The C^*-algebras of the Heisenberg Group and of thread-like Lie groups.

Jean Ludwig and Lyudmila Turowska

Abstract

We describe the C^*-algebras of the Heisenberg group H_n, $n \geq 1$, and the thread-like Lie groups G_N, $N \geq 3$, in terms of C^*-algebras of operator fields.

1 Introduction and notation

Let H_n be the Heisenberg group of dimension $2n+1$. It has been known for a long time that the C^*-algebra, $C^*(H_n)$, of H_n is an extension of an ideal J isomorphic to $C_0(\mathbb{R}^*, \mathcal{K})$ with the quotient algebra isomorphic to $C^*(\mathbb{R}^{2n})$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable Hilbert space, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $C_0(\mathbb{R}^*, \mathcal{K})$ is the C^*-algebra of continuous functions vanishing at infinity from $\mathbb{R}^*$ to $\mathcal{K}$.
We obtain an exact characterization of this extension giving a linear mapping from $C^*(\mathbb{R}^{2n})$ to $C^*(H_n)/J$ which is a cross section of the quotient mapping $i : C^*(H_n) \to C^*(H_n)/J$. More precisely, realizing $C^*(H_n)$ as a C^*-subalgebra of the C^*-algebra $\mathcal{F}_n$ of all operator fields $(F = F(\lambda))_{\lambda \in \mathbb{R}}$ taking values in $\mathcal{K}$ for $\lambda \in \mathbb{R}^*$ and in $C^*(\mathbb{R}^{2n})$ for $\lambda = 0$, norm continuous on $\mathbb{R}^*$ and vanishing as $\lambda \to \infty$, we construct a linear map ν from $C^*(\mathbb{R}^{2n})$ to $\mathcal{F}_n$, such that the C^*-subalgebra is isomorphic to the C^*-algebra $D_\nu(H_n)$ of all $(F = F(\lambda))_{\lambda \in \mathbb{R}} \in \mathcal{F}_n$ such that

$$\|F(\lambda) - \nu(F(0))\|_{\text{op}} \to 0,$$

where $\| \cdot \|_{\text{op}}$ is the operator norm on $\mathcal{K}$. The constructed mapping ν is an almost homomorphism in the sense that

$$\lim_{\lambda \to 0} \|\nu(f \cdot h)(\lambda) - \nu(f)(\lambda) \circ \nu(h)(\lambda)\|_{\text{op}} = 0.$$

Moreover, any such almost homomorphism $\tau : C^*(\mathbb{R}^2) \to \mathcal{F}_n$ defines a C^*-algebra, $D_\tau(H_n)$, which is an extension of $C_0(\mathbb{R}^*, \mathcal{K})$ by $C^*(\mathbb{R}^{2n})$. A question we left unanswered : what mappings τ give the C^*-algebras which are isomorphic to $C^*(H_n)$. We note that the condition

$$\lim_{\lambda \to 0} \|\tau(h)(\lambda)\|_{\text{op}} = \|h\|_{C^*(\mathbb{R}^{2n})}, \text{ for all } h \in C^*(\mathbb{R}^{2n}),$$

which is equivalent to the condition that the topologies of $D_\tau(H_n)$ and that of $C^*(H_n)$ agree, is not the right condition: there are examples of splitting extensions of type $D_\tau(H_n)$ with the same spectrum as $C^*(H_n)$ (see [De] and Example 2.23) while it is known that $C^*(H_n)$ is a non-splitting extension.
We note that another characterisation of $C^*(H_n)$ as a C^*-algebra of operator fields is given without proof in a short paper by Gorbachev [Gor].

The second part of the paper deals with the C^*-algebra of thread-like Lie groups G_N, $N \geq 3$. The group G_3 is the Heisenberg group of dimension 3 treated in the first part of the paper. The groups G_N are nilpotent Lie groups and their unitary representations can be described using the Kirillov orbit method. The topology of the dual space $\widehat{G_N}$ has been investigated in details in [ALS]. In particular, it was shown that like for the Heisenberg group G_3 the topology of $\widehat{G_N}$, $N \geq 3$ is not Hausdorff. It is known that $\widehat{G_3} = \mathbb{R}^* \cup \mathbb{R}^2$ as a set with natural topology on each pieces, the limit set when $\lambda \in \mathbb{R}^*$ goes to 0 is the whole real plane $\mathbb{R}^2$. The topology of $\widehat{G_N}$, becomes more complicated with growth of the dimension N. Using a description of the limit sets of converging sequences $(\pi_k) \in \widehat{G_N}$ obtained in [AKLSS] and [ALS] we give a characterisation of the C^*-algebra of G_N in the spirit of one for the Heisenberg group H_n. Namely, parametrising $\widehat{G_N}$ by a set $S_N^{gen} \cup \mathbb{R}^2$, where S_N^{gen} consists of element $\ell \in \mathfrak{g}_N^*$ corresponding to non-characters (here $\mathfrak{g}_N$ is the Lie algebra of G_N), we realize $C^*(G_N)$ as a C^*-algebra of operator fields $(A = A(\ell))$ on $S_N^{gen} \cup \{0\}$, such that $A(\ell) \in \mathcal{K}$, $\ell \in S_N^{gen}$, $A(0) \in C^*(\mathbb{R}^2)$ and $(A = A(\ell))$ satisfy for each converging sequence in the dual space the generic, the character and the infinity conditions (see Definition 3.12).
We shall use the following notation. $L^p(\mathbb{R}^n)$ denote the space of (almost everywhere equivalence classes) p-integrable functions for $p = 1, 2$ with norm $\| \cdot \|_p$. By $\|f\|_\infty$ we denote the supremum norm $\sup_{x\in\Omega} |f(x)|$ of a continuous function f vanishing at infinity from a locally compact space Ω to $\mathbb{C}$. $\mathcal{D}(\mathbb{R}^n)$ is the space of complex-valued C^∞ functions with compact support and $\mathcal{S}(\mathbb{R}^n)$ is the space of Schwartz functions, i.e. rapidly decreasing complex-valued C^∞ functions on $\mathbb{R}^n$. The space of Schwartz functions on the groups H_n and G_N (see [CG]) will be denoted by $\mathcal{S}(H_n)$ and $\mathcal{S}(G_N)$ respectively. We use the usual notation $B(H)$ for the space of all linear bounded operators on a Hilbert space H with the operator norm $\| \cdot \|_{\text{op}}$.

Keywords. Heisenberg group, thread-like Lie group, unitary representation, C^*-algebra.

2000 Mathematics Subject Classification: 22D25, 22E27, 46L05.

2 The C^*-algebra of the Heisenberg group H_n

Let H_n be the $2n + 1$ dimensional Heisenberg group, which is defined as to be the Lie group whose underlying variety is the vector space $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and on which the multiplication is given by

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - x' \cdot y)),$$

where $x \cdot y = x_1y_1 + \cdots + x_ny_n$ denotes the Euclidean scalar product on $\mathbb{R}^n$. The center of H_n is the subgroup $\mathcal{Z} := \{0_n\} \times \{0_n\} \times \mathbb{R}$ and the commutator subgroup $[H_n, H_n]$ of H_n is given by $[H_n, H_n] = \mathcal{Z}$. The Lie algebra $\mathfrak{g}$ of H_n has the basis

$$\mathcal{B} := \{X_j, Y_j, j = 1 \cdots, n, Z = (0_n, 0_n, 1)\},$$

where $X_j = (e_j, 0_n, 0), Y_j = (0_n, e_j, 0), j = 1, \cdots, n$ and e_j is the j'th canonical basis vector of $\mathbb{R}^n$, with the non trivial brackets

$$[X_i, Y_j] = \delta_{i,j} Z.$$

2.1 The unitary dual of H_n.

The unitary dual $\widehat{H}_n$ of H_n can be described as follows.

2.1.1 The infinite dimensional irreducible representations

For every $\lambda \in \mathbb{R}^*$, there exists a unitary representation π_λ of H_n on the Hilbert space $L^2(\mathbb{R}^n)$, which is given by the formula

$$\pi_\lambda(x,y,t)\xi(s) := e^{-2\pi i\lambda t - 2\pi i\frac{\lambda}{2}x\cdot y + 2\pi i\lambda s\cdot y}\xi(s-x),\ \ s \in \mathbb{R}^n, \xi \in L^2(\mathbb{R}^n), (x,y,t) \in H_n.$$

It is easily seen that π_λ is in fact irreducible and that π_λ is equivalent to π_ν if and only if $\lambda = \nu$.
The representation π_λ is equivalent to the induced representation $\tau_\lambda := \mathrm{ind}_P^{H_n}\chi_\lambda$, where $P = \{0_n\} \times \mathbb{R}^n \times \mathbb{R}$ is a polarization at the linear functional $\ell_\lambda((x,y,t)) := \lambda t, (x,y,t) \in \mathfrak{g}$ and where χ_λ is the character of P defined by $\chi_\lambda(0_n, y, t) = e^{-2\pi i\lambda t}$.
The theorem of Stone-Von Neumann tells us that every infinite dimensional unitary representation of H_n is equivalent to one of the π_λ's. (see [CG]).

2.1.2 The finite dimensional irreducible representations

Since H_n is nilpotent, every irreducible finite dimensional representation of H_n is one-dimensional, by Lie's theorem.
Any one-dimensional representation is a unitary character $\chi_{a,b}$, $(a,b) \in \mathbb{R}^n \times \mathbb{R}^n$, of H_n, which is given by

$$\chi_{a,b}(x,y,t) = e^{-2\pi i(a\cdot x + b\cdot y)}, (x,y,t) \in H_n.$$

For $f \in L^1(H_n)$, let

$$\hat{f}(a,b) := \chi_{a,b}(f) = \int_{H_n} f(x,y,t)e^{-2\pi i(x\cdot a + y\cdot b)}dxdydt,\ \ a,b \in \mathbb{R}^n,$$

and

$$\|f\|_{\infty,0} := \sup_{a,b\in\mathbb{R}^n} |\chi_{a,b}(f)| = \|\hat{f}\|_\infty.$$

2.2 The topology of $\widehat{C^*(H_n)}$

Let $C^*(H_n)$ denote the full C^*-algebra of H_n. We recall that $C^*(H_n)$ is obtained by the completion of $L^1(H_n)$ with respect to the norm

$$\|f\|_{C^*(H_n)} = \sup \| \int f(x,y,t)\pi(x,y,t)dxdydt\|_{\mathrm{op}},$$

where the supremum is taken over all unitary representations π of H_n.

Definition 2.1. Let

$$\rho = \mathrm{ind}_{\mathcal{Z}}^{H_n} 1$$

be the left regular representation of H_n on the Hilbert space $L^2(H_n/\mathcal{Z})$. Then the image $\rho(C^*(H_n))$ is just the C^*-algebra of $\mathbb{R}^{2n}$ considered as an algebra of convolution operators on

$L^2(\mathbb{R}^{2n})$ and $\rho(C^*(H_n))$ is isomorphic to the algebra $C_0(\mathbb{R}^{2n})$ of continuous functions vanishing at infinity on $\mathbb{R}^{2n}$ via the Fourier transform. For $f \in L^1(H_n)$ we have $\widehat{\rho(f)}(a,b) = \hat{f}(a,b,0)$, $a,\, b \in \mathbb{R}^n$.

