On the Duality Between Low-Rate and High-Rate Sampling

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Abstract — **Lattices are considered for sampling and interpolation of stationary stochastic processes. The error variance of a linear unbiased estimator (interpolator) is calculated as a function of the lattice used and the covariance function of the process, and the optimal estimator is characterized. It is shown that in any dimension, the optimal lattice for very low rate sampling is the dual of the one for very high rate.**

I. Introduction

We consider the problem in multidimensional signal processing where a signal is to be discretized, stored, and reconstructed. We assume that the signal is a realization of a zero-mean wide-sense stationary stochastic process $(Z(\boldsymbol{x}); \boldsymbol{x} \in \mathbb{R}^d)$ with finite second moments. We assume that the (multidimensional) covariance function $R(\boldsymbol{x}) = \mathbb{E}[Z(\boldsymbol{y})Z(\boldsymbol{y}+\boldsymbol{x})]$ is known and denote the spectral density, i.e., the Fourier transform of R, by f. The process is not required to be band-limited in any direction.

From a geometrical point of view, the multidimensional signal should be sampled as uniformly as possible, in order to gain as much information as possible about the signal everywhere in the relevant region. No part of the region should lie very far from the closest sample point, since this would cause a relatively large uncertainty in the estimate of the signal in that part. Therefore, sampling on a *lattice* is considered [1].

A d-dimensional *lattice* $\Lambda = \Lambda(\boldsymbol{B})$ is a subset of $\mathbb{R}^d$ of the form $\{\boldsymbol{u} = \boldsymbol{B}^T\boldsymbol{w} : \boldsymbol{w} \in \mathbb{Z}^n\}$ where $\boldsymbol{B}$, the *generator matrix,* is a matrix with n linearly independent rows [2, Ch. 1]. The *packing radius* $\rho(\boldsymbol{B})$ is half the minimum distance between two points of the lattice, and the *kissing number* $\tau(\boldsymbol{B})$ is the number of lattice points at distance 2ρ. The *Voronoi region* $\Omega(\boldsymbol{B})$ of a lattice is the set of all vectors in $\mathbb{R}^d$ that are at least as close to the origin $\boldsymbol{0}$ as to any other lattice point. In the frequency domain, an important role is played by the *dual lattice* of $\Lambda(\boldsymbol{B})$, scaled by 2π, which we denote by $\Lambda(\boldsymbol{A})$. Any matrix $\boldsymbol{A}$ of the same dimensions as $\boldsymbol{B}$ that satisfies $\boldsymbol{A}^T\boldsymbol{B} = 2\pi\boldsymbol{I}$ can serve as generator for the dual lattice, such as $\boldsymbol{A} = 2\pi(\boldsymbol{B}^{-1})^T$ if $\boldsymbol{B}$ is square.

II. The interpolation error

Let $\widehat{Z}(\boldsymbol{x})$ be the best linear unbiased estimator of $Z(\boldsymbol{x})$ based on observations $(Z(\boldsymbol{u}); \boldsymbol{u} \in \Lambda(\boldsymbol{B}))$ on a lattice. Here, "best" means minimizing the mean square error $\sigma^2_{\Lambda(\boldsymbol{B})}(\boldsymbol{x}) \stackrel{\text{def}}{=} \mathbb{E}[(Z(\boldsymbol{x}) - \widehat{Z}(\boldsymbol{x}))^2]$. It can be shown [1] that the average interpolation error, $\sigma^2_{\Lambda(\boldsymbol{B})} \stackrel{\text{def}}{=} (\operatorname{vol}\Omega(\boldsymbol{B}))^{-1} \int_{\Omega(\boldsymbol{B})} \sigma^2_{\Lambda(\boldsymbol{B})}(\boldsymbol{x})d\boldsymbol{x}$, is given by

$$\sigma^2_{\Lambda(\boldsymbol{B})} = R(\boldsymbol{0}) - \frac{1}{(2\pi)^d} \int_{\Omega(\boldsymbol{A})} \frac{\sum_{\boldsymbol{\lambda}\in\Lambda(\boldsymbol{A})} f^2(\boldsymbol{\omega}+\boldsymbol{\lambda})}{\sum_{\boldsymbol{\lambda}\in\Lambda(\boldsymbol{A})} f(\boldsymbol{\omega}+\boldsymbol{\lambda})} d\boldsymbol{\omega}. \quad (1)$$

