_\varepsilon$ then

$$\left| x - \frac{\gamma_i(z)}{|\gamma_i(z)|} \right| < k\varepsilon^{\alpha}. \tag{1}$$

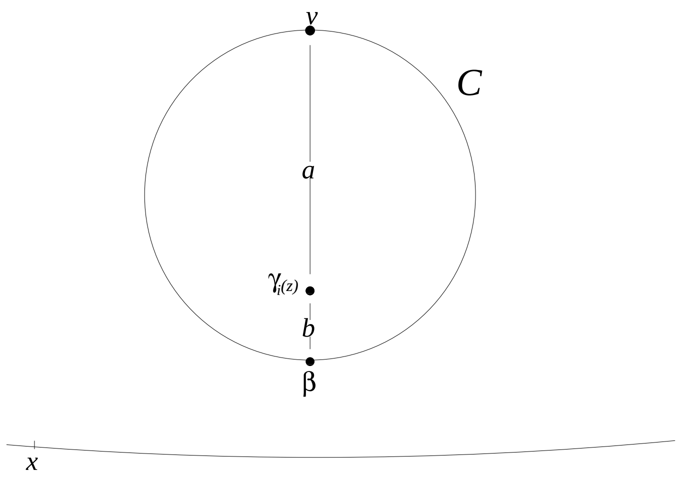

FIGURE 1. $\gamma_i(0)$ lies on the hyperbolic sphere C centered at $y = \gamma_i(z)$, ν, β and y lies in the unit ball on the ray from the origin through y where ν is the point on the sphere that is closest to the origin and β is point furthest away. a is the Euclidean distance from y to ν, and similarly b is $|\beta - y|$.

We know that $\gamma_i(0)$ lies on the hyperbolic sphere

$$C = \{\zeta \in B; d(\zeta, \gamma_i(z)) = \delta\}.$$

Let us now make an Euclidean estimate how far $\gamma_i(0)$ can be from $\gamma_i(z)$ by computing the two extremal distances to C from $\gamma_i(z)$. Let a be the distance from $\gamma_i(z)$ to ν, the point closest to the origin in C, and let b be the distance to the point β furthest away from the origin in C. See Figure 1.

We have that

$$\delta = d(\gamma_i(z), \nu) = d(0, \gamma_i(z)) - d(0, \nu) = \log\left(\frac{(1-|\nu|)}{1-|\gamma_i(z)|}\frac{1+|\gamma_i(z)|}{(1+|\nu|)}\right).$$

Hence we get the following rough estimate.

$$\frac{1}{2}e^{\delta}(1-|\gamma_i(z)|) < 1-|\nu| < 2e^{\delta}(1-|\gamma_i(z)|).$$

We can then estimate a.

$$a = |\gamma_i(z)| - |\nu| = (1-|\nu|) - (1-|\gamma_i(z)|) <$$

$$< 2e^{\delta}(1-|\gamma_i(z)|) - (1-|\gamma_i(z)|) < \varepsilon(2e^{\delta}-1). \tag{2}$$

Similarly, we have for b.

$$b = |\beta| - |\gamma_i(z)| < (1-|\gamma_i(z)|) - \frac{1}{2e^{\delta}}(1-|\gamma_i(z)|) < \varepsilon(1-\frac{1}{2e^{\delta}}). \tag{3}$$

Let us define θ to be $\arctan \frac{a}{|\gamma_i(z)|}$, then we could estimate

$$\left|x - \frac{\gamma_i(0)}{|\gamma_i(0)|}\right| < \left|x - \frac{\gamma_i(z)}{|\gamma_i(z)|}\right| + \theta.$$

But from equation (2) and since we have chosen $\varepsilon < \frac{1}{2e^{\delta}}$, we can estimate θ.

$$\theta = \arctan\frac{a}{|\gamma_i(z)|} \le \frac{a}{|\gamma_i(z)|} < \frac{a}{1-\varepsilon} < \frac{\varepsilon(2e^{\delta}-1)}{1-\varepsilon} < \varepsilon 2e^{\delta}.$$

So for $i \in J_\varepsilon$ we have, using equation (1),

$$\left|x - \frac{\gamma_i(0)}{|\gamma_i(0)|}\right| < k\varepsilon^{\alpha} + \varepsilon 2e^{\delta}. \tag{4}$$

From equation (3) we have

$$1 - |\gamma_i(0)| > \varepsilon - b > \varepsilon - \varepsilon(1-\frac{1}{2e^{\delta}}) = \frac{\varepsilon}{2e^{\delta}}. \tag{5}$$

We aim to find a K such that $x \in L(0:K,\alpha)$, i.e.

$$K > \left|x - \frac{\gamma_i(0)}{|\gamma_i(0)|}\right|\Big(1-|\gamma_i(0)|\Big)^{-\alpha}. \tag{6}$$

Let us therefore study the right hand side of this expression. From equations (4) and (5) we have

$$\left|x - \frac{\gamma_i(0)}{|\gamma_i(0)|}\right|\Big(1-|\gamma_i(0)|\Big)^{-\alpha} < \Big(k\varepsilon^{\alpha} + \varepsilon 2e^{\delta}\Big)\Big(\frac{\varepsilon}{2e^{\delta}}\Big)^{-\alpha} = (2e^{\delta})^{\alpha}(k + 2e^{\delta}\varepsilon^{1-\alpha}). \tag{7}$$

Since $\alpha \le 1$ we have that $\varepsilon^{1-\alpha} \le 1$. Therefore by picking $K = (2e^{\delta})^{\alpha}(k+2e^{\delta})$ equation (6) will be satisfied, and hence $x \in L(0:K,\alpha)$. This proves that $\mathcal{L}(\alpha)$ is independent of the base point of the orbit if $\alpha \le 1$, which ends the first part of the proof.

To prove the statement about the strong α–limit set, let us suppose that

$$x \in \bigcap_{k>0} L(z:k,\alpha) \text{ and } \alpha < 1. \tag{8}$$

We aim to show that $x \in \bigcap_{k>0} L(0 : k, \alpha)$. It is therefore enough to show that for any given $\kappa > 0$, $x \in L(0 : \kappa, \alpha)$.

Note from (8) that $x \in L(z, k, \alpha)$ trivially holds with the special choice of $k = \frac{\kappa}{2^{\alpha+1} e^{\alpha\delta}}$, where as above, $\delta = d(0, z)$. Thus, $x \in I(\gamma_i(z) : k, \alpha)$ for infinitely many indices i. Let us denote this set of indices by J. Now pick

$$\varepsilon = \left(\frac{k}{2e^\delta}\right)^{\frac{1}{1-\alpha}}.$$

Since

$$L(0 : c_1, \alpha) \subset L(0 : c_2, \alpha), \text{ if } c_1 < c_2,$$

we can without loss of generality assume that $\kappa < 2(2e^\delta)^{2\alpha}$. That gives us that $k < (2e^\delta)^\alpha$ and $\varepsilon < \frac{1}{2e^\delta}$, hence we can use the estimate in equation (7). Let $i \in J_\varepsilon$ then

$$\left| x - \frac{\gamma_i(0)}{|\gamma_i(0)|} \right| \Big(1 - |\gamma_i(0)|\Big)^{-\alpha} < (2e^\delta)^\alpha (k + 2e^\delta \varepsilon^{1-\alpha}) = 2k(2e^\delta)^\alpha = \kappa.$$

Thus as in equation (6), we have that for every $i \in J_\varepsilon$, which are infinitely many, $x \in I(\gamma_i(0) : \kappa, \alpha)$. Hence $x \in L(0 : \kappa, \alpha)$ which ends the proof. □

Remark 1.7. It is easy to see that $\mathcal{L}_s(1)$ is not independent of the base point. Let for example Γ be a Fuchsian group generated by a single hyperbolic generator in the unit ball. Then we have that the limit set consist only of the two fixed points which are both in Λ_c, however the points are in $\mathcal{L}_s(1)$ if and only if the base point is taken to be the origin.