Definition 2.2. Define for $C^*(H_n)$ the *Fourier transform* $F(c)$ of c by

$$F(c)(\lambda) := \pi_\lambda(c) \in B(L^2(\mathbb{R}^n)), \lambda \in \mathbb{R}^*$$

and

$$F(c)(0) := \rho(c) \in C^*(\mathbb{R}^{2n}).$$

2.2.1 Behavior on $\mathbb{R}^*$

As for the topology of the dual space, it is well known that $[\pi_\lambda]$ tends to $[\pi_\nu]$ in $\widehat{H}_n$ if and only if λ tends to ν in $\mathbb{R}^*$, where $[\pi]$ denotes the unitary equivalence class of the unitary representation π. Furthermore, if λ tends to 0, then the representations π_λ converge in the dual space topology to all the characters $\chi_{a,b}, a, b \in \mathbb{R}^n$.
Let us compute for $f \in L^1(H_n)$ the operator $\pi_\lambda(f)$. We have for $\xi \in L^2(\mathbb{R}^n)$ and $s \in \mathbb{R}^n$ that

$$\begin{aligned} \pi_\lambda(f)\xi(s) &= \int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} f(x,y,t)\pi_\lambda(x,y,t)\xi(s)dxdydt \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} f(x,y,t)e^{-2\pi i\lambda t-\frac{2\pi i\lambda}{2}x\cdot y+2\pi i\lambda s\cdot y}\xi(s-x)dxdydt \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} f(s-x,y,t)e^{-2\pi i\lambda t-\frac{2\pi i\lambda}{2}(s-x)\cdot y+2\pi i\lambda s\cdot y}\xi(x)dxdydt \\ &= \int_{\mathbb{R}^n} \hat{f}^{2,3}(s-x,-\frac{\lambda}{2}(s+x),\lambda)\xi(x)dx. \end{aligned} \tag{2.1}$$

Here

$$\hat{f}^{2,3}(s,u,\lambda) = \int_{\mathbb{R}^n\times\mathbb{R}} f(s,y,t)e^{-2\pi i(y\cdot u+\lambda t)}dydt,\ (s,u,\lambda) \in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$$

denotes the partial Fourier transform of f in the variables y and t.
Hence $\pi_\lambda(f)$ is a kernel operator with kernel

$$f_\lambda(s,x) := \hat{f}^{2,3}(s-x,-\frac{\lambda}{2}(s+x),\lambda),\ s,x \in \mathbb{R}^n. \tag{2.2}$$

If we take now a Schwartz-functions $f \in \mathcal{S}(H_n)$, then the operator $\pi_\lambda(f)$ is Hilbert-Schmidt and its Hilbert-Schmidt norm $\|\pi_\lambda(f)\|_{\text{H.S.}}$ is given by

$$\|\pi_\lambda(f)\|^2_{\text{H.S.}} = \int_{\mathbb{R}^2} |f_\lambda(s,x)|^2dxds = \int_{\mathbb{R}^2} |\hat{f}^{2,3}(s,\lambda x,\lambda)|^2dsdx < \infty. \tag{2.3}$$

Proposition 2.3. *For any $c \in C^*(H_n)$ and $\lambda\in \mathbb{R}^*$, the operator $\pi_\lambda(c)$ is compact , the mapping $\mathbb{R}^* \to B(L^2(\mathbb{R}^n)) : \lambda \mapsto \pi_\lambda(c)$ is norm continuous and tending to 0 for λ going to infinity.*

Proof. Indeed, for $f \in \mathcal{S}(H_n)$, the compactness of the operator $\pi_\lambda(f)$ is a consequence of (2.3) and by (2.1) we have the estimate:

$$\begin{aligned}
\|\pi_\lambda(f) - \pi_\nu(f)\|^2_{H.S} &= \int_{\mathbb{R}^n\times\mathbb{R}^n} |\hat{f}^{2,3}(s-x, -\frac{\lambda}{2}(s+x), \lambda) - \hat{f}^{2,3}(s-x, -\frac{\nu}{2}(s+x), \nu)|^2 ds dx \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} |\hat{f}^{2,3}(s, \lambda x, \lambda) - \hat{f}^{2,3}(s, \nu x, \nu)|^2 ds dx
\end{aligned}$$

Hence, since f is a Schwartz function, this expression goes to 0 if λ tends to ν by Lebesgue's theorem of dominated convergence. Therefore the mapping $\lambda \mapsto \pi_\lambda(f)$ is norm continuous. Furthermore, the Hilbert-Schmidt norms of the operators $\pi_\lambda(f)$ go to 0, when λ tends to infinity. The proposition follows from the density of $\mathcal{S}(H_n)$ in $C^*(H_n)$. □

2.2.2 Behavior at 0

Let us now see the behavior of $\pi_\lambda(f)$ for Schwartz functions $f \in \mathcal{S}(H_n)$, as λ tends to 0. Choose a Schwartz-function η in $\mathcal{S}(\mathbb{R}^n)$ with L^2-norm equal to 1. For $u = (a, b)$ in $\mathbb{R}^n \times \mathbb{R}^n$, $\lambda \in \mathbb{R}^*$, we define the function $\eta(\lambda, a, b)$ by

(2.4) $$\eta(\lambda, a, b)(s) := |\lambda|^{n/4} e^{2\pi i a\cdot s}\eta(|\lambda|^{1/2}(s + \frac{b}{\lambda}))\ s \in \mathbb{R}^n.$$

and let $\eta_\lambda(s) = |\lambda|^{n/4}\eta(|\lambda|^{1/2}s),\ s \in \mathbb{R}^n$.
Let us compute

$$\begin{aligned}
c_{\lambda,u,u'}(x, y, t) &= \langle \pi_\lambda(x, y, t)\eta(\lambda, u), \eta(\lambda, u')\rangle \\
&= \int_{\mathbb{R}^n} e^{-2\pi i\lambda t - 2\pi i(\lambda/2)x\cdot y} e^{2\pi i\lambda s\cdot y}\eta(\lambda, u)(s - x)\overline{\eta(\lambda, u')(s)} ds \\
&= |\lambda|^{n/2} e^{-2\pi i\lambda t - 2\pi i(\lambda/2)x\cdot y} \int_{\mathbb{R}^n} e^{2\pi i\lambda s\cdot y - 2\pi i a\cdot x} e^{2\pi i(a-a')\cdot s} \\
&\qquad \eta(|\lambda|^{1/2}(s - x + \frac{b}{\lambda}))\overline{\eta(|\lambda|^{1/2}(s + \frac{b'}{\lambda}))} ds \qquad (2.5) \\
&= |\lambda|^{n/2} e^{-2\pi i\lambda t - 2\pi i(\lambda/2)x\cdot y} e^{-2\pi i b\cdot y} e^{-2\pi i a\cdot x} \int_{\mathbb{R}^n} e^{2\pi i\lambda s\cdot y} e^{2\pi i(a-a')\cdot(s - \frac{b}{\lambda})} \\
&\qquad \eta(|\lambda|^{1/2}(s - x)\overline{\eta(|\lambda|^{1/2}(s + \frac{b'-b}{\lambda}))} ds.
\end{aligned}$$

Hence for $u = u'$ we get

$$\begin{aligned}
c_{\lambda,u,u}(x, y, t) &= e^{-2\pi i\lambda t - 2\pi i\frac{\lambda}{2}x\cdot y} e^{-2\pi i a\cdot x - 2\pi i b\cdot y} \int_{\mathbb{R}^n} e^{2\pi i(\mathrm{sign}\lambda)|\lambda|^{1/2}s\cdot y}\eta(s - |\lambda|^{1/2}x)\overline{\eta(s)} ds \\
&\to e^{-2\pi i a\cdot x - 2\pi i b\cdot y} \int_{\mathbb{R}^n} \eta(s)\overline{\eta(s)} ds = e^{-2\pi i a\cdot x - 2\pi i b\cdot y}.
\end{aligned}$$

It follows also that the convergence of the coefficients $c_{\lambda,u,u}$ to the characters $\chi_{a,b}$ is uniform in u and uniform on compacta in (x, y, t) since

$$\begin{aligned}
|c_{\lambda,u,u}(x, y, t) - \chi_{a,b}(x, y, t)| &= |\int_{\mathbb{R}^n} (e^{-2\pi i\lambda t - 2\pi i\frac{\lambda}{2}x\cdot y} e^{2\pi i(\mathrm{sign}\lambda)|\lambda|^{1/2}s\cdot y} \\
&\quad \eta(s - |\lambda|^{1/2}x)\overline{\eta(s)} - |\eta(s)|^2) ds| \to 0 \\
&\quad \text{as } \lambda \to 0.
\end{aligned}$$

Proposition 2.4. *For every $u = (a, b) \in \mathbb{R}^n \times \mathbb{R}^n, c \in C^*(H_n)$, we have that*

$$\lim_{\lambda \to 0} \|F(c)(\lambda)\eta(\lambda, u) - \widehat{F(c)(0)}(u)\eta(\lambda, u)\|_2 = 0$$

uniformly in (a, b).

Proof. For $c \in C^*(H_n)$ we have that
$\|F(c)(\lambda)\eta(\lambda, u) - \widehat{F(c)(0)}(u)\eta(\lambda, u)\|_2^2 = \|\pi_\lambda(c)\eta(\lambda, u) - \chi_{a,b}(c)\eta(\lambda, u)\|_2^2 =$

$$\begin{aligned}
&= \langle \pi_\lambda(c)\eta(\lambda, u) - \chi_{a,b}(c)\eta(\lambda, u), \pi_\lambda(c)\eta(\lambda, u) - \chi_{a,b}(c)\eta(\lambda, u)\rangle \\
&= \langle \pi_\lambda(c^* * c)\eta(\lambda, u), \eta(\lambda, u)\rangle - \overline{\chi_{a,b}(c)}\langle \pi_\lambda(c)\eta(\lambda, u), \eta(\lambda, u)\rangle \\
&- \chi_{a,b}(c)\overline{\langle \pi_\lambda(c)\eta(\lambda, u), \eta(\lambda, u)\rangle} + |\chi_{a,b}(c)|^2 \\
&\to |\chi_{a,b}(c)|^2 - |\chi_{a,b}(c)|^2 - |\chi_{a,b}(c)|^2 + |\chi_{a,b}(c)|^2 = 0.
\end{aligned}$$

□

2.3 A C^*-condition

The aim of this section is to obtain a characterization of the C^*-algebra $C^*(H_n)$ as a C^*-algebra of operator fields ([Lee1, Lee2]).
Let us first define a larger C^*-algebra $\mathcal{F}_n$.

