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A note on a paper of Harris concerning the asymptotic approximation to the eigenvalues of -y″ + qy = λy, with boundary conditions of general form

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Abstract

In this article, we derive an asymptotic approximation to the eigenvalues of the linear differential equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (a, b)$$

with boundary conditions of general form, when q is a measurable function which has a singularity in (a, b) and which is integrable on subsets of (a, b) which exclude the singularity.

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1. Introduction

Consider the linear differential equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (a, b), \tag{1.1}$$

where λ is a real parameter and q is real-valued function which has a singularity in (a, b). According to [1], an eigenvalue problem may be associate with (1.1) by imposing the boundary conditions

$$y(a)\cos\alpha - y'(a)\sin\alpha = 0, \quad \alpha \in [0, \pi), \tag{1.2}$$

$$y(b)\cos\beta - y'(b)\sin\beta = 0, \quad \beta \in [0, \pi). \tag{1.3}$$

In [2], Atkinson obtained an asymptotic approximation of eigenvalues where y satisfies Dirichlet and Neumann boundary conditions in (1.1). Here, we find asymptotic approximation of eigenvalues for all boundary condition of the forms (1.2) and (1.3). To achieve this, we transform (1.1) to a differential equation all of whose coefficients belong to $L_1[a, b]$. Then we employ a Prüfer transformation to obtain an approximation of the eigenvalues. In this way, many basic properties of singular problems can be inferred from the corresponding regular ones. In [3], Harris derived an asymptotic approximation to the eigenvalues of the differential Equation (1.1), defined on the interval $[a, b]$, with boundary conditions of general form. But, he demands the condition, $q \in L^1[a, b]$. Atkinson and Harris found asymptotic formulae for the eigenvalues of spectral problems associated with linear differential equations of the form (1.1), where $q(x)$ has a singularity of the form αx^{-k} with $1 \le k < \frac{4}{3}$ and $1 \le k < \frac{3}{2}$ in [2,4] respectively. Harris and Race [5] generalized those results for the case $1 \le k < 2$. In [6], Harris and Marzano derived asymptotic estimates for the eigenvalues of (1.1) on $[0, a]$ with periodic and semi-periodic boundary conditions. The reader can find the related results in [7-10]. We consider $q(x) = Cx^{-K}$ where $1 \le K < 2$ and an asymptotic approximation to the eigenvalues of (1.1) with boundary conditions of general form. Our technique in this article follows closely the technique used in [2-5]. Let $U = [a, 0) \cup (0, b]$ and $q \in L_{1,\mathrm{Loc}}(U)$. As Harris did in [[5], p. 90], suppose that there exists some real function f on $[a, 0) \cup (0, b]$ in $AC_{\mathrm{Loc}}([a, 0) \cup (0, b])$ which regularizes (1.1) in the following sense. For f which can be chosen in Section 2, define quasi-derivatives, $y^{[i]}$ as follows:

$$y^{[0]} := y, \quad y^{[1]} := y' + fy,$$

y is a solution of (1.1) with boundary conditions (1.2) and (1.3) if and only if

$$\begin{pmatrix} y^{[0]} \\ y^{[1]} \end{pmatrix} = \begin{pmatrix} -f & 1 \\ f' + q - f^2 - \lambda & f \end{pmatrix} \begin{pmatrix} y^{[0]} \\ y^{[1]} \end{pmatrix} \tag{1.4}$$

The object of the regularization process is to chose f in such way that

$$f \in L^1(a, b) \quad \text{and} \quad -F := q - f^2 + f' \in L^1(a, b). \tag{1.5}$$

Having rewritten (1.1) as the system (1.4), we observe that, for any solution y of (1.1) with $\lambda > 0$, according to [2,4], we can define a function $\theta \in AC(a, b)$ by

$$\tan\theta = \frac{\lambda^{\frac{1}{2}} y}{y^{[1]}}.$$

When $y^{[1]} = 0$, θ is defined by continuity [[5], p. 91]. It makes sense to mention that one can find full discussions and nice examples about the choice of f in [2,4,5]. Atkinson in [2] noticed that the function θ satisfies the differential equation

$$\theta' = \lambda^{\frac{1}{2}} - f\sin(2\theta) + \lambda^{-\frac{1}{2}} F \sin^2(\theta). \tag{1.6}$$

