

Output-Feedback MRAC with Reference Model Tolerance of Nonlinearly Perturbed Delayed Plants

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Abstract: A novel approach within the conventional MRAC framework for solving the problem of robust adaptive output tracking for a class of uncertain nonlinear dynamical systems with unknown time-varying state delays is proposed. A MRAC problem formulation in addition to the traditional asymptotic zero error tracking specification makes the problem statement contain an additional explicit requirement, namely the tracking objective is described by a set of admissible reference trajectories which is called a performance tube. As the design parameter which specifies the size of the performance tube, we select the reference model input signal than can be changed within specified bounds. An additional optimization task is formulated to find the best trajectory with a cost function that penalizes the deviation of the control signal from a given signal.

Keywords: Direct adaptive control; Nonlinear Time-delay systems; Reference model tolerance.

1. INTRODUCTION

Model reference adaptive control (MRAC) is one of the main approaches to adaptive control, see e.g. the popular textbooks Narendra and Annaswamy (1989); Åström and Wittenmark (1995); Krstić et al. (1995); Ioannou and Sun (1996); Tao (2003); Ioannou and Fidan (2006). Most model reference adaptive control design techniques have only paid attention to control problem solutions for one particular performance index - the tracking error which is the difference between the plant output and the reference model output. The reference model is chosen to generate a *single desired trajectory* that the plant output has to follow.

In many applications, however, it may be permissible to have an admissible set of reference trajectories which may lead to a possibly significant improvement of performance. Such a definition of reference trajectories appears in flight control design, in the control of space structures and in industrial process control. While this reasoning makes sense, to date the real stumbling-block to its practical use is the lack of tractable algorithmic methods to guarantee for performance. The problem of how to determine a formal mechanism for the selection of an admissible set of reference trajectories is obviously important. This paper aims to open new directions to address this problem. It is shown that the desired result can be achieved by forming an additional optimization task and solve it. The optimality criterion penalizes the deviation of the control signal from a given signal.

In Mirkin and Mirkin (1999) we used an interval reference model to reduce the cost of control. The controller selects on-line a suitable reference trajectory from an a priori known admissible set. To find the best trajectory is formulated as an additional optimization task. The main idea of this method

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follows from the fact that if the input matrix B of a linear plant with parameter uncertainties in the usual state-feedback MRAC formulation is known and equal the input matrix B_r of a reference model, then the basic tracking error equation which is used for stability analysis and the adaptation algorithms design does not depend explicitly on the command signal of the reference model, see e.g. (Ioannou and Sun, 1996, p.325). Therefore it is possible to consider the command input of the reference model as a design parameter and to change it within allowed limits. Its variability does not change neither the stability properties nor the synthesis of the adaptation algorithms. And these limits in turn determine the set of admissible reference trajectories.

But the result in Mirkin and Mirkin (1999) is applicable only to state-feedback control of linear systems with parametric uncertainty and without disturbances. Unfortunately it requires an exact knowledge of the input matrix which moreover has to equal to the reference model input matrix. It turns out to be quite difficult overcome that restriction. That research direction requires effective output-feedback designs to handle enlarged classes of nonlinear systems with disturbances, and time delays.

Therefore, in the current work, we focus on the problem of output-feedback MRAC when the tracking objective is described by an admissible set of reference trajectories called a performance tube, by which we propose a way to overcome the above mentioned difficulties. We consider a class of dynamic systems with unknown parameters and subject to unknown time-varying delays, nonlinear perturbations and external disturbances with unknown bounds.

2. DESCRIPTION OF THE APPROACH

The proposed design approach is based on the concept of goal adaptation as a way to ensure additional desirable properties of the closed-loop controlled systems. Goal adaptation technique as a way to formalize additional design requirements was intro-

duced in our previous papers, see e.g. the review paper Mirkin (2001). In the framework of this concept, we now propose a new paradigm of *performance shaping* in the traditional MRAC setting.

The approach is as follows: Let us assume that it is possible to change, within prescribed bounds, a part of the reference signal r to the reference model, or some of the parameters of its operator W_r . Let us call these *the tolerance control* u_c . Hence, tuning u_c in prescribed bounds we form a admissible set of the reference trajectories which we will call - a *performance tube*. The performance tube has the property that each of its reference trajectory satisfies the specification. The adaptive control algorithm chooses, on-line, a desired trajectory from the performance tube based on measurements of the error $e(t)$ by varying the tolerance control component u_c . The main problems here are to determine a formal mechanism how to choose a reference trajectory from the performance tube, and how to find u_c such that the resulting problem is tractable.

To achieve the stated aims we formulate our goal in the form of two control objectives - primary and secondary: (i) to achieve that the plant output asymptotically exact follows the output of any reference trajectory from a performance tube; (ii) to receive some additional benefit by varying the tolerance control $u_c(t)$ within specified limits.

The proposed design procedure has two phases. In the first phase, to achieve (i) we find a control law parametrization in order to ensure asymptotical exact model tracking for any admissible $u_c(t)$. In the second phase, to get (ii), an optimization problem is formulated and solved.

