Statistical features of drift wave plasma turbulence

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Outline

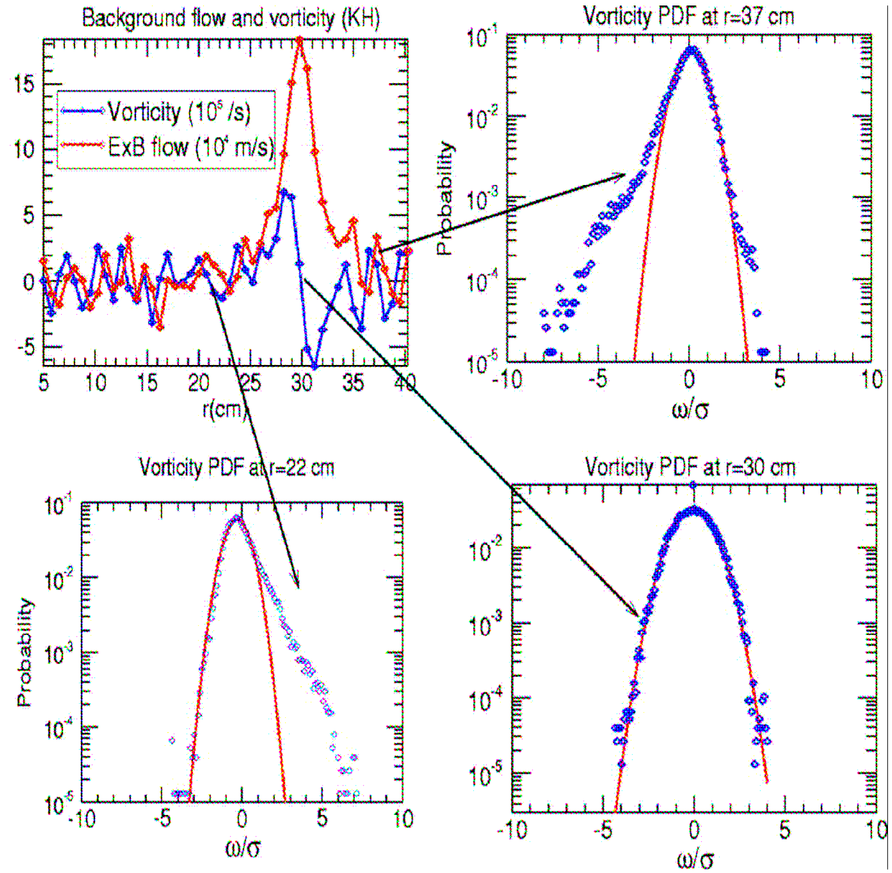
- Goals and Motivation
- Introduction to Intermittency and Probability Density Functions
- Theory overview – Instantons and coherent structures
- Results and recent developments
- Summary

Goals and motivation

- Goal: Investigate the statistical properties of heat flux in toroidal ion-temperature-gradient turbulence with adiabatic electrons. We will compare the predicted PDFs with the resulting PDFs from non-linear GK simulations in s-α geometry and stellarator geometry.
- The PDF tails are found to be qualitatively and quantitatively different from Gaussian distributions.
- Motivation: There is theoretical and experimental evidence that for understanding transport (involving many scales and amplitudes) a probabilistic description is needed.
- Intermittent systems are badly described by mean field theory and the quasi-linear transport coefficients are invalid.
- Note that the term ”intermittent” will be used for all phenomena that exhibit strong non-linear features.

Probability distribution function

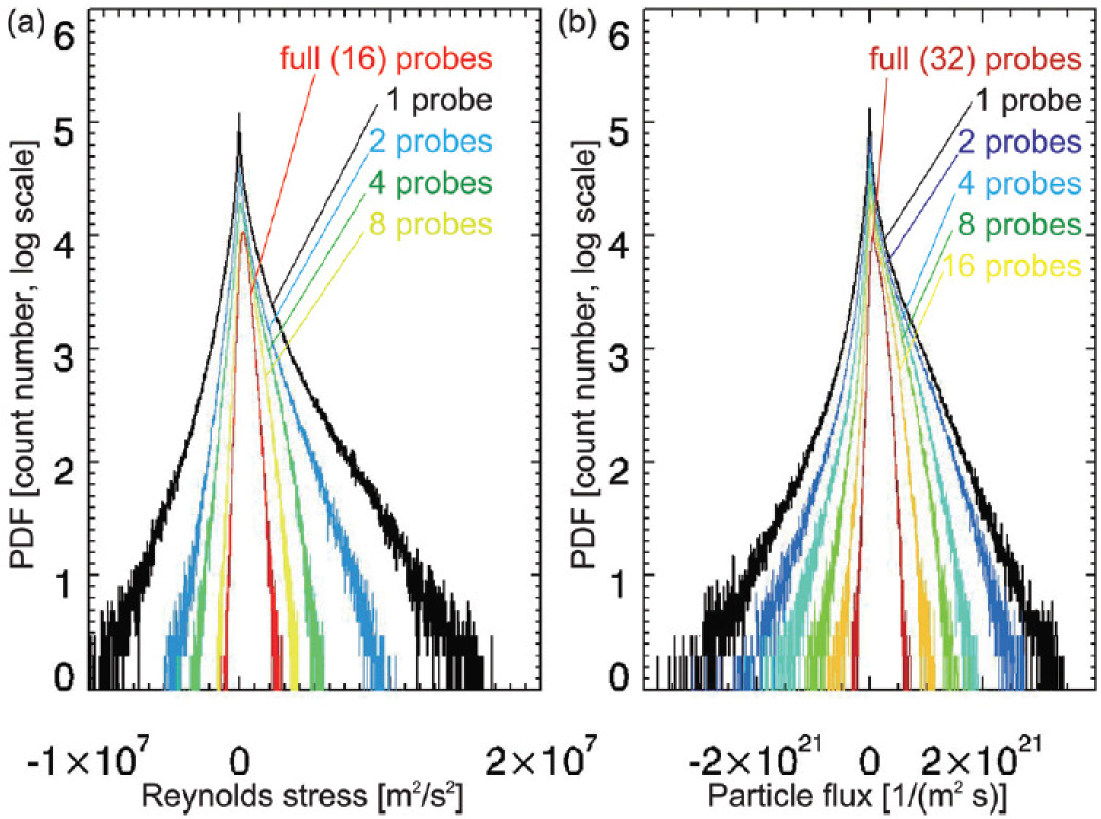
- Near the center, the PDF is often close to Gaussian but reveals a significant deviation from Gaussianity at the tails (intermittency - the events contributing to the tails are strongly non-linear.).
- Rather than a transport coefficient, a flux PDF is required in order to substantially characterize the transport process.
- PDF tail – rare events, but large amplitude (e.g. large heat load on the wall.)