2. The case $0 < \alpha \leq \frac{1}{2}$

If the geodesic immediately runs out on a parabolic cusp then $\varphi(t) \sim t$. If the geodesics runs out inside the Dirichlet domain to a point on the boundary that is not in the limit set, then we will also have $\varphi(t) \sim t$.

Hence the φ function will not be of any help for classifying points in $\mathcal{L}(\alpha)$ when $0 < \alpha \leq \frac{1}{2}$. Let us quickly turn to the next case.

3. The case $\frac{1}{2} \leq \alpha < 1$

Theorem 3.1. *Suppose that ξ is on the unit sphere. We have the following two equivalences.*

- $\xi \in \mathcal{L}(\alpha)$, *for* $\frac{1}{2} < \alpha \leq 1$, *if and only if*

$$\liminf_{t\to\infty} (\alpha(\varphi(t) + t) - t) < \infty.$$

- $\xi \in \mathcal{L}_s(\alpha)$, *for* $\frac{1}{2} \leq \alpha < 1$, *if and only if*

$$\liminf_{t\to\infty} (\alpha(\varphi(t) + t) - t) = -\infty.$$

Corollary 3.2. *Let* $\frac{1}{2} < \alpha < 1$*.*

$$\text{If } \xi \in \mathcal{L}(\alpha) \text{ then } \liminf_{t\to\infty} \frac{\varphi(t)}{t} \le \frac{1-\alpha}{\alpha};$$

$$\text{and if } \liminf_{t\to\infty} \frac{\varphi(t)}{t} < \frac{1-\alpha}{\alpha} \text{ then } \xi \in \mathcal{L}_s(\alpha).$$

Before we plunge into the proofs, let us turn our attention to an auxiliary sequence related to the function φ.

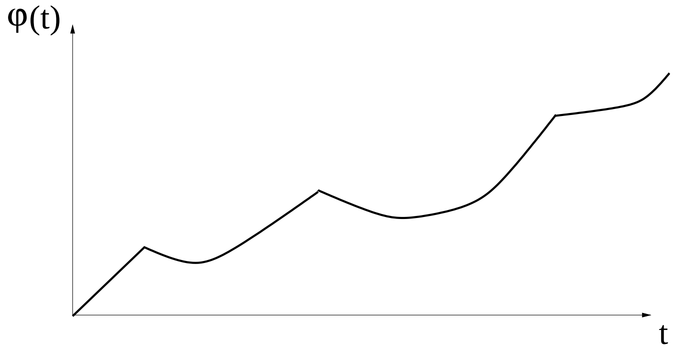

FIGURE 2. An example of a graph of φ.

3.1. **The sequence of generalized local minima for $\varphi(t)$.** Let the sequence $\{\varphi(t_i)\}$ be the sequence of *generalized* local minimas for the function φ in the following sense. Let us follow the geodesic mapped to the unit ball, where it will be the ray from the origin to the boundary point ξ, and let D_i be the ith fundamental domain that we visit on our way from 0 to ξ. D_i is copy of the Dirichlet domain around the origin mapped by γ_i. Let now φ_i be the (hyperbolic) distance to the ray from the single orbit point $\gamma_i(0)$ in D_i, and let t_i be the distance from the origin to the point on the ray that is closest to $\gamma_i(0)$; see Figure 3.

When the closest point on the ray lies inside D_i, $\varphi_i = \varphi(t_i)$ will be a local minimum, since we travel with unit speed along the ray; but in the situation schematically depicted in Figure 3 we will get a *generalized* local minimum point outside the graph of $\varphi(t)$, see Figure 4.

Remark 3.3. Note that such a generalized local minima φ_i will never be smaller than $\varphi(t_i)$ since the closest orbit point is $\gamma_{i-1}(0)$ for every point in D_{i-1}.

We will repeatedly use this auxiliary sequence $\{t_i, \varphi_i\}$ of generalized local minimas.

Lemma 3.4. *If*

$$0 < \liminf_{t\to\infty} \frac{\varphi(t)}{t} < 1,$$

then

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} = \liminf_{i\to\infty} \frac{\varphi_i}{t_i}.$$

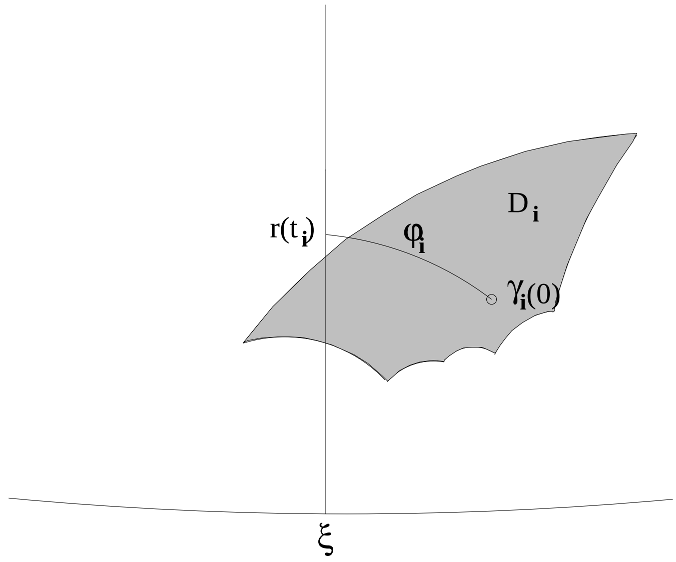

FIGURE 3. A situation where the closest point (on the ray, $(t, r(t))$, towards ξ) to $\gamma_i(0)$ lies outside the fundamental domain D_i. Compare to φ_3 in Figure 4 below.

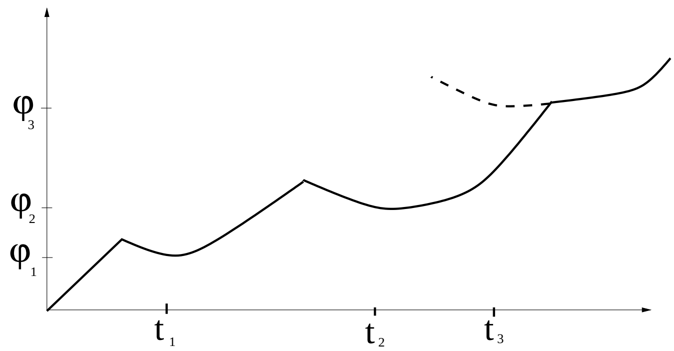

FIGURE 4. An example of the $\varphi(t)$ graph and the sequence of generalized local minimas at $\varphi_i = \varphi(t_i)$. Note that $\varphi_1 = \varphi(t_1), \varphi_2 = \varphi(t_2)$, but $\varphi_3 > \varphi(t_3)$.

