't Hooft operators in the boundary

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We consider a topologically twisted maximally supersymmetric Yang-Mills theory on a four-manifold of the form $V = W \times \mathbb{R}_+$. 't Hooft disorder operators localized in the boundary component at finite distance of V are relevant for the study of knot theory on the three-manifold W and have recently been constructed for a gauge group of rank one. We extend this construction to an arbitrary gauge group G. For certain values of the magnetic charge of the 't Hooft operator, the solutions are obtained by embedding the rank-one solutions in G and can be given in closed form.

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I. INTRODUCTION

Maximally supersymmetric Yang-Mills theory in four dimensions admits a topological twisting[1] which leads to localization equations of the form

$$F - \phi \wedge \phi + *d_A\phi = 0 \qquad d_A(*\phi) = 0, \tag{1.1}$$

together with[2]

$$d_A\sigma = 0 \qquad [\phi, \sigma] = 0 \qquad [\sigma, \bar{\sigma}] = 0. \tag{1.2}$$

Here, d_A is the covariant exterior derivative associated with a connection A with field strength $F = dA + A \wedge A$ on the gauge bundle E (a principal G-bundle over the four-manifold V on which the theory with gauge group G is defined). The other bosonic fields are a one-form ϕ and a complex zero-form σ with values in the vector bundle $\mathrm{ad}(E)$ associated to E via the adjoint representation of G. There is a Lie product understood in the $\phi \wedge \phi$ term, and $*$ denotes the Hodge duality operator induced from the Riemannian structure on V.

As described in Ref. [2], on an open four-manifold V of the form

$$V = W \times \mathbb{R}_+, \tag{1.3}$$

these equations are relevant to the theory defined on a stack of coincident $D3$-branes terminating on a $D5$-brane. They must then be supplemented by suitable boundary conditions at both ends of V. These have been described in Ref. [2] and further elaborated in Ref. [6]. With $0 < y < \infty$ a linear coordinate on $\mathbb{R}_+$, the boundary conditions at infinity state that

$$A + i\phi \to \rho \tag{1.4}$$

as $y \to \infty$, where ρ is a fixed flat connection on the complexification $E_{\mathbb{C}}$ of E. The boundary conditions at finite distance are related to an embedding of the tangent frame bundle of W as a sub-bundle of $\mathrm{ad}(E)$ via a "principal embedding" of SO(3) in G [7]. Denoting the corresponding images of the vielbein and the spin connection of W as e and ω, respectively, we have the "Nahm-pole" behavior

$$A \to \omega \qquad \phi - \frac{1}{y}e \to 0 \tag{1.5}$$

as $y \to 0^+$.

For a generic closed curve γ in $V = W \times \mathbb{R}_+$, it is not possible to construct a line operator supported on γ and invariant under the topological supersymmetry. But such operators do exist for γ of the form

$$\gamma = K \times \{0\}, \tag{1.6}$$

where K is a closed curve in W. 't Hooft operators of that kind are relevant for the gauge-theory approach to knot theory developed in Ref. [2] and aimed at making contact with the invariants given by the Jones polynomial [8] and Khovanov homology [9]. These operators are labeled by the highest weight w of a representation of the Langlands dual $G^\vee$ of G. On the complement of K in W, the solution is equivalent to the solution in the absence of the 't Hooft operator up to a "large" gauge transformation. The topological class of this gauge transformation is determined by w, and for nontrivial w, it cannot be extended over K. Together with the requirement that the solution be nonsingular in the interior of V, this determines the asymptotic boundary behavior completely.

For the case when G is of rank one, i.e. $G = \mathrm{SU}(2)$ or $G = \mathrm{SO}(3)$, explicit model solutions with these properties were determined in Ref. [2] for arbitrary weights w. The purpose of this paper is to analyze the case of a general G. We hope that this may be useful for performing explicit calculations along the lines of Ref. [3].

In the next section, we will describe an Ansatz that respects the symmetries of the problem, and in Sec. III, we will discuss how the required boundary behavior determines a particular solution. We will arrive at a fairly good qualitative understanding, although it is only for

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[1]This particular twisting is an element of a $\mathbb{C}P^1$ family of inequivalent twistings [1,2]; the generalization has also been used in Ref. [3]. There are also two further unrelated possible twistings [4,5].
[2]This second set of equations (1.2) typically forces σ to vanish identically and will not be considered further in this paper.

certain special weights w that exact solutions (obtained by embedding of the rank-one solutions) can be given in closed form.