Perez PoP 2006

$$P(Z) \propto \exp(-\xi \langle \omega \rangle)$$ Falcovich PRE 2011

Non-Gaussian statistical properties of the azimuthally averaged momentum and particle fluxes driven by turbulence



Nagashima et al Phys. Plasmas, **18** 070701 (2011)

$$P(Z) \propto \exp(-\xi \langle \Pi \rangle^{X})$$

- The exponent varies from 0.6 (point-wise) to 1.2 (averaged), as the spatial length of averaging becomes longer and converges to a constant value up to 1.75.
- The PDF of point-wise flux represents "microscopic" dynamics. The averaging over spatial range larger than that of the flux event causes mixing among the different flux events.

PDF tail of a general moment using the instanton method

The PDF tails of moment (m) and with the order of the highest non-linear interaction term (n) Kim & Anderson PoP 2008.

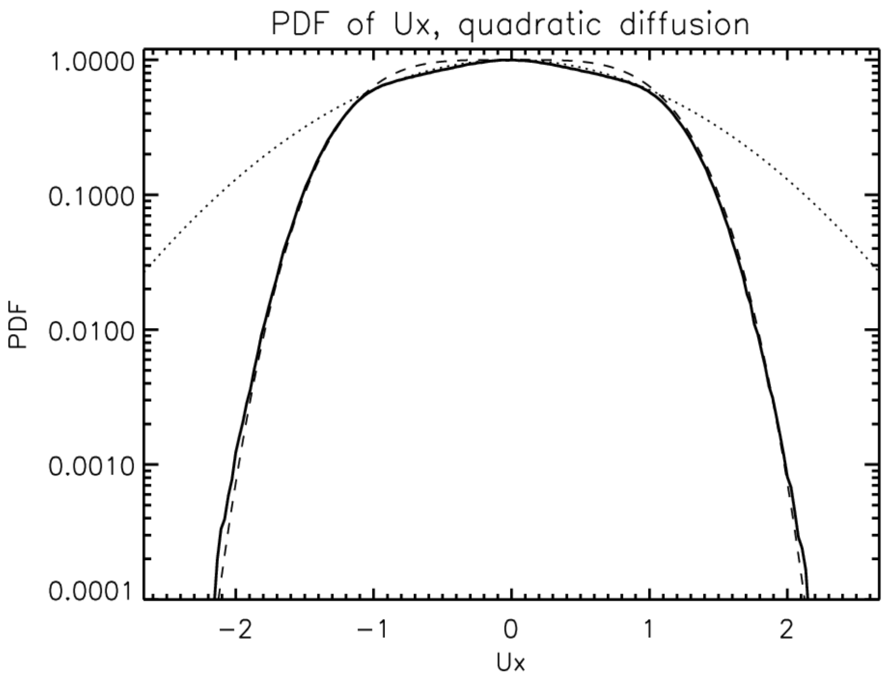
$$\frac{\partial \phi}{\partial t} + K\phi^n = f \qquad P(Z) \propto \exp(-\xi Z^s) \qquad s = \frac{n+1}{m}$$

Examples:

1. Linear system with PDF tails of first moment (n,ϕ) – Gaussian s=2.
2. Linear system with PDF tails of flux (n*v) – s=1 (Carreras 1996)
3. Hasegawa–Mima system with PDF tails of momentum flux – s=3/2 (Kim et al 2002, confirmed in experiments Yan et al 2007 and Nagashima 2011)
4. Burgers turbulence with velocity differences – s=3 (Cheklov 1995, Gurarie 1996, Balkovsky 1997,)

The PDF tails can be calculated provided that the integral mean value over the considered coherent structure is non-zero. A coherent structure For the HM system is the modon.

Previous results Quadratic diffusion



E. Kim et al Phys., 16 052304 Plasmas 2009

Solid line: Non-linear numerical calculation.
Dotted line: Gaussian fit
Dashed line: A fit to the PDF.

$$\frac{\partial u_x}{\partial t} = \frac{\partial^2}{\partial x^2}(D(u)u_x) + f$$

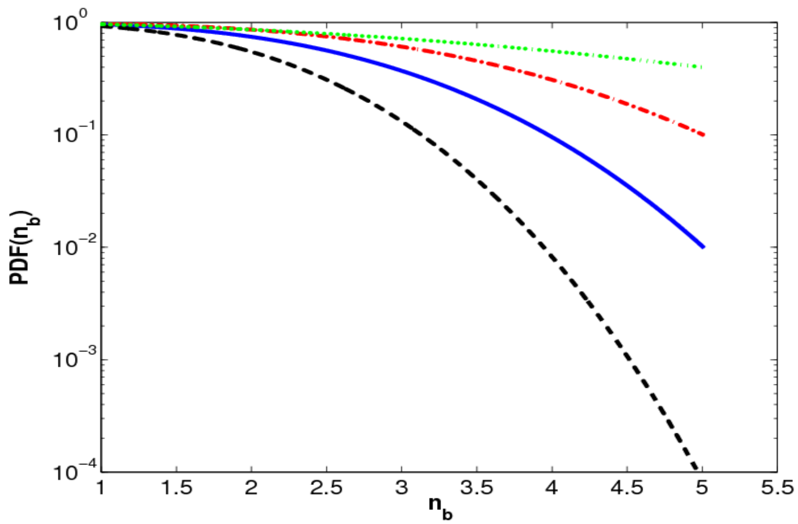
$$D(u) = \nu + \beta u_x^2$$

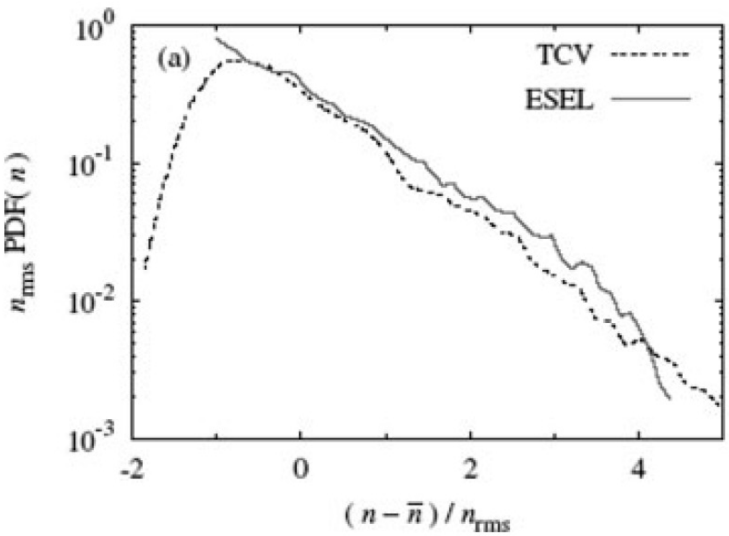
$$P(U_x) \propto e^{-cU_x^4}$$

The same PDF may be found Using the Fokker-Planck PDF equation.

Close to 0 the PDF is close to Gaussian whereas the tails are strongly intermittent.
There is a cross-over between occurs roughly at the expected critical gradient $u_{xc} \cong \mathrm{sqrt}(\nu/\beta) = 0.98$.