Proof. We can assume that $\{\varphi_i\}$ is an infinite sequence. If not, there exists an integer I such that φ_I is the last generalized local minimum. That is, the ray, from the origin in the unit ball towards the point ξ, would never leave the fundamental domain D_I. Let $r(t)$ be the ray towards $\xi \in \partial B$ such that $d(r(t), 0) = t$. Let $t > t_I$ and consider the hyperbolic triangle in D_I with corners in $r(t), r(t_I)$, and $\gamma_I(0)$. The side lengths are $t - t_I, \varphi_I$, and $\varphi(t)$. Note that the angle at $r(t_I)$ is $\frac{\pi}{2}$. From the triangle inequality we have that

$$t - t_I \leq \varphi(t) \leq t - t_I + \varphi_I.$$

Hence,

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} = 1,$$

which is not allowed. Thus we have that $\{\varphi_i\}$ is an infinite sequence, and then

$$\liminf_{i\to\infty} \frac{\varphi_i}{t_i}$$

exists.

Due to Remark 3.3 we have immediately that

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} \leq \liminf_{i\to\infty} \frac{\varphi_i}{t_i}. \tag{9}$$

Let us first assume that $\varphi(t_i) = \varphi_i$ for all i, i.e. the generalized local minimas are true local minimas for $\varphi(\cdot)$. (At the very end of this proof we will treat the general case.) In this situation it we have that for a given i, there is a δ_i such that $t_i + \delta_i$ is a gives a local minimum of the function $\frac{\varphi(\cdot)}{\cdot}$. Note that the line from the origin to the point $(t_i + \delta_i, \varphi(t_i + \delta_i))$ will be a tangent to the graph of φ.

As above, let us look at a right angled triangle with corners in $r(t_i + \delta_i)$, $r(t_i)$, and $\varphi_i(0)$. The side lengths are then δ_i, φ_i, and $\psi_i := \varphi(t_i + \delta_i)$.

We will use the hyperbolic version of Pythagoras theorem, c.f. [3, p. 146].

$$\cosh \psi_i = \cosh \delta_i \cosh \varphi_i. \tag{10}$$

From the assumption and (9), we have that

$$0 < \liminf_{t\to\infty} \frac{\varphi(t)}{t} \leq \liminf_{i\to\infty} \frac{\varphi_i}{t_i}.$$

Thus $\varphi_i \to \infty$. Hence

$$\cosh \varphi_i \approx \frac{e^{\varphi_i}}{2}$$

if i is large.

Since $\psi_i > \varphi_i$ we have the following approximation of (10) for large index i.

$$e^{\psi_i} \approx \cosh \delta_i e^{\varphi_i}.$$

Hence,

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} = \liminf_{i\to\infty} \frac{\psi_i}{t_i + \delta_i} = \liminf_{i\to\infty} \frac{\varphi_i + \log(\cosh \delta_i)}{t_i + \delta_i}. \tag{11}$$

Let us now separately study two cases.

(1) If

$$\limsup_{i\to\infty} \delta_i < \infty$$

then from (11)

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} = \liminf_{i\to\infty} \frac{\frac{\varphi_i}{t_i} + \frac{\log(\cosh \delta_i))}{t_i}}{1 + \frac{\delta_i}{t_i}} = \liminf_{i\to\infty} \frac{\varphi_i}{t_i}.$$

(2) If

$$\limsup_{i\to\infty} \delta_i = \infty,$$

then there are infinitely many indices i such that

$$\cosh \delta_i \approx \frac{e^{\delta_i}}{2}.$$

From (11) we have that

$$\liminf_{t\to\infty}\frac{\varphi(t)}{t}=\liminf_{i\to\infty}\frac{\varphi_i+\delta_i-\log 2}{t_i+\delta_i}=\liminf_{i\to\infty}\frac{\varphi_i/t_i+\delta_i/t_i}{1+\delta_i/t_i}. \tag{12}$$

To simplify the notation, let

$$k=\liminf_{i\to\infty}\frac{\varphi_i}{t_i},\ \text{and}\ f(x)=\frac{k+x}{1+x}.$$

We have then that

$$\liminf_{t\to\infty}\frac{\varphi(t)}{t}\geq\liminf_{i\to\infty}f(\frac{\delta_i}{t_i}).$$

We note also that $0<k\leq 1$, and that

$$f'(x)>0 \text{ if and only if } k<1.$$

If $k=1$ then $f(x)\equiv 1$ and thus

$$\liminf_{t\to\infty}\frac{\varphi(t)}{t}\geq 1$$

which is not allowed. Hence we have that $k<1$ which gives us that

$$\liminf_{t\to\infty}\frac{\varphi(t)}{t}\geq\liminf_{i\to\infty}f(\frac{\delta_i}{t_i})\geq f(0)=k=\liminf_{i\to\infty}\frac{\varphi_i}{t_i}.$$

We are now done under the assumption that $\varphi(t_i)=\varphi_i$ for all i.

Finally, let us treat the situation when $\varphi(t_i)<\varphi_i$, schematically depicted in Figure 4, for $i=3$, above. Let us first concentrate on the graph consisting on the dotted arc continuation when t is such that $r(t)\in D_i$. In other words, let us look at the quotients $d(r(t),\gamma_i(0))/t$. By following the arguments above, starting at the point where we made the assumption that $\varphi(t_i)=\varphi_i$, we conclude that

$$\liminf_{i\to\infty}\frac{\varphi_i}{t_i}=\liminf_{i\to\infty}\inf_t\frac{d(r(t),\gamma_i(0))}{t}.$$

On the other hand,

$$\inf_t\frac{d(r(t),\gamma_i(0))}{t}\leq\inf_{r(t)\in D_i}\frac{d(r(t),\gamma_i(0))}{t}=\inf_{r(t)\in D_i}\frac{\varphi(t)}{t}.$$

Thus

$$\liminf_{i\to\infty}\frac{\varphi_i}{t_i}\leq\liminf_{i\to\infty}\inf_{r(t)\in D_i}\frac{\varphi(t)}{t}=\liminf_{t\to\infty}\frac{\varphi(t)}{t}.$$

□

We will also need the following variant of the above lemma.

Lemma 3.5. *Let* $\frac{1}{2}<\alpha<1$*. If* $\liminf_{t\to\infty}\varphi(t)=\infty$ *and if*

$$\liminf_{t\to\infty}(\alpha(\varphi(t)+t)-t)<\infty,$$

then

$$\liminf_{t\to\infty}(\alpha(\varphi(t)+t)-t)=\liminf_{i\to\infty}(\alpha(\varphi_i+t_i)-t_i).$$

Proof. Since the proof is completely analogous to the proof of Lemma 3.4 above, we just give a short outline.

Let us for simplicity define $\Phi(t) := \alpha(\varphi(t)+t)-t$. With the same triangle argument as above, we see that if $\{\varphi_i\}$ is finite, then

$$\alpha(2t-t_I)-t \leq \Phi(t) \leq \alpha(2t-t_I+\varphi_I)-t,$$

and since $\alpha > \frac{1}{2}$ we have that

$$\liminf_{t\to\infty} \Phi(t) = \infty$$

so we can safely assume that the sequence $\{\varphi_i\}$ is infinite.