3. FORMAL PROBLEM STATEMENT

3.1 Model

Let the uncertain single-input, $u(t)$, single-output, $y(t)$ plant with time delays appropriately initialized be given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_\tau x(t - \tau_x(t)) + bu(t) + bf(y(t), y(t - \tau_y(t)), t) \\ &\quad + bd(t) \\ y(t) &= c^T x(t) \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $y(t), u(t) \in \mathbb{R}$, $f(y(t), y(t - \tau_y(t)), t) \in \mathbb{R}$ and $d(t) \in \mathbb{R}$ are, respectively, the state, output, control input, nonlinear delayed perturbation and external disturbance. The constant matrices A , A_τ and the vectors b and c of appropriate dimensions have unknown elements. The time-delays $\tau_x(t)$ and $\tau_y(t)$ are nonnegative differentiable functions, satisfying

$$\begin{aligned} 0 \leq \tau_x(t) \leq \tau_{xmax}, \quad \dot{\tau}_x(t) \leq \tau_x^* < 1 \\ 0 \leq \tau_y(t) \leq \tau_{ymax}, \quad \dot{\tau}_y(t) \leq \tau_y^* < 1 \end{aligned} \quad (2)$$

where τ_{xmax} , τ_x^* , τ_{ymax} and τ_y^* are some unknown, positive constants. Hence, the time delays are uncertain within unknown upper bounds.

3.2 Specification

The primary control objective is asymptotically exact tracking by the output $y(t)$ of the output $y_r(t)$ of a stable reference model

$$\dot{x}_r(t) = A_r x_r(t) + b_r(r_n(t) + u_c(t)) \quad y_r(t) = c_r^T x_r(t), \quad (3)$$

where $x_r \in \mathbb{R}^n$, $r_n(t), u_c(t), y_r(t) \in \mathbb{R}$, i.e. $\lim_{t \rightarrow \infty} e(t) = 0$, where $e(t) = y(t) - y_r(t)$. In (3) the "fixed" part $r_n(t)$ of the reference input is in the form of a standard reference signal

of adaptive control theory, Ioannou and Sun (1996), and the tolerance component of the command signal $u_c(t)$ may vary within specified given limits

$$u_c(t) \in [u_c^-(t) \ u_c^+(t)]. \quad (4)$$

which determine the performance tube of the reference model (3).

The transfer function of the reference model is expressed as

$$W_r(s) = c_r^T (sI_n - A_r)^{-1} b_r = k_r \frac{N_r(s)}{D_r(s)} \quad (5)$$

where $N_r(s), D_r(s)$ are monic polynomials and k_r is a constant.

The second control objective is formalized as the following optimization task:

$$\begin{aligned} \text{minimize (w.r.t. } u_c) \quad & J(u_c) = u^2(t) \\ \text{subject to} \quad & u_c^-(t) \leq u_c(t) \leq u_c^+(t). \end{aligned} \quad (6)$$

3.3 Main assumptions

The following assumptions are made on the plant (1) and the reference model (3): **(A1)** When there are no terms with time delays τ_x , τ_y and external disturbance, the plant (1) can be described by

$$y = W_0(s)u, \quad W_0(s) = c^T (Is - A)^{-1} b = k_p \frac{N(s)}{D(s)} \quad (7)$$

where $W_0(s)$ is the transfer function associated with the undelayed, undisturbed plant; $D(s)$ is a monic polynomial of degree n ; the polynomial $N(s)$ is monic and Hurwitz and of degree $n - 1$, i.e. the plant is minimum phase; the high-frequency plant gain k_p is constant with known sign; **(A2)** The transfer function $W_r(s)$ is strictly positive real (SPR); **(A3)** $|d(t)| \leq d^*$, where d^* is unknown and **(A4)** $A_\tau = ba_\tau^* T$. These assumptions are the same as in Mirkin and Gutman (2010b).

In addition to (A1)-(A4) we include the following assumption for the nonlinear function $f(y(t), y(t - \tau_y(t)), t)$. **(A5)** The function $f(y(t), y(t - \tau_y(t)), t)$ is assumed to be bounded by

$$\begin{aligned} f(y(t), y(t - \tau_y(t)), t) \leq & \sum_{j=1}^{p_1} \xi_{1j}^* |y(t) \phi_j(y(t))| \\ & + \sum_{j=1}^{p_2} \xi_{2j}^* |y(t - \tau_y(t)) \phi_j(y(t - \tau_y(t)))| \end{aligned} \quad (8)$$

where p_1 and p_2 are known; $\phi_j(\star)$ and $\phi_j(\star)$, ($j = 1, \dots, p$) are known bounded nonlinear functions and ξ_{1j}^* and ξ_{2j}^* are unknown positive constants.

Remark 1. The minimum phase assumption is fundamental in MRAC schemes, see Ioannou and Sun (1996); Tao (2003). The assumption that the relative degree be one focusses on the simplest case amenable to Lyapunov designs; however, the paper's main idea of the *new control parametrization* can be extended to higher relative degree, for which case it is required to use "error augmentation" and/or "tuning error normalization", see Ioannou and Sun (1996).

4. CONTROLLER DESIGN

We are looking for a control strategy in the form

$$u(t) = u_a(t) + u_c(t) \quad (9)$$

where $u_a(t)$ is an adaptive control part and $u_c(t)$ is the tolerance component. Following the two phase design procedure, see

Section 2, the two types of control laws $u_a(t)$ and $u_c(t)$ will be derived separately. In the adaptive control design phase, we are looking for an adaptive control component parametrization to guarantee the asymptotic error $e(t)$ to be equal to zero for any the command signal of the reference model (3). In the tolerance control design phase, we derive the tolerance control, $u_c(t)$ in order to minimize the performance index (6). So, the adaptive control strategy modifies the controller parameters whereas the tolerance control component modifies the reference model input.