Previous results – Blob density PDF





Time evolution of the blob density.

$$\partial_t n_b + K(\partial_x[n_b \partial_y(\frac{1}{n_t}\partial_y n_b)] - \partial_y[n_b \partial_x(\frac{1}{n_t}\partial_x n_b)]) = f$$

$$P(n) \propto e^{-cn^3}$$

Fitting experimental data we find:

$$P(n) \propto e^{-cn^\Gamma} \quad \Gamma = 2.5 - 4.0$$

Using the exponential form a sheath potential we find an extreme value distribution also known as the Gumbel distribution:

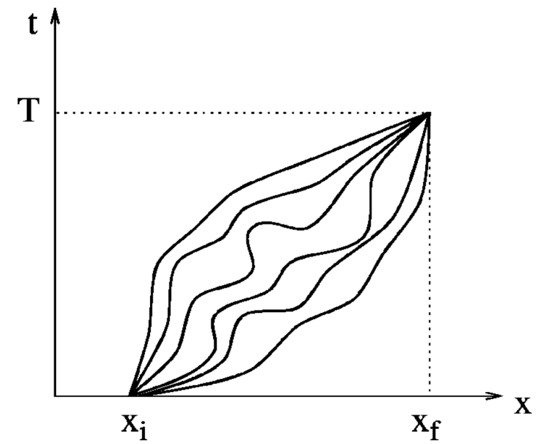
$$P(n) \propto e^{-\alpha n - \beta e^{-n}}, \ \alpha \text{ and } \beta \text{ are constants}$$

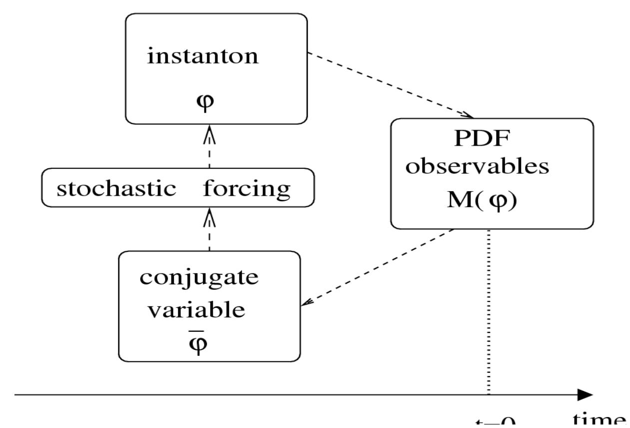
Anderson et al PoP **15** 122303 (2008)
Anderson et al PPCF **52** 012001 (2010)

Instanton method

Kim and Anderson PoP (2008)

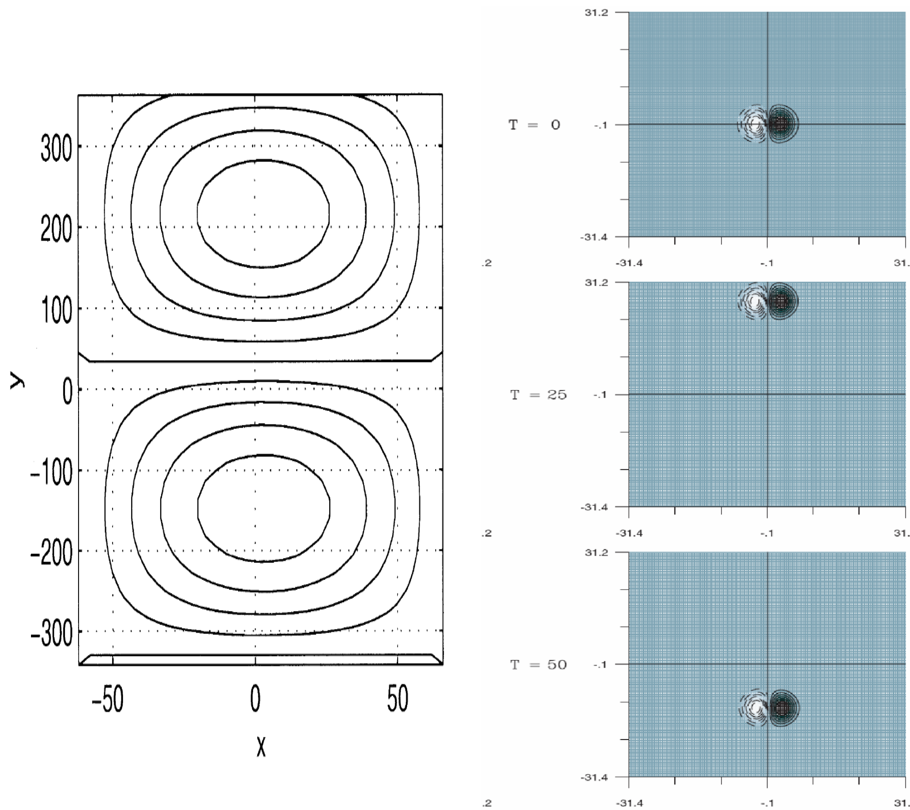
- The instanton method is a non-perturbative way of calculating the Probability Distribution Function tails.
- The PDF tail is viewed as the transition amplitude from a state with no fluid motion to a final state governed by the coherent structure.
- The creation of the coherent structure is associated with the bursty event.
- The optimum path is found by using the saddle-point method.





Coherent structures

- Coherent structures are major players in transport dynamics through the formation of avalanche-like events with large amplitude.
- There are several examples of coherent structures (c.f. modon or bipolar vortex soliton) to the non-linear governing equations.
- Strong theoretical evidence that a probabilistic formulation is needed to characterize the problem.



Left:Dastgeer IEEE TPS 2003, Right: Waelbroeck et al PPCF 46 1331 (2004)

Previous work

Modon and Coherent structures

- Modon solution in Rossby wave turbulence Larichev et al Dokl. Akad. Nauk SSSR 231, 1077 (1976)
- ITG modon solution Shukla et al Phys. Lett. A 136, 59 (1989) and Hong et al Phys. Fluids B 3, 615 (1991)
- ITG Modon stability, coherent structures and invariants Waelbroeck et al PPCF 46, 1331 (2004)

PDF tails using path-integral formulation

- Burgers turbulence (bi-fractal turbulence with ramps and shocks) Gurarie et al Phys. Rev. E 54, 4908 (1996)
- Hasegawa-Mima turbulence for local Reynolds stress Kim et al Phys. Rev. Lett. 88 225002 (2002) and PoP 9, 71 (2002)
- ITG mode turbulence for local Reynolds stress Kim et al, NF 43, 961 (2003)

Physics model for the ions with adiabatic electrons

Ion continuity and energy equation

$$\frac{\partial\tilde{\phi}}{\partial t}-\left(\frac{\partial}{\partial t}-\alpha_i\frac{\partial}{\partial y}\right)\nabla_\perp^2\tilde{\phi}+\frac{\partial\tilde{\phi}}{\partial y}-\varepsilon_n g_i\frac{\partial}{\partial y}\left(\tilde{\phi}+\tau(\tilde{\phi}+\tilde{T}_i)\right)=\left[\tilde{\phi},\nabla_\perp^2\tilde{\phi}\right]$$
$$+\tau\left[\tilde{\phi},\nabla_\perp^2(\tilde{\phi}+\tilde{T}_i)\right]$$

$$\frac{\partial\tilde{T}_i}{\partial t}-\frac{5}{3}\varepsilon_n g_i\frac{\partial\tilde{T}_i}{\partial y}+\left(\eta_i-\frac{2}{3}\right)\frac{\partial\tilde{\phi}}{\partial y}-\frac{2}{3}\frac{\partial\tilde{\phi}}{\partial t}=-\left[\tilde{\phi},\tilde{T}_i\right]$$

