Effect of space in the game "war of attrition"

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Spatial dynamics has in many cases been invoked as a mechanism that can promote the evolution of coexistence and cooperation, although the precise conditions for this to occur have not yet been characterised. In an effort to address this question we have analyzed an alternative version of the theoretical game "war of attrition," which exhibits unusual behavior: The well-mixed system exhibits quasistationary coexistence and a relaxation time that scales exponentially with the system size, while the spatial system shows a relaxation time that is considerably smaller and scales with a power $\alpha \approx 1.4$ of the system size. Further, the spatial system exhibits a first-order phase transition in the strategy distribution at a consolation prize of $k \approx 1/3$. Close to this point the relaxation time diverges with an exponent $\gamma \approx 1.2$. This analysis shows that the effect of space is highly dependent on the type of game considered.

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I. BACKGROUND

Maynard Smith and Price presented the game called *the war of attrition*, which was inspired by a standoff situation where two animals do not fight over territory, but rather wait each other out [1]. Assuming that the two animals have strategies (i.e., maximal waiting times) T_1 and T_2, where $T_1 > T_2$, the game will finish after T_2 time units has passed, when the shorter waiting time has been reached, and the corresponding player gives up. The winner, i.e., the player with strategy T_1 prepared to wait the longest, will get a unity payoff, while the loser has to go for the less attractive territory and therefore receives a consolation payoff of $k \leqslant 1$. However, both have to pay a price for the time T_2 spent in the game, which means that the total payoff for the winner is $1 - T_2$, while that for the loser is $k - T_2$. Maynard Smith and Price found the evolutionary stable stategy (ESS) of this game as a mixed strategy of waiting times described by the formula $P(t) = 1/(1-k)e^{-t/(1-k)}$. Since then, many authors have analyzed variants of this game, for example, multiple player situations [2] and asymmetric predator-prey versions [3].

II. MOTIVATION

In the above-mentioned studies the population of players was considered well mixed, or in the context of statistical mechanics, a mean-field approach was used. The players were paired together in a random fashion, averaging out any possible spatial structure and local correlations. However, the dynamics of many biological systems depends crucially on spatial separation, and this has also been shown to be true in game theoretic models. In the "prisoner's dilemma" game [4] the well-mixed case always gives rise to defection as the dominant strategy, while in a spatial configuration cooperation can persist, and coexistence of strategies is observed [5]. In a game similar to the rock-scissors-paper game modeled on bacterial competition, a similar phenomenon is observed [6]. When interactions occur locally the model shows coexistence, while well-mixed dynamics leads to the dominance of a single strategy. The introduction of a spatial component can thus have a positive impact on the level of coexistence. Motivated by this, we have investigated a spatially extended version of the war of attrition.

As a starting point of our study we have taken an extension of the war of attrition, which was introduced by Eriksson *et al.* [7], in which the cost function, instead of being given explicitly as the time spent in the game, was given only implicitly. This was achieved by considering the social dynamics of a population of players, where the cost for playing emerges implicitly, as the games missed out by players already engaged in a waiting contest. Players with long waiting times run the risk of getting caught in very long games, which, because of the constant payoff for winning a single game, results in low payoff per unit time. An analytic expression for the ESS was found and the evolutionary dynamics turned out to be significantly different for consolation prizes below or above a certain break-point around $k \approx 0.17$. For $k \gtrsim 0.17$ the population converged to a stable distribution of waiting times, while for smaller consolation prizes oscillatory dynamics were observed.

III. MODEL

In the model of Eriksson *et al.* the strategy space was given by the non-negative real numbers, but in an effort to simplify the system we have considered a fully coarse-grained strategy space that contains only two possible strategies: impatient players I and stubborn players S. The impatient players have a waiting time equal to zero, while the stubborn players are ready to wait essentially an infinite amount of time. This means that an S player always beats an I player and that the game finishes instantly (as the I player gives up instantly). The game between two I players also finishes instantly, this time with a draw, while the game between two S players continues *ad infinitum* and hence does not produce a winner. The only way to end such a game is for an I player to challenge one of the two occupied S players. This means that we do not need to keep track of the never-ending games between S players; instead the games finish instantly without any payoffs.