Definition 2.5. Let $\mathcal{F}_n$ be the family consisting of all operator fields $(F = F(\lambda))_{\lambda \in \mathbb{R}}$ satisfying the following conditions:

1. $F(\lambda)$ is a compact operator on $L^2(\mathbb{R}^n)$ for every $\lambda \in \mathbb{R}^*$,
2. $F(0) \in C^*(\mathbb{R}^{2n})$,
3. the mapping $\mathbb{R}^* \to B(L^2(\mathbb{R}^n)) : \lambda \mapsto F(\lambda)$ is norm continuous,
4. $\lim_{\lambda \to \infty} \|F(\lambda)\|_{\text{op}} = 0$.

Proposition 2.6. *$\mathcal{F}_n$ is a C^*-algebra.*

Proof. The proof is straight forward. □

Proposition 2.7. *The Fourier transform $F : C^*(H_n) \to \mathcal{F}_n$ is an injective homomorphism.*

Proof. It is clear from the definition of F and Proposition 2.3 that F is a homomorphism with values in $\mathcal{F}_n$. If $F(c) = 0$, then for each irreducible representation π of $C^*(H_n)$, $\pi(c) = 0$. Hence $c = 0$. □

Lemma 2.8. *Let $\xi \in \mathcal{S}(\mathbb{R}^n)$. Then, for any $\lambda \in \mathbb{R}^*$,*

$$\xi = \frac{1}{|\lambda|^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \langle \xi, \eta(\lambda, u)\rangle \eta(\lambda, u) du,$$

where $\eta(\lambda, u)$ is given by (2.4), the integral converging in $L^2(\mathbb{R}^n)$.

Proof. Let $\xi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n} \langle \xi, \eta(\lambda,a,b)\rangle \eta(\lambda,a,b)(x) da db \\
= &\int_{\mathbb{R}^n\times\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \xi(s) e^{-2\pi i a\cdot s}\overline{\eta_\lambda(s+\frac{b}{\lambda}))ds}\right) e^{2\pi i a\cdot x}\eta_\lambda(x+\frac{b}{\lambda}) da db \\
&\text{(by Fourier's inversion formula)} \\
= &\int_{\mathbb{R}^n\times\mathbb{R}^n} \xi(x)\overline{\eta_\lambda(x+\frac{b}{\lambda})}\eta_\lambda(x+\frac{b}{\lambda}) db = |\lambda|^n \xi(x)
\end{aligned}$$

giving $\xi = \frac{1}{|\lambda|^n}\int_{\mathbb{R}^n\times\mathbb{R}^n} \langle \xi, \eta(\lambda,a,b)\rangle\eta(\lambda,a,b) da db$.
Furthermore, since ξ is a Schwartz function, it follows that the mapping

$$(a,b) \to \langle \xi, \eta(\lambda,a,b)\rangle = |\lambda|^{n/4}\int_{\mathbb{R}^n} \xi(s)e^{-2\pi i a\cdot s}\overline{\eta(|\lambda|^{1/2}(s+\frac{b}{\lambda}))}ds$$

is also a Schwartz function in the variables a, b. Hence the integral $\int_{\mathbb{R}^n\times\mathbb{R}^n} \langle \xi, \eta(\lambda,a,b)\rangle\eta(\lambda,a,b) da db$ converges in $\mathcal{S}(\mathbb{R}^n)$ and hence also in $L^2(\mathbb{R}^n)$.

□

Remark 2.9. By Lemma 2.8,

$$\pi_\lambda(f)\xi = \frac{1}{|\lambda|^n}\int_{\mathbb{R}^{2n}} \pi_\lambda(f)\eta(\lambda,u)\langle \xi, \eta(\lambda,u)\rangle du = \frac{1}{|\lambda|^n}\int_{\mathbb{R}^{2n}} \pi_\lambda(f)\circ P_{\eta(\lambda,u)}\xi du$$

for any $f \in C^*(H_n)$, where $P_{\eta(\lambda,u)}$ is the orthogonal projection onto the one dimensional subspace $\mathbb{C}\eta(\lambda,u)$.

Definition 2.10. For a vector $0 \neq \eta \in L^2(\mathbb{R}^n)$, we let P_η be the orthogonal projection onto the one dimensional subspace $\mathbb{C}\eta$.
Define for $\lambda \in \mathbb{R}^*$ and $h \in C^*(\mathbb{R}^{2n})$ the linear operator

$$\nu_\lambda(h) \quad := \quad \int_{\mathbb{R}^{2n}} \hat{h}(u)P_{\eta(\lambda,u)}\frac{du}{|\lambda|^n}. \tag{2.6}$$

Proposition 2.11.

1. *For every $\lambda \in \mathbb{R}^*$ and $h \in \mathcal{S}(\mathbb{R}^{2n})$ the integral (2.6) converges in operator norm.*
2. *$\nu_\lambda(h)$ is compact and $\|\nu_\lambda(h)\|_{op} \leq \|h\|_{C^*(\mathbb{R}^{2n})}$.*
3. *The mapping $\nu_\lambda : C^*(\mathbb{R}^{2n}) \to \mathcal{F}_n$ is involutive, i.e. $\nu_\lambda(h^*) = \nu_\lambda(h)^*, h \in C^*(\mathbb{R}^{2n})$, where by ν_λ we denote also the extension of ν_λ to $C^*(\mathbb{R}^{2n})$.*

Proof. Since $\|P_{\eta(\lambda,u)}\|_{\text{op}} = \|\eta(\lambda,u)\|_2^2 = 1$, we have that

$$\|\nu_\lambda(h)\|_{\text{op}} \quad = \quad \|\int_{\mathbb{R}^{2n}} \hat{h}(u)P_{\eta(\lambda,u)}\frac{du}{|\lambda|^n}\|_{\text{op}} \leq \int_{\mathbb{R}^{2n}} |\hat{h}(u)|\frac{du}{|\lambda|^n} = \frac{\|\hat{h}\|_1}{|\lambda|^n}.$$

Hence the integral $\int_{\mathbb{R}^{2n}} \hat{h}(u)P_{\eta(\lambda,u)}\frac{du}{|\lambda|^n}$ converges in operator norm for $h \in \mathcal{S}(\mathbb{R}^{2n})$.

We compute $\nu_\lambda(h)$ applied to a Schwartz function $\xi \in \mathcal{S}(\mathbb{R}^n)$:

$$\begin{aligned}
\nu_\lambda(h)\xi(x) &= \int_{\mathbb{R}^{2n}} \hat{h}(u)\langle \xi, \eta(\lambda, u)\rangle \eta(\lambda, u)(x)\frac{du}{|\lambda|^n} \\
&= \int_{\mathbb{R}^{2n}} \hat{h}(u)\left(\int_{\mathbb{R}^n} \xi(r)\overline{\eta}_\lambda(r+\frac{b}{\lambda})e^{-2\pi i a\cdot r}dr\right)e^{2\pi i a\cdot x}\eta_\lambda(x+\frac{b}{\lambda})\frac{dadb}{|\lambda|^n} \\
&= \int_{\mathbb{R}^n} \hat{h}^2(-,b)*(\xi\overline{\eta}_{\lambda,b})(x)\eta_\lambda(x+\frac{b}{\lambda})\frac{db}{|\lambda|^n} \qquad (2.7)\\
&\quad \text{(where } \eta_{\lambda,b}(s) := \eta_\lambda(s+\frac{b}{\lambda}), s\in\mathbb{R}^n) \\
&= \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^n} \hat{h}^2(x-s,b)\xi(s)\overline{\eta}_\lambda(s+\frac{b}{\lambda})\eta_\lambda(x+\frac{b}{\lambda})\frac{db}{|\lambda|^n}ds \\
&= \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^n} \hat{h}^2(x-s,|\lambda|^{1/2}b)\xi(s)\overline{\eta}(|\lambda|^{1/2}s+\mathrm{sign}\lambda\cdot b)\eta(|\lambda|^{1/2}x+\mathrm{sign}\lambda\cdot b)dbds
\end{aligned}$$

The kernel function $h_\lambda(x,s)$ of $\nu_\lambda(h)$ is in $\mathcal{S}(\mathbb{R}^{2n})$ if $h\in\mathcal{S}(\mathbb{R}^{2n})$. In particular $\nu_\lambda(h)$ is a compact operator and we have the following estimate for the Hilbert-Schmidt norm, $\|\cdot\|_{H.S}$, of $\nu_\lambda(h)$:

$$\begin{aligned}
\|\nu_\lambda(h)\|_{H.S}^2 &= \int_{\mathbb{R}^n\times\mathbb{R}^n} |\int_{\mathbb{R}^n} \hat{h}^2(x-s,b)\overline{\eta}_\lambda(s+\frac{b}{\lambda})\eta_\lambda(x+\frac{b}{\lambda})\frac{db}{|\lambda|^n}|^2 dsdx \\
&\le \int_{\mathbb{R}^{3n}} |\hat{h}^2(x-s,\lambda(b-x))|^2|\eta_\lambda(s-x+b)|^2 dbdxds \\
&= \int_{\mathbb{R}^{3n}} |\hat{h}^2(x,\lambda(b+s))|^2|\eta_\lambda(b)|^2 dbdxds \\
&= \int_{\mathbb{R}^{2n}} |\hat{h}^2(x,\lambda s)|^2 dxds < \infty.
\end{aligned}$$

Let us show that $\|\nu_\lambda(h)\|_{\mathrm{op}} \le \|\hat{h}\|_\infty$. Indeed

$$\begin{aligned}
\|\nu_\lambda(h)\xi\|_2^2 &= \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} \hat{h}^2(-,b)*(\xi\overline{\eta}_{\lambda,b})(x)\eta_\lambda(x+\frac{b}{\lambda})\frac{db}{|\lambda|^n}|^2 dx \\
&\le \frac{1}{|\lambda|^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\hat{h}^2(-,b)*(\xi\overline{\eta}_{\lambda,b})(x)|^2 dbdx \\
&\le \frac{\|\hat{h}\|_\infty^2}{|\lambda|^n}\int_{\mathbb{R}^n} \|\xi\eta_{\lambda,b}\|_2^2 db \\
&= \frac{\|\hat{h}\|_\infty^2}{|\lambda|^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi(x)\eta_\lambda(x+\frac{b}{\lambda})|^2 dxdb \\
&= \|\hat{h}\|_\infty^2\|\xi\|_2^2.
\end{aligned}$$

Let $h\in S(\mathbb{R}^{2n})$. Then $\overline{\hat{h}} = \hat{h^*}$. This gives

$$\begin{aligned}
\nu_\lambda(h)^* = (\int_{\mathbb{R}^{2n}} \hat{h}(u)P_{\eta(\lambda,u)}\frac{du}{|\lambda|^n})^* &= \int_{\mathbb{R}^{2n}} \overline{\hat{h}(u)}P_{\eta(\lambda,u)}\frac{du}{|\lambda|^n} \\
&= \int_{\mathbb{R}^{2n}} \hat{h^*}(u)P_{\eta(\lambda,u)}\frac{du}{|\lambda|^n} = \nu_\lambda(h^*).
\end{aligned}$$