4.1 Adaptive part parametrization

We propose the following structure of $u_a(t)$:

$$\begin{aligned} u_a(t) &= \theta^T(t)\omega(t) + \theta_I(t) + u_{nc}(t) \\ u_{nc}(t) &= -\sum_{j=0}^{p_1} k_{1j}e(t)\phi_j^2(e(t)) - \sum_{j=0}^{p_2} k_{2j}e(t)\phi_j^2(e(t)) \\ \dot{\theta}(t) &= -\text{sgn}(\rho^*)\Gamma\omega(t)e(t) \\ \dot{\theta}_I(t) &= -\text{sgn}(\rho^*)\gamma_I e(t) \end{aligned} \quad (10)$$

where $\theta(t) = [\theta_e(t) \ \theta_1^T(t) \ \theta_2^T(t)]^T \in \mathbb{R}^{2n-1}$ and $\theta_I \in \mathbb{R}^1$ are the matrix and scalar adaptation gains respectively; the signal vector $\omega(t) = [e(t) \ x_1^T(t) \ x_2^T(t)]^T \in \mathbb{R}^{2n-1}$ and $x_1 \in \mathbb{R}^{n-1}$ and $x_2 \in \mathbb{R}^{n-1}$ are the states of the auxiliary filters

$$\begin{aligned} \dot{x}_1(t) &= A_f x_1(t) + b_f u(t), \ x_1(0) = 0 \\ \dot{x}_2(t) &= A_f x_2(t) + b_f y(t), \ x_2(0) = 0 \end{aligned} \quad (12)$$

where $A_f \in \mathbb{R}^{(n-1) \times (n-1)}$ is Hurwitz and b_f is a constant vector such that (A_f, b_f) is controllable. The positive scalars $k_{1j} > 0$, $k_{2j} > 0$, $j = 1, \dots, p$, $\gamma_I > 0$ and the matrix $\Gamma = \Gamma^T > 0$ are design parameters.

The control components $\theta^T(t)\omega(t)$, $\theta_I(t)$ and $u_{nc}(t)$ are used simultaneously to maintain robust tracking performance with $e(t) \rightarrow 0$ in face of parametric uncertainties, bounded external disturbances and nonlinear delayed perturbations, respectively.

Remark 2. The tracking objective is achieved by output feedback only. As in Mirkin and Gutman (2010b), but in contrast to standard MRAC schemes, the feedback signal $\omega(t)$ is used in the tracking error instead of the plant output. Moreover, unlike the standard schemes, we remove the adjustable feed-forward term driven by the reference model input. However as an extension to Mirkin and Gutman (2010b), we consider a rather general class of plants and therefore, to handle the plant non-linearities, a new feedback compensator term $u_{nc}(t)$ is incorporated, with adaptive error feedback. Hence the adaptive controller parametrization is a new, also in the case when $u_c(t) = 0$, i.e. in the case we do have the additional optimization problem.

Remark 3. For adaptation a proportional-integral-time delay (PITD) type algorithm can be chosen which may have better transient adaptation performance than the traditional I and PI schemes, see e.g. Mirkin and Gutman (2010b).

4.2 Error Equation

By using the conventional technique of model reference adaptive control Ioannou and Sun (1996); Tao (2003) the tracking error $e(t) = y(t) - y_r(t)$ for any $u(t)$ can be expressed as

$$\begin{aligned} e &= W_r(s)\rho^* \left[u - \theta_e^* y - \theta_1^{*T} x_1 - \theta_2^{*T} x_2 - \theta_r^*(r_n(t) + u_c(t)) \right. \\ &\quad + H(s)d(t) + H(s)f(y, y(t - \tau_y(t)), t) + a_\tau^{*T} x(t - \tau_x(t)) \\ &\quad \left. - \theta_1^{*T} H_f(s) a_\tau^{*T} x(t - \tau_x(t)) \right] \end{aligned} \quad (13)$$

where $\rho^* = \theta_r^{*-1} = k_p k_r^{-1}$, $\theta_1^*, \theta_2^* \in \mathbb{R}^{n-1}$, $\theta_e^* \in \mathbb{R}$, $\theta_r^* \in \mathbb{R}$ are so-called matching parameters, $H(s) = 1 - \theta_1^{*T} H_f(s)$ with $H_f(s) = (sI - A_f)^{-1} b_f = \frac{[s^{n-2} \dots s \ 1]^T}{\lambda(s)} \in \mathbb{R}^{n-1}$, $\lambda(s) = s^{n-1} + \dots + \lambda_1 s + \lambda_0$ is a monic Hurwitz polynomial and A_f, b_f are from (12).

To find a suitable error equation parametrization, we manipulate the last term of (13). Firstly, we introduce a new dynamical system

$$z(t) = \theta_1^{*T} H_f(s) [a_\tau^{*T} x(t - \tau_x(t))] = \theta_z^{*T} z_x(t) \quad (14)$$

where $\theta_z^{*T} = [\theta_{11}^* a_\tau^{*T}, \theta_{12}^* a_\tau^{*T}, \dots, \theta_{1(n-1)}^* a_\tau^{*T}]$ and

$$z_x(t) = H_n(s) [x(t - \tau_x(t))] \quad (15)$$

$$H_n(s) = \frac{[I_{n \times n} s^{n-2}, \dots, I_{n \times n} s, I_{n \times n}]^T}{\lambda(s)} \quad (16)$$

Here $\theta_z^* \in \mathbb{R}^{n(n-1)}$, $z_x \in \mathbb{R}^{n(n-1)}$, $H_n(s) \in \mathbb{R}^{(n-1) \times n}$ and $I_{n \times n}$ is the $n \times n$ identity matrix.

Secondly we decompose the signals $z_{xj}(t)$ in (15) into two components $z_x(t) = z_e(t) + z_r(t)$ where

$$\begin{aligned} z_e(t) &= H_n(s) [e_x(t - \tau_x(t))] \quad z_r(t) = H_n(s) [x_r(t - \tau_x(t))] \\ e_x(t - \tau_x(t)) &= x(t - \tau_x(t)) - x_r(t - \tau_x(t)) \end{aligned} \quad (17)$$

where $x_r(t) \in \mathbb{R}^n$ is the state of the reference model (3) with the state space triple (A_r, b_r, c_r) .

