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ON THE MAXIMAL FUNCTION FOR THE MEHLER KERNEL.

Peter Sjögren

1. INTRODUCTION.

Let $Nu = -\Delta u + x\cdot \operatorname{grad} u$ be the well-known number operator for the quantum-mechanical harmonic oscillator in $\mathbb{R}^n$. In $\mathbb{R}^{n+1}_+ = \{(x,t): x\in\mathbb{R}^n,\ t>0\}$, the initial-value problem

$$-\frac{\partial u}{\partial t} = Nu$$

$$u(x,0) = f(x)$$

is solved by

$$u(x,t) = e^{-tN}f(x) = M_\lambda f(x) = \int M_\lambda(x,y)f(y)dy$$

with $\lambda = e^{-t}$. Here

$$M_\lambda(x,y) = (2\pi(1-\lambda^2))^{-n/2}\exp\left(-\frac{(y-\lambda x)^2}{2(1-\lambda^2)}\right)$$

is the Lebesgue measure form of the Mehler kernel, and $(e^{-tN})_{t>0}$ is the Hermite semigroup, whose infinitesimal generator is $-N$. The n-dimensional Hermite polynomials

$$H_m(x) = \prod_1^n H_{m_i}(x_i), \quad m = (m_1,\dots,m_n)\in\mathbb{N}^n ,$$

are defined so as to be orthogonal with respect to the canonical Gaussian measure γ, whose density is $\gamma(x) = (2\pi)^{-n/2}\exp(-|x|^2/2)$. In terms of these polynomials, M_λ is conveniently expressed:

$$M_\lambda\ \Sigma a_m H_m = \Sigma\lambda^{|m|}\ a_m H_m, \quad |m| = \Sigma m_i .$$

The operators M_λ are bounded and of norm 1 on L^p_γ, $1 \le p \le \infty$, and they are self-adjoint on L^2_γ. Further, they are given by a positive kernel and leave constant functions invariant. This makes the maximal theorem from semigroup theory (see Stein [3, III.3]) applicable. Hence, the operator

$$M^* f(x) = \sup_{0<\lambda<1} |M_\lambda f(x)|$$

is bounded on L^p_γ, $1 < p < \infty$. This works even in infinite dimension.

The one-dimensional case is studied by Muckenhoupt [1], who also shows that M^* maps L^1_γ into $L^{1,\infty}_\gamma$ (i.e., weak L^1_γ). We shall prove the same thing in arbitrary finite dimension. Of course, estimates for M^* imply convergence results for $M_\lambda f$ as $\lambda \to 1$ $(t \to 0)$.

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<u>Theorem</u>. <u>For each finite dimension</u> n, <u>the operator</u> M^* <u>is bounded from</u> L^1_γ <u>into</u> $L^{1,\infty}_\gamma$.

We need some notation for the proof. If $D \subset \mathbb{R}^n \times \mathbb{R}^n$, we let $D^x = \{y: (x,y) \in D\}$ for $x \in \mathbb{R}^n$, and slightly abusively, $D^y = \{x: (x,y) \in D\}$ for $y \in \mathbb{R}^n$. By $c > 0$ and $C < \infty$, we denote various constants, and $f \sim g$ means $c \le f/g \le C$.

2. First part of the proof.

Notice that

$$\gamma(x) \sim \gamma(y) \quad \text{for} \quad |x-y| < C/|y| \tag{2.1}$$

when y stays away from 0. We first study M^* when x is near y in this sense, setting for $R > 0$

$$N_R = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x| \le R \text{ and } |y| \le R, \text{ or } |y| \ge R/2 \text{ and } |x-y| \le R/|y|\}.$$

Lemma 1. *The operator*

$$f \to \sup_{0<\lambda<1} \left| \int_{N_R^x} M_\lambda(x,y) f(y)\,dy \right|$$

maps L^1_γ *boundedly into* $L^{1,\infty}_\gamma$, *for any* $R < \infty$.