We will consider the dynamics of a population of N players both in the well-mixed (WM) case where encounters are

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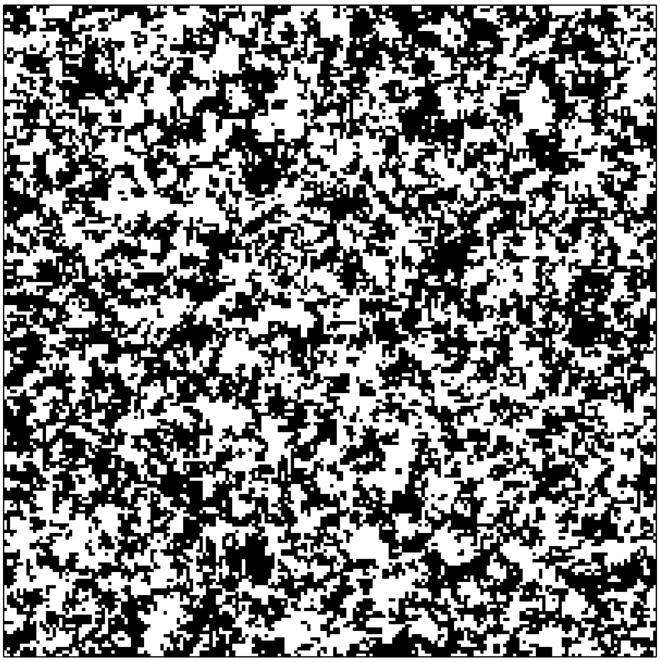

FIG. 1. Distribution of impatient players (white) and stubborn players (black) on 200 × 200 lattice when the consolation prize $k = 1/3$. Periodic boundary conditions have been imposed.

completely random and a spatial system where the players are located on a square lattice of dimension two and only interact with their neighbors in a von Neumann neighborhood.

In the spatial system each iteration of the model starts by picking two neighboring agents and letting them play one round of the game. At the end of the game a reproduction event occurs proportional to the payoffs received by the agents. As before the winner receives a unity payoff, while the loser gets the consolation prize k. The dynamics of the model can be summarized in the following way:

$$\begin{aligned} &(I,I) \to (I,I) + I, \\ &(S,I) \to (S,I) + \begin{cases} I & \text{with probability } k/(k+1) \\ S & \text{with probability } 1/(k+1), \end{cases} \\ &(S,S) \to (S,S), \end{aligned} \quad (1)$$

where the players in brackets represent the agents involved in the game and the offspring replaces a random agent in the neighborhood of the parent agent (i.e., not necessarily the losing agent). In the (I,I) case the parent is chosen with equal probability among the two players since the game ends with a draw. From this formulation it is clear that the model is equivalent to a stochastic two-state two-dimensional cellular automata with a neighborhood consisting of second nearest (von Neumann) neighbors. An example of a spatial configuration arising through the dynamics of the system is shown in Fig. 1, which is the result of a random initial condition with equal numbers of I and S players, which has been iterated 10^8 time steps with the consolation prize $k = 1/3$.

In the WM model, the two interacting agents are chosen at random in the population and the offspring replaces a randomly chosen player among the entire population conserving the population size. That is, the dynamics of the spatial and well-mixed models are identical up to the neighborhood structure. As there is no spatial configuration in the WM model, the state of the system is fully determined by specifying the number N_s of S players (or I players) and the transitions from one state to another only depends on this variable (and not on the history). This means that the WM model is equivalent to a Markov chain with N states, a fact we will make use of later on. It is also worth mentioning that both the spatial and the WM model contain two absorbing states: only S players or only I players. Summarizing, the model contains two types of strategies, stubborn and impatient players, and two parameters, the population size N and the consolation prize $0 < k < 1$.