Then, using (14) and (17), we obtain from (13) the *basic error equation*

$$\begin{aligned} e &= W_r(s)\rho^* \left[u - \theta_e^* e(t) - \theta_1^{*T} x_1(t) - \theta_2^{*T} x_2(t) - \theta_r^*(r_n + u_c) \right. \\ &\quad - \theta_e^* y_r(t) + a_\tau^{*T} x_r(t - \tau_x(t)) - \theta_z^{*T} z_r(t) + a_\tau^{*T} e_x(t - \tau_x(t)) \\ &\quad \left. - \theta_z^{*T} z_e(t) + H(s)[d(t)] + H(s)[f(y, y(t - \tau_y(t)), t)] \right] \end{aligned} \quad (18)$$

Remark 4. Note that $e_x(t - \tau_x(t))$ and $z_e(t)$ are not available for measurement, and we shall use them only for analysis, and not for the implementation.

Introducing the parameter vector $\theta^* = [\theta_e^* \ \theta_1^{*T} \ \theta_2^{*T}]^T \in \mathbb{R}^{2n-1}$ the basic tracking error equation (18) can be expressed as

$$\begin{aligned} e &= W_r(s)\rho^* \left[u(t) - \theta^{*T} \omega(t) - \theta_r^*(r_n(t) + u_c(t)) - \theta_e^* y_r(t) \right. \\ &\quad - \theta_z^{*T} z_e(t) + a_\tau^{*T} x_r(t - \tau_x(t)) - \theta_z^{*T} z_r(t) + a_\tau^{*T} e_x(t - \tau_x(t)) \\ &\quad \left. + H(s)[d(t)] + H(s)[f(y, y(t - \tau_y(t)), t)] \right] \end{aligned} \quad (19)$$

4.3 Stability Proof

To prove stability, the usual way of MRAC for delay free systems is used, see e.g. Ioannou and Sun (1996). The augmented vector $\hat{x}(t) = [x \ x_1 \ x_2]^T$ is introduced, and the state of the corresponding non-minimal realization $\hat{c}^T (sI - \hat{A})^{-1} \hat{b} \theta_r^*$ of W_r is denoted by $\hat{x}_r(t)$. Then we can write the following state space representation for (19)

$$\begin{aligned} \frac{d\hat{e}(t)}{dt} &= \hat{A}\hat{e}(t) + \bar{b}\rho^* \left\{ u(t) - \theta^* \omega(t) + a_\tau^* \tilde{l}^T \hat{e}(t - \tau_x(t)) \right. \\ &\quad - \theta_z^* C_e \hat{z}_e(t) - \theta_r^* (r_n(t) + u_c(t)) - \theta_e^* y_r(t) \\ &\quad + a_\tau^* x_r(t - \tau_x(t)) - \theta_z^* H_n(s) [x_r(t - \tau_x(t))] \\ &\quad \left. + H(s)[d(t)] + H(s)[f(y, y(t - \tau_y(t)), t)] \right\} \\ \frac{d\hat{z}_e(t)}{dt} &= A_e \hat{z}_e(t) + B_e \tilde{l}^T \hat{e}(t - \tau_x(t)), \quad z_e(t) = C_e \hat{z}_e(t) \end{aligned} \quad (20)$$

where the triple (A_e, B_e, C_e) is a minimal state space realization of the stable transfer matrix $H_n(s)$ in (17), $\bar{b} = \hat{b}\theta_r^*$, and $0_{n \times (n-1)}$ is the zero $n \times (n-1)$ matrix.

Because $\hat{c}^T (sI - \hat{A})^{-1} \hat{b}\theta_r^* = W_r$ is SPR, there exists a matrix $P = P^T > 0$ satisfying

$$\hat{A}^T P + P\hat{A} + \zeta \zeta^T + \nu Q = 0, \quad P\hat{b}\theta_r^* = P\bar{b} = \hat{c}, \quad (21)$$

where ζ is a vector, $Q = Q^T > 0$ is any positive definite matrix and $\nu > 0$ is some selective constant, all of which is implied by the Meyer-Kalman-Yakubovich lemma (Ioannou and Sun, 1996, pp.129-130). Since A_e in (20) is stable, it also holds that

$$A_e^T P_z + P_z A_e + Q_z = 0 \quad (22)$$

where $P_z = P_z^T > 0$ and $Q_z = Q_z^T > 0$.

For the stability analysis we use the following Lyapunov-Krasovskii type functional

$$\begin{aligned} V &= \sum_{i=1}^4 V_i, \quad V_1 = \hat{e}^T P \hat{e} + \hat{z}_e^T P_z \hat{z}_e \\ V_2 &= \frac{\nu}{2} \int_{t-\tau_x(t)}^t \hat{e}^T(s) Q \hat{e}(s) ds + \sum_{j=1}^{p_2} \varsigma_j \int_{t-\tau_y(t)}^t e^2(s) \varphi_j^2(e(s)) ds \\ V_3 &= |\rho^*| \tilde{\chi}^T(t) \Gamma^{-1} \tilde{\chi}(t) \\ V_4 &= |\rho^*| \gamma_f^{-1} (\theta_f(t) + \theta_f^* \text{sgn}(e(t)))^2 \end{aligned} \quad (23)$$

where $\Gamma > 0$ is from (11) and

$$\tilde{\chi}(t) = \tilde{\theta}(t) + \theta_0 \quad (24)$$

The signum function $\text{sgn}(e(t))$ is defined such that $\text{sgn}(e(t)) = 1$, if $e(t) > 0$; $\text{sgn}(e(t)) = 0$, if $e(t) = 0$; and $\text{sgn}(e(t)) = -1$, if $e(t) < 0$.