Proof. We cover $\mathbb{R}^n$ with $B(0,R)$ together with a sequence of balls of type $B(z, CR/|z|)$, $|z| \ge R/2$, with bounded overlap, so that $(x,y) \in N_R$ implies that one of these balls contains x and y. Hence, it is enough to verify that the restriction of M^* to a ball of this type is uniformly of weak type (1,1) for γ or, equivalently in view of (2.1), for Lebesgue measure. Because of the bounded overlap, we may then add these estimates and obtain the lemma.

So take $g \ge 0$ in $L^1(B)$ (Lebesgue measure), with $B = B(z, CR/|z|)$. Now if $\sqrt{1-\lambda^2} \ge 1/|z|$ and $x, y \in B$, we can estimate $M_\lambda(x,y)$ by $(1-\lambda^2)^{-n/2} \le C|B|^{-1}$, $C = C(R,n)$, where $|\cdot|$ means Lebesgue measure. Hence,

$$\int_B M_\lambda(x,y) g(y)\,dy \le C|B|^{-1} \int_B g\,dy \le Cg^*(x),$$

g^* denoting the Hardy-Littlewood maximal function. And if $\sqrt{1-\lambda^2} < 1/|z|$ and $x, y \in B$, then $|y-\lambda x|$ differs from $|x-y|$ by at most $(1-\lambda)|x| < C\sqrt{1-\lambda^2}$. Considering separately the cases when $|y-\lambda x|$ is or is not much larger than $\sqrt{1-\lambda^2}$, we see that

$$\exp\left(-\frac{|y-\lambda x|^2}{2(1-\lambda^2)}\right) \le C \exp\left(-\frac{c|y-x|^2}{1-\lambda^2}\right)$$

for some c. But then $\int M_\lambda(x,y) g(y)\,dy$ is bounded in B by a convolution of g with a normalized contraction of the integrable radial decreasing kernel $C\exp(-c|.|^2)$ and thus by $Cg^*(x)$, see [4, III.2.2]. Lemma 1 follows since the case of $B(0,R)$ is similar.

Outside N_R, we shall estimate M^* by the operator defined by the pointwise sup kernel

$$M(x,y) = \sup_{0<\lambda<1} M_\lambda(x,y).$$

Lemma 2. *For some* R, *the operator*

$$f \mapsto \int_{\mathbb{R}^n \setminus N_R^x} M(x,y) f(y)\,dy$$

maps L^1_γ *into* $L^{1,\infty}_\gamma$.

This would clearly imply the theorem.

We must estimate M and need some notation. If $y \neq 0$, let $\eta = |y|$ and $e = y/\eta$, and set $x = \xi e + v$ where v is orthogonal to e. By a and A we mean, respectively, $\min(\xi,\eta)$ and $\max(\xi,\eta)$. Of course, $\xi_+ = \max(\xi,0)$.

Lemma 3. *Given a small* $\beta > 0$, *we may choose* R *so that the following estimates hold when* $(x,y) \notin N_R$ *for some* c *and* C *depending only on* β, R, *and* n.

(a) *If* $|x-y| \geq \beta \max(|x|,|y|)$, *then*

$$M(x,y) \leq C \min\left(1, e^{\xi_+^2/2 - \eta^2/2}\right).$$

(b) *If* $|x-y| < \beta \max(|x|,|y|)$ *and* $|v| < A-a$, *then*

$$M(x,y) \leq C\left(\frac{A}{A-a}\right)^{n/2} \exp\left(-\frac{cA|v|^2}{A-a}\right) \min(1, e^{\xi^2/2 - \eta^2/2}).$$