Let $\lambda > 0$ and the n-th eigenvalue λ_n of (1.1-1.3), then according to [[1], Theorem 2], Dirichlet and non-Dirichlet boundary conditions can be described as bellow:

$$\begin{cases} \text{in Case 1 } (\alpha = 0, \beta = 0): & \theta(b,\lambda) - \theta(a,\lambda) = (n+1)\pi; \\ \text{in Case 2 } (\alpha = 0, \beta \neq 0): & \theta(b,\lambda) - \theta(a,\lambda) = (n+\frac{1}{2})\pi - \lambda^{-\frac{1}{2}}\cot\beta + O\left(\lambda^{-\frac{3}{2}}\right); \\ \text{in Case 3 } (\alpha \neq 0, \beta = 0): & \theta(b,\lambda) - \theta(a,\lambda) = (n+\frac{1}{2})\pi + \lambda^{-\frac{1}{2}}\cot\alpha + O\left(\lambda^{-\frac{3}{2}}\right); \\ \text{in Case 4 } (\alpha \neq 0, \beta \neq 0): & \theta(b,\lambda) - \theta(a,\lambda) = n\pi + \lambda^{-\frac{1}{2}}(\cot\alpha - \cot\beta) + O\left(\lambda^{-\frac{3}{2}}\right). \end{cases}$$

It follows from (1.5-1.6) that large positive eigenvalues of either the Dirichlet or non-Dirichlet problems over $[a, b]$ satisfy

$$\lambda^{\frac{1}{2}} = \frac{\theta(b) - \theta(a)}{(b-a)} + O(1). \tag{1.7}$$

Our aim here is to obtain a formula like (1.7) in which the $O(1)$ term is replaced by an integral term plus and error term of smaller order. We obtain an error term of $o\left(\lambda^{\frac{-N}{2}}\right)(N \geq 1)$. To achieve this we first use the differential Equation (1.6) to obtain estimates for $\theta(b)$ - $\theta(a)$ for general λ as $\lambda \to \infty$.

2. Statement of result

We define a sequence $\xi_j(t)$ for $j = 1, ..., N + 1$, $t \in [a, b]$ by

$$\begin{aligned} \xi_1(t) &:= \left|\int_0^t\right| f(s)\,|+|\,F(s)\,|ds| \\ \xi_j(t) &:= \left|\int_0^t (|f(s)| + |F(s)|)\xi_{j-1}(s)ds\right| \end{aligned} \tag{2.1}$$

and note that in view of $f, F \in L(a, b)$,

$$\xi_j(t) \leq c\xi_{j-1}(t) \quad \text{for} \quad t \in [a,b], \quad 2 \leq j \leq N+1 \tag{2.2}$$

Suppose that for some $N \geq 1$,

$$\begin{aligned} &f'\xi_{N+1}, \quad f^2\xi_N, \quad fF\xi_N \in L[a,b]; \\ &f(t)\xi_{N+1}(t) \to 0 \quad \text{as} \quad t \to 0. \end{aligned} \tag{2.3}$$

We define a sequence of approximating functions a

$$\theta_0(x) := \theta(a) + \lambda^{\frac{1}{2}}(x-a); \tag{2.4}$$

$$\theta_j(0) := \theta(0); \tag{2.5}$$

$$\theta_{j+1}(x) := \theta(a) + \lambda^{\frac{1}{2}}(x-a) - \int_a^x f\sin(2\theta_j(t))dt + \lambda^{-\frac{1}{2}}\int_a^x F\sin^2(\theta_j(t))dt. \tag{2.6}$$

for $j = 0, 1, 2, ...$ and for $a \leq x \leq b$. We measure the closeness of the approximation in the next result. Thus

$$\theta'_{j+1} = \lambda^{\frac{1}{2}} - f\sin(2\theta_j) + \lambda^{-\frac{1}{2}}F\sin^2(\theta_j) \tag{2.7}$$

The following lemma appears in [2,5].