Anderson et al PoP 9, 4500 (2002)

$$\varepsilon_n=2L_n/R,\quad \alpha_i=\tau(1+\eta_i),\ \eta_{\mathrm{i}}=L_n/L_{Ti},\quad \tau=T_i/T_e$$
$$\tilde{\phi}=(L_n/\rho_s)e\delta\phi/T_e,\ \tilde{T}_i=(L_n/\rho_s)\delta T_i/T_0$$

The model for calculating the PDF tails

The PDF for global Reynolds stress can be defined as:

$$P(R) = \langle \delta(v_x v_y - R) \rangle = \int d\lambda e^{i\lambda R} \langle e^{-i\lambda v_x v_y} \rangle = \int d\lambda e^{i\lambda R} I_\lambda$$

The integrand can be re-written as a path-integral:

$$I_\lambda = \int D\phi D\bar{\phi} e^{-S_\lambda}$$

Here, the effective action can be written:

$$S_\lambda = \text{Drift wave action} + \text{Forcing}$$

$$+ \text{Initial conditions} = const * \lambda^3$$

The forcing is a Gaussian with a delta-correlation in time and the ion temperature effects are included using a linear relation ship between ϕ and T_i.

Instanton (saddle-point) solutions

An estimate of the exponential can be made (Kim and Diamond NF 961 (2003)):

$$S_\lambda \text{ as } \lambda \to \infty \text{ as } \lambda\phi\phi \propto \phi\overline{\phi} \propto T\overline{\phi}\phi^2 \propto T\overline{\phi}^2, \text{from which it follows:}$$

$$\phi \propto \lambda, \overline{\phi} \propto \lambda^2, \mathrm{T} \propto \lambda^{-1} \text{ and } S_\lambda \propto \lambda^3$$

- The path-integral can be solved in the large λ limit using the saddle-point method.
- Assuming that the modon is the coherent structure that is mostly contributing the intermittent state.

$$\phi(x, y, t) = \psi(x, y - Ut)F(t)$$

$$\frac{\delta S_\lambda}{\delta F} = 0$$

$$\frac{\delta S_\lambda}{\delta \overline{F}} = 0$$

The function F is associated with the instanton time dependency. F=0 at the initial point and F ≠ 0 in the long time limit. The action is expressed in the modon solution and the variations of F and the conjugate variables to F are computed.

The analytically predicted PDF tails of heat flux from the two-fluid model

Using the instanton method (on the two fluid model for ϕ and T_i) we can only predict the right tail whereas here we assume that the PDF is symmetric.

$$P(Q) = \frac{N}{2b}\exp\{-\frac{1}{b}(\frac{Q-\mu}{Q_0})^{3/2}\}$$

The same exponent has been found in experiments at CSDX by Yan et al. (2007)

This gives the probability of having a normalized heat flux Q.

$$b = b_0(\frac{R}{L_n} + 2\langle g_i\rangle\beta - U - \langle k_\perp^2\rangle(U + \frac{R}{L_{Ti}}))$$

$$\beta = 2 + \frac{2}{3}\frac{R/L_n - U}{U + 10/3\,\tau\langle g_i\rangle}$$

$$\langle f\rangle = \frac{\int \phi\, f\, \phi d\theta}{\int \phi^2 d\theta} \quad \text{with } \phi(\theta) \text{ taken from GENE}$$

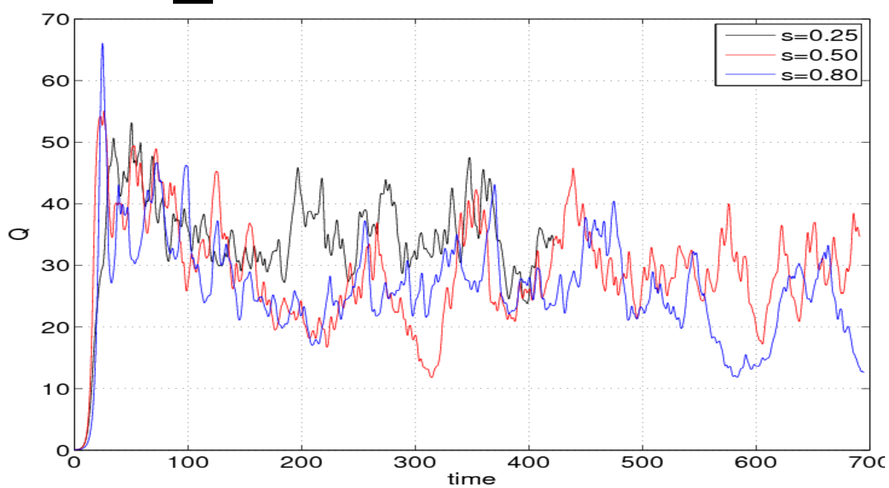
$$\langle g_i\rangle = A + Bs \qquad \langle k_\perp^2\rangle = k_y^2(1 + Cs^2) \qquad s = \frac{r}{q}\frac{dq}{dr}$$

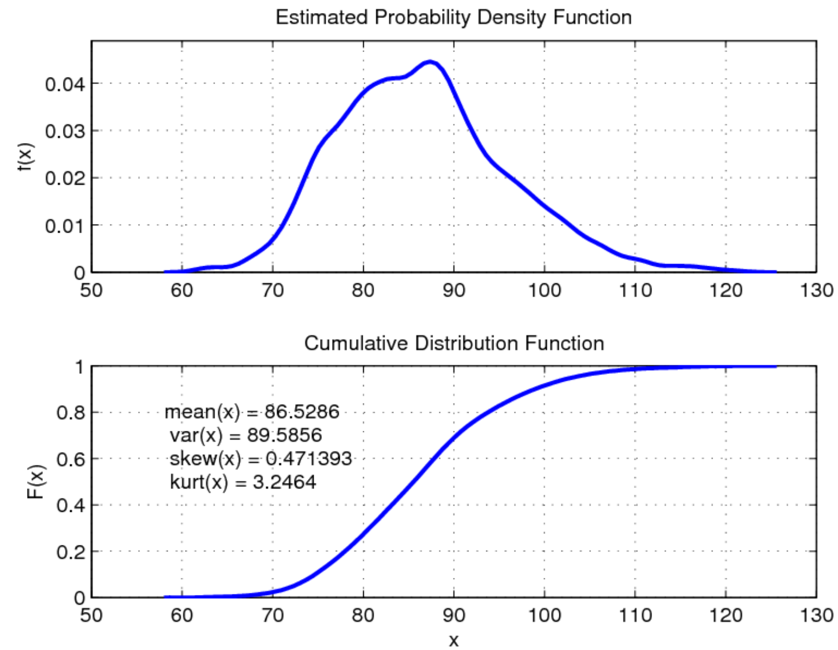
R/L_n and R/L_{Ti} normalized gradients in density and temperature, U is the speed of the modon soliton solution and s is the magnetic shear.
U=1 is assumed hereafter.