□

Theorem 2.12. *Let $a \in C^*(H_n)$ and let A be the operator field $A = F(a)$, i. e.*

$$A(\lambda) = \pi_\lambda(a), \lambda \in \mathbb{R}^*, A(0) = \rho(a) \in C^*(\mathbb{R}^{2n}).$$

Then

$$\lim_{\lambda \to 0} \|A(\lambda) - \nu_\lambda(A(0))\|_{\text{op}} = 0.$$

Proof. Let $f \in \mathcal{S}(H_n)$, $\xi \in L^2(\mathbb{R}^n)$, $\eta \in \mathcal{S}(\mathbb{R}^n)$, $\|\eta\|_2 = 1$. Then by (2.1) and (2.7)

$$\begin{aligned}
((\pi_\lambda(f) - \nu_\lambda(\rho(f))\xi)(x) &= \int_{\mathbb{R}^n} \hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s), \lambda)\xi(s)ds \\
&- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}^{2,3}(x-s, b, 0)\xi(s)\overline{\eta}_\lambda(s+\frac{b}{\lambda})\eta_\lambda(x+\frac{b}{\lambda})\frac{db}{|\lambda|^n}ds \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s), \lambda)\eta_\lambda(b)\overline{\eta}_\lambda(b)\xi(s)dbds \\
&- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}^{2,3}(x-s, \lambda(b-x), 0)\xi(s)\overline{\eta}_\lambda(s-x+b)\eta_\lambda(b)dbds.
\end{aligned}$$

Let

$$\begin{aligned}
u_\lambda(x,b) &= \int_{\mathbb{R}^n} \xi(s)\overline{\eta}_\lambda(s-x+b)(\hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s), \lambda) - \hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s), 0))ds, \\
v_\lambda(x,b) &= \int_{\mathbb{R}^n} \xi(s)\overline{\eta}_\lambda(s-x+b)(\hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s), 0) - \hat{f}^{2,3}(x-s, \lambda(b-x), 0))ds
\end{aligned}$$

and

$$w_\lambda(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s), \lambda)\xi(s)\eta_\lambda(b)(\overline{\eta}_\lambda(b) - \overline{\eta}_\lambda(s-x+b))dbds.$$

We have

$$((\pi_\lambda(f) - \nu_\lambda(\rho(f))\xi)(x) = \int_{\mathbb{R}^n} u_\lambda(x,b)\eta_\lambda(b)db + \int_{\mathbb{R}^n} v_\lambda(x,b)\eta_\lambda(b)db + w_\lambda(x). \tag{2.8}$$

Thus to prove $\|\pi_\lambda(f) - \nu_\lambda(\rho(f))\|_{\text{op}} \to 0$ as $\lambda \to 0$ it is enough to show that $\|u_\lambda\|_2 \leq \delta_\lambda\|\xi\|_2$, $\|v_\lambda\|_2 \leq \omega_\lambda\|\xi\|_2$ and $\|w_\lambda\|_2 \leq \epsilon_\lambda\|\xi\|_2$, where δ_λ, ω_λ, $\epsilon_\lambda \to 0$ as $\lambda \to 0$.
We have

$$\hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s)), \lambda) - \hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s), 0) = \lambda\int_0^1 \partial_3\hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s)), t\lambda)dt$$

and

$$\begin{aligned}
\hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s)), 0) - \hat{f}^{2,3}(x-s, \lambda(b-x), 0) &= \lambda(\frac{1}{2}(s-x) - (s-x+b)) \\
&\times \int_0^1 \partial_2\hat{f}^{2,3}(x-s, \lambda(b-x) + t(\lambda(\frac{1}{2}(s-x) - (s-x+b)), 0)dt.
\end{aligned}$$

Hence, since $f \in \mathcal{S}(H_n)$, there exists a constant $C > 0$ such that

$$|f^{2,3}(x-s, -\frac{\lambda}{2}(x+s)), \lambda) - \hat{f}^{2,3}(x-s, -\frac{\lambda}{2}(x+s), 0)| \leq |\lambda|\frac{C}{(1+\|x-s\|)^{2n+1}},$$

and

$$|\hat{f}^{2,3}(x-s,-\frac{\lambda}{2}(x+s)),0)-\hat{f}^{2,3}(x-s,\lambda(b-x),0)|$$
$$\leq |\lambda|(\|s-x+b\|+\|s-x\|)\frac{C}{(1+\|x-s\|)^{4n+1}}$$

for all $\lambda\in\mathbb{R}^*$, $x,s\in\mathbb{R}^n$. Therefore we see that

$$\begin{aligned}
\|u_\lambda\|_2^2 &= \int_{\mathbb{R}^n\times\mathbb{R}^n}|u_\lambda(x,b)|^2dxdb\\
&\leq \int_{\mathbb{R}^n\times\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|\xi(s)\eta_\lambda(s-x+b)||\lambda|\frac{C}{(1+\|x-s\|)^{2n+1}}ds\Big)^2dxdb\\
&\leq |\lambda|^2C'\int_{\mathbb{R}^{3n}}\frac{|\xi(s)|^2}{(1+\|x-s\|)^2}|\eta_\lambda(s-x+b)|^2dbdxds\\
&\leq C''|\lambda|^2\|\xi\|_2^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
\|v_\lambda\|_2^2 &= \int_{\mathbb{R}^n\times\mathbb{R}^n}|v_\lambda(x,b)|^2dxdb\\
&\leq \int_{\mathbb{R}^n\times\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|\xi(s)\eta_\lambda(s-x+b)||\lambda|(\|s-x+b\|+\|s-x\|)\\
&\qquad\frac{C}{(1+\|x-s\|)^{4n+1}}ds\Big)^2dbdx\\
&\leq C'\int_{\mathbb{R}^{3n}}|\xi(s)\eta_\lambda(s-x+b)|^2|\lambda|^2(\|s-x+b\|+\|s-x\|)^2\\
&\qquad\frac{1}{(1+\|x-s\|)^{4n+1}}dsdbdx\\
&\leq 2C'\int_{\mathbb{R}^{3n}}|\xi(s)|\lambda|^{n/4}\eta(|\lambda|^{1/2}(s-x+b))|^2|\lambda|^2\|s-x+b\|^2\frac{dsdbdx}{(1+\|x-s\|)^{4n+1}}\\
&+ 2C'\int_{\mathbb{R}^{3n}}|\xi(s)|\lambda|^{n/4}\eta(|\lambda|^{1/2}(s-x+b))|^2|\lambda|^2\|s-x\|^2\frac{dsdbdx}{(1+\|x-s\|)^{4n+1}}\\
&\leq 2C'|\lambda|\int_{\mathbb{R}^{3n}}|\xi(s)|\lambda|^{n/4}\tilde{\eta}(|\lambda|^{1/2}(s-x+b))|^2\frac{dsdbdx}{(1+\|x-s\|)^{4n+1}}\\
&+ 2C'|\lambda|^2\int_{\mathbb{R}^{3n}}|\xi(s)|\lambda|^{n/4}\eta(|\lambda|^{1/2}(s-x+b))|^2\frac{dsdbdx}{(1+\|x-s\|)^{4n-1}}\\
&\leq C''|\lambda|(\|\tilde{\eta}\|_2^2+|\lambda|\|\eta\|_2^2)\|\xi\|_2^2,
\end{aligned}$$

for some constants $C',C''>0$, where the function $\tilde{\eta}$ is defined by $\tilde{\eta}(s):=\|s\|\eta(s),\ s\in\mathbb{R}$. Since $\eta\in\mathcal{S}(\mathbb{R}^n)$, we can use the same arguments to see that

$$\begin{aligned}
\|w_\lambda\|_2^2 &= \int_{\mathbb{R}^n}|w_\lambda(x)|^2dx\\
&\leq \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\xi(s)|\eta_\lambda(b)||\lambda|^{n/4+1/2}(\|s-x\|)\frac{C}{(1+\|x-s\|)^{4n+1}}dbds\Big)^2dx\\
&\leq C'|\lambda|^{n/2+1}\int_{\mathbb{R}^{3n}}|\xi(s)|^2|\eta_\lambda(b)|^2\frac{\|x-s\|^2dsdxdb}{1+\|x-s\|^{4n+1}}\leq C''|\lambda|^{n/2+1}\|\xi\|_2^2\|\eta\|_2^2.
\end{aligned}$$

We have proved therefore $\|\pi_\lambda(f) - \nu_\lambda(\rho(f))\| \to 0$ as $\lambda \to 0$ for $f \in \mathcal{S}(H_n)$. Since $\mathcal{S}(H_n)$ is dense in $C^*(H_n)$, the statement holds for any $a \in C^*(H_n)$.

□

Definition 2.13. *For $\eta \in \mathcal{S}(\mathbb{R}^n)$ we define the linear mapping $\nu_\eta := \nu : C^*(\mathbb{R}^{2n}) \to \mathcal{F}_n$ by*

$$\nu(h)(\lambda) = \nu_\lambda(h), \lambda \in \mathbb{R}^* \text{ and } \nu(h)(0) = h.$$

Proposition 2.14. *The mapping $\nu : C^*(\mathbb{R}^{2n}) \to \mathcal{F}_n$ has the following properties:*

1. $\|\nu\| = 1$.
2. *For every $h, h' \in C^*(\mathbb{R}^{2n})$, we have that*

$$\lim_{\lambda\to 0} \|\nu_\lambda(h \cdot h') - \nu_\lambda(h) \circ \nu_\lambda(h')\|_{\text{op}} = 0$$

   and also

$$\lim_{\lambda\to 0} \|\nu_\lambda(h^*) - \nu_\lambda(h)^*\|_{\text{op}} = 0.$$

3. *For $(a, b) \in \mathbb{R}^{2n}$ and $h \in C^*(\mathbb{R}^{2n})$ we have that*

$$\lim_{\lambda\to 0} \|\nu(h)(\lambda)\eta(\lambda, a, b) - \hat{h}(a, b)\eta(\lambda, a, b)\|_2 = 0.$$

4. $\lim_{\lambda\to 0} \|\nu(h)(\lambda)\| = \|\hat{h}\|_\infty$.

