The binary reflected Gray code is optimal for M-PSK

Erik Agrell, Johan Lassing, Erik G. Ström, and Tony Ottosson
Dept. Signals and Systems, Chalmers University of Technology
SE-412 96 Göteborg, Sweden
johan.lassing@s2.chalmers.se

Abstract — **This paper is concerned with the problem of selecting a binary labeling for the signal constellation in an M-PSK communication system. A good starting point is labelings having the Gray property, but this is not altogether enough, since the number of distinct Gray labelings that result in different bit error probability grows rapidly with increasing constellation size. By introducing a recursive Gray labeling construction method called expansion, the paper answers the natural question of what labeling, among all possible constellation labelings (not only Gray), that will give the lowest possible average probability of bit errors. Under certain assumptions on the channel, the answer is that the labeling originally proposed by Gray, the binary reflected Gray code, is the optimal labeling for M-PSK systems, which has, surprisingly, never been proved before.**

For ML detection and equally likely symbols, the average bit error probability for M-PSK ($M = 2^m$) is given by

$$P_b = \frac{1}{\log_2 M} \sum_{k=1}^{M-1} \bar{d}\,(k)\, P(k), \tag{1}$$

where $\bar{d}(k)$ is the average distance spectrum (ADS, see Def. 1) of the bit-to-symbol mapping used and $P(k)$ is the probability of detecting symbol k after transmission of symbol 0. $P(k)$ depends on the channel, but is independent of the bit-to-symbol mapping used. If $P(k)$ is decreasing rapidly with k, then P_b is determined only by the ADS. The paper finds the bit-to-symbol mapping that results in the slowest growing ADS (see Def. 2) and the main result is captured by the following theorem.

Theorem 1—Optimality of BRGC for M-PSK: The binary reflected Gray code (BRGC) of order m is the optimal labeling for 2^m-PSK, in the sense of Definition 2. The labeling is unique up to cyclic shifts and reflection of the codeword sequence, permutation of codeword coordinates, and binary inversion of any coordinates.

The discussion in the paper focuses on binary, cyclic labelings and the following definitions are used to quantify optimality.

Definition 1—Average Distance Spectrum: The average distance spectrum (ADS), $\bar{d}(k)$, of a binary labeling $\mathcal{C} = (\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{M-1})$ is the average number of bit positions that differ in codewords separated by k steps, averaged cyclically over all the codewords, i.e.,

$$\bar{d}(k) \triangleq \frac{1}{M} \sum_{l=0}^{M-1} d_H\left(\mathbf{c}_l, \mathbf{c}_{(l+k) \bmod M}\right) \tag{2}$$

for all $k \in \mathbb{Z}$, where $d_H(\mathbf{a}, \mathbf{b})$ is the Hamming distance between the binary vectors $\mathbf{a}$ and $\mathbf{b}$.

Definition 2—Optimal ADS: The ADS $\bar{d}(k)$ of a binary labeling $\mathcal{C}_1$ is said to be superior to the ADS $\bar{h}(k)$ of a binary labeling $\mathcal{C}_2$ of the same order, if the following relations hold for some integer $j > 0$,

$$\begin{aligned} \bar{d}(i) &= \bar{h}(i), \qquad 0 \le i < j \\ \bar{d}(j) &< \bar{h}(j). \end{aligned}$$

If $\bar{d}(i) = \bar{h}(i)$ for all integers i, $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to have equivalent ADS. The ADS is said to be *optimal* if it is superior or equivalent to the ADS of any other binary labeling of the same order.

The proof of Theorem 1 relies on a construction method proposed for the binary reflected Gray code, called *labeling expansion*. For codes of order $m \ge 2$, a labeling $\mathcal{C}_m$ is generated by expansion from a labeling $\mathcal{C}_{m-1}$ by taking $\mathcal{C}_{m-1} = (\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{M/2-1})$, repeating each codeword once to obtain a new sequence of M vectors $(\mathbf{c}_0, \mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_1 \ldots, \mathbf{c}_{M/2-1}, \mathbf{c}_{M/2-1})$. In the final step, an extra coordinate is appended to each codeword, taken in turn from the vector $(0, 1, 1, 0, 0, 1, 1, 0, \ldots, 0, 1, 1, 0)$ of length M. The proof is inductive in nature and an outline of the proof is given below (the full proofs of theorems and lemmas are found in [1]). First we derive a recursion for the ADS of an expanded labeling (valid for all k):

$$\bar{d}_m(4k) = \bar{d}_{m-1}(2k) \tag{3}$$

$$\bar{d}_m(4k+2) = \bar{d}_{m-1}(2k+1) + 1 \tag{4}$$

$$\bar{d}_m(2k+1) = \frac{1}{2}\bar{d}_{m-1}(k) + \frac{1}{2}\bar{d}_{m-1}(k+1) + \frac{1}{2} \tag{5}$$

Next, we establish that any Gray code of order $m \ge 3$ has an ADS that satisfies $\bar{d}(1) = 1$, $\bar{d}(2) = 2$, and $\bar{d}(3) \ge 2$ and that if the Gray code is obtained by expansion we have $\bar{d}(3) = 2$. The final step of the proof is to show that expansion will result in a labeling having optimal ADS for any order by examining the structure of the recursions (3)–(5).

As a concluding example, we may consider the Gray codes of order $m = 4$ used for 16-PSK systems. There are nine types of Gray codes of order 4 that produce distinct ADS. In Figure 1, (1) is evaluated for two 16-PSK systems communicating over a Gaussian channel using Gray labelings. The better performing system uses the optimal labeling (the BRGC of order 4) and the worse performing system uses the labeling with worst ADS (for Gray codes of order 4). The difference in performance is difficult to observe at bit-energy-to-noise ratios γ_b above 0 dB, but can be clearly seen for lower γ_b. For comparison, the performance of a system using natural binary coded decimal (NBCD) labeling is also shown.

The above result that the BRGC is optimal can be extended to hold also for M-PAM and M-QAM systems [1].

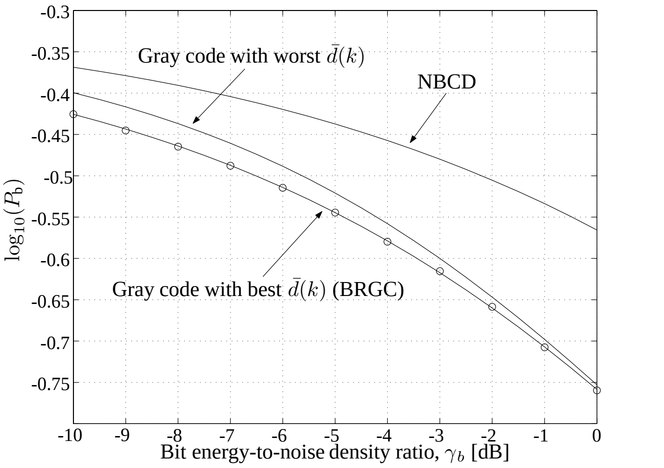

Fig. 1: The bit error probability of three 16-PSK systems communicating over a Gaussian channel; one using the Gray code with worst ADS, one using the Gray code with optimal ADS and one using the NBCD labeling. The rings indicate computer simulated values for the optimal Gray code.

References

[1] E. Agrell, J. Lassing and E. G. Ström and T. Ottosson, “On the optimality of the binary reflected Gray code,” *IEEE Transactions on Information Theory*, submitted, Dec. 2003.