II. THE ANSATZ

We take $W = \mathbb{C} \times \mathbb{R}$ so that

$$V = \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+, \tag{2.1}$$

which we endow with the standard metric

$$ds^2 = |dz|^2 + dx^2 + dy^2. \tag{2.2}$$

(Here z, x, and y are standard coordinates on the three factors.) The 't Hooft operator will be localized along

$$K = \{0\} \times \mathbb{R} \times \{0\}, \tag{2.3}$$

i.e. at $z = y = 0$.

By a choice of gauge and a certain vanishing theorem [1,2,10], the components of A and ϕ, respectively, in the direction of $\mathbb{R}_+$ vanish. Furthermore, we make the Ansatz that the component of A in the direction of $\mathbb{R}$ vanishes and that the solution is invariant under translations along $\mathbb{R}$. The remaining variables are thus

$$A = A_z dz + A_{\bar{z}} d\bar{z} \qquad \phi = \phi_z dz + \phi_{\bar{z}} d\bar{z} + \phi_x dx \tag{2.4}$$

and depend on z, $\bar{z}$, and y only. In terms of the components of A and ϕ, Eqs. (1.1) read

$$\partial_y A_{\bar{z}} = D_{\bar{z}} \phi_x \quad D_{\bar{z}} \phi_z = 0 \quad \partial_y \phi_z = -[\phi_x, \phi_z], \tag{2.5}$$

together with

$$-\partial_y \phi_x = 2F_{z\bar{z}} + \frac{1}{2}[\phi_z, \phi_{\bar{z}}]. \tag{2.6}$$

We postpone the treatment of the "moment-map" equation (2.6) for a while, and start by considering the "holomorphic" equations (2.5). They can be solved by temporarily interpreting ϕ_x as the component of the gauge field in the y direction and are then invariant under gauge transformations with a parameter valued in the complexification $G_{\mathbb{C}}$ of G. Their content is that the covariant derivatives in the y and $\bar{z}$ directions annihilate ϕ_z and commute with each other, so the general solution is

$$\phi_z = g\varphi g^{-1} \quad \phi_x = -\partial_y g g^{-1} \quad A_{\bar{z}} = -\partial_{\bar{z}} g g^{-1}. \tag{2.7}$$

Here, $\varphi = \varphi(z)$ is an arbitrary holomorphic function with values in the Lie algebra of $G_{\mathbb{C}}$, and the gauge transformation parameter $g = g(z, \bar{z}, y)$ is an arbitrary function with values in $G_{\mathbb{C}}$.

Away from the locus $z = 0$, the Nahm-pole boundary condition (1.5) corresponding to a principal embedding requires φ to lie in the "regular nilpotent orbit" (see, e.g Ref. [11]). At $z = 0$, φ must then lie in the closure of the regular nilpotent orbit, but it may define a more special nilpotent conjugacy class. To describe the possibilities, we choose a Cartan torus T with Lie algebra $\mathbf{t}$ in G and a principal embedding of SO(3) in G with standard generators J^1, J^2, J^3 such that $J^3 \in \mathbf{t}$. The commutation relations of $J^+ = J^1 + iJ^2$, $J^- = J^1 - iJ^2$, and J^3 are

$$[J^3, J^+] = J^+ \quad [J^3, J^-] = -J^- \quad [J^+, J^-] = 2J^3. \tag{2.8}$$

We now take

$$\varphi = hJ^+h^{-1}, \tag{2.9}$$

where

$$h: \mathbb{C}^* \to T_{\mathbb{C}} \tag{2.10}$$

is a holomorphic homomorphism such that φ has no pole at $z = 0$. (Here, $T_{\mathbb{C}}$ is the complexification of T.) This means that

$$h = \exp(w \log z), \tag{2.11}$$

where w is an element of the weight lattice of the Langlands dual group $G^\vee$ (normalized so that $\exp(2\pi i w) = 1$) subject to a certain non-negativity condition. In fact, there is a one-to-one correspondence (up to conjugation) between such w and highest-weight representations of $G^\vee$. A solution with this φ defines what we mean by a 't Hooft operator in the corresponding representation inserted at $z = 0$ in the boundary $y = 0$.