Proof. (1) follows from Proposition 2.11.
To prove (2) we take for $h, h' \in \mathcal{S}(\mathbb{R}^{2n})$ two elements $f, f' \in \mathcal{S}(H_n)$, such that $\rho(f) = h, \rho(f') = h'$. Then $\rho(f * f') = h \cdot h'$ and

$$\begin{aligned}
\|\nu_\lambda(h \cdot h') - \nu_\lambda(h) \circ \nu_\lambda(h')\|_{\text{op}} &\leq \|\nu_\lambda(h \cdot h') - \pi_\lambda(f * f')\|_{\text{op}} \\
&+ \|\nu_\lambda(h) \circ \nu_\lambda(h') - \pi_\lambda(f) \circ \pi_\lambda(f')\|_{\text{op}} \\
&\leq \|\nu_\lambda(h \cdot h') - \pi_\lambda(f * f')\|_{\text{op}} \\
&+ \|f'\|_{C^*(H_n)}\|\nu_\lambda(h) - \pi_\lambda(f)\|_{\text{op}} \\
&+ \|h\|_{C^*(\mathbb{R}^{2n})}\|\nu_\lambda(h') - \pi_\lambda(f')\|_{\text{op}}.
\end{aligned} \tag{2.9}$$

Hence, by Theorem 2.12, $\lim_{\lambda\to 0} \|\nu_\lambda(f * f') - \nu_\lambda(f) \circ \nu_\lambda(f')\|_{\text{op}} = 0$. Furthermore

$$\begin{aligned}
\|\nu_\lambda(h^*) - \nu_\lambda(h)^*\|_{\text{op}} &\leq \|\nu_\lambda(h^*) - \pi_\lambda(f^*)\|_{\text{op}} + \|\nu_\lambda(h)^* - \pi_\lambda(f)^*\|_{\text{op}} \to 0 \\
&\quad \text{as } \lambda \to 0.
\end{aligned}$$

We conclude by the usual approximation argument.
For assertion (3), using Propositions 2.4 and Theorem 2.12, it suffices to take for $h \in C^*(\mathbb{R}^{2n})$ an element $c \in C^*(H_n)$, for which $\rho(c) = h$.
The last statement follows from Proposition 2.11 and assertion (3). □

Definition 2.15. *Let $D_\nu(H_n)$ be the subspace of the algebra $\mathcal{F}_n$, consisting of all the fields $(F(\lambda))_{\lambda\in\mathbb{R}} \in \mathcal{F}_n$, such that*

$$\lim_{\lambda\to 0} \|F(\lambda) - \nu_\lambda(F(0))\|_{\text{op}} = 0.$$

Our main theorem of this section is the following characterisation of $C^*(H_n)$.

Theorem 2.16. *The Heisenberg C^*-algebra $C^*(H_n)$ is isomorphic to $D_\nu(H_n)$.*

Proof. First we show that $D_\nu(H_n)$ is a $*$-subalgebra of $\mathcal{F}_n$. Indeed if $F, F' \in D_\nu(H_n)$, then

$$\begin{aligned}\|\nu_\lambda(F(0)+F'(0)) - \pi_\lambda(F+F')\|_{\mathrm{op}} &\leq \|\nu_\lambda(F(0)) - \pi_\lambda(F)\|_{\mathrm{op}} + \|\nu_\lambda(F'(0)) - \pi_\lambda(F')\|_{\mathrm{op}} \to 0 \\ &\quad \text{as } \lambda \to 0.\end{aligned}$$

and since $\lim_{\lambda\to 0} \|\nu_\lambda(F \cdot F'(0)) - \nu_\lambda(F(0)) \circ \nu(F'(0))\|_{\mathrm{op}} = 0$ it follows that

$$\|\nu_\lambda(F(0) \cdot F'(0)) - \pi_\lambda(F \cdot F')\|_{\mathrm{op}} \quad \to \quad 0.$$

Proposition 2.14 tells us that $D_\nu(H_n)$ is also invariant under the involution $*$.
In order to see that $D_\nu(H_n)$ is closed, let $F \in \mathcal{F}_n$ be contained in the closure of $D_\nu(H_n)$. Let $\varepsilon > 0$. Choose $F' \in D_\nu(H_n)$, such that $\|F - F'\|_{\mathcal{F}_n} < \varepsilon$. In particular, $\|F(0) - F'(0)\|_{C^*(\mathbb{R}^2)} < \varepsilon$. Thus there exists $\lambda_0 > 0$, such that

$$\|\pi_\lambda(F') - \nu_\lambda(F'(0))\|_{\mathrm{op}} < \varepsilon$$

for all $|\lambda| < |\lambda_0|$, whence

$$\begin{aligned}\|\pi_\lambda(F) - \nu_\lambda(F(0))\|_{\mathrm{op}} &= \|\pi_\lambda(F) - \pi_\lambda(F') + \pi_\lambda(F') - \nu_\lambda(F'(0)) + \nu_\lambda(F'(0)) - \nu_\lambda(F(0))\|_{\mathrm{op}} \\ &\leq 3\varepsilon, \text{ for } |\lambda| < |\lambda_0|.\end{aligned}$$

Hence $D_\nu(H_n)$ is a C^*-subalgebra of $\mathcal{F}_n$.
Let $I_0 := \{F \in \mathcal{F}_n, F(0) = 0\}$ and let $I_{00} = \{F \in I_0; \lim_{\lambda\to 0} \|F(\lambda)\|_{\mathrm{op}} = 0\}$. Then I_0 and I_{00} are closed two sided ideals of $\mathcal{F}_n$ and it follows from the definition of $\mathcal{F}_n$ that I_{00} is just the algebra $C_0(\mathbb{R}^*, \mathcal{K})$. It is clear that $D_\nu(H_n) \cap I_0 = I_{00}$. But $D_\nu(H_n) \cap I_0$ is the kernel in $D_\nu(H_n)$ of the homomorphism $\delta_0 : \mathcal{F}_n \to \mathbb{C}^*(\mathbb{R}^{2n}); F \mapsto F(0)$.
Since $\mathrm{im}(\nu) \subset D_\nu(H_n)$, the canonical projection $D_\nu(H_n) \to C^*(\mathbb{R}^{2n}) : F \mapsto F(0)$ is surjective and has the ideal I_{00} as its kernel. Thus $D_\nu(H_n)/I_{00} = C^*(\mathbb{R}^{2n})$ and therefore $D_\nu(H_n)$ is an extension of I_{00} by $C^*(\mathbb{R}^{2n})$. Moreover,

$$D_\nu(H_n) \quad = \quad I_{00} + \mathrm{im}(\nu).$$

Since for every irreducible representation π of $D_\nu(H_n)$, we have either $\pi(I_{00}) \neq 0$, and then $\pi = \pi_\lambda$ for some $\lambda \in \mathbb{R}^*$ or $\pi = 0$ on I_{00} and then π must be a character of $C^*(\mathbb{R}^{2n})$. Hence $\hat{D}_\nu(H_n) = \hat{H}_n$ as sets. That topologies of these spaces agree follows from the equality

$$\lim_{\lambda\to 0} \|\tau(h)(\lambda)\|_{\mathrm{op}} = \|\hat{h}\|_\infty, \ \forall h \in C^*(\mathbb{R}^{2n}),$$

which is due to Proposition 2.14.
By Theorem 2.12, $F(\mathcal{S}(H_n)) \subset D_\nu(H_n)$. Hence the C^*- algebra $C^*(H_n)$ can be injected into $D_\nu(H_n)$.
Since $D_\nu(H_n)$ is a type I algebra and the dual spaces of $D_\nu(H_n)$ and of $C^*(H_n)$ are the same, we have that $F(C^*(H_n))$ is equal to $D_\nu(H_n)$ by the Stone -Weierstrass theorem (see [Di]). □

Remark 2.17. Another characterisation of the C^*-algebra $C^*(H_n)$ is given (without proof) in a short paper by Gorbachev [Gor]. For $n = 1$ and $\lambda \in \mathbb{R}^*$ he defines an operator-valued measure μ_λ on $\mathbb{R}^2$ given on the product of two intervals $[s,t] \times [e,d]$ by $\mu_\lambda([s,t] \times [e,d]) = P^{\frac{e}{\lambda},\frac{d}{\lambda}} F P^{s,t} F^{-1}$, where $P^{s,t}$ is the multiplication operator by the characteristic function of $[t,s]$ on $L^2(\mathbb{R})$ and F is the Fourier transform on $L^2(\mathbb{R})$. For $f \in C_0(\mathbb{R}^2)$, $\lambda \in \mathbb{R}^*$ let

$$y(f)(\lambda) = \int_{\mathbb{R}^2} f(a,b) d\mu_\lambda(a,b)$$

and $y(f)(0) = f$. Gorbachev states that $C^*(H_1)$ is isomorphic to the C^*-algebra of operator fields $B = \{B(\lambda) = y(f)(\lambda) + a, \lambda \in \mathbb{R}^*, B(0) = f, f \in C_0(\mathbb{R}^2), a \in C_0(\mathbb{R}^*, \mathcal{K})\}$.

2.4 Almost homomorphisms and Heisenberg property

Definition 2.18. A bounded mapping $\tau : C^*(\mathbb{R}^{2n}) \to \mathcal{F}_n$ is called an *almost homomorphism* if

$$\begin{aligned}
&\lim_{\lambda\to 0} \|\tau_\lambda(\alpha h + \beta f) - \alpha\tau_\lambda(h) - \beta\tau_\lambda(f)\|_{\text{op}} = 0, \\
&\lim_{\lambda\to 0} \|\tau_\lambda(h \cdot h') - \tau_\lambda(h) \circ \tau_\lambda(h')\|_{\text{op}} = 0, \\
&\lim_{\lambda\to 0} \|\tau_\lambda(h^*) - \tau_\lambda(h)^*\|_{\text{op}} = 0, \ \alpha, \beta \in \mathbb{C}, f, h \in C^*(\mathbb{R}^{2n}).
\end{aligned}$$

The mapping ν from the previous section is an example of such almost homomorphism.
Let τ be an arbitrary almost homomorphism such that $\tau(f)(0) = f$ for any $f \in C^*(\mathbb{R}^{2n})$. We define as before $D_\tau(H_n)$ to be the subspace of the algebra $\mathcal{F}_n$, consisting of all the fields $F = (F(\lambda))_{\lambda\in\mathbb{R}} \in \mathcal{F}_n$, such that

$$\lim_{\lambda\to 0} \|F(\lambda) - \tau_\lambda(F(0))\|_{\text{op}} \ = \ 0.$$

Using the same arguments as the one in the proof of Theorem 2.16 one can easily prove the following

Proposition 2.19. *The subspace $D_\tau(H_n)$ of the C^*-algebra $\mathcal{F}_n$ is itself a C^*-algebra. The algebra $D_\tau(H_n)$ is an extension of $C_0(\mathbb{R}^*, \mathcal{K})$ by $C^*(\mathbb{R}^{2n})$, i.e., $C_0(\mathbb{R}^*, \mathcal{K})$ is a closed $*$-ideal in $D_\tau(H_n)$ such that $D_\tau(H_n)/C_0(\mathbb{R}^*, \mathcal{K})$ is isomorphic to $C^*(\mathbb{R}^{2n})$.*

Definition 2.20. We say that an almost homomorphism $\tau : C^*(\mathbb{R}^{2n}) \to \mathcal{F}_n$ has the *Heisenberg* property, if the C^*-algebra $D_\tau(H_n)$ is isomorphic to $C^*(H_n)$.