Lemma 2.1. *If* $g \in Ł^1$ *then for any* j *and* $a \leq x \leq b$

$$\int_a^x g(t)\sin(2\theta_j(t))dt = o(1)$$

as $\lambda \to \infty$.

By using Lemmas 5.1 and 5.2 of [5] we conclude the following lemma

Lemma 2.2. *There exists a suitable constant C such that*

$$\left|a_{j+1} - \theta_j\right| \le C \sup_{a\le x\le b} \left|\theta - \theta_j\right| \xi_{j+1}(x)$$

Now, we prove an elementary lemma.

Lemma 2.3. *If* $g \in Ł^1$ *and* $\theta(x) - \theta_j(x) = \lambda^{-\frac{1}{2}} \int_a^x g\{\sin^2(\theta(t)) - \sin^2(\theta_j(t))\}dt$ *then* $\left|\theta(x) - \theta_{j+1}(x)\right| \le \lambda^{-\frac{1}{2}} \sup_{a\le x\le b} \left|\theta(x) - \theta_j(x)\right| \int_a^x g dt$

Proof.

$$\begin{aligned}\theta(x) - \theta_{j+1}(x) &= \lambda^{-\frac{1}{2}} \int_a^x g\{\sin^2(\theta(t)) - \sin^2(\theta_j(t))\}dt \\ &= \frac{1}{2}\lambda^{-\frac{1}{2}} \int_a^x g\{\cos(2\theta_j(t)) - \cos(2\theta(t))\}dt \\ &= -\lambda^{-\frac{1}{2}} \int_a^x g \sin(\theta_j(t) - \theta(t)) \sin(\theta_j(t) + \theta(t))\}dt \\ &\le \lambda^{-\frac{1}{2}} \sup_{a\le x\le b} \left|\theta(x) - \theta_j(x)\right| \int_a^x g dt\end{aligned}$$

Remark 2.4. *Lemma 2.2 shows that if* $\left|\theta(x) - \theta_j(x)\right| = o\left(\lambda^{\frac{-j}{2}}\right)$ *then* $\left|\theta(x) - \theta_{j+1}(x)\right| = o\left(\lambda^{\frac{-(j+1)}{2}}\right)$

Lemma 2.5. *There exists a suitable constant C such that*

$$\int_a^x f(\sin(2\theta_j(t)) - \sin(2\theta(t)))dt \le C\lambda^{-\frac{1}{2}} \sup_{a\le x\le b} \left|\theta(x) - \theta_j(x)\right|, \quad x \in (a, b),$$

Proof.

$$\begin{aligned}\int_a^x f(\sin(2\theta(t))(1) - \sin(2\theta_j(t))(1))dt &= \lambda^{-\frac{1}{2}} \int_a^x f\{\sin(2\theta)\theta' - \sin(2\theta_j)\theta'_j\}dt \\ &+ \lambda^{-\frac{1}{2}} \int_a^x f^2\{\sin^2(2\theta) - \sin(2\theta_j)\sin(2\theta_{j-1})\}dt \\ &- \lambda^{-1} \int_a^x fF\{\sin(2\theta)\sin^2(\theta) - \sin(2\theta_j)\sin^2(\theta_{j-1})\}dt \\ &=: I_1 + I_2 - I_3.\end{aligned}$$

But

$$I_1 = \lambda^{\frac{-1}{2}} [f(t)(\sin^2(\theta(t)) - \sin^2(\theta_j(t)))]_a^x - \lambda^{-\frac{1}{2}} \int_a^x f'(t)\{\sin^2(\theta) - \sin^2(\theta_j)\}dt$$

By using Lemma 2.1 we have

$$I_1 \le C_1 \lambda^{-\frac{1}{2}} \sup_{a \le x \le b} \left| \theta(x) - \theta_j(x) \right| .$$