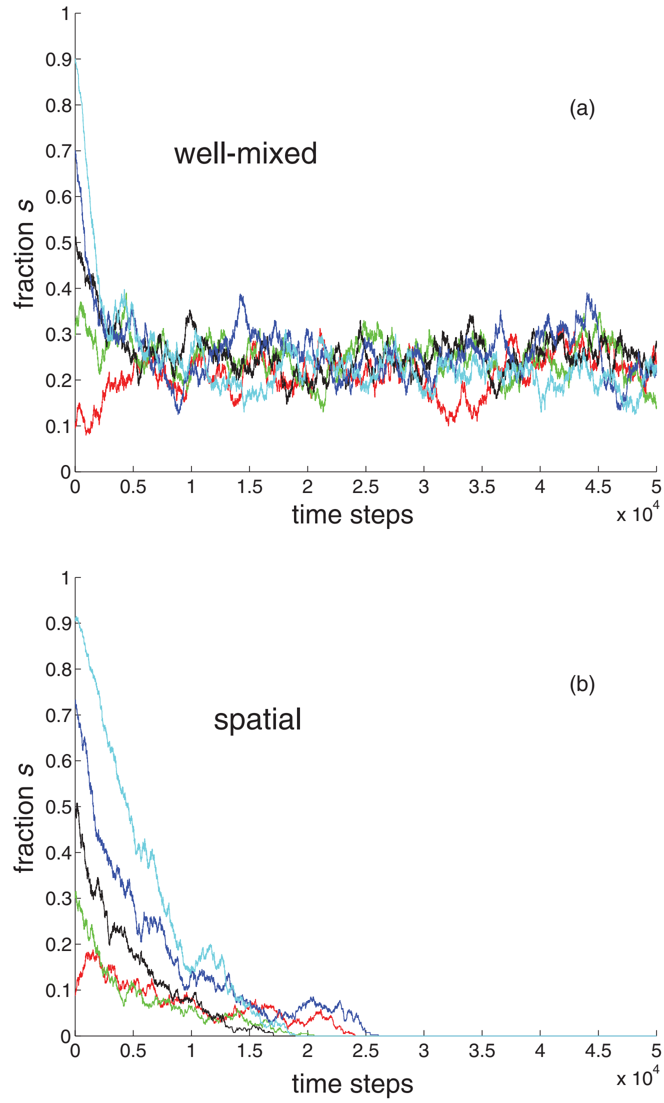

FIG. 2. (Color online) Fraction of stubborn players as a function of time for different initial conditions in both the (a) well-mixed and (b) spatial setups. Here the population size is $N = 400$ and the consolation prize $k = 0.6$.

IV. RESULTS

Figure 2 shows the fraction $s = N_s/N$ of S players as a function of time over 5×10^4 time steps in the (a) well-mixed and (b) spatial cases for a range of different initial conditions. The population size is set to $N = 400$ and $k = 0.6$. From

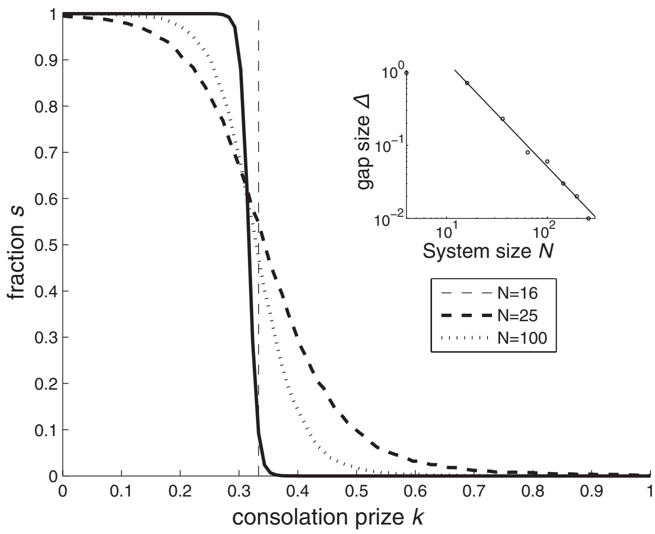

FIG. 3. Fraction of S players in the spatial model after 10^{10} iterations for $N = 16$, 25, and 100. The dashed vertical line corresponds to $k = \frac{1}{3}$. The inset shows that the size Δ of the region where coexistence occurs ($0 < s < 1$) scales as $\Delta \sim N^{-1.44}$.