The vector

$$\theta_0 = \left[\frac{r_0}{2\rho^*}, 0, \dots, 0 \right]^T \quad (25)$$

has the same dimension as θ . The unknown vector θ_f^* and the selective parameters $r_0 > 0$, $\varsigma_j > 0$ will be defined later.

Using (21) and (22), the time derivatives $\dot{V}_i(t)$, $i = 1, \dots, 4$ of (23) along (20) can be written

$$\begin{aligned} \dot{V}_1(t)|_{(20)} &= -\nu \hat{e}^T(t) Q \hat{e}(t) - \hat{e}^T(t) \zeta \zeta^T \hat{e}(t) - \hat{z}_e^T(t) Q_z \hat{z}_e(t) \\ &\quad + 2\hat{e}^T(t) P \bar{b} \rho^* u_a(t) + 2\hat{e}^T(t) P \bar{b} \rho^* a_\tau^* \tilde{l}^T \hat{e}(t - \tau_x(t)) \\ &\quad - 2\rho^* \hat{e}^T(t) P \bar{b} \theta^* \omega(t) - 2\hat{e}^T(t) P \bar{b} \rho^* \theta_z^* C_e \hat{z}_e(t) \\ &\quad + 2\hat{z}_e^T(t) P_z B_e \tilde{l}^T \hat{e}(t - \tau_x(t)) \\ &\quad - 2\hat{e}^T(t) P \bar{b} \rho^* \theta_e^* y_r(t) + 2\hat{e}^T(t) P \bar{b} \rho^* H(s)[d(t)] \\ &\quad - 2\hat{e}^T(t) P \bar{b} \rho^* \theta_z^* H_n(s) [x_r(t - \tau_x(t))] \\ &\quad + 2\hat{e}^T(t) P \bar{b} \rho^* a_\tau^* x_r(t - \tau_x(t)) \\ &\quad + 2\hat{e}^T(t) P \bar{b} \rho^* H(s)[f(y, y(t - \tau_y(t)), t)] \\ &\quad - 2\hat{e}^T(t) P \bar{b} \rho^* [\theta_r^* r(t) + \theta_r^* u_c(t) - u_c(t)] \end{aligned} \quad (26)$$

In view the known fact that for any vectors x, y , and any positive-definite matrix S of appropriate dimensions, it holds that $2x^T y \leq x^T S x + y^T S^{-1} y$ by which we can estimate some of the terms in (26) as follows

$$\begin{aligned} 2\hat{e}^T(t) P \bar{b} \rho^* a_\tau^* \tilde{l}^T \hat{e}(t - \tau_x(t)) &\leq \hat{e}^T(t) P \bar{b} \Psi_1 \bar{b}^T P \hat{e}^T(t) \\ &\quad + \hat{e}^T(t - \tau_x(t)) S \hat{e}(t - \tau_x(t)) \\ -2\hat{e}^T(t) P \bar{b} \rho^* \theta_z^* C_e \hat{z}_e(t) &\leq \hat{e}^T(t) P \bar{b} \Psi_2 \bar{b}^T P \hat{e}^T(t) \\ &\quad + \hat{z}_e^T(t) S \hat{z}_e(t) \\ 2\hat{z}_e^T(t) P_z B_e \tilde{l}^T \hat{e}(t - \tau_x(t)) &\leq \hat{e}^T(t - \tau_x(t)) \Psi_3 \hat{e}(t - \tau_x(t)) \\ &\quad + \hat{z}_e^T(t) S \hat{z}_e(t) \end{aligned} \quad (27)$$

where

$$\begin{aligned} \Psi_1 &= \rho^{*2} a_\tau^* \tilde{l}^T S^{-1} \hat{l} a_\tau^*, \quad \Psi_2 = \rho^{*2} \theta_z^* C_e S^{-1} C_e^T \theta_z^*, \\ \Psi_3 &= \hat{l} B_e^T P_z^T S^{-1} P_z B_e \tilde{l}^T \end{aligned} \quad (28)$$

Using boundedness of the reference signals, the correction control input $u_c(t)$ and the external disturbance $d(t)$ ($|r_n(t)| \leq r^*$, $|y_r(t)| \leq y_r^*$, $|u_c(t)| \leq \delta = \frac{1}{2}[|u_c^+ + u_c^-| + |u_c^+ - u_c^-|]$ and $|d(t)| \leq d^*$), the stability of the transfer functions $H_f(s)$ and $H(s)$ and in view of (8) we can write the following estimates for the other terms of (26):

$$\begin{aligned} \Xi_{eq} &= -2\hat{e}^T(t) P \bar{b} \rho^* [\theta_r^* r(t) + \theta_r^* u_c(t) - u_c(t)] \\ &\quad - 2\hat{e}^T(t) P \bar{b} \rho^* \theta_e^* y_r(t) \\ &\quad + 2\hat{e}^T(t) P \bar{b} \rho^* a_\tau^* x_r(t - \tau_x(t)) + 2\hat{e}^T(t) P \bar{b} \rho^* H(s)[d(t)] \\ &\quad - 2\hat{e}^T(t) P \bar{b} \rho^* \theta_z^* H_n(s) [x_r(t - \tau_x(t))] \\ &\leq 2|e(t)| |\rho^*| \left(|\theta_r^*| r^* + |\theta_r^*| \delta + \delta + |\theta_e^*| y_r^* + \|a_\tau^*\| x_r^* \right. \\ &\quad \left. + \|\theta_z^*\| \|H_n(s)\|_\infty x_r^* + \|H(s)\|_\infty d^* \right) \\ &\leq 2|e(t)| |\rho^*| \eta_1 \end{aligned} \quad (29)$$

where

$$\begin{aligned} \eta_1 &= |\theta_r^*| r^* + |\theta_r^*| \delta + \delta + |\theta_e^*| y_r^* + \|a_\tau^*\| x_r^* \\ &\quad + \|\theta_z^*\| \|H_n(s)\|_\infty x_r^* + \|H(s)\|_\infty d^* \end{aligned}$$