Remark 2.21. As for the mapping ν we have that the dual spaces of $D_\tau(H_n)$ and of H_n coincides as sets. The necessary and the sufficient conditions for them to coincide as topological spaces is

$$\lim_\lambda \|\tau_\lambda(h)\|_{\text{op}} = \|\hat{h}\|_\infty, \quad h \in C^*(\mathbb{R}^n).$$

Remark 2.22. Using the notion of Busby invariant for a C^*-algebra extension and the pullback algebra ([W]), one can show that any extension $\mathcal{B} \subset \mathcal{F}_n$ of $C_0(\mathbb{R}^*, \mathcal{K})$ by $C^*(\mathbb{R}^{2n})$ is isomorphic to $D_\tau(H_n)$ for some almost homomorphism τ. The Busby invariant of such extension is $b : C^*(\mathbb{R}^{2n}) \to C_b(\mathbb{R}^*, B(H))/C_0(\mathbb{R}^*, \mathcal{K})$, $b(h) = \tau(h) + C_0(\mathbb{R}^*, \mathcal{K})$.

Question.
What mappings τ give us C^*-algebras $D_\tau(H_n)$, which are isomorphic to $C^*(H_n)$?

Using a procedure described in [De] one can construct families of C^*-algebras of type $D_\tau(H_n)$ which are isomorphic to $D_\nu(H_n)$ and therefore to $C^*(H_n)$.
Next example shows that there is no topological obstacle for a C^*-algebra of type $D_\tau(H_n)$ to be non-isomorphic to $C^*(H_n)$. Namely, there is a C^*-algebras $D_\tau(H_n)$ with the spectrum equal to $\widehat{H}_n$ and such that $D_\tau(H_n) \not\simeq C^*(H_n)$.
We recall first that if $\mathcal{A}$, $\mathcal{C}$ are C^*-algebras, then an extension of $\mathcal{C}$ by $\mathcal{A}$ is a short exact sequence

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0 \tag{2.10}$$

of C^*-algebras. One says that the exact sequence splits if there is a cross-section $*$-homomorphism $s : \mathcal{C} \to \mathcal{B}$ such that $\beta \circ s = I_{\mathcal{C}}$.
It is known that the extension

$$0 \to C_0(\mathbb{R}^*, \mathcal{K}) \to C^*(H_n) \to C^*(\mathbb{R}^{2n}) \to 0 \tag{2.11}$$

does not split (see [R] and references therein) while there exists a large number of splitting extensions $\mathcal{B}$ and therefore non-isomorphic to $C^*(H_n)$ such that $\hat{\mathcal{B}} = \hat{H}_n$ (see [De, VII.3.4]).
Here is a concrete example inspired by [De].

Example 2.23. Let $\{\xi_Z\}_{Z\in\mathbb{Z}^{2n}}$ be an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^n)$. Let $P_Z, Z \in \mathbb{Z}^{2n}$, be the orthogonal projection onto the one-dimensional $\mathbb{C}\xi_Z$. We define a homomorphism ν from $C^*(\mathbb{R}^{2n})$ to $\mathcal{F}_n$ by

$$\nu(\varphi)(\lambda) := \sum_{Z\in\mathbb{Z}^{2n}} \hat{\varphi}(|\lambda|^{1/2}Z)P_Z, \lambda \in \mathbb{R}^*, \nu(\varphi)(0) := \varphi,\ \varphi \in C^*(\mathbb{R}^{2n}).$$

We note that since for each $\lambda \neq 0$ and each compact subset $K \subset \mathbb{R}^{2n}$, the set $\{Z \in \mathbb{Z}^{2n} : |\lambda|^{1/2} \in K\}$ is finite and since $\hat{\varphi} \in C_0(\mathbb{R}^{2n})$, one can easily see that $\nu(\varphi)(\lambda)$ is compact. Moreover

$$\|\nu(\varphi)(\lambda)\|_{\text{op}} = \sup_{Z\in\mathbb{Z}^{2n}} |\varphi(|\lambda|^{1/2}Z)|.$$

Since we can find for every vector $u \in \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}^*$ a vector $Z_\lambda \in \mathbb{Z}^{2n}$, such that $\lim_{\lambda\to 0} |\lambda|^{1/2}Z_\lambda = u$, we see that

$$\lim_{\lambda\to 0} \|\nu_\lambda(\varphi)\|_{\text{op}} = \|\hat{\varphi}\|_\infty = \|\varphi\|_{\mathbb{C}^*(\mathbb{R}^{2n})}. \tag{2.12}$$

3 The C^*-algebra of the thread-like Lie groups G_N

For $N \geq 3$, let $\mathfrak{g}_N$ be the N-dimensional real nilpotent Lie algebra with basis $X_1, \dots, X_N$ and non-trivial Lie brackets

$$[X_N, X_{N-1}] = X_{N-2}, \dots, [X_N, X_2] = X_1.$$

The Lie algebra $\mathfrak{g}_N$ is $(N-1)$-step nilpotent and is a semi-direct product of $\mathbb{R}X_N$ with the abelian ideal

$$\mathfrak{b} := \sum_{j=1}^{N-1} \mathbb{R}X_j. \tag{3.13}$$

Let

$$\mathfrak{b}_j := \operatorname{span}\{X_i, i = 1, \cdots, j\}, 1 \le j \le N-1.$$

Note that $\mathfrak{g}_3$ is the three dimensional Heisenberg Lie algebra. Let $G_N := \exp(\mathfrak{g}_N)$ be the associated connected, simply connected Lie group. Let also $B_j := \exp(\mathfrak{b}_j)$ and $B := \exp(\mathfrak{b})$. Then for $3 \le M \le N$ we have $G_M \simeq G_N/B_{N-M}$.

3.1 The unitary dual of G_N

In this section we describe the unitary irreducible representations of G_N up to a unitary equivalence.
For $\xi = \sum_{j=1}^{N-1} \xi_j X_j^* \in \mathfrak{g}_N^*$, the coadjoint action is given by

$$\mathrm{Ad}^*(\exp(-\mathrm{t}\mathrm{X}_\mathrm{N}))\xi = \sum_{\mathrm{j}=1}^{\mathrm{N}-1} \mathrm{p}_\mathrm{j}(\xi, \mathrm{t})\mathrm{X}_\mathrm{j}^*,$$

where, for $1 \le j \le N-1$, $p_j(\xi, t)$ is a polynomial in t defined by

$$p_j(\xi, t) = \sum_{k=0}^{j-1} \frac{t^k}{k!}\xi_{j-k}.$$

Moreover, if $\xi_j \neq 0$ for at least one $1 \le j \le N-2$, then $\mathrm{Ad}^*(\mathrm{G}_\mathrm{N})\xi$ is of dimension two, and $\mathrm{Ad}^*(\mathrm{G}_\mathrm{N})\xi = \{\mathrm{Ad}^*(\exp(\mathrm{t}\mathrm{X}_\mathrm{N}))\xi + \mathbb{R}\mathrm{X}_\mathrm{N}^*, \mathrm{t} \in \mathbb{R}\}$. We shall always identify $\mathfrak{g}_N^*$ with $\mathbb{R}^N$ via the mapping $(\xi_N, \ldots, \xi_1) \to \sum_{j=1}^N \xi_j X_j^*$ and the subspace $V = \{\xi \in \mathfrak{g}_N^* : \xi_N = 0\}$ with the dual space of $\mathfrak{b}$. For $\xi \in V$ and $t \in \mathbb{R}$, let

$$t \cdot \xi = \mathrm{Ad}^*(\exp(\mathrm{t}\mathrm{X}_\mathrm{N}))\xi$$

$$= \Big(0, \xi_{N-1} - t\xi_{N-2} + \ldots + \frac{1}{(N-2)!}(-t)^{N-2}\xi_1, \ldots, \xi_2 - t\xi_1, \xi_1\Big). \tag{3.14}$$

As in [AKLSS], we define the function $\widehat{\xi}$ on $\mathbb{R}$ by

$$\widehat{\xi}(t) := (t \cdot \xi)_{N-1} = \xi_{N-1} - t\xi_{N-2} + \ldots + \frac{1}{(N-2)!}(-t)^{N-2}\xi_1. \tag{3.15}$$

Then the mapping $\xi \to \widehat{\xi}$ is a linear isomorphism of V onto P_{N-2}, the space of real polynomials of degree at most $N-2$. In particular, $\xi_k \to \xi$ coordinate-wise in V as $k \to \infty$ if and only if $\widehat{\xi}_k(t) \to \widehat{\xi}(t)$ for all $t \in \mathbb{R}$. Also, the mapping $\xi \to \widehat{\xi}$ intertwines the Ad^*-action and translation in the following way:

$$\widehat{t \cdot \xi}(s) = (s \cdot (t \cdot \xi))_{N-1}$$

$$= ((s+t) \cdot \xi)_{N-1} = \widehat{\xi}(s+t)$$

for $\xi \in V$ and $s, t \in \mathbb{R}$.

By Kirillov's orbit picture of the dual space of a nilpotent Lie group, we can describe the irreducible unitary representations of G_N in the following way (see [CG] for details). For any non-constant polynomial $p = \hat{\ell} \in P_{N-2}$ we consider the induced representation $\pi_\ell = \mathrm{ind}_B^G \chi_\ell$, where χ_ℓ denotes the unitary character of the abelian group B defined by:

$$\chi_\ell(\exp(U)) = e^{-2\pi i \langle \ell, U\rangle}, U \in \mathfrak{b}.$$

Since $\mathfrak{b}$ is abelian of codimension 1, it is a polarization at ℓ and so π_ℓ is irreducible. Every infinite dimensional irreducible unitary representation of G_N arises in this manner up to equivalence.