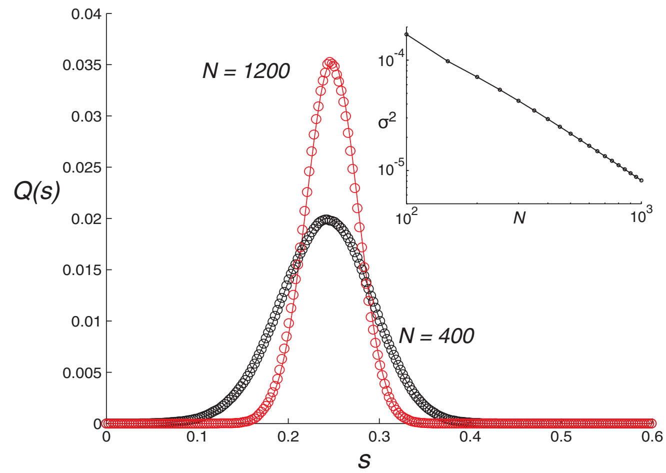

FIG. 4. (Color online) Quasistationary distribution $Q(s)$ of the fraction of S players in the well-mixed model when $N = 400,1200$ and $k = 0.6$. The circles show the values obtained from simulation and the solid lines display the results obtained from the nonabsorbing Markov chain. The inset shows that for the consolation prize fixed at $k = 0.6$ the variance of the distribution scales as $\sigma^2 \sim N^{-1.3}$.

this we can see that the WM model does not immediately drift into an absorbing state but instead exhibits coexistence and fluctuates around some intermediate fraction of stubborn players, while the spatial model rapidly converges to a state that contains only I players. This suggests an increased propensity for a single strategy to dominate the population in the spatial system. However, since there are two absorbing states ($s = 0$ and 1) for both the models and the system will eventually end up in one of them, we are essentially observing a difference in transient behavior.

In order to investigate if the rapid convergence to the absorbing state is a general feature of the spatial model, we simulate the system, with the consolation prize k ranging from 0 to 1, until it reaches an absorbing state or a maximum of 10^{10} time steps. The final fraction of S players is shown in Fig. 3 for $N = 16$, 25, and 100. For the largest system size the dynamics exhibit a sharp threshold at $k \approx 1/3$. For $k < 0.3$ the system always ends up containing only S players, while for $k > 0.36$ the system is absorbed into the opposite state, i.e., the size of the interval of k values where coexistence occurs is given by $\Delta \approx 0.36–0.3 = 0.06$. For smaller N the transition is smoother and it turns out that the width Δ of the region where coexistence occurs (i.e., where $0 < s < 1$) scales as $\Delta \sim N^{-1.44}$ (Fig. 3, inset). The nature of this seeming phase transition will be investigated further, but first we explore the dynamics of the well-mixed model.

In the well-mixed system the time to absorption is considerably longer, which implies that the behavior of the system can be characterized by looking at the quasistationary distribution $Q(s)$, the probability that the system contains a fraction s of stubborn players, given that it has not reached fixation. Such a distribution can be obtained by considering the Markov chain that describes the dynamics of the WM model [given in Eq. (2)] and modifying it by adding infinitesimal escape probabilities ε out of the two absorbing states. The stationary distribution (independent of ε) for this related process can then be computed using standard numerical techniques [8]. Two such quasistationary distributions are plotted in Fig. 4, one of which ($N = 400$) describes the behavior of the simulations shown in Fig. 2(a). The inset of Fig. 4 shows that as the system grows the quasistationary distribution becomes narrower or, more precisely, that the variance of the distribution scales as $\sigma^2 \sim N^{-\alpha}$, where $\alpha \approx 1.3$.

The simulation results presented in Fig. 2(a) suggest that the well-mixed model attains some quasistationary state around which it fluctuates. In order to find an analytical expression for the quasistationary state we view the WM model as a Markov chain with the number of S players N_s as the state variable. For a given N and N_s we can, by viewing Eq. (1) as a transition between different states, derive the following transition probabilities:

$$P(N_s \to N_s + 1) = \frac{2s(1-s)^2}{(1+k)},$$
$$P(N_s \to N_s - 1) = \frac{2s^2(1-s)k}{(1+k)} + s(1-s)^2. \tag{2}$$

The quasistationary state is given by the value of s for which the probability of increasing and decreasing the number of S players is equal and can thus be found by setting $P(N_s \to N_s + 1) = P(N_s \to N_s - 1)$ and solving for s. By doing this we obtain the quasistationary solution as

$$s^\star = \frac{1-k}{1+k}. \tag{3}$$

It is straightforward to show that $s^\star$ also is an ESS in the two-player game with the payoff matrix $\pi_{SS} = 0, \pi_{SI} = 1, \pi_{IS} = k$, and $\pi_{II} = \frac{1+k}{2}$ and that $s^\star$ is a stable fixed point for the corresponding replicator equation.