For the term $\Xi_f = 2\hat{e}^T(t) P \bar{b} \rho^* H(s)[f(\star)]$ we have

$$\begin{aligned} \Xi_f &\leq 2|e(t)| |\rho^*| \|H(s)\|_\infty \left(\sum_{j=1}^{p_1} \xi_{1j}^* |e(t)| |\phi_j(e(t))| \right. \\ &\quad + \sum_{j=1}^{p_2} \xi_{2j}^* |e(t - \tau_y(t))| |\phi_j(e(t - \tau_y(t)))| \\ &\quad + \sum_{j=1}^{p_1} \xi_{1j}^* |y_r(t)| |\phi_j(y_r(t))| \\ &\quad \left. + \sum_{j=1}^{p_2} \xi_{2j}^* |y_r(t - \tau_y(t))| |\phi_j(y_r(t - \tau_y(t)))| \right) \\ &\leq 2|e(t)| |\rho^*| \|H(s)\|_\infty \left(\sum_{j=1}^{p_1} \xi_{1j}^* |e(t)| |\phi_j(e(t))| \right. \\ &\quad \left. + \sum_{j=1}^{p_2} \xi_{2j}^* |e(t - \tau_y(t))| |\phi_j(e(t - \tau_y(t)))| + \eta_2 \right) \end{aligned} \quad (30)$$

where the unknown constant

$$\eta_2 = \sum_{j=1}^{p_1} \xi_{1j}^* y_r^* \phi_j^* + \sum_{j=1}^{p_2} \xi_{2j}^* y_r^* \phi_j^*$$

Using (2) for the time derivative $\dot{V}_2(t)$ of (23) we obtain

$$\begin{aligned} \dot{V}_2(t)|_{(20)} &\leq \frac{\bar{v}}{2} \hat{e}^T(t) Q \hat{e}(t) - \frac{\bar{v}}{2} \hat{e}^T(t - \tau_x(t)) Q \hat{e}(t - \tau_x(t)) \\ &\quad - \sum_{j=1}^{p_2} 2\zeta_j^* e^2(t - \tau_y(t)) \phi_j^2(e(t - \tau_y(t))) \\ &\quad + \sum_{j=1}^{p_2} \zeta_j^* e^2(t) \phi_j^2(e(t)) \end{aligned} \quad (31)$$

where $\bar{v} = \frac{v\bar{\tau}_x}{2}$, $\bar{\tau}_x = 1 - \tau_x^* > 0$ and $\zeta_j^* = (1 - \tau_y^*) \zeta_j$.

For the derivative $\dot{V}_3(t)$ in view of (25) we have

$$\begin{aligned} \dot{V}_3(t)|_{(20)} &= -2\rho^* \bar{\theta}^T(t) \omega(t) e(t) - r_0 \hat{e}^T(t) P \bar{b} \bar{b}^T P \hat{e}(t) \\ &= -2\rho^* \bar{\theta}^T(t) \omega(t) e(t) - r_0 e^2(t) \end{aligned} \quad (32)$$

Here, we used the fact that $e(t) = \hat{c}^T \hat{e}(t)$, see (20).

For convenience, let Q from (21), Q_z from (22) and r_0 from (25) be $Q = Q_1 + Q_2$, $Q_z = Q_{z1} + Q_{z2}$, $r_0 = r_{01} + r_{02} + r_{03}$ with $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $Q_{z1} = Q_{z1}^T > 0$, $Q_{z2} = Q_{z2}^T > 0$ and $r_{01} > 0$, $r_{02} > 0$, $r_{03} > 0$. Then, combining $-r_{01}e^2(t)$ and $-r_{02}e^2(t)$ from (32) and the first two terms from (30), completing the squares, and dropping negative terms, we obtain

$$\begin{aligned} \sum_{j=1}^{p_1} \left(-r_{01} \frac{1}{p_1} e^2(t) + 2|e(t)| |\rho^*| \|H(s)\|_\infty \xi_{1j}^* |e(t)| |\phi_j(e(t))| \right) \\ \leq \sum_{j=1}^{p_1} \frac{1}{r_{01}} \zeta_{1j} e^2(t) \phi_j^2(e(t)) \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^{p_2} \left(-r_{02} \frac{1}{p_2} e^2(t) \right. \\ \left. + 2|e(t)| |\rho^*| \|H(s)\|_\infty \xi_{2j}^* |e(t - \tau_y(t))| |\phi_j(e(t - \tau_y(t)))| \right) \\ \leq \sum_{j=1}^{p_2} \frac{1}{r_{02}} \zeta_{2j} e^2(t - \tau_y(t)) \phi_j^2(e(t - \tau_y(t))) \end{aligned} \quad (33)$$

where $\zeta_{j1} = p_1 (\rho^* \|H(s)\|_\infty \xi_{1j}^*)^2$ and $\zeta_{j2} = p_2 (\rho^* \|H(s)\|_\infty \xi_{2j}^*)^2$

where $\zeta_{j1} > 0$ and $\zeta_{j2} > 0$ are some appropriate unknown constants.