We aim to find the lattice that minimizes $\sigma^2_{\Lambda(\boldsymbol{B})}$ among all lattices with equal volume $\operatorname{vol}\Omega(\boldsymbol{B})$. Note that $1/\operatorname{vol}\Omega(\boldsymbol{B})$ is the sampling rate, that is, the limit of the number of points in $\Lambda(\boldsymbol{B}) \cap D$ divided by $\operatorname{vol} D$ as the domain D is extended in all directions.

The following two theorems characterize the two extreme cases where the sampling rate tends to zero and infinity, resp., assuming an isotropic covariance function R. Instead of rescaling the lattice, however, we can equivalently rescale the covariance function R and its Fourier transform f. Letting the sampling rate go to zero is equivalent to concentrating the covariance near the origin and letting the sampling rate go to infinity is equivalent to concentrating the spectral density near the origin. In this sense, the low-rate case corresponds to processes whose realizations look rough and the high-rate case to processes that look smooth. In [3], we state both theorems rigorously and prove that the error in the approximations is of smaller order.

Theorem 1 *For $R(\boldsymbol{x}) = R_0(\beta\|\boldsymbol{x}\|)$ with $\beta \to \infty$,*

$$\sigma^2_{\Lambda(\boldsymbol{B})} \approx 1 - \frac{R^{*2}(\boldsymbol{0})}{\operatorname{vol}\Omega(\boldsymbol{B})} \left(1 + \tau R^2(2\rho\boldsymbol{e}) - \tau R(2\rho\boldsymbol{e}) \frac{R^{*2}(2\rho\boldsymbol{e})}{R^{*2}(\boldsymbol{0})}\right) \quad (2)$$

where

$$R^{*2}(\boldsymbol{x}) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} R(\boldsymbol{y})R(\boldsymbol{x}+\boldsymbol{y})d\boldsymbol{y}, \quad (3)$$

$\rho = \rho(\boldsymbol{B})$, $\tau = \tau(\boldsymbol{B})$, and $\boldsymbol{e}$ is an arbitrary unit vector. If $R_0(r) \sim C\exp(-r^p)$ for some $p > 0$ as $r \to> \infty$, for a given $\operatorname{vol}\Omega(\boldsymbol{B})$, (2) is minimal for the lattice with the greatest $\rho(\boldsymbol{B})$.

Theorem 2 *For $f(\boldsymbol{\omega}) = f_0(\alpha\|\boldsymbol{\omega}\|)$ with $\alpha \to \infty$,*

$$\sigma^2_{\Lambda(\boldsymbol{B})} \approx \frac{\tau}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{f(\boldsymbol{\omega})f(2\rho\boldsymbol{e}-\boldsymbol{\omega})}{f(\boldsymbol{\omega}) + f(2\rho\boldsymbol{e}-\boldsymbol{\omega})} d\boldsymbol{\omega} \quad (4)$$

where now $\rho = \rho(\boldsymbol{A})$, $\tau = \tau(\boldsymbol{A})$ and $\boldsymbol{e}$ is again an arbitrary unit vector. If $f_0(r) \sim C\exp(-r^p)$ for some $p > 0$ as $r \to \infty$, for a given $\operatorname{vol}\Omega(\boldsymbol{B})$, (4) is minimal for the lattice with the greatest $\rho(\boldsymbol{A})$.

In conclusion, the asymptotically optimal lattice for low-rate sampling is the one that solves the *packing problem*, which is to minimize $\rho(\boldsymbol{B})$ [2, Ch. 1], and the dual of this lattice is asymptotically optimal for high-rate sampling.

References

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