Next we assume that $\Phi(t)$ has a local minimum at $t_i+\delta_i$. Similarly to (11) we have that

$$\liminf_{t\to\infty} \Phi(t) = \liminf_{i\to\infty} \alpha(\varphi_i+\log(\cosh\delta_i)+t_i+\delta_i)-t_i-\delta_i.$$

Now we treat the two cases.

(1) If $\limsup \delta_i < \infty$ then

$$\liminf_{t\to\infty} \Phi(t) = \liminf_{i\to\infty} \alpha(\varphi_i+t_i)-t_i.$$

(2) If $\limsup \delta_i = \infty$ then

$$\liminf_{t\to\infty} \Phi(t) = \liminf_{i\to\infty} \alpha(\varphi_i+t_i)-t_i+\delta_i(2\alpha-1) \geq \liminf_{i\to\infty} \alpha(\varphi_i+t_i)-t_i,$$

since $\alpha > \frac{1}{2}$.

□

3.2. **The proof of Theorem 3.1.**

Proof. For the first result, note that we already know that $\xi \in \mathcal{L}(1) = \Lambda_c$ if and only if $\liminf_{t\to\infty} \varphi(t) < \infty$. That takes care of the case $\alpha = 1$. From now on in the proof, we will assume that $\alpha < 1$ and that $\liminf_{t\to\infty} \varphi(t) = \infty$.

We have that $\xi \in \mathcal{L}(\alpha)$ if and only if there is a $k > 0$ such that $\xi \in L(0 : k, \alpha)$. That is equivalent to say that for infinitely many γ_i in Γ we have

$$\left|\xi - \frac{\gamma_i}{|\gamma_i|}\right| < k(1-|\gamma_i|)^\alpha.$$

This can be expressed using the notion in Figure 5 as $R_i < kh_i^\alpha$ or as

$$r_i \sin(\theta_i) < k(r_i\cos(\theta_i))^\alpha \tag{13}$$

for infinitely many i.

We note from Figure 5 that φ_i is small if and only if the angle θ_i is small. Therefore $\xi \in \mathcal{L}_s(\alpha)$ if and only if (13) holds for infinitely many such local minima points in $\{t_i\}$.

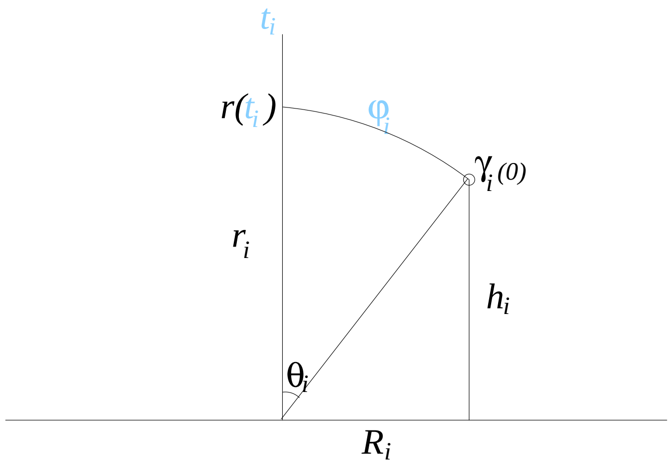

FIGURE 5. The geometric relations in the upper half space between θ_i, φ_i and t_i, where φ_i is the shortest hyperbolic distance from the orbit point to the vertical line, i.e. the hyperbolic distance along the circular arc from the orbit point to $r(t_i)$ on the vertical line. The hyperbolic distance from $r(t_i)$ to the image of the origin in the upper half plane is t_i. (The shaded tone of φ_i and t_i indicates that they are the only hyperbolic distances in the figure.)

Let us now give estimates for the components in (13). From a standard calculation, we have that

$$t_i = \log\left(\frac{2-r_i}{r_i}\right).$$

Since $r_i \ll 1$ we have the following estimate.

$$r_i \approx 2\exp(-t_i). \tag{14}$$

On page 162 in [3], relations between φ_i and the angle θ_i in Figure 5 can be found.

$$\sin(\theta_i) = \tanh(\varphi_i), \tag{15}$$

$$\cos(\theta_i) = \frac{1}{\cosh(\varphi_i)} \quad \text{and} \tag{16}$$

$$\tan(\theta_i) = \sinh(\varphi_i). \tag{17}$$

Since we assumed that ξ is not a conical limit point, we know that $\liminf_{i\to\infty} \varphi_i = \infty$ and thus we can make the following two estimates of (15) and (16).

$$\sin(\theta_i) \approx 1. \tag{18}$$

$$\cos(\theta_i) \approx \frac{2}{\exp(\varphi_i)}. \tag{19}$$

Using the estimates in equations (14), (18) and (19) in the condition (13), we obtain the following relation.

$$\xi \in \mathcal{L}(\alpha) \text{ if and only if } \Big(2\exp(-t_i)\Big)^{1-\alpha} < K\Big(2\exp(-\varphi_i)\Big)^{\alpha},$$

for infinitely many indices i and for some constant K. Let $C = \log(K) + (2\alpha - 1)\log 2$. We can write the above inequality as

$$-t_i(1-\alpha) < -\alpha\varphi_i + C.$$

Hence $\xi \in \mathcal{L}(\alpha)$ if and only if there is a C such that

$$\alpha(\varphi_i + t_i) - t_i < C$$

for infinitely many indices i. Thus,

$$\xi \in \mathcal{L}(\alpha) \text{ if and only if } \liminf_{i\to\infty}(\alpha(\varphi_i + t_i) - t_i) < \infty. \tag{20}$$

Which is, thanks to Lemma 3.5, equivalent to the following statement.

$$\xi \in \mathcal{L}(\alpha) \text{ if and only if } \liminf_{t\to\infty}(\alpha(\varphi(t) + t) - t) < \infty.$$

(Recall that we could make the assumption that $\liminf_{t\to\infty}\varphi(t) = \infty$ after treating the case $\alpha = 1$ separately in the beginning. Hence all the assumptions in Lemma 3.5 are fulfilled.) We are done with the proof of the first statement.

To prove the second statement let us treat the case $\alpha = \frac{1}{2}$ separately at the end.

We note that $\xi \in \mathcal{L}_s(\alpha)$ if and only if $\xi \in L(0 : k, \alpha)$ for *all* $k > 0$. Using the above arguments we will in this situation have the analogue to condition (13) as $\xi \in \mathcal{L}_s(\alpha)$ if and only if

$$r_i \sin(\theta_i) < k(r_i \cos(\theta_i))^\alpha \tag{21}$$

for infinitely many i and for all $k > 0$. Using equations (14), (18) and (19) the above inequality is equivalent to the following.

$$-t_i(1-\alpha) < -\alpha\varphi_i - C,$$

for infinitely many indices i and for all $C < \infty$. Thus, $\xi \in \mathcal{L}_s(\alpha)$ if and only if

$$\liminf_{i\to\infty}(\alpha(\varphi_i + t_i) - t_i) = -\infty. \tag{22}$$

which is by Lemma 3.5 equivalent to

$$\xi \in \mathcal{L}(\alpha) \text{ if and only if } \liminf_{t\to\infty}(\alpha(\varphi(t) + t) - t) = -\infty.$$

For the case $\alpha = \frac{1}{2}$, we note that the assumption $\alpha > \frac{1}{2}$ in Lemma 3.5 is used only to make sure that we have an infinite sequence $\{\varphi_i\}$.