Let us describe the representation $\pi_\ell, \ell \in \mathfrak{b}^*$, explicitly. The Hilbert space $\mathcal{H}_\ell$ of the representation π_ℓ is the space $L^2(G_N/B, \chi_\ell)$ consisting of all measurable functions $\tilde{\xi} : G_N \to \mathbb{C}$, such that $\tilde{\xi}(gb) = \chi_\ell(b^{-1})\tilde{\xi}(g)$ for all $b \in B$ and all $g \in G$ outside some set of measure of Lebesgue measure 0 and such that the function $|\tilde{\xi}|$ is contained in $L^2(G_N/B)$. We can identify the space $L^2(G_N/B, \chi_\ell)$ in an obvious way with $L^2(\mathbb{R})$ via the isomorphism $U : \xi \mapsto \tilde{\xi}$ where $\tilde{\xi}(\exp(sX_N)b) := \chi_\ell(b^{-1})\xi(s)$, $s \in \mathbb{R}$, $b \in B$. Hence for $g = \exp(tX_N)b$ and $\xi \in L^2(\mathbb{R})$ we have an explicit expression for the operator $\pi_\ell(g)$:

$$\begin{aligned}
(3.16) \qquad \pi_\ell(g)\xi(s) &= \tilde{\xi}(g^{-1}\exp(sX_N)) \\
&= \tilde{\xi}(b^{-1}\exp((s-t)X_N)) \\
&= \tilde{\xi}(\exp((s-t)X_N)(\exp((t-s)X_N)b^{-1}\exp((s-t)X_N))) \\
&= \chi_\ell(\exp((t-s)X_N)b\exp((s-t)X_N))\xi(s-t) \\
&= e^{-2\pi i \ell(\mathrm{Ad}(\exp((t-s)X_N)\log(b))}\xi(s-t), s \in \mathbb{R}.
\end{aligned}$$

We can parametrize the orbit space $\mathfrak{g}_N^*/G_N$ in the following way. First we have a decomposition

$$\mathfrak{g}_N^*/G_N = \bigcup_{j=1}^{N-2} {\mathfrak{g}_N^*}^j/G_N \bigcup X^*,$$

where

$${\mathfrak{g}_N^*}^j := \{\ell \in \mathfrak{g}_N^*, \ell(X_i) = 0, i = 1, \cdots, j-1, \ell(X_j) \neq 0\}$$

and where

$$X^* := \{\ell \in \mathfrak{g}_N^*, \ell(X_j) = 0, j = 1, \cdots, N-2\}$$

denotes the characters of G_N. A character of the group G_N can be written as $\chi_{a,b}, a, b \in \mathbb{R}$, where

$$\chi_{a,b}(x_N, x_{N-1},, \cdots, x_1) := e^{-2\pi i a x_N - 2\pi i b x_{N-1}},\ (x_N,, \cdots, x_1) \in G_N.$$

For any $\ell \in {\mathfrak{g}_N^*}^j, N-2 \geq j \geq 1$ there exists exactly one element ℓ_0 in the G_N-orbit of ℓ, which satisfies the conditions

$$\ell_0(X_j) \neq 0, \ell_0(X_{j+1}) = 0, \ell_0(X_N) = 0.$$

We can thus parametrize the orbit space $\mathfrak{g}_N^*/G_N$, and hence also the dual space $\widehat{G_N}$, with the sets

$$S_N := \bigcup_{j=1}^{N-2} S_N^j \bigcup X^*,$$

where $S_N^j := \mathcal{S}_N \cap \mathfrak{g}^{*j} = \{\ell \in {\mathfrak{g}_N^*}^j, \ell(X_k) = 0, k = 1, \cdots, j-1, j+1, \ell(X_j) \neq 0\}$. Let

$$S_N^{gen} := \bigcup_{j=1}^{N-2} S_N^j$$

be the family of points in S_N, whose G_N-orbits are of dimension 2.

3.2 The topology of $\widehat{G_N}$

The topology of the dual space of G_N has been studied in detail in the papers [ALS] and [AKLSS] based on the methods developed in [LRS] and [L]. We need the following description of the convergence of sequences $(\pi_k)_k$ of representations in $\widehat{G_N}$.
Let $(\pi_k)_k$ be a sequence in $\widehat{G_N}$. It is said to be *properly convergent* if it is convergent and all cluster points are limits. It is known (see [LRS]) that any convergent sequence has a properly convergent subsequence.

Proposition 3.1. *Suppose that $(\pi_k = \pi_{\ell_k})_k, (\ell_k \in S_N^{\rm gen}, k \in \mathbb{N})$ is a sequence in $\widehat{G_N}$ that has a cluster point. Then there exists a subsequence, (also indexed by the symbol k for simplicity), called with perfect data such that $(\pi_k)_k$ is properly converging and such that the polynomials p_k, $k \in \mathbb{N}$, associated to π_k have the following properties: The polynomials p_k have all the same degree d. Write*

$$p_k(t) := c_k \prod_{j=1}^{d} (t - a_j^k) = \hat{\ell}_k(t), t \in \mathbb{R}, \ell_k \in V.$$

There exist $0 < m \leq 2d$, real sequences $(t_i^k)_k$ and polynomials q_i of degree $d_i \leq d$, $i = 1, \cdots, m$, such that

1. $\lim_{k\to\infty} p_k(t + t_i^k) \to q_i(t)$, $t \in \mathbb{R}$, $1 \leq i \leq m$ *or equivalently* $\lim_{k\to\infty} t_i^k \cdot \ell_k \to \ell^i$, *where* ℓ^i *in* V *such that* $\hat{\ell}^i(t) = q_i(t)$.
2. $\lim_{k\to\infty} |t_i^k - t_{i'}^k| = +\infty$, *for all* $i \neq i' \in \{1, \cdots, m\}$.
3. *If* $C = \{i \in \{1, \cdots, m, \}, \ell^i$ *is a character* $\}$ *then for all* $i \in C$
   (a) $\lim_{k\to\infty} |t_i^k - a_j^k| = +\infty$ *for all* $j \in \{1, \cdots, d\}$;
   (b) *there exists an index* $j(i) \in \{1, \cdots, d\}$ *such that* $|t_i^k - a_{j(i)}^k| \leq |t_i^k - a_j^k|$ *for all* $j \in \{1, \cdots, d\}$; *let*
   $$\rho_i^k := |t_i^k - a_{j(i)}^k|;$$
   (c) *there exists a subset* $L(i) \subset \{1, \cdots, m\}$, *such that* $\lim_{k\to\infty} \frac{|t_i^k - a_j^k|}{\rho_i^k}$ *exists in* $\mathbb{R}$ *for every* $j \in L(i)$ *and such that* $\lim_{k\to\infty} \frac{|t_i^k - a_j^k|}{\rho_i^k} = +\infty$ *for* $j \notin L(i)$;
   (d) *the polynomials* $(t_i^k + s\rho_i^k) \cdot p_k$ *in* t *converge uniformly on compacta to the constants*
   $$\lim_{k\to\infty} (t_i^k + s\rho_i^k) \cdot p_k(t) = p^i(s), s \in \mathbb{R},$$
   and these constants define a real polynomial of degree $\#L(i)$ *in* s.

(e) *If* $i' \neq i \in C$, *then* $L(i) \cap L(i') = \emptyset$.

4. *Let* $D = \{1, \cdots, m\} \setminus C$ *and write* $\rho_i^k := 1$ *for* $i \in D$. *For* $i \in D$, *let*

$$J(i) := \{1 \leq j \leq d, \lim_{k \to \infty} |t_i^k - a_j^k| = \infty\}.$$

Suppose that $(t_k)_k$ *is a real sequence, such that* $\lim_{k\to\infty} t_k \cdot \ell_k \to \ell$ *in* $\mathfrak{g}_N^*$, *then*

(a) *if* ℓ *is a non-character, then the sequence* $(|t_k - t_i^k|)_k$ *is bounded for some* $i \notin C$;

(b) *if* ℓ *is a character, then* $\lim_{k\to\infty} \left|\frac{t_k - a_j^k}{t_i^k - a_j^k}\right|$ *exists for some* $i \in C$ *and some* $j \in L(i)$ *and* $\ell_{|\mathfrak{b}} = q_i(s) X_{N-1}^*$ *for some* $s \in \mathbb{R}$.

5. *Take any real sequence* $(s_k)_k$, *such that* $\lim_{k\to\infty} |s_k| = +\infty$, *and such that for any* $i \in D$, $j \in J(i)$, $\frac{s_k}{|t_i^k - a_j^k|} \to 0$, *and for* $i \in C$, $j \notin L(i)$, $\frac{s_k \rho_i^k}{|a_j^k - t_i^k|} \to 0$ *and* $\frac{s_k}{\rho_i^k} \to 0$ *as* $k \to \infty$. *Let*

$$S_k := (\bigcup_{i=1}^{m} [t_i^k - s_k \rho_i^k, t_i^k + s_k \rho_i^k]); \ T_k := \mathbb{R} \setminus S_k, k \in \mathbb{N}.$$

Then for any sequence $(t_k)_k$, $t_k \in T_k$, *we have* $t_k \cdot l_k \to \infty$.

We say that the sequence $(s_k)_k$ *is adapted to the sequence* (ℓ_k).

Proof. We may assume that $(\pi_k)_k$ is properly convergent with limit set L. We can also assume, by passing to a subsequence, that each p_k has degree d. By [L] the number of non-characters in L is finite. Let this subset of non-characters be denoted by L^{gen}. If L^{gen} is non-empty by passing further to a subsequence we may assume the sequence $(\pi_k)_k$ converges i_σ-times to each non-character $\sigma \in L^{gen}$ (see p.34, [AKLSS] for the definition of m-convergence and p.253 [ALS]). Let $s = \sum_{\sigma \in L^{gen}} i_\sigma$. Then there exist non-constant polynomials $q_1, \ldots, q_s$ of degree $d_i \leq d$, $i = 1, \ldots, s$, and sequences $(t_1^k)_k, \ldots, (t_s^k)_k$ such that the conditions (1) and (2) are fulfilled and for each $\sigma \in L^{gen}$ there are i_σ equal polynomials amongst $q_1, \ldots, q_s$ corresponding to σ. Then if $(t_k)_k$ is a real sequence such that $t_k \cdot \ell_k \to \ell$, ℓ is a non-character, then ℓ corresponds to some $\sigma \in L^{gen}$ and we may assume that $\hat{\ell} = q_i$ for some $i \in 1, \ldots, s$. It follows from the definition of i_σ-convergence that the sequences $(t_k \cdot \ell_k)$ and $(t_i^k \cdot \ell_k)$ are not disjoint implying $|t_k - t_i^k|$ is bounded and therefore (4a).
If $(\pi_k)_k$ has a character as a limit point then passing if necessary to a subsequence we can find a maximal family of real sequences $(t_l^k)_k$, $l < s \leq m \leq d$, constant polynomials q_l, non-negative sequences $(\rho_l^k)_k$ and polynomials p_l satisfying $(1) - (4)$ (see Definition 6.4 and the discussion before in [ALS]).
The condition (4b) follows from the maximality of the family of sequences $(t_l^k)_k$ and the proof of Proposition 6.2, [ALS].
Suppose now that we have a sequence $(t_k)_k$ such that $t_k \in T_k$ for every k and such that some subsequence (also indexed by k for simplicity of notations) $(t_k \cdot \ell_k)_k$ converges to an $\ell \in \mathfrak{g}^*$. By condition (5) then either for some $i \in D$, the sequence $(t_k - t_i^k)_k$ is bounded, i.e $t_k \in [t_i^k - s_k \rho_i^k, t_i^k + s_k \rho_i^k]$ for k large enough, which is impossible, or we have an $i \in C$, such

that $\lim_{k\to\infty}\left|\frac{t_k-a_j^k}{t_i^k-a_j^k}\right|$ exists for some $j\in L(i)$. But then

$$\begin{aligned}\frac{|t_k-t_i^k|}{\rho_i^k} &\leq \frac{|t_k-a_j^k|}{\rho_i^k}+\frac{|t_i^k-a_j^k|}{\rho_i^k}\\ &= \frac{|t_k-a_j^k|}{|t_i^k-a_j^k|}\frac{|t_i^k-a_j^k|}{\rho_i^k}+\frac{|t_i^k-a_j^k|}{\rho_i^k}\end{aligned}$$

and so the sequence $(\frac{|t_k-t_i^k|}{\rho_i^k})_k$ is bounded, i.e $t_k\in[t_i^k-s_k\rho_i^k,t_i^k+s_k\rho_i^k]\subset S_k$ for k large enough, a contradiction. Hence $\lim_k t_k\cdot\ell_k=\infty$ whenever $t_k\in T_k$ for large k. □