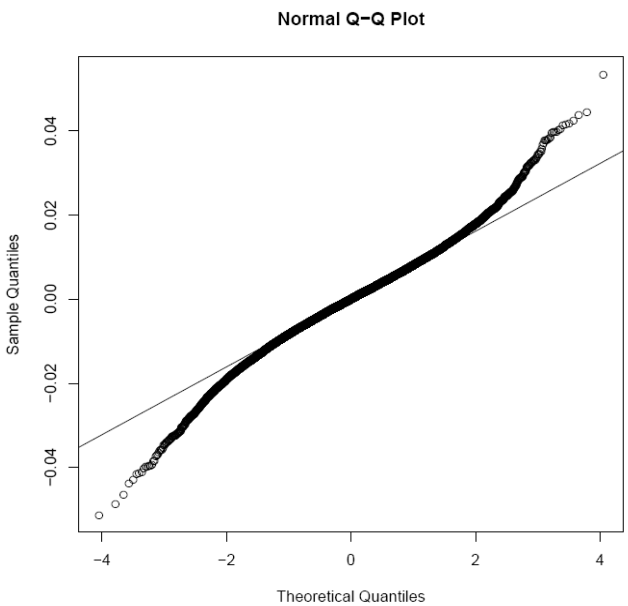
The nonlinear Gyro-Kinetic code GENE is used for the simulations

- Eulerian solver for the Vlasov-Maxwell equations on $(x,y,z,v_{||},\mu)$ grid (initial value or eigenvalue mode).
- Includes multi-species (fully gyro-kinetic), collision operators, electromagnetic effects.
- Excellent scaling up to at least 32K processors on BlueGene/L.
- Flux-tube domain for stellarators and global for tokamak.

Statistical analysis of time traces of radial heat flux



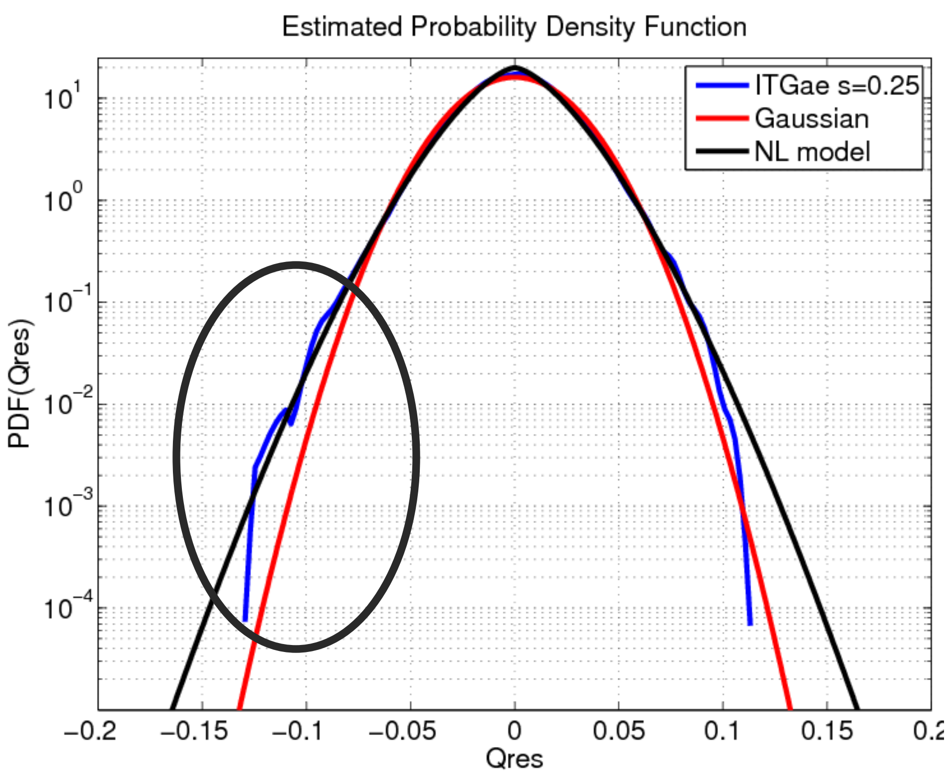


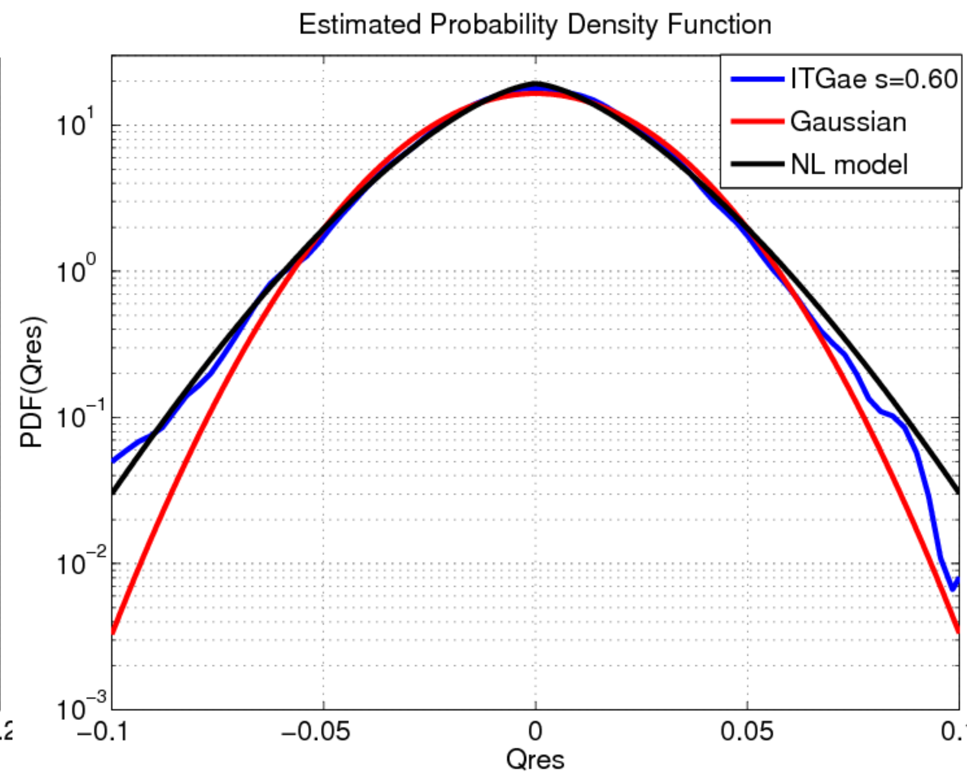


1. Key element: Remove autocorrelations from time traces.
2. Create ARIMA model using Box-Jenkins method.
3. Residuals systematically manifest long non-Gaussian tails.