As an example, we consider the case where $G = \mathrm{SU}(n)$ so that $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$. We choose T and $T_{\mathbb{C}}$ to consist of diagonal unimodular $n \times n$ matrices with complex entries that are of unit modulus or just nonzero, respectively. An arbitrary holomorphic homomorphism $h: \mathbb{C}^* \to T_{\mathbb{C}}$ is then of the form

$$h = \begin{pmatrix} z^{w_1} & 0 & \dots & 0 \\ 0 & z^{w_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z^{w_n} \end{pmatrix} \tag{2.12}$$

with integers $w_1, w_2, \dots, w_n$ subject to

$$w_1 + w_2 + \dots + w_n = 0. \tag{2.13}$$

Defining the principal embedding by

$$J^3 = \frac{1}{2}\begin{pmatrix} n-1 & 0 & \dots & 0 \\ 0 & n-3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -(n-1) \end{pmatrix} \qquad J^+ = \begin{pmatrix} 0 & \sqrt{1(n-1)} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{2(n-2)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{(n-1)1} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad J^- = (J^+)^\dagger, \tag{2.14}$$

we get

$$\varphi = \begin{pmatrix} 0 & \sqrt{1(n-1)}z^{w_1-w_2} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2(n-2)}z^{w_2-w_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{(n-1)1}z^{w_{n-1}-w_n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.15}$$

so regularity at $z = 0$ amounts to the non-negativity conditions

$$w_1 \geq w_2 \geq \ldots \geq w_n. \tag{2.16}$$

The number of saturated inequalities in Eq. (2.16) determines precisely which nilpotent orbit appears at $z = 0$; the trivial case when $w_1 = w_2 = \ldots = w_n = 0$ gives the regular nilpotent orbit and of course corresponds to a trivial 't Hooft operator.

We now return to the general case and turn our attention to the remaining moment-map equation (2.6). Together with the boundary conditions, this will determine g uniquely up to an ordinary G-valued gauge transformation. By exploiting this gauge symmetry, it is sufficient to consider g of the form

$$g = e^{u-(w+J^3)\log|z|}, \tag{2.17}$$

where $u = u(z, \bar{z}, y)$ is an element of the Lie algebra $\mathbf{t}$ of the Cartan torus T of G.[3] We then have

$$\begin{aligned} \phi_x &= -\partial_y u \\ \phi_z &= |z|^{-1} e^{u+(w/2)\log(z/\bar{z})} J^+ e^{-u-(w/2)(\log(z/\bar{z})} \\ A_{\bar{z}} &= -\partial_{\bar{z}} u + \frac{1}{2}(w + J^3)\bar{z}^{-1}, \end{aligned} \tag{2.18}$$

and the moment-map equation (2.6) reads[4]

$$(4\partial_z\partial_{\bar{z}} + \partial_y^2)u = |z|^{-2}\frac{1}{2}[e^u J^+ e^{-u}, e^{-u} J^- e^u]. \tag{2.19}$$

This equation is invariant under rotations of the z-plane around the origin and also under scaling of y and z by a common real positive factor[5]. We seek a model solution that is invariant under such transformations, which means that u may only depend on z, $\bar{z}$, and y in the combination

$$s = |z|/y. \tag{2.20}$$

With this Ansatz, the moment-map equation is equivalent to a system of ordinary differential equations:

$$\left(\left(s\frac{d}{ds}\right)^2 + \left(s^2\frac{d}{ds}\right)^2\right)u = \frac{1}{2}[e^u J^+ e^{-u}, e^{-u} J^- e^u]. \tag{2.21}$$

There is clearly a $2r$-dimensional space of bulk solutions, where r is the rank of G. In the next section, we will discuss the relevant solution picked out by the boundary conditions.

III. THE SOLUTION

In the vicinity of the two-dimensional surface in V right above the locus of the 't Hooft operator, we have $s \to 0^+$. In that limit, the general solution to Eq. (2.21) behaves as

$$u = \alpha \log s + \beta + \mathcal{O}(s), \tag{3.1}$$

for some parameters α and β in $\mathbf{t}$, that must be chosen such that

$$e^u J^+ e^{-u} = \mathcal{O}(s). \tag{3.2}$$

In fact, regularity of g in this limit requires, according to (2.17), that

$$\alpha = w + J^3 \tag{3.3}$$

so that

[3] Since there is no factor of i in the exponent, g is not an element of T or even of G but only of $G_{\mathbb{C}}$.

[4] Note that the right hand side is an element of $\mathbf{t}$ and, in particular, commutes with the element $e^{(w/2)\log(z/\bar{z})}$ of T.

[5] These transformations generate the subgroup of the conformal group of V that leaves the boundary and the locus of the 't Hooft operator invariant.

$$e^u J^+ e^{-u} = s e^{w \log s + \beta} J^+ e^{-w \log s - \beta} = \mathcal{O}(s) \tag{3.4}$$

by the non-negativity condition on the weight w. For a given w, the boundary condition as $s \to 0^+$ thus leaves us with a codimension r space of solutions to Eq. (2.21) parametrized by β.