The expression in Eq. (3) together with the fraction obtained from simulating the WM model for a range of different values of k is shown in Fig. 5. From an initial condition of $s = 0.5$ a system of $N = 400$ agents was iterated for 5×10^5 time steps and the final fraction of S players was averaged over

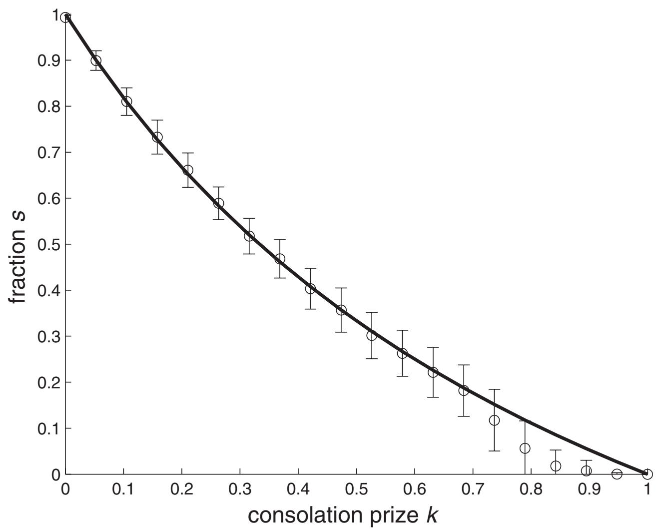

FIG. 5. Quasistationary state as a function of the consolation prize k. The solid line corresponds to the analytically derived expression $s^\star = (1-k)/(1+k)$ and the circles correspond to values obtained from simulation. The error bars show standard error. Note that for k values larger than 0.7, the simulated values are all lower than the analytic values.

20 simulations. From this plot we can observe that Eq. (3) gives a good approximation of the behavior of the system, although deviations occur for k close to unity. This is a finite-size effect, where the impatient players might become fixed due to fluctuations in the system. These are greater for small values of s since impatient players both spread and decay more rapidly than the stubborn players.

In order to quantify the long-term behavior of the system we analyze the relaxation time T_r, i.e., the time it takes for the system to reach one of the absorbing states (all S or all I players), as a function of the model parameters N and k in both the spatial and WM setups. For example, in Fig. 2(b) it corresponds to the average number of time steps that elapse before the fraction of S players reaches zero. The results are summarized in Fig. 6, where Figs. 6(a) and 6(c) correspond to the spatial system and Figs. 6(b) and 6(d) correspond to the nonspatial system. Starting with the dependence on system size, we observe two things. First, the relaxation time is much larger for the WM system compared to the spatial system. For $N = 400$ it takes on average 10^{12} time steps for the WM system to reach an absorbing state, while the corresponding value for the spatial system is 3×10^4. Second, the relaxation time scales with N in different ways in the two cases: For the spatial system we have that $T_r \sim N^\alpha$ for $\alpha \approx 1.4$, while for the WM system we see that $T_r \sim e^{\beta N}$ for $\beta \approx 0.03$. When it comes to the dependence on the consolation prize k in the spatial system [Fig. 6(b)] we observe that the relaxation time diverges in the vicinity of the transition point as $T_r \sim |k - k_c|^{-\gamma}$ for $\gamma \approx 1.2$, where we have used $k_c = 1/3$. In the well-mixed case we also observe a sharp increase of the relaxation time, but at a lower value of $k \approx 0.2$ and with a broader peak.

A. Phase transition

From the results presented so far it is clear the spatial component in the model has a significant impact on the dynamics. The WM system exhibits coexistence of the two strategies at a fraction $s^\star = s^\star(k)$ of S players, while the spatial system rapidly (compared to the relaxation time for the WM system) goes into one of the absorbing states. The transition between the absorbing states shown in Fig. 3 is quite sharp and suggests a phase transition in the fraction of S players at $k_c = 1/3$. In order to characterize this transition further, we measure the spatial correlations at the critical point $k = 1/3$ [9] using the pair correlation function $G(r)$. We find that $G(r) \sim e^{-r/\xi}$, with a correlation length $\xi \approx 1.2$. Hence the length scale in the system does not diverge at the critical point. This can be seen in Fig. 1, which shows that spatial distribution of strategies for $k = 1/3$, where a typical length scale of the S and I domains is present even though the system is at the critical point. The temporal autocorrelation [9] in contrast exhibits a scaling of the form $C(\tau) \sim \tau^{-\delta}$ for $\delta \approx 0.2$ and $\tau < 10$. Here τ is measured in generations (corresponding to N games), suggesting long-time memory effects in the system.