Anderson and Xanthopolous PoP 2011

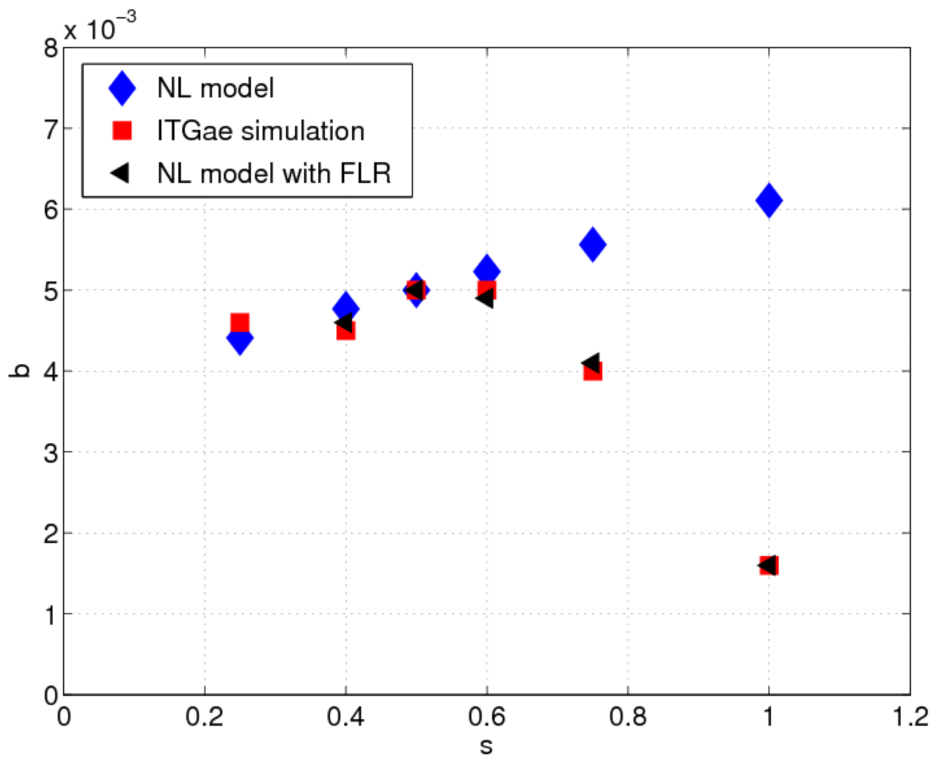
Result 1: PDFs at different magnetic shears

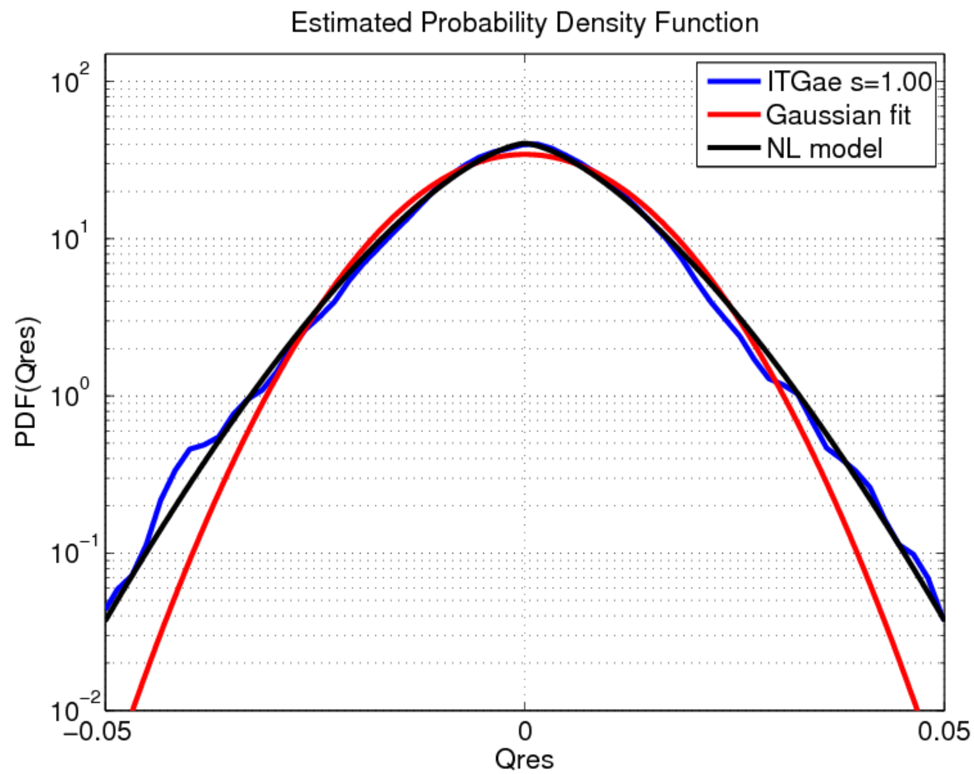




The analytically predicted PDFs are in good agreement with the numerically estimated PDFs for several orders of magnitude. In particular the NL model fits are considerably better than Gaussian fits (using the variance and mean from the distributions). $R/L_{Ti} = 9$, $R/L_n = 2$, $\tau=1$, $k = 0.5$.