Applying Lemmas 2.1 and 2.2 we have

$$\begin{aligned} I_2 :&= \lambda^{-\frac{1}{2}} \int_a^x f^2(t)\{\sin(2\theta) - \sin(2\theta_j)\} \sin(2\theta) dt \\ &+ \lambda^{-\frac{1}{2}} \int_a^x f^2(t)\{\sin(2\theta) - \sin(2\theta_j)\} \sin(2\theta_j) dt \\ &+ \lambda^{-\frac{1}{2}} \int_a^x f^2(t)\{\sin(2\theta_j) - \sin(2\theta_{j-1})\} \sin(2\theta_j) dt \\ &\le C_2 \lambda^{-\frac{1}{2}} \sup_{a \le x \le b} \left| \theta(x) - \theta_j(x) \right| \end{aligned}$$

Finally, using Lemma 2.1, we conclude

$$\begin{aligned} I_3 :&= \lambda^{-1} \int_a^x fF\{\sin(2\theta) - \sin(2\theta_j)\}\sin^2(\theta) dt \\ &+ \lambda^{-1} \int_a^x fF(\sin(\theta) - \sin(\theta_{j-1}))(\sin(\theta) + \sin(\theta_{j-1})) \sin(2\theta_j) dt \\ &\le C_3 \lambda^{-\frac{1}{2}} \sup_{a \le x \le b} \left| \theta(x) - \theta_j(x) \right| \end{aligned}$$

This ends the proof of Lemma 2.5.

Theorem 2.6. *Suppose that* (2.3) *hold for some positive integer N, then*

$$\theta(b) - \theta(a) - (b-a)\lambda^{\frac{1}{2}} = -\int_a^b f \sin(2\theta_N(x)) dx + \left(\lambda^{\frac{-1}{2}}\right) \int_a^b F\sin^2(\theta_N) dx + o\left(\lambda^{\frac{-N}{2}}\right)$$

as $\lambda \to \infty$.

Proof. We integrate (1.5) over $[a, x]$ and obtain

$$\theta(x) - \theta(a) = \lambda^{\frac{1}{2}}(x-a) - \int_a^x f \sin(2\theta(t)) dt + \lambda^{-\frac{1}{2}} \int_a^x F\sin^2(\theta(t)) dt$$

In particular

$$\theta(b) - \theta(a) = \lambda^{\frac{1}{2}}(b-a) - \int_a^b f \sin(2\theta(t)) dt + \lambda^{-\frac{1}{2}} \int_a^b F\sin^2(\theta(t)) dt$$

and so,

$$\theta(b)-\theta(a)-(b-a)\lambda^{\frac{1}{2}} = -\int_a^b f\sin(2\theta_N(x))dx + \left(\lambda^{\frac{-1}{2}}\right)\int_a^b F\sin^2(\theta_N)dx$$
$$+\int_a^b f\{\sin(2\theta_N(x)-\sin(2\theta(x)))\}dx$$
$$+\left(\lambda^{\frac{-1}{2}}\right)\int_a^b F\{\sin^2(\theta)-\sin^2(\theta_N)\}dx.$$

We need to prove that two last terms are $o\left(\lambda^{\frac{-N}{2}}\right)$ as $\lambda\to\infty$. Applying Lemmas 2.2 and 2.4 we have

$$I := \int_a^b f(x)\{\sin(2\theta_N(x)-\sin(2\theta(x)))\}dx + \left(\lambda^{\frac{-1}{2}}\right)\int_a^b F(x)\{\sin^2(\theta)-\sin^2(\theta_N)\}dx$$
$$\leq C\lambda^{-\frac{1}{2}}\sup_{a\leq x\leq b}\left|\theta(x)-\theta_N(x)\right| + C\left(\lambda^{\frac{-1}{2}}\right)\int_a^b F\sup_{a\leq x\leq b}\left|\theta(x)-\theta_N(x)\right|dx$$

When $N = 1$, applying Lemma 2.5, $\left|\theta(x)-\theta_1(x)\right| = o\left(\lambda^{\frac{-j}{2}}\right)$. Now By using Lemma 2.3 and induction we achieve that $I = o\left(\lambda^{\frac{-N}{2}}\right)$ as $\lambda\to\infty$.

Remark 2.7. *By using the discussions of choice of f in* [5], *the condition* (2.3) *let us to consider q as the form* $q(x) \sim x^{-K}$ *where* $1 \leq K < 2$.

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Competing interests
The author declares that they have no competing interests.

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