Applying (27), (29) and (30) to (26), and using (31), (32) and (33) yields the next equation for the time derivative of $V_\Sigma = V_1 + V_2 + V_3$,

$$\begin{aligned} \dot{V}_\Sigma|_{(20)} &\leq -\hat{e}^T(t) \frac{v}{2} Q \hat{e}(t) - \hat{e}^T(t) \zeta \zeta^T \hat{e}(t) - \hat{z}_e^T(t) (Q_z - 2S) \hat{z}_e(t) \\ &\quad - \hat{e}^T(t) P \bar{b} (r_{03} I - \Psi_1 - \Psi_2) \bar{b}^T P \hat{e}(t) + 2e(t) \rho^* \theta_I(t) \\ &\quad - \hat{e}^T(t - \tau_x(t)) (\bar{v} Q - S - \Psi_3) \hat{e}(t - \tau_x(t)) \\ &\quad - \sum_{j=1}^{p_1} \left(\frac{1}{r_{01}} \zeta_{1j} - 2\rho^* k_{1j} \right) e^2(t) \phi_j^2(e(t)) \\ &\quad - \sum_{j=1}^{p_2} \left(\frac{1}{r_{02}} \zeta_{2j} - 2\zeta^* k_{2j} \right) e^2(t - \tau_y(t)) \phi_j^2(e(t - \tau_y(t))) \end{aligned} \quad (34)$$

Then, selecting values for $r_{01}, r_{02}, r_{03}, Q_2, Q_{z2}$ and ζ_j of (31) from the inequalities

$$\begin{aligned} r_{03} &> \lambda_{\max}(\Psi_1 + \Psi_2), \lambda_{\min}(\bar{v} Q_2) > \lambda_{\max}(S + \Psi_3), \\ \lambda_{\min}(Q_{z2}) &> \lambda_{\max}(2S), r_{01} > \frac{\zeta_{1j}}{2\rho^* k_{1j}}, r_{02} > \frac{\zeta_{2j}}{2\zeta^* k_{2j}} \end{aligned} \quad (35)$$

we obtain from (34)

$$\begin{aligned} \dot{V}_\Sigma|_{(20)} &\leq -\hat{e}^T(t) \frac{v}{2} Q_1 \hat{e}(t) - \hat{e}^T(t) \zeta \zeta^T \hat{e}(t) - \hat{z}_e^T(t) Q_{z1} \hat{z}_e(t) \\ &\quad - \hat{e}^T(t - \tau_x(t)) \bar{v} Q_1 \hat{e}(t - \tau_x(t)) \\ &\quad + 2\rho^* |e(t)| \theta^* + 2\rho^* e(t) \theta_I(t) \end{aligned} \quad (36)$$

where $\theta^* = (\eta_1 + \|H(s)\|_\infty \eta_2) \text{sgn}(\rho^*)$, with η_1 from (29) and η_2 from (30).

Remark 5. We note that the coefficient matrices Q, Q_z and S and the scalar parameters r_{01}, r_{02}, r_{03} and ζ_j are used only for analysis and do not influence the control law. Controller gains adjust automatically to counter the non-desirable effects of delayed states, parameter uncertainties, a nonlinear perturbation and an external disturbance.

Now we write the time derivative $\dot{V}_4(t)$ in view of (9). For the time derivative \dot{V}_4 we consider two cases: when $e(t) \neq 0$ and $e(t) = 0$

$$\begin{aligned} \dot{V}_4(t)|_{(20)}^{e(t) \neq 0} &= 2|\rho^*| \gamma_I^{-1} (\theta_I(t) + \text{sgn}(e(t)) \theta_I^*) \dot{\theta}_I(t) \\ &= -2\rho^* |e(t)| \theta^* - 2\rho^* e(t) \theta_I(t) \\ \dot{V}_4(t)|_{(20)}^{e(t) = 0} &= 0 \end{aligned} \quad (37)$$

Using (36) and (37) we obtain for \dot{V} in (23)

$$\begin{aligned} \dot{V}|_{(20)} &\leq -\hat{e}^T(t) \frac{v}{2} Q_1 \hat{e}(t) - \hat{e}^T(t) \zeta \zeta^T \hat{e}(t) - \hat{z}_e^T(t) Q_{z1} \hat{z}_e(t) \\ &\quad - \hat{e}^T(t - \tau_x(t)) \bar{v} Q_1 \hat{e}(t - \tau_x(t)) \end{aligned} \quad (38)$$

This implies, e.g. Ioannou and Sun (1996); Tao (2003) that V and, therefore, $\hat{e}(t), e(t), \hat{z}_e(t), \theta_I(t), \bar{\theta}, \theta \in L_\infty$. This fact is central to the remainder of the stability analysis, which follows directly using the steps in Ioannou and Sun (1996).

4.4 Tolerance component

Using (3), (9) and (10) it is easy to show that the optimization problem (6) has the following solution

$$u_c^{\text{opt}} = \begin{cases} v(t), & \text{if } v(t) \in [u_c^-(t), u_c^+(t)]; \\ \text{argmin}(J(u_c^-), J(u_c^+)), & \text{if } v(t) \notin [u_c^-(t), u_c^+(t)] \end{cases} \quad (39)$$

where $v(t) = -\theta^T(t) \omega(t) - \theta_I(t) - u_{nc}(t)$.

We summarize the main result as

Theorem 1. Consider the closed-loop system defined by the plant in (1), the controller in (9), and the updating algorithm in (11) with the reference model as in (3). Then for the bounded disturbance $d(t)$ and the nonlinearity $f(\cdot)$ satisfying the inequality (8), and for any delays $\tau_x(t)$ and $\tau_y(t)$ that satisfy (2), the following properties hold: (i) all signals of the closed-loop system are bounded, (ii) $\lim_{t \rightarrow \infty} e(t) = 0$ and (iii) the secondary control objective (6) is achieved.