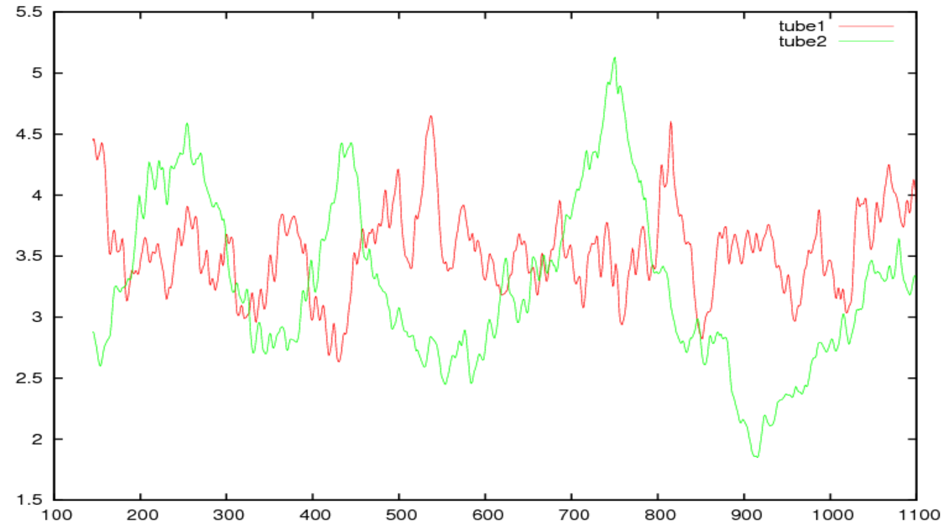
The PDF shear scaling

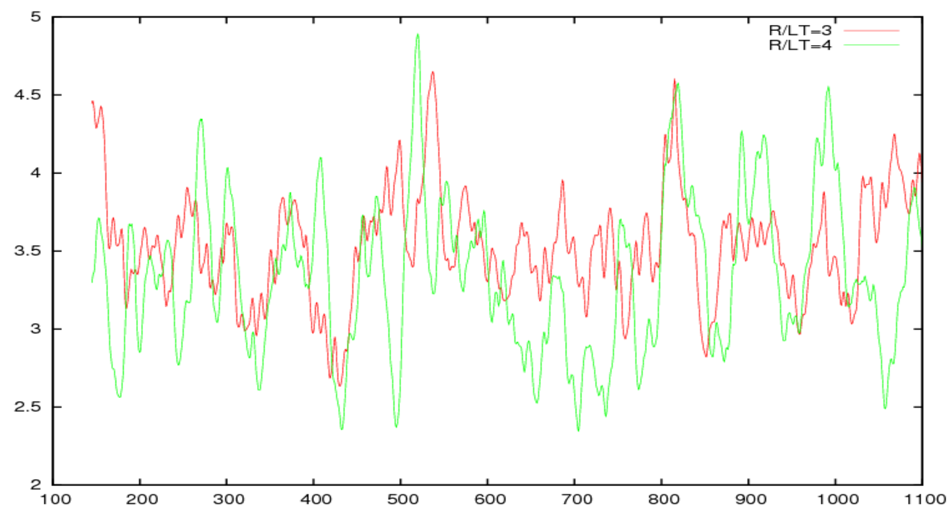




We fix the constant $b_0/Q^{1.5}_0$ at s=0.50. The GENE FLR is obtained by using the eigenfunctions from the simulations and estimate ky ≈0.45.

Result 2: Effects of geometry and spectrum in NCSX





- Time series are rescaled with a constant factor.
- Changes affect mean and variance of the data but NOT the skewness and kurtosis
- The PDF tails are retained (intermittency) and the ARIMA model is preserved.
- Simulations for different flux tubes on the same stellarator surface (keeping gradients, resolution etc constant).
- Tube1 has kurt=3.02 (no tails seen) while Tube2 has kurt=4.19, with visible tails.
- The intermittency for Tube2 is depicted on the kurtosis ONLY when the auto-correlations are removed.

Analytical model including spectrum

- Use the same PDFs and compute the expectation value of the radial heat flux.
- Use a Gaussian forcing for the spectrum.

$$P(Q)=\frac{N}{2b}\exp\{-\frac{1}{b}(\frac{Q-\mu}{Q_0})^{3/2}\}$$

$$\langle Q\rangle(k_x,k_y)=\int dQ\,Q\,P(Q)\propto b^{1/3}$$

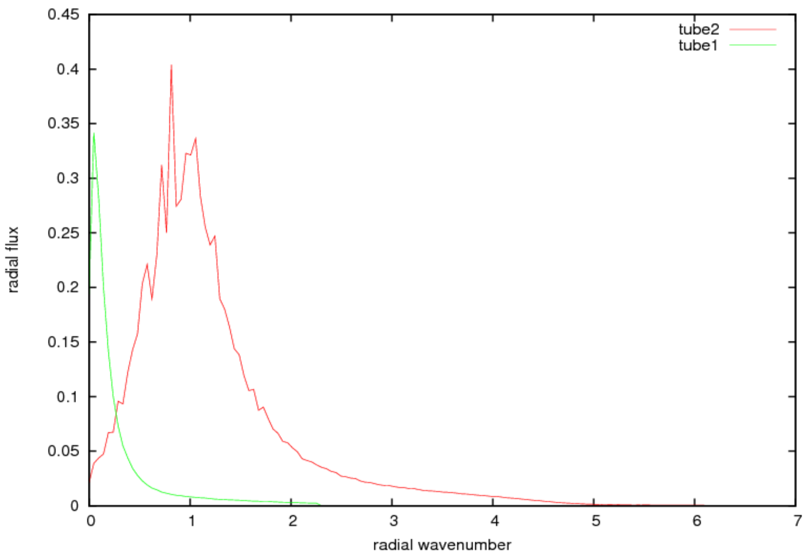
$$b(k_x,k_y)=\frac{f(k_x,k_y)}{k_y\alpha(1+k_\perp^2+\frac{\alpha}{k_y k_\perp^2})}$$