Taken together these results suggest that the spatial system exhibits a first-order phase transition, where the order parameter, the fraction of S players, changes discontinuously at the critical point $k_c = 1/3$, at which the correlation length remains bounded. In contrast we observe a divergence in the relaxation time with $T_r \sim |k - k_c|^{-\gamma}$ and a scale-free decay in the temporal autocorrelation function.

V. DISCUSSION

An explanation as to why the relaxation time in the spatial system is so much shorter compared to the WM system is that stubborn players are confined into regions within which they can enter only extremely long games, in our setup approximated as zero payoff games, while they can be exploited by (or exploit depending on the value of k) impatient players at the domain boundaries. In fact, on a one-dimensional lattice a domain of I players will always expand into surrounding S players independent of the value of k, showing that coexistence is impossible in this case. In the WM model there are no boundaries to impede the dynamics, but counterintuitively this leads to a lower degree of competition, which is due to the fact that the lack of spatial structure balances the competition between the strategies in favor of the weaker strategy. In the continuum limit of the well-mixed model, when the number of players $N \to \infty$ (i.e., the replicator equation) the coexistence is realized by an internal stable fixed point whose position is given by $s^\star$ in Eq. (3). When $s > s^\star$ the impatient players have a higher fitness, while for $s < s^\star$ the stubborn players are favored.

The location of the critical point k_c, which from both the fraction (Fig. 3) and the relaxation time plot [Fig. 6(c)] seems to occur at $k \approx 1/3$, can be arrived at by inspecting Eq. (3). If we (falsely) assume that all possible neighborhoods are equally frequent in the spatial system and further that the system is at the critical point where both strategies are equally frequent, i.e., $s = 1/2$, then by plugging this value into Eq. (3) we can solve for k and find that $k_c = 1/3$. This should not be taken as a proof that the phase transition occurs at the derived value but rather as a conjecture supported by numerical evidence.