(c) *If* $|x-y| < \beta \max(|x|,|y|)$ *and* $|v| \geq A-a$, *then*

$$M(x,y) \leq C(A/|v|)^{n/2} \exp(-cA|v|) \min(1, e^{\xi^2/2 - \eta^2/2}).$$

Proof. For x and y fixed, $x \neq y$, it is easily seen that $M_\lambda(x,y)$ takes its sup in $0 < \lambda < 1$ for some λ_{max} in $[0,1[$. The derivative $\partial M_\lambda(x,y)/\partial\lambda$ equals a positive factor times

$$\begin{aligned} U &= n\lambda(1-\lambda^2) + (x-\lambda y)\cdot(y-\lambda x) \\ &= n\lambda(1-\lambda^2) - \lambda|v|^2 + (\xi-\lambda\eta)(\eta-\lambda\xi) = I - II + P. \end{aligned} \tag{2.2}$$

Here the last product is

$$P = (A-\lambda a)(a-\lambda A) = ((1-\lambda)a + A-a)((1-\lambda)A - (A-a)). \tag{2.3}$$

If we replace η and v by 0 here, we see that then $\lambda_{max} = (a/A)_+$ and so

$$\sup_{0<\lambda<1} \exp\left(-\frac{(\eta-\lambda\xi)^2}{2(1-\lambda^2)}\right) = \min\left(1,\ e^{\xi_+^2/2 - \eta^2/2}\right). \tag{2.4}$$

In case (a), we conclude from (2.2) that

$$\begin{aligned} U &\le n - (x-y)\cdot(y-x) + (1-\lambda)(|x| + |y|)^2 \\ &\le n - |x-y|^2 + 4(1-\lambda)\max(|x|,|y|)^2 < 0 \end{aligned}$$

if R is large and λ close to 1. Hence, λ_{max} is bounded away from 1, and (2.4) gives the estimate in (a).

In case (b), notice that $a > A/2$ and $A-a > RA^{-1}/2$ because $|x-y| > R/|y|$. We see from (2.3) that

$$P \sim (1-\lambda)^2A^2 \quad \text{for } 1-\lambda > 4(A-a)A^{-1}. \tag{2.5}$$

So if $1-\lambda$ is much larger than $(A-a)A^{-1}$, then $II < P$ and $U > 0$. And if $1-\lambda < (A-a)A^{-1}/2$, we see by estimating I and P that

$$U < n(A-a)A^{-1} - (A-a)^2/2 < 0$$

for suitable R. Hence, $1-\lambda_{max} \sim (A-a)A^{-1}$, and (b) follows from (2.4).

To prove (c), we may assume $|v| > B(A-a)$ for any fixed B, since the contrary case is covered by the method of (b). Notice that $|v| > RA^{-1}/2$. It is enough to show that

$$1 - \lambda_{max} \sim |v|A^{-1}. \tag{2.6}$$

For $1-\lambda < c|v|A^{-1}$, we have

$$I/II < 2n(1-\lambda)/|v|^2 < cA^{-1}/|v| < 1/2.$$

Since (2.5) remains valid, (2.6) follows if we can exclude $1-\lambda_{max} \le 4(A-a)A^{-1}$. But $1-\lambda \le 4(A-a)A^{-1}$ implies $P < C(A-a)^2 < II/2$, and thus $U < 0$, if B is large enough. This completes the proof of (c) and Lemma 3.

3. Proof of Lemma 2.

We introduce sets forming a disjoint partition of $\mathbb{R}^n \times \mathbb{R}^n \setminus N_R$ if $\beta > 0$ is small. Let $\alpha(x,y)$ denote the angle between non-zero x and y, satisfying $0 \le \alpha(x,y) \le \pi$, and define

$$D_1 = \{(x,y) \notin N_R:\ \xi \le \eta, \text{ and } \alpha(x,y) \ge \pi/4\}$$

$$D_2 = \{(x,y) \notin N_R:\ \xi > \eta, \text{ and } |x-y| \ge \beta \max(|x|,|y|)\}$$

$$D_3 = \{(x,y) \notin N_R:\ |x-y| < \beta \max(|x|,|y|) \text{ or both } \xi \le \eta \text{ and } \alpha(x,y) < \pi/4\}.$$

Take an $f \ge 0$ in L^1_γ. We write

$$\int_{\mathbb{R}^n \setminus N_R^x} M(x,y) f(y)\,dy = \int_{D_1^x} + \int_{D_2^x} + \int_{D_3^x}$$

and estimate these three terms.