Suppose now that $\alpha = \frac{1}{2}$ and that $\xi \in \mathcal{L}_s(\frac{1}{2})$, then we will show that $\{\varphi_i\}$ has to be infinite. The idea of that argument is taken from the proof of Theorem 2.4.10 in [7] which says that a conical limit point can not appear on the boundary on a Dirichlet domain.

Let us study the unit ball tessellated by images of the Dirichlet domain around the origin. We know that the ray to ξ visits such a domain only once. Since the number of local minimas is finite, we conclude that there is a domain, F_i, where the ray finally enters and then never leaves on its way to ξ, i.e. there is a C such that $c\xi \in F_i$ for every $c > C$. Recall that every point in this open domain F_i has the property that it is closer to the orbit point $\gamma_i(0)$ in it than to any other orbit point. Let us for simplicity map the whole picture by the mapping γ_i^{-1} so that $F_0 = \gamma_i^{-1}(F_i)$ is a Dirichlet domain centered at the origin. Let us now study the ray from the origin to $\xi_0 = \gamma_i^{-1}(\xi)$. We see that this ray is in F_0 and we parameterize it by $c\xi_0$ for $c \in (0,1)$. Let us now construct an open hyperbolic ball centered at $c\xi_0$ with radius $d(c\xi, \gamma_i(0))$. That is let

$$B_c = \{z : d(z, c\xi_0) < d(c\xi_0, 0)\}.$$

We note that B_c does not contain any orbit points and the same is true for the union

$$\hat{B} = \bigcup_{c\in(0,1)} B_c.$$

We note that $\hat{B}$ is a horo ball with Euclidean radius $\frac{1}{2}$ and tangent to the unit ball at ξ_0.

Finally, let us map $\hat{B}$ back to $\gamma_i(\hat{B})$ which is tangent to ξ has a radius greater than or equal to $\frac{1-|\gamma_i(0)|}{2}$ and contains no orbit points. We conclude then from the Definition 1.5 that $\xi \notin \mathcal{L}_s(\frac{1}{2})$. Hence we conclude that the sequence $\{\varphi_i\}$ is infinite.

Furthermore, since $\xi \in \mathcal{L}_s(\frac{1}{2})$ we have that

$$\liminf_{i\to\infty}(\varphi_i - t_i) = -\infty < \infty$$

from the above reasoning leading to Equation (22) which is still valid. We can now use the proof for Lemma 3.5, since $\{\varphi_i\}$ is infinite, to obtain that

$$\liminf_{t\to\infty}(\varphi(t) - t) = -\infty.$$

On the other hand, assume that

$$\liminf_{t\to\infty}(\varphi(t) - t) = -\infty.$$

If $\{\varphi_i\}$ is finite, then from the beginning of the proof of Lemma 3.5 we have (since $\alpha = \frac{1}{2}$) that $\frac{1}{2}(\varphi(t) - t) = \Phi(t) \geq -\frac{1}{2}t_I$ which contradicts the assumption. Hence we conclude that $\{\varphi_i\}$ is infinite. As above we can use the proof of Lemma 3.5 to see that

$$\liminf_{i\to\infty}(\varphi_i - t_i) = -\infty,$$

which is equivalent to $\xi \in \mathcal{L}_s(\frac{1}{2})$. $\square$

3.3. The proof of Corollary 3.2.

Proof. Suppose $\xi \in \mathcal{L}(\alpha)$ then we have from the first part in Theorem 3.1 that

$$\liminf_{t\to\infty}(\alpha(\varphi(t)+t)-t)<\infty.$$

Hence there is a $K<\infty$ such that

$$\liminf_{t\to\infty}(\varphi(t)-\frac{1-\alpha}{\alpha}t)=K.$$

Thus

$$\liminf_{t\to\infty}\frac{\varphi(t)}{t}\leq\frac{1-\alpha}{\alpha}.$$

For the second part, suppose that

$$a:=\liminf_{t\to\infty}\frac{\varphi(t)}{t}<\frac{1-\alpha}{\alpha}.$$

From Lemma 3.4 we have that $a=\frac{\varphi_i}{t_i}$. Let $\delta:=\frac{1-\alpha}{\alpha}-a$.

We have that for every $\varepsilon>0$, there is an infinite set of indices $J=\{j\}$ such that

$$\frac{\varphi(t_j)}{t_j}-a<\varepsilon \text{ for all } j\in J.$$

This will especially be valid for our choice of $\varepsilon=\frac{\delta}{2}$. (Note that $\delta>0$ by the assumption above.) Hence we have that

$$\varphi(t_j)<t_j(a+\varepsilon) \text{ for all } j\in J.$$

Thus

$$\varphi(t_j)-\frac{1-\alpha}{\alpha}t_j<t_j(a+\varepsilon-\frac{1-\alpha}{\alpha})=t_j(\varepsilon-\delta)=-\frac{\delta}{2}t_j.$$

We conclude, using Lemma 3.5 that

$$\liminf_{t\to\infty}\alpha(\varphi(t)+t)-t)=-\infty,$$

which is equivalent to $\xi\in\mathcal{L}_s(\alpha)$ by the latter part of Theorem 3.1. □

3.4. A global result.

By using Theorem 3.1 together with a Borel–Cantelli type lemma in [7] we can get a global result for the limit sets $\mathcal{L}_s(\alpha)$ and their corresponding Poincaré series.

Corollary 3.6. *Let γ_θ be a geodesic on $\mathcal{B}/\Gamma$ starting at the reference point x_0 in the θ direction, where θ is on the unit sphere and let φ_θ be the φ distance function for γ_θ. Let $|\cdot|$ be the $(n-1)$–dimensionally Lebesgue measure on the unit sphere, and let $\frac{1}{2}<\alpha\leq 1$. If*

$$|\{\theta:\liminf_{t\to\infty}(\alpha(\varphi_\theta(t)+t)-t)<\infty\}|>0$$

then the Poincaré type series

$$\sum_{\gamma_i\in\Gamma}(1-|\gamma_i(0)|)^{(n-1)\alpha}=\infty.$$

Proof. If $\liminf_{t\to\infty}(\alpha(\varphi_\theta(t)+t)-t)<\infty$ then we have from Theorem 3.1 that when γ_θ is transformed into the unit ball, it ends at a point in $\mathcal{L}(\alpha)$. Hence if

$$|\{\theta : \liminf_{t\to\infty}(\alpha(\varphi_\theta(t)+t)-t)<\infty\}|>0$$

then $|\mathcal{L}(\alpha)|>0$. Now Theorem 2.1.1 in [7] tells us then that if $|\mathcal{L}(\alpha)|>0$, then

$$\sum_{\gamma_i\in\Gamma}(1-|\gamma_i(0)|)^{(n-1)\alpha}=\infty.$$

□

Remark 3.7. In [4] it was showed, in the Kleinian case (n=3), that the Hausdorff-dimension of

$$\{\theta : \liminf_{t\to\infty}\varphi_\theta(t)<\infty\}$$

equals the critical exponent $\delta(\Gamma)$. Recall that the critical exponent can be defined in the following way:

$$\delta(\Gamma)=\inf\{s : \sum_{\gamma_i\in\Gamma}(1-|\gamma_i(0)|)^s<\infty\}.$$

4. An ladder-like example

Let us study a Riemann surface that looks like a ladder or a "one dimensional jungle gym". Our surface is an infinitely long body with evenly distributed "holes". See Figure 6.