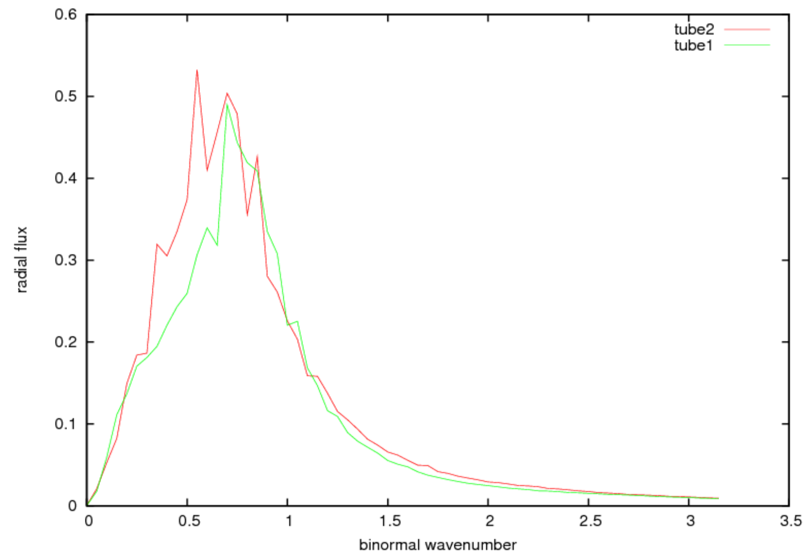
$$\alpha=1-U(1+k_\perp^2)$$

$$f(k_x,k_y)=\kappa_0\exp(-((k_x-k_{x0})^2-(k_y-k_{y0})^2)/k_w^2)$$

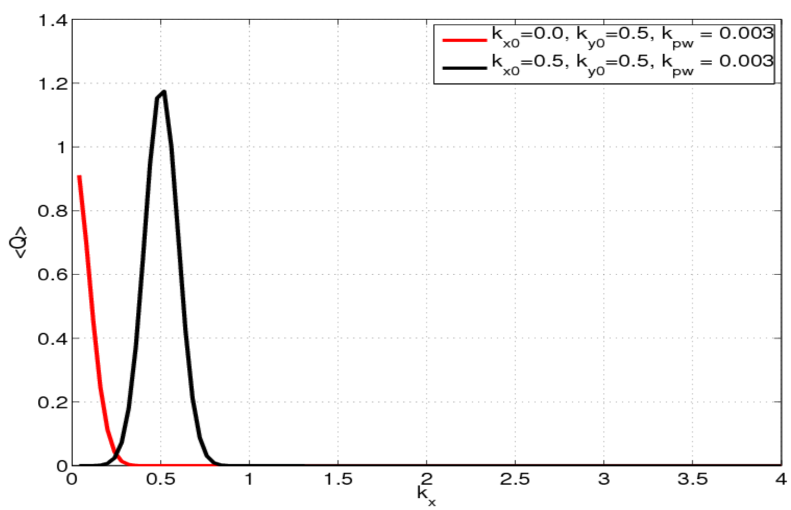
Numerical and analytical spectrum

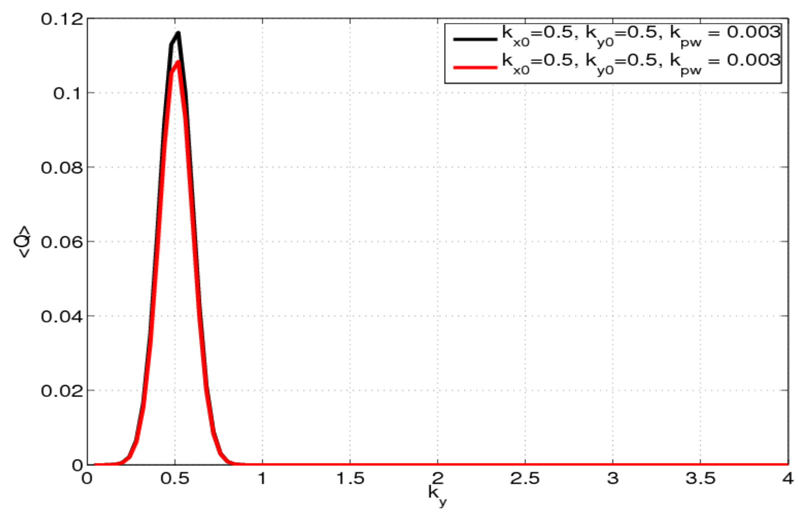
Numerical k_x, and k_y spectrum



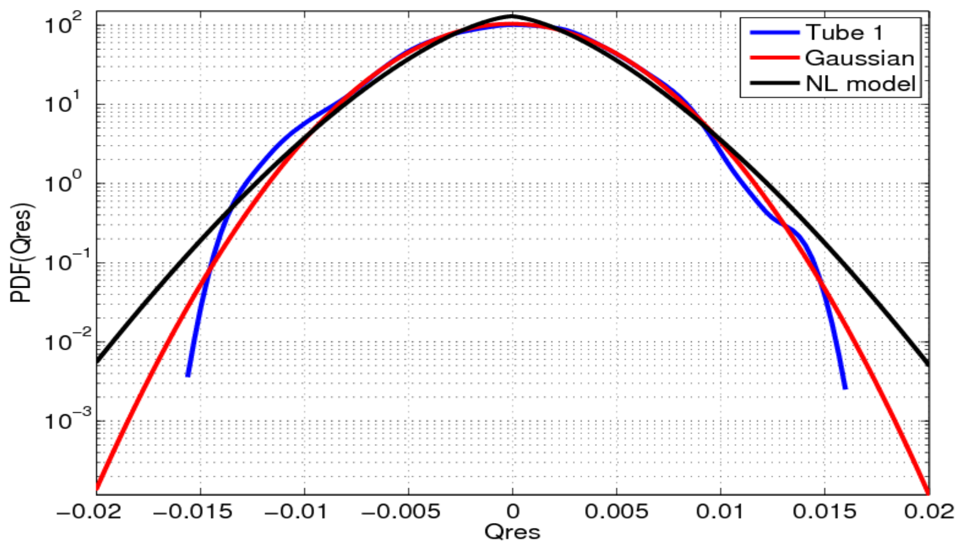


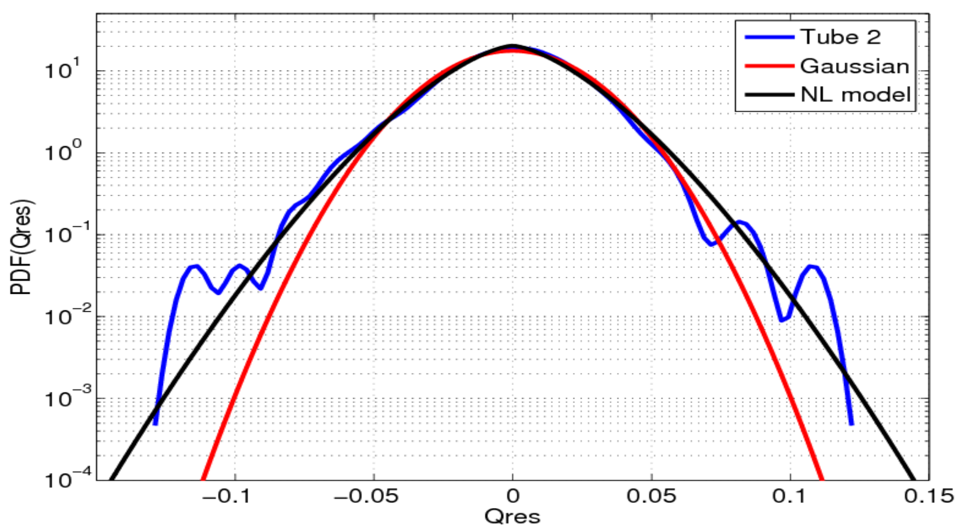
Modeled k_x, and k_y spectrum





Effects of spectrum and geometry on the PDF tails





- The PDF computed on tube 1 (kurt=3.02) is very close to a Gaussian whereas significant tails are visible on tube 2 (kurt=4.19).
- Fixing the spectrum using the simulations we find the ratios:

$$\frac{b(\text{tube } 2)}{b(\text{tube } 1)} = 16.0714 \quad \text{simulation}$$

$$\frac{b(\text{tube } 2)}{b(\text{tube } 1)} = 16.0548 \quad \text{NL model}$$

Future work

- PDF scaling of gradients R/L_{Ti} and R/L_n.
- Reproduce the PDFs with non-linear fluid simulations and check the shear scaling.
- Would trapped particle effects change the PDFs? E.g. using kinetic electrons in the GENE runs. Is the same exponential form of the PDFs still valid?
- A stochastic transport model for tail optimisation in stellarators.

Summary

- We have found good agreement between GK simulations and analytical estimated PDFs of heat flux in ITG turbulence with adiabatic electrons.
- The PDFs have been shown to have manifestly enhanced tails compared to Gaussian distributions.
- We have applied Box-Jenkins methods to remove all auto-correlations from the heat flux time traces.
- The intermittency of a simulation can be attributed to the tails which are quantified via the kurtosis, only when the auto-correlations are removed from the system.
- A convenient way of starting the comparison between different cases is to apply rescaling of data. This means simply taking an original series and multiplying by a constant factor. This has a direct effect on the mean and variance of the data but NOT the skewness or kurtosis. The tails are retained and the arima model is exactly the same as before, so there is no loss of information.

Summary 2

- The NL analytical model having inherently fat tails can reproduce the PDFs in a shear scaling with good agreement for low to moderate magnetic shears and reproduces the spectrum of the simulation.
- Simulations indicate that different tubes on the same stellarator surface manifest different degree of intermittency and this can be understood using the NL model.
- The differences CANNOT be attributed to ZFs since the tubes are on the same surface and the ZF effect is equal for both (ZFs are a surface property). The same goes for the gradient.

Thank You

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