The first two terms turn out to be in L^1_γ. For

$$\int \gamma(x)\,dx \int_{D_1^x} M(x,y) f(y)\,dy = \int f(y)\,dy \int_{D_1^y} M(x,y)\gamma(x)\,dx,$$

and the integral over D_1^y here can be estimated by $C\gamma(y)$ if we use Lemma 3(a) and the fact that $\xi_+^2 \leq |x|^2/2$ in D_1^y. As to D_2, we arrive similarly at the integral

$$\int_{D_2^y} e^{-|x|^2/2}\, dx \leq C \int_y^\infty e^{-\xi^2/2}\, d\xi \leq C\gamma(y) .$$

Before dealing with D_3, we divide $\mathbb{R}^n$ into disjoint cubes Q_i centered at x_i, $i = 1,2,\dots$, such that

$$c \min(1, 1/|x_i|) \leq \operatorname{diam} Q_i \leq \min(1, 1/|x_i|).$$

Choose the enumeration so that $|x_i|$ is nondecreasing in i.

If χ denotes the characteristic function of D_3, we set $M_3 = \chi M$. Since we do not want our kernel to vary too much within a Q_i, we let

$$\bar{M}_3(x,y) = \sup\{M_3(x',y):\ x \text{ and } x' \text{ in the same } Q_i\}.$$

Notice that the estimates of Lemma 3 hold also for $\bar{M}_3$, with new constants. Clearly $\bar{M}_3 f(x) = \int \bar{M}_3(x,y) f(y)dy$ dominates $M_3 f(x)$ and is constant in each Q_i.

Given $\alpha > 0$, we shall construct a subset E of $\{x: \bar{M}_3 f(x) > \alpha\}$ such that

$$\gamma\{\bar{M}_3 f > \alpha\} \leq C\gamma(E) \tag{3.1}$$

and

$$U(y) \leq C\gamma(y) \text{ in } \mathbb{R}^n. \tag{3.2}$$

Here

$$U(y) = \int_E \bar{M}_3(x,y)\gamma(x)dx.$$

This would yield

$$\gamma\{M_3 f > \alpha\} \le C\gamma(E) \le C\alpha^{-1}\int_E \bar{M}_3 f(x)\gamma(x)dx$$

$$= C\alpha^{-1}\int f(y)U(y)dy \le C\alpha^{-1}\|f\|_{L^1_\gamma},$$

and thus complete the proof of Lemma 2. This method is similar to that used for Theorem 1 in [2].

The set E will be constructed as the union of certain Q_j, which will be selected inductively. To obtain (3.2), we must not select too many Q_j close to each other. Therefore, we associate with each Q_j a forbidden region F_j, defined as the union of those Q_i, $i > j$, which intersect the set $Q_j + K_j$, where K_j is the cone $\{x: \alpha(x,y) \le \pi/4$ for some $y \in Q_j\}$.

The first step of the construction consists of selecting Q_1 if and only if it intersects, and thus is contained in, $\{\bar{M}_3 f > \alpha\}$. At the ith step, Q_i is selected if and only if it intersects $\{\bar{M}_3 f > \alpha\}$ and is not forbidden, i.e., it is not contained in F_j for any Q_j already selected. Then E is defined as the union of those Q_i selected.

To verify (3.1), we observe that $\{\bar{M}_3 f > \alpha\}$ is contained in the union of those Q_j selected and the corresponding F_j. The Q_j selected of course have total γ-measure $\gamma(E)$. So (3.1) follows if we verify that $\gamma(F_j) \le C\gamma(Q_j)$. When $|x_j| \le C$, we have $\gamma(Q_j) \sim \gamma(\mathbb{R}^n)$, so assume the contrary. Let H_s be the hyperplane $\{x: x\cdot x_j/|x_j| = x_j + s\}$. Then $F_j \cap H_s$ is empty for $s \le -C/|x_j|$, and has $(n-1)$-dimensional Lebesgue measure at most $C\max(s, 1/|x_j|)^{n-1}$ for $s > -C/|x_j|$. On $F_j \cap H_s$, we see that