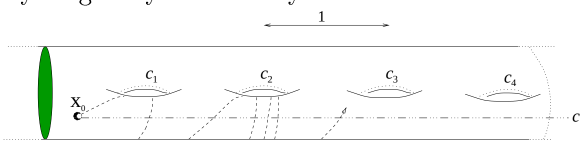

Figure 6. A one dimensional jungle gym.

Let us for simplicity assume that the distance between the centers of two consecutive holes are 1 and that the shortest closed arc around in such a hole also has unit length.

Let us use this jungle gym construction together with Corollary 3.2 to give simple examples of $\xi\in\mathcal{L}(\alpha)\setminus\mathcal{L}(\alpha')$, where $\frac{1}{2}<\alpha<\alpha'<1$.

Corollary 4.1. *Let* Γ *be given by the above jungle gym construction and* ξ *by the geodesic making* $N(j)$ *turns in hole* j *twisting out in a consecutive way. Let us assume that* $N(j)\geq 1$ *and that the limit–average of the number of turns is bounded, i.e.*

$$\bar{N}:=\limsup_{i\to\infty}\frac{1}{i}\sum_{j=1}^{i}N(j)<\infty.$$

Then we have that

$$\xi\in\mathcal{L}_s(\frac{\bar{N}}{\bar{N}+1})\setminus\mathcal{L}(\frac{\bar{N}+1}{\bar{N}+2}).$$

Proof. We have that $\xi \notin \mathcal{L}(1)$ since the geodesic does not return at all.

Let us give estimates of the local minimum of $\varphi(t)$ about where the geodesic has made $N(i)$ turns in hole i. Let us denote that "ending" local minimum by $\varphi_{i_e} = \varphi(t_{i_e})$. By the use of the "little ordo" $o(\cdot)$ function we can immediately give the following estimate:

$$i_e - o(i_e) < \varphi_{i_e} < i + o(i_e).$$

Note that we have the following approximation of t_{i_e}.

$$t_{i_e} \approx \sqrt{2} i_e + \sum_{j=1}^{i} (N(j) - 1).$$

We will use the following rather rough estimate.

$$i_e + \sum_{j=1}^{i} (N(j) - 1) - o(i_e) < t_{i_e} < i_e + \sum_{j=1}^{i} N(j) + o(i_e).$$

We see from the construction that

$$\liminf_{i\to\infty} \frac{\varphi_i}{t_i} = \liminf_{i\to\infty} \frac{\varphi_{i_e}}{t_{i_e}}.$$

Hence we have from the estimates above that

$$\liminf_{i\to\infty} \frac{1}{1 + \frac{1}{i}\sum_{j=1}^{i} N(j)} < \liminf_{i\to\infty} \frac{\varphi_i}{t_i} < \liminf_{i\to\infty} \frac{1}{\frac{1}{i}\sum_{j=1}^{i} N(j)}.$$

Thus we see that

$$\frac{1}{1+\bar{N}} < \liminf_{i\to\infty} \frac{\varphi_i}{t_i} < \frac{1}{\bar{N}}.$$

Now, if $x = \frac{1-\alpha}{\alpha}$, then $\alpha = \frac{1}{x+1}$. We can then use Corollary 3.2 and conclude that

$$\xi \in \mathcal{L}_s\left(\frac{\bar{N}}{\bar{N}+1}\right) \setminus \mathcal{L}\left(\frac{\bar{N}+1}{\bar{N}+2}\right),$$

which is the sought after expression. $\square$

5. The case $\alpha = 1$

It is well known that if the geodesic $g(t)$ returns infinitely often to a compact neighborhood of x_0, then the limit point ξ is in the non-tangentially limit set Λ_c. Let us try to be a little more precise about this.

Proposition 5.1. *Let the set $L(\cdot : \cdot, \cdot)$ be as in Definition 1.2.*

$$\xi \in L(0 : \sinh(K), 1) \text{ if and only if } \liminf_{t\to\infty} \varphi(t) < K.$$

Furthermore,

$$\xi \in \mathcal{L}_s(1) \text{ if and only if } \liminf_{t\to\infty} \varphi(t) = 0.$$

Proof. Suppose that $\xi \in L(0 : \sinh(K), 1)$. Using the notation from Figure 5 we know that there are infinitely many orbit points $\gamma_i(0)$ such that $R_i < \sinh(K)h_i$. We can reformulate this as there are infinitely many $\gamma_i(0)$ such that

$$\tan(\theta_i) < \sinh(K).$$

Using equation (17) we end up with that for infinitely many indices i, $\sinh(\varphi_i) < \sinh(K)$ and thus $\varphi_i < K$ for infinitely many i. Thus

$$\xi \in L(0 : \sinh(K), 1) \text{ if and only if } \liminf_{i\to\infty} \varphi_i < K,$$

which by Lemma 5.2 below, gives us the first equivalence:

$$\xi \in L(0 : \sinh(K), 1) \text{ if and only if } \liminf_{t\to\infty} \varphi(t) < K.$$

(Note that we can use Lemma 5.2 since we safely can assume that $\{\varphi_i\}$ is infinite.)

To get the second statement, we argue as above and conclude that $\xi \in \mathcal{L}_s(1)$ if and only if for every $K > 0$ there are infinitely many indices i such that $\varphi_i < K$. Hence by using Lemma 5.2,

$$\xi \in \mathcal{L}_s(1) \text{ if and only if } \liminf_{t\to\infty} \varphi(t) = 0.$$

□

Lemma 5.2. *If the sequence $\{\varphi_i\}$ is infinite, then*

$$\liminf_{i\to\infty} \varphi_i = \liminf_{t\to\infty} \varphi(t).$$

Proof. We have from Remark 3.3 that,

$$\liminf_{i\to\infty} \varphi_i \geq \liminf_{i\to\infty} \varphi(t_i) \geq \liminf_{t\to\infty} \varphi(t).$$

On the other hand, we can use a similar argument as we did in the latter part of the proof of Lemma 3.4 above to obtain the following inequality.

$$\varphi_i = \inf_t d(r(t), \gamma_i(0)) \leq \inf_{r(t)\in D_i} d(r(t), \gamma_i(0)) = \inf_{r(t)\in D_i} \varphi(t).$$

Hence,

$$\liminf_{i\to\infty} \varphi_i \leq \liminf_{i\to\infty} \inf_{r(t)\in D_i} \varphi(t) = \liminf_{t\to\infty} \varphi(t).$$

□

6. The case $\alpha > 1$

When $\xi \in \mathcal{L}(\alpha)$ for $\alpha > 1$ we have immediately that $\xi \in \Lambda_c$ and thus that there exists a bounded subsequence of $\{\varphi_i\}$. But we can say more than this.