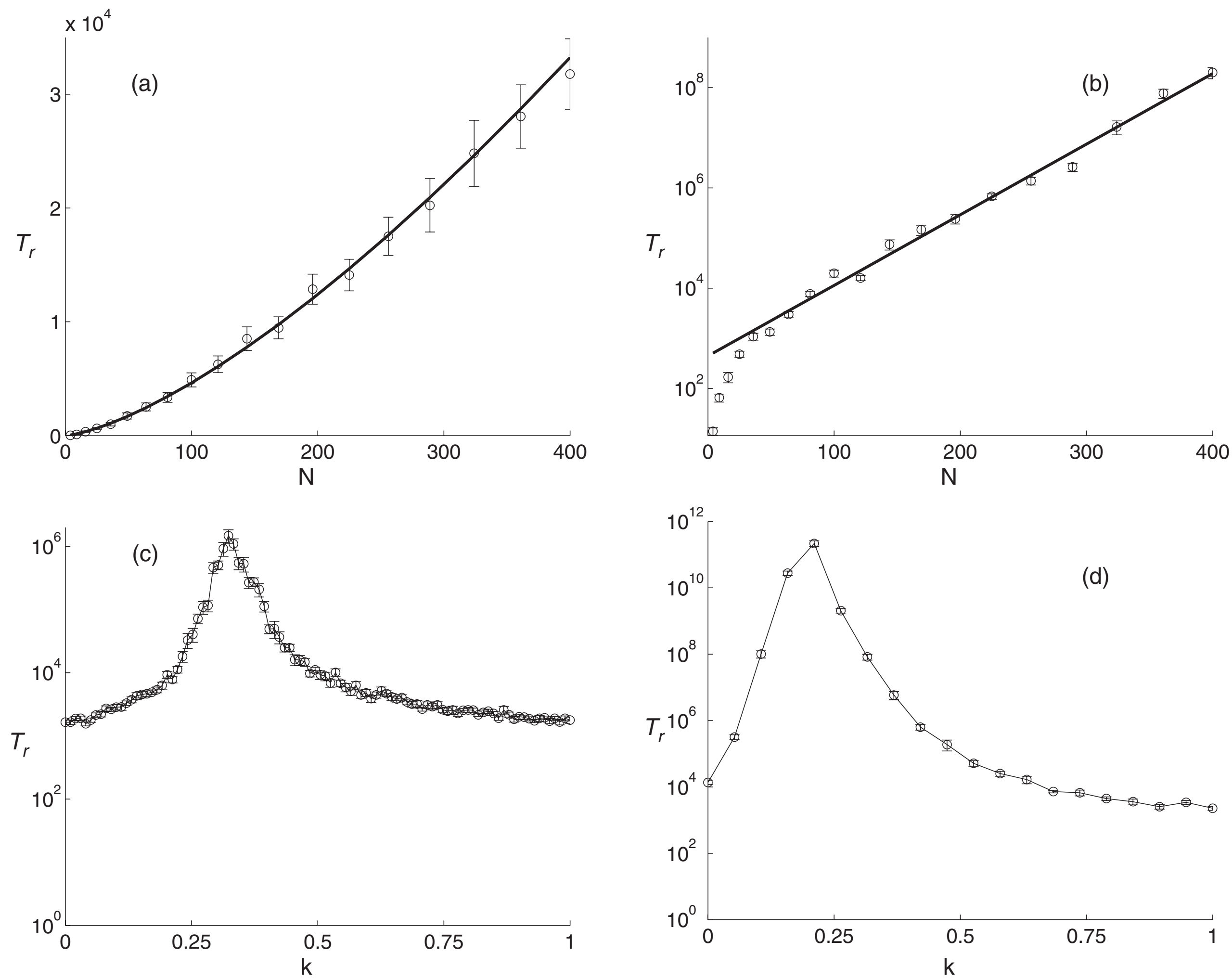

FIG. 6. Relaxation times as a function of the population size N and k for both (a) and (c) spatial and (b) and (d) well-mixed games. In (a) and (b) the consolation prize is fixed at $k = 0.6$ and in (b) and (d) the population size is fixed at $N = 400$. Error bars show standard error.

The observation that a spatial component inhibits coexistence of different strategies is contrary to what has been seen in most game theoretic models (see, for example, Ref. [10]), although some instances have been reported [11–14]. All these examples belong to the so-called directed percolation universality class described in Chap. 6 in Ref. [15]. The snowdrift (or hawk-dove) game is also known to exhibit less diversity in a spatial setting [16]. Interestingly, that system also shows a phase transition in its spatial version, although the properties of that transition have not yet been properly characterized.

VI. CONCLUSION

In conclusion, we have in this paper introduced and analyzed the dynamics of a simplified version of the war of attrition in the framework of statistical mechanics. Our analysis has shown that the spatial system exhibits dominance of a single strategy and a first-order phase transition in the strategy distribution depending on the consolation prize k, while the well-mixed model displays a larger degree of coexistence for which we derived an expression for the strategy distribution. These results emphasize the effect of spatial interactions in game theoretical models and the usefulness of applying the concepts of statistical mechanics to models such as these.

ACKNOWLEDGMENTS

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[1] J. Maynard Smith and G. R. Price, Nature (London) **246**, 15 (1973).
[2] J. Haigh and C. Cannings, Acta Appl. Math. **14**, 59 (1989).
[3] D. Hugie, Anim. Behav. **67**, 823 (2004).
[4] R. Axelrod and W. Hamilton, Science **211**, 1390 (1981).
[5] M. Nowak and R. May, Nature (London) **359**, 826 (1992).
[6] B. Kerr, M. Riley, M. Feldman, and B. Bohannan, Nature (London) **418**, 171 (2002).

[7] A. Eriksson, K. Lindgren, and T. Lundh, J. Theor. Biol. **230**, 319 (2004).
[8] J. Darroch and E. Seneta, J. Appl. Probab. **2**, 88 (1965).
[9] C. Sinha, Physica D **39**, 352 (1989).
[10] D. Helbing, A. Szolnoki, M. Perc, and G. Szabó, New J. Phys. **12**, 083005 (2010).
[11] Y.-C. Ni, H. P. Yin, C. Xu, and P. M. Hui, Eur. Phys. J. B **80**, 233 (2011).
[12] Y.-Y. Yu, C. Xu, G.-Q. Gu, and P. M. Hui, Phys. Rev. E **77**, 011106 (2008).
[13] D. Hernández and D. Zanette, Eur. Phys. J. B **82**, 361 (2011).
[14] G. Szabó, A. Szolnoki, M. Varga, and L. Hanusovszky, Phys. Rev. E **82**, 026110 (2010).
[15] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, 2005).
[16] C. Hauert and M. Doebeli, Nature (London) **428**, 643 (2004).