Remark 6. Theorem 1 shows that the stability of the closed-loop system and the controller parameters are completely independent of the value of the plant time-delays $\tau_x(t)$ and $\tau_y(t)$. The controller is also robust to an external disturbance.

5. SIMULATION RESULTS

We illustrate the results by a simple unstable second order time delay system.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -0.2 & -0.1 \end{bmatrix} \begin{bmatrix} x_1(t - \tau_x(t)) \\ x_2(t - \tau_x(t)) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} (u(t) + f(\star) + d(t))$$

$$f(\star) = 0.5y^4(t) + 0.8y(t)y^2(t - \tau_y(t)) + 0.9y^3(t - \tau_y(t))$$

$$y(t) = [0.6 \ 0.5]x(t), \quad x(0) = [1 \ 1]^T \quad (40)$$

All plant parameters, the nonlinearity $f(\star)$, the disturbance $d(t) = 0.2\sin(0.2t)$, the delays $\tau_x(t) = 4.5 + 3\sin(t)$ and $\tau_y(t) = 4.5 - 3\sin(t/2)$ are unknown to the controller. The parameter values of the adaptive controller (9) in our simulation were chosen as $A_f = -1$, $b_f = 1$ and $\Gamma = 90I$. The command signal $r_n(t)$ is ± 0.5 plus a square signal with amplitude 0.1 and basic frequency 0.5 rad/s. $A_r = \begin{bmatrix} -4 & -1 \\ 1 & 0 \end{bmatrix}$, $b_r = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $c_r = [0.5 \ 0.5]^T$. The component $u_{nc}(t)$ from (10) is $u_{nc}(t) = -9e(t)|e^3(t)|^2 - 9e^3(t)$.

The simulation results are shown in Figures 1–2, where we show the time responses of the plant output $y(t)$ (blue lines) and the output of the reference model $y_r(t)$ (red lines) for two different values of the command signal $r_n(t)$: $r_n(t) = -0.5 + 0.1\sin 2t$ (Fig. 1) and $r(t) = 0.5 + 0.1\sin 2t$ (Fig. 2). The variables $y_r^+(t)$ and $y_r^-(t)$ (black lines) define the admissible range for the reference trajectories. The graphs show how the adaptive controller finds an output trajectory within the tolerance envelope $y_r^- \leq y(t) \leq y_r^+$ such that the criterion (6) is minimized. In both graphs we have included also the reference trajectory $y_m(t)$ (green lines) for the case with $u_c(t) = 0$ in (4).

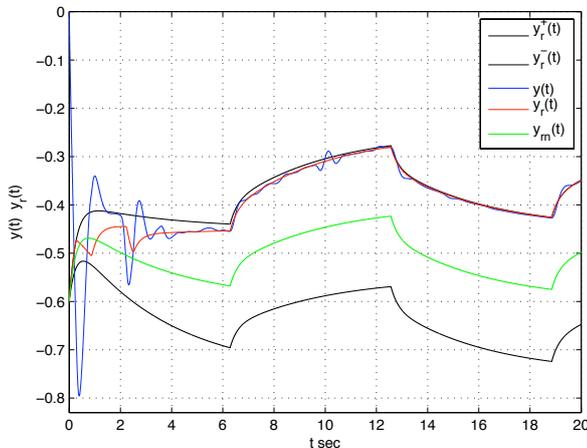


Fig. 1. Simulation of the adaptive control system for the nonlinear plant (40) and the controller (9). The graph shows how the adaptive controller finds an output trajectory within the admissible range $y_r^- \leq y_r(t) \leq y_r^+$.

6. CONCLUDING REMARKS

In this paper an initial study is presented to develop into the conventional MRAC framework an approach for solving the problem of robust adaptive output tracking for a class of uncertain nonlinear dynamical systems with unknown time-varying state delays. In common with Mirkin and Gutman (2010a), we restrict the attention to SISO systems. However, here, in contrast to Mirkin and Gutman (2010a), the presence of time-varying state delays is a distinguishing feature of the underlying

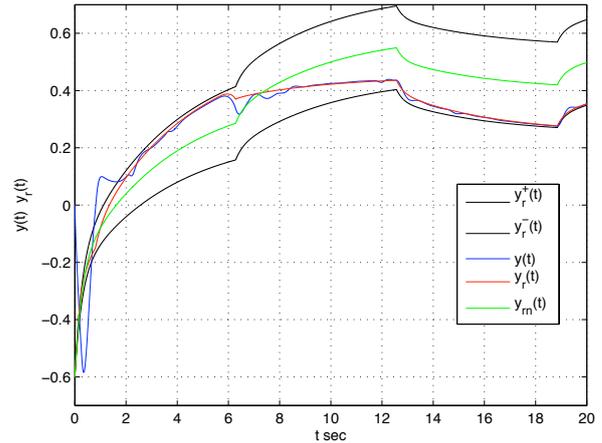


Fig. 2. Simulation of the adaptive control system for the nonlinear plant (40) and the controller (9). The graph shows how the adaptive controller finds an output trajectory within the admissible range $y_r^- \leq y_r(t) \leq y_r^+$.

system class. In addition to the traditional asymptotic zero error tracking specification, the MRAC problem statement includes an additional explicit requirement. The tracking objective is described by an admissible set of reference trajectories. The best trajectory is found by optimization, with a cost function that penalizes control deviations. The suggested robust adaptive controller (i) guarantees that the tracking errors tend to zero; (ii) minimizes the control cost; and (iii) includes switching logic. To ensure the asymptotical zero tracking error, a new feedback compensator with adaptive error feedback to compensate for plant non-linearities is incorporated. The central ideas of this paper can be generalized to several other adaptive control problems.

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