In the vicinity of the boundary of V, we have $s \to \infty$. In that limit, the Nahm-pole boundary condition requires that

$$u = J^3 \log s + \mathcal{O}(s^{-1}). \tag{3.5}$$

Linearizing Eq. (2.21) around such a solution gives the equation

$$\left(\left(s\frac{d}{ds}\right)^2 + \left(s^2\frac{d}{ds}\right)^2\right)\tilde{u} = s^2\left(\frac{1}{2}[J^-, [J^+, \tilde{u}]] + \frac{1}{2}[J^+, [J^-, \tilde{u}]] + \mathcal{O}(s^{-1})\tilde{u}\right) \tag{3.6}$$

for the first order deviation $\tilde{u}$. To analyze this equation, we note that

$$\frac{1}{2}[J^-,[J^+,\tilde{u}]] + \frac{1}{2}[J^+,[J^-,\tilde{u}]] = [J^3,[J^3,\tilde{u}]] + \frac{1}{2}[J^-,[J^+,\tilde{u}]] + \frac{1}{2}[J^+,[J^-,\tilde{u}]] \tag{3.7}$$

is given by the adjoint action of the SO(3) quadratic Casimir operator

$$C = J^3 J^3 + \frac{1}{2}J^+ J^- + \frac{1}{2}J^- J^+ \tag{3.8}$$

on $\tilde{u}$. The eigenvalues of this action of C are of the form $j(j+1)$, where the r possible integer values of the spin j are those that appear in the decomposition of the adjoint representation of G under the principally embedded SO(3). These possible j-values (known as the exponents) are given in Table I for all simple G. The spin j component $\tilde{u}_j$ of $\tilde{u}$ should thus obey

$$\left(\left(s\frac{d}{ds}\right)^2 + \left(s^2\frac{d}{ds}\right)^2\right)\tilde{u}_j = s^2(j(j+1) + \mathcal{O}(s^{-1}))\tilde{u}_j. \tag{3.9}$$

Two linearly independent solutions behave as s^j and s^{-j-1}, respectively, for large s. Only the latter is acceptable in view of Eq. (3.5), which leaves us with a codimension r space of solutions of Eq. (2.21).

Taking the conditions in both limits $s \to 0^+$ and $s \to \infty$ into account should generically give a discrete set of solutions to Eq. (2.21). Indeed, for a given weight w, we expect to find a unique solution. The singular behavior of this scale and rotationally invariant model solution defines the 't Hooft operator, but further nonsingular terms are allowed to appear when the 't Hooft operator is inserted in a more complicated configuration.

TABLE I. Dimensions and exponents of simple Lie algebras.

algebra	dimension	exponents
A_r	$r^2 + 2r$	$1, \ldots, r$
B_r	$2r^2 + r$	$1, 3, \ldots, 2r - 1$
C_r	$2r^2 + r$	$1, 3, \ldots, 2r - 1$
D_r	$2r^2 - r$	$1, 3, \ldots, 2r - 3, r - 1$
E_6	78	1, 4, 5, 7, 8, 11
E_7	133	1, 5, 7, 9, 11, 13, 17
E_8	248	1, 7, 11, 13, 17, 19, 23, 29
F_4	52	1, 5, 7, 11
G_2	14	1, 5

When w is a multiple of J^3, i.e. when

$$w = kJ^3 \tag{3.10}$$

for some non-negative integer k, the model solution is given by embedding the rank-one solution of Eq. [2] in G and can be given in closed form: We then have

$$u = fJ^3, \tag{3.11}$$

where the real function f obeys

$$\left(\left(s\frac{d}{ds}\right)^2 + \left(s^2\frac{d}{ds}\right)^2\right)f = e^{2f}. \tag{3.12}$$

This ordinary differential equation has a two-dimensional space of solutions, but imposing that

$$f = (k+1)\log s + \text{finite} \tag{3.13}$$

as $s \to 0^+$ and

$$f = \log s + \mathcal{O}(s^{-1}) \tag{3.14}$$

as $s \to \infty$ determines f uniquely:

$$f = \log \frac{2(k+1)s^{k+1}}{(\sqrt{1+s^2}+1)^{k+1} - (\sqrt{1+s^2}-1)^{k+1}}. \tag{3.15}$$

For a more general weight w, it appears that the model solution can only be determined numerically.

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