Proposition 6.1. *Suppose that $\alpha > 1$ then we have the following two equivalences.*

- *$\xi \in \mathcal{L}(\alpha)$ if and only if there exists a $K < \infty$ such that*

$$\liminf_{t\to\infty} \varphi(t)e^{(\alpha-1)t} < K.$$

- *$\xi \in \mathcal{L}_s(\alpha)$ if and only if*

$$\liminf_{t\to\infty} \varphi(t)e^{(\alpha-1)t} = 0.$$

Proof. We have that $\xi \in \mathcal{L}(\alpha)$ if and only if there is a $k < \infty$ such that ξ is in infinitely many $L(0 : k, \alpha)$. With the notion from Figure 5 that translates into $R_i < kh_i^\alpha$ for infinitely many i, or

$$r_i \sin(\theta_i) < k\Big(r_i \cos(\theta_i)\Big)^\alpha \text{ for infinitely many } i.$$

If we now use equations (14), (15) and (16) we have that $\xi \in \mathcal{L}(\alpha)$ if and only if there is a k such that for infinitely many i

$$2e^{-t_i}\tanh(\varphi_i) < k\left(2e^{-t_i}\frac{1}{\cosh(\varphi_i)}\right)^\alpha. \tag{23}$$

Since we know that $\alpha > 1$ we have that every cone with vertex at ξ has infinitely many orbit points inside even if the opening angle is very small. Thus we have that

$$\liminf_{i\to\infty} \theta_i \to 0$$

and hence using equation (15)

$$0 = \liminf_{i\to\infty} \sin(\theta_i) = \liminf_{i\to\infty} \tanh(\varphi_i).$$

Thus we have that

$$\liminf_{i\to\infty} \varphi_i = 0.$$

Using that fact in equation (23) we obtain the following asymptotic relation. $\xi \in \mathcal{L}(\alpha)$ if and only if there is a k such that

$$\liminf_{i\to\infty} \varphi_i < k2^{\alpha-1}e^{-t_i(\alpha-1)}.$$

We conclude that $\xi \in \mathcal{L}(\alpha)$ if and only if there is a $K < \infty$ such that

$$\liminf_{i\to\infty} \varphi_i e^{t_i(\alpha-1)} < K. \tag{24}$$

We will now show that the above inequality is equivalent to

$$\liminf_{t\to\infty} \varphi(t)e^{t(\alpha-1)} < K', \tag{25}$$

for some K'.

In order to simplify the notation, let

$$f(t) = \varphi(t)e^{t(\alpha-1)}, g_i(t) = d(r(t), \gamma_i(0)), \text{ and } f_i(t) = g_i(t)e^{t(\alpha-1)}.$$

Note that

$$f_i(t) = f(t) \text{ when } r(t) \in D_i, \tag{26}$$

and that $g_i(t_i) = \varphi_i$ is a local minimum for g_i.

On the other hand, f_i has not a local minimum at t_i but at $t_i - \delta_i$ for some positive δ_i. Hence, after differentiation, we get that

$$g_i'(t_i - \delta_i) = -(\alpha - 1)g_i(t_i - \delta_i). \tag{27}$$

So again by Pythagoras' Theorem we have that

$$\cosh(g_i(t_i - \delta_i)) = \cosh(\delta_i)\cosh(\varphi_i).$$

And by differentiation with respect to δ_i we get that

$$-\sinh(g_i(t_i - \delta_i))g_i'(t_i - \delta_i) = \sinh(\delta_i)\cosh(\varphi_i). \tag{28}$$

Combining (27) and (28) gives us

$$\sinh(g_i(t_i - \delta_i))g_i(t_i - \delta_i)(\alpha - 1) = \sinh(\delta_i)\cosh(\varphi_i). \tag{29}$$

From (28) we see that $\frac{d}{dx}g(t_i - x)$ decreases from 0 to $-\cosh(\varphi_i)$ as x goes from 0 to ∞. Thus,

$$-gi'(t_i - \delta_i) \leq \cosh(\varphi_i). \tag{30}$$

That estimate together with (27) gives us that

$$g_i(t_i - \delta_i) \leq \frac{\cosh(\varphi_i)}{\alpha - 1}. \tag{31}$$

We have from (28) that

$$\delta_i \leq \sinh(\delta_i) = \frac{-\sinh(g_i(t_i - \delta_i))g_i'(t_i - \delta_i)}{\cosh(\varphi_i)} \leq \sinh(g_i(t_i - \delta_i)).$$

Using (31) we get

$$\delta_i \leq \sinh(\frac{\cosh(\varphi_i)}{\alpha - 1}). \tag{32}$$

Since

$$f_i(t_i - \delta_i) = g_i(t_i - \delta_i)e^{(t_i - \delta_i)(\alpha - 1)} = \frac{g_i(t_i - \delta_i)}{\varphi_i}f_i(t_i)e^{-\delta_i(\alpha - 1)},$$

we have the following estimate for the function f_i.

$$f_i(t_i - \delta_i) \leq f_i(t_i)e^{-\delta_i(\alpha - 1)}. \tag{33}$$

We have that

$$\liminf_{t\to\infty} f(t) \le \liminf_{i\to\infty} f_i(t_i).$$

We show now an inverse inequality to prove the equivalence between (24)and (25).

$$\liminf_{t\to\infty} f(t) = \liminf_{i\to\infty} \inf_{r(t)\in D_i} f(t) = (26) = \liminf_{i\to\infty} \inf_{r(t)\in D_i} f_i(t) \ge \liminf_{i\to\infty} f_i(t_i - \delta_i) \ge (33) \ge$$

$$\ge \liminf_{i\to\infty} f_i(t_i)\exp(-\delta_i(\alpha-1)) \ge (32) \ge \liminf_{i\to\infty} f_i(t_i)\exp\Big(-\sinh(\frac{\cosh(\varphi_i)}{\alpha-1})(\alpha-1)\Big) \ge$$

$$\ge \Big(\liminf_{i\to\infty} f_i(t_i)\Big)\exp\Big(-\sinh(\frac{\cosh(\liminf_{i\to\infty}\varphi_i)}{\alpha-1})(\alpha-1)\Big).$$

We can assume that $\liminf_{i\to\infty}\varphi_i = 0$, since otherwise we have, from Proposition 5.1 and Lemma 5.2, that

$$\xi \notin \mathcal{L}_s(1) \supset \mathcal{L}(\alpha) \text{ for all } \alpha > 1.$$

Hence

$$\liminf_{t\to\infty} f(t) \ge \liminf_{i\to\infty} f_i(t_i)\exp\Big(-\sinh(\frac{1}{\alpha-1})(\alpha-1)\Big).$$

So if we let

$$K' = K\exp\Big(-\sinh(\frac{1}{\alpha-1})(\alpha-1)\Big),$$

we have that (24)and (25) are equivalent. This ends the proof of the first part.

To prove the second part we only have to note that $\xi\in\mathcal{L}(\alpha)$ if and only if for every $k>0$, ξ is in infinitely many $L(0:k,\alpha)$. Following the same arguments as above we obtain that $\xi\in\mathcal{L}(\alpha)$ if and only if

$$\liminf_{i\to\infty}\varphi_i e^{t_i(\alpha-1)} = 0, \tag{34}$$

where last expression is equivalent, with the same reasoning as above (with $K=0$), to

$$\liminf_{t\to\infty}\varphi(t)e^{t(\alpha-1)} = 0.$$

□

6.1. **The point– and line– transitive sets.** Note from Remark 1.7 that the results in Proposition 6.1 depends on the choice of base point $x_0 \in B/\Gamma$. Let us now allow ourself to vary the base point letting $\varphi_a(t)$ be as $\varphi(t)$ above, except that we replace x_0, the image of the origin, by x_a, the image of $a\in B$. Then it is easy to see, using the definitions on pp. 26, 27 in [7] together with Figure 5, that ξ is a point transitive limit point ($\xi\in T_p$) if and only if

$$\liminf_{t\to\infty}\varphi_a(t) = 0, \text{ for all } a\in B,$$

and that ξ is a line transitive limit point, T_l, if and only if

$$\liminf_{t\to\infty}(\varphi_a(t)+\varphi_b(t)) = 0, \text{ for all pairs } a,b\in B.$$

Remark 6.2. We have trivially that $T_l \subset T_p$. Furthermore, $T_l \neq \emptyset$ if Γ is of the first kind, and $T_p = \emptyset$ if Γ is of the second kind. See for example Theorems 2.2.2 and 2.3.3 in [7].