$$\gamma(x) \le e^{-(s+|x_j|)^2/2} \le e^{-|x_j|^2/2 - |x_j|s}.$$

Hence,

$$\gamma(F_j) \le C\int_{-C/|x_j|}^{\infty} \max(s, 1/|x_j|)^{n-1} e^{-|x_j|^2/2 - |x_j|s}\, ds$$

$$\leq C|x_j|^{-n} e^{-|x_j|^2/2} \sim \gamma(Q_j),$$

and (3.1) follows.

To show (3.2), we fix y and may assume $|y| \geq R/2$ since $D_3^y = \emptyset$ and $U(y) = 0$ otherwise. Let S_y denote the support of $\overline{M}_3(\cdot,y)$, which is the union of those Q_j intersecting D_3^y. For $v \perp e = y/\eta$, $\eta = |y|$, we let $\ell = \ell_v$ denote the line $\{s\,e+v\colon s \in \mathbb{R}\}$, and set

$$I(v) = \int_{\ell_v \cap E \cap S_y} \overline{M}_3(s\,e+v,y)\gamma(s\,e+v)ds$$

so that

$$U(y) = \int I(v)dv, \tag{3.3}$$

the integral taken over $e^{\perp} = \mathbb{R}^{n-1}$.

Assume z belongs to some $Q_j \subset E \cap S_y$. Then Q_j intersects D_3^y and so e is in K_j. Therefore, F_j includes any Q_i, $i > j$, intersecting the ray $\{z + te\colon t > 0\}$. It follows that $\ell_v \cap E \cap S_y$ is contained in an interval $J = \{s\,e+v\colon \xi \leq s \leq \xi + C\min(1,1/|\xi|)$. The point $x = \xi e+v$ is in or near D_3^y. We shall estimate $I(v)$ by means of Lemma 3, and consider the same cases (a), (b), (c) as in this lemma. Let a and A be as there. Notice that the estimates for $M(x,y)$ of Lemma 3 still hold of we replace x by any point in J.

(a) Lemma 3(a) gives

$$I(v) \leq C\min(1,1/|\xi|)e^{\xi_+^2/2 - \eta^2/2 - |x|^2/2} \leq Ce^{-\eta^2/2 - |v|^2/2}. \tag{3.4}$$

(b) Here $A \sim a \sim \eta$ and $A-a > 1/\eta$. Lemma 3(b) gives

$$I(v) \leq C\eta^{-1}\left(\frac{A}{A-a}\right)^{n/2} \exp\left(-\frac{c\,A}{A-a}|v|^2\right)e^{-|x|^2/2} \min(1,e^{\xi^2/2 - \eta^2/2})$$

$$\leq C\eta^{-1}\left(\frac{\eta}{A-a}\right)^{n/2} \exp\left(-\frac{c\,\eta}{A-a}|v|^2\right)e^{-\eta^2/2}.$$

Varying A-a, we see that this expression takes its maximum when $A-a \sim \eta|v|^2$. Such a value of A-a is compatible with $A-a > 1/\eta$ only when $\eta|v| > 1$, and otherwise the largest admissible value of the expression occurs when $A-a \sim 1/\eta$. In both cases, we get

$$I(v) \leq C\eta^{-1} \min(|v|^{-n}, \eta^{n})\ e^{-\eta^2/2}. \tag{3.5}$$

(c) Here $|v| > 1/\eta$, and Lemma 3(c) gives

$$I(v) \leq C\eta^{-1}(\eta/|v|)^{n/2}\ \exp(-\eta|v|)\ e^{-\eta^2/2}.$$

Estimating $\exp(-\eta|v|)$ by $C(\eta|v|)^{-n/2}$, we see that (3.5) holds also in this case.

Applying now (3.4-5) to (3.3), we obtain (3.2), and the proof is complete.

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