7. A question about the archipelago of Γ

On page 300 in [6], the *archipelago* of a discrete group Γ is defined. Let $B_j := \{z \in B; d(z, \gamma_j(0)) < r_\Gamma, \gamma_j \in \Gamma \setminus \{I\}\}$. By the fact that Γ is discrete it is possible to find an $r_\Gamma > 0$ such that the balls B_j do not intersect each other. Let us fix such an r_Γ and let $E := \bigcup_j B_j$. That is, E is the "fattened" orbit of Γ and we call it **the archipelago of** Γ.

Definition 7.1. *We define the set* $\mathfrak{N}$ *to be*

$$\mathfrak{N} = \{x \in \partial B : \textit{ the archipelago is not minimally thin at } x\}.$$

From [6, Section 5 and 6] we have that

$$\Lambda_c \subset \mathfrak{N} \subset \mathcal{L}_s(\alpha),$$

when $\alpha < 1$. We have also $\mathcal{L}_s(1) \subseteq \mathcal{L}(1) = \Lambda_c$. Furthermore $\mathfrak{N}$ and Λ_c have the same Hausdorff dimension and in the case where Γ is geometrically finite, $\mathfrak{N} = \Lambda_c$, see Theorem 5.4, and Corollary 6.1 in [6].

The following question was raised at the end of Section 5 in [6] p 310: *Is in fact* $\mathfrak{N} = \Lambda_c$*?* We will answer this question negatively in Section 9 below.

8. A generalized version of thinness

Definition 8.1. *The set E is β–thin at y if there is a measure μ such that*

$$\liminf_{x \to y, x \in E} k_\beta * \mu(x) > k_\beta * \mu(y),$$

where $k_\beta(x)$ is the Riesz kernel $|x|^{\beta - n}$. To find out more about this type of thinness, see for example [2, pp. 155–158]. (Note that we here used β instead of α as the parameter in an attempt to avoid confusion.)

Now let the set E be the archipelago Γ. What can be said about the β–thinness of E if the sequence $\{\varphi_i\}$ is known? We have immediately that if there is a bounded subsequence of $\{\varphi_i\}$ then $\xi \in \Lambda_c$ and thus in $\mathfrak{N}$ due to Proposition 4.14 in [6] which then would imply that E is not β–thin at ξ (since *minimal thinness* is 0–thinness). The follwing result gives a more precise statement.

Proposition 8.2. *Let $\{\varphi_i\}$ and ξ be as above and let $\beta \in [0, 1)$. The archipelago of Γ is not β–thin at ξ if*

$$\sum e^{-(n-\beta)\varphi_i} = \infty.$$

Proof. Let E be the archipelago of Γ. Let $\{Q_k\}$ be a Whitney decomposition of the unit ball. Using the estimates in Lemma 4.11 in [6] and Corollary 7.4.3 on p. 155 in [2] we obtain that E is β–thin at ξ if and only if

$$\bigcup_{Q_k \cap E \neq \emptyset} Q_k \text{ is } \beta\text{–thin at } \xi.$$

Corollary 7.4.3 (iv) in [2] tells us then that E is β–thin at ξ if and only if

$$\sum_{Q_k \cap E \neq \emptyset} \left(\frac{\operatorname{diam}(Q_k)}{\operatorname{dist}(Q_k, \xi)} \right)^{\mathrm{n}-\beta} < \infty.$$

Thus from Lemma 4.11 in [6] and (7) and (8) in its proof we have that

$$\sum_{Q_k \cap E \neq \emptyset} \left(\frac{\operatorname{diam}(Q_k)}{\operatorname{dist}(Q_k, \xi)} \right)^{\mathrm{n}-\beta} \geq C \sum_{\{\gamma_j \in \Gamma\}} \left(\frac{|1 - |\gamma_j(0)||}{|\xi - \gamma_j(0)|} \right)^{\mathrm{n}-\beta} \geq C \sum_{\{\varphi_i\}} \left(\frac{h_i}{R_i} \right)^{\mathrm{n}-\beta},$$

with the notation from Figure 5 above. Since

$$\frac{h_i}{R_i} = \cos(\theta_i) = \frac{1}{\cosh(\varphi_i)},$$

we conclude that E is not β–thin at ξ if

$$\sum_{\{\varphi_i\}} e^{-(\mathrm{n}-\beta)\varphi_i} = \infty.$$

$\square$

Corollary 8.3.

$$\textit{If } \sum e^{-n\varphi_i} = \infty \textit{ then } \xi \in \mathfrak{M}.$$

The corollary follows immediately from the above proposition since 0–thinness is minimal thinness. We will use Corollary 8.3 to give a concrete example in Section 9 below, of a Fuchsian group with a limit point $\xi \in \mathfrak{M} \setminus \Lambda_c$.

What can we say about *rarefiedness*? Nothing in general is the negative answer. In Lemma 6.3 in [6] we see that if ξ is a fixed point for a parabolic element in the Fuchsian group Γ then the archipelago is not rarefied at ξ although $\{\varphi_i\}$ could even be empty.

9. A counter example

Let us again study the “jungle gym” in figure 6. Where we assumed that the distance between the centers of two consecutive holes are 1 and that the shortest closed arc around in such a hole also has unit length.

Given a starting point, x_0, we can completely determine the geodesic, and thus the related limit point ξ on the unit sphere for the underlying discrete group Γ, by the number of turns $g(t)$ makes in each hole. Let us suppose that the holes are visited in strict order going to the “right” for example. Thus φ is increasing. Let us denote the number of turns in the j-th hole by N_j. We will show that if we choose N_j to be the upper integer part of $\exp(2j)/j$ then will ξ be in $\mathfrak{M}$ but not in Λ_c.

Note that in this set up $\varphi_j \approx j$ and thus $\varphi_j \to \infty$ and hence $\xi \notin \Lambda_c$. From Corollary 8.3 it is sufficient to show that with the choice of N_j as above $\sum_{\{\varphi_i\}} e^{-2\varphi_i} = \infty$.

$$\sum_{\{\varphi_i\}} e^{-2\varphi_i} \geq \sum_{\{\text{ hole}_j\}} N_j e^{-2\varphi_j} \approx \sum_{\{\text{ hole}_j\}} N_j e^{-2j} \geq$$

$$\geq \sum_{\{\text{ hole}_j\}} \frac{e^{2j}}{j} e^{-2j} = \sum_{\{\text{ hole}_j\}} \frac{1}{j} = \infty.$$

Hence we conclude that $\Lambda_c \neq \mathfrak{N}$.

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