Example 3.2. Let us consider the Heisenberg group G_3. Then $S_3=S_3^1\cup\mathbb{R}^2$. Let $(\ell_k)\in S_3^1$. Then $\ell_k=\lambda_k X_1^*$, $\lambda_k\in\mathbb{R}^*$. The associated polynomials are $p_k(t)=-\lambda_k t$ ($d=1$, $c_k=-\lambda_k$, $a_1^k=1$). Assume that (ℓ_k) is a sequence with perfect data. Then either π_{ℓ_k} converges to π_ℓ, $\ell\in S_3^1$, or π_{ℓ_k} converges to a character and in this case $\lambda_k\to 0$ as $k\to\infty$. We shall consider now the second case. So we have $m=1$ and $\ell^1=X_2^*$ with the corresponding polynomial $q_1(t)=1$ and $t_1^k=-1/\lambda_k$ and thus $\rho_1^k=1/|\lambda_k|$. The polynomial $p^1(s)$ is the limit

$$\lim_{k\to\infty}p_k(t_1^k+s\rho_i^k+t)=\lim_{k\to\infty}(-\lambda_k)(-1/\lambda_k+s/|\lambda_k|+t)=\lim_{k\to\infty}(1-\text{sign}\lambda_k s-\lambda_k t).$$

Since (ℓ_k) is a sequence with perfect data, the sign of λ_k is constant, implying $q_1 1(s)=1+\epsilon s$, where $\epsilon=\pm 1$. A real sequence (s_k) is adapted to (ℓ_k) if and only if $s_k\to\infty$ and $s_k|\lambda_k|\to 0$.

3.3 A C^*-condition

Let $C^*(G_N)$ be the full C^*-algebra of G_N that is the completion of the convolution algebra $L^1(G_N)$ with respect to the norm

$$\|f\|_{C^*(G_N)}=\sup_{\ell\in\mathcal{S}_N}\|\int_{G_N}f(g)\pi_\ell(g)dg\|_{\text{op}}.$$

Definition 3.3. Let $f\in L^1(G_N)$. Define the function $\hat{f}^2$ on $\mathbb{R}\times\mathfrak{b}^*$ by

$$\hat{f}^2(s,\ell):=\int_B f(s,u)e^{-2\pi i\ell(\log(u))}du, s\in\mathbb{R},\ell\in\mathfrak{b}^*.$$

We denote by $L_c^1(G_N)$ the space of functions $f\in L^1(G_N)$, for which $\hat{f}^2$ is contained in $C_c^\infty(\mathbb{R}\times\mathfrak{b}^*)$, the space of compactly supported C^∞-functions on $\mathbb{R}\times\mathfrak{b}^*$. The subspace $L_c^1(G_N)$ is dense in $L^1(G_N)$ and hence in the full C^*-algebra $C^*(G_N)$ of G_N.

Proposition 3.4. *Take $f\in L_c^1(G_N)$ and let $\ell\in S_N^{gen}$. Then the operator $\pi_\ell(f)$ is a kernel operator with kernel function*

$$f_\ell(s,t)=\hat{f}^2(s-t,t\cdot\ell), s,t\in\mathbb{R}.$$

Proof. Indeed, for $\xi \in L^2(\mathbb{R}), s \in \mathbb{R}$, we have that

$$
\begin{aligned}
\pi_\ell(f)\xi(s) &= \int_{G_N} f(g)\pi_\ell(g)\xi dg \\
&= \int_{\mathbb{R}}\int_B f(t,b)e^{-2\pi i\ell(\mathrm{Ad}(\exp((t-s)X_N)\log(b))}\xi(s-t)dbdt(\text{ by } 3.16) \\
&= \int_{\mathbb{R}}\int_B f(s-t,b)e^{-2\pi i(\mathrm{Ad}^*(\exp(tX_N)(\ell)(\log(b))}\xi(t)dbdt \qquad (3.17)\\
&= \int_{\mathbb{R}} \hat{f}^2(s-t,\mathrm{Ad}^*(\exp(tX_N(\ell))\xi(t)dt \\
&= \int_{\mathbb{R}} \hat{f}^2(s-t,t\cdot\ell)\xi(t)dt.
\end{aligned}
$$

□

Definition 3.5. Let $\mathfrak{c} :=$ span $\{X_1,,\cdots,X_{N-2}\}$. Then $\mathfrak{c}$ is an abelian ideal of $\mathfrak{g}_N$, the algebra $\mathfrak{g}_N/\mathfrak{c}$ is abelian and isomorphic to $\mathbb{R}^2$ and $C := \exp(\mathfrak{c})$ is an abelian closed normal subgroup of G_N.
Let

$$\rho = \mathrm{ind}_C^{G_N} 1$$

be the left regular representation of G_N on the Hilbert space $L^2(G_N/C)$. Then the image $\rho(C^*(G_N))$ is the C^*-algebra of $\mathbb{R}^2$ considered as algebra of convolution operators on $L^2(\mathbb{R}^2)$ and hence $\rho(C^*(G_N))$ is isomorphic to the algebra $C_0(\mathbb{R}^2)$ of continuous functions vanishing at infinity on $\mathbb{R}^2$. As for the Heisenberg algebra we have that if $f \in L^1(G_N)$ then the Fourier transform $\widehat{\rho(f)}(a,b)$ of $\rho(f) \in C^*(\mathbb{R}^2)$ equals $\hat{f}(a,b,0,\ldots,0)$.

Our aim is to realize the C^*-algebra $C^*(G_N)$ as a C^*-algebra of operator fields.

Definition 3.6. For $a \in C^*(G_N)$ we define the Fourier transform $F(a)$ of a as operator field

$$F(a) := \{(A(\ell) := \pi_\ell(a), \ell \in S_N^{gen}, A(0) := \rho(a) \in C^*(\mathbb{R}^2)\}.$$

Remark 3.7. We observe that the spaces $S_N^j, j = 1,\cdots,N-2$, are Hausdorff spaces if we equip them with the topology of $\widehat{G_N}$. Indeed, let $(\ell_k)_k$ be a sequence in S_N^j, such that the sequence of representations $(\pi_{\ell_k})_k$ converges to some π_ℓ with $\ell \in S_N^j$. Then the numerical sequence $(\lambda_k := \ell_k(X_k))_k$ converges to $\lambda := \ell(X_j) \neq 0$. Suppose now that the same sequence (π_{ℓ_k}) converges to some other point $\pi_{\ell'}$. Then there exists a numerical sequence $(t_k)_k$ such that $\mathrm{Ad}^*(\exp(t_kX_N))\ell_{k|\mathfrak{b}}$ converges to $\ell'_{|\mathfrak{b}}$. In particular $-\lambda_k t_k = \mathrm{Ad}^*(\exp(t_kX_N))\ell_k(X_{j+1}) \overset{k\to\infty}{\to} \ell'(X_{j+1})$. Hence the sequence $(t_k)_k$ converges to some $t \in \mathbb{R}$ and $\pi_{\ell'} = \pi_\ell$. Similarly, we see from (3.17) that for $f \in L_c^1(G_N)$, the mapping $\ell \to \pi_\ell(f)$ is norm continuous when restricted to the sets $S_N^j, j = 1,\cdots,N-2$, since for the sequence $(\pi_{\ell_k})_k$ above, the functions f_{ℓ_k} converge in the L_2-norm to f_ℓ.

Definition 3.8. Define for $t, s \in \mathbb{R}$ the selfadjoint projection operator on $L^2(\mathbb{R})$ given by

$$M_{t,s}\xi(x) := 1_{(t-s,t+s)}(x)\xi(x), x \in \mathbb{R}, \xi \in L^2(\mathbb{R}),$$

where $1_{(a,b)}, a, b \in \mathbb{R}$, denotes the characteristic function of the interval $(a,b) \subset \mathbb{R}$.

We put for $s \in \mathbb{R}$
$$M_s := M_{0,s}.$$
More generally, for a measurable subset $T \subset \mathbb{R}$, we let M_T be the multiplication operator with the characteristic function of the set T. For $r \in \mathbb{R}$, let $U(r)$ be the unitary operator on $L^2(\mathbb{R})$ defined by
$$U(r)\xi(s) \quad := \quad \xi(s+r), \xi \in L^2(\mathbb{R}), s \in \mathbb{R}.$$

Definition 3.9. Let $(\pi_{\ell_k})_k$ be a properly converging sequence in $\widehat{G}_N$ with perfect data $((t_i^k)_k, (\rho_i^k), (s_i^k))$. Let $i \in C$ and let $\eta \in \mathcal{D}(\mathbb{R}^n)$ such that η has L^2-norm 1. Define for $\rho_i^k, k \in \mathbb{N}, i \in C$, and $u = (a,b) \in \mathbb{R}^2$ the Schwartz function
$$\eta(i,k,u)(s) \quad := \quad \eta(s_k p^i \left(\frac{s}{\rho_i^k}\right) + s_k b) e^{2\pi i a \cdot s}, s \in \mathbb{R}.$$
By Example 3.2, for $N = 3$ we have
$$\eta(1,k,u) = \eta(\pm s_k |\lambda_k| s + s_k(1+b)) e^{2\pi i a \cdot s}.$$
Let $P_{i,k,u}$ be the operator of rank one defined by
$$P_{i,k,u}\xi \quad := \quad \langle \xi, \eta(i,k,u)\rangle \eta(i,k,u), \xi \in L^2(\mathbb{R}).$$

Definition 3.10. For an element $\varphi \in \mathcal{S}(\mathbb{R}^2)$ let
$$\nu(\varphi)(i,k) \quad := s_k \textstyle\int_{\mathbb{R}^2} \hat{\varphi}(a,-b) P_{i,k,u} da db, k \in \mathbb{N}, i \in C.$$
Then for $\varphi \in \mathcal{S}(\mathbb{R}^2), \xi \in L^2(\mathbb{R}), s \in \mathbb{R}$, we have that
$$\begin{aligned}
\nu(\varphi)(i,k)(\xi)(s) \quad &:= \quad s_k \int_{\mathbb{R}^2} \hat{\varphi}(a,-b)(P_{i,k,u}\xi)(s) du \\
&= \quad s_k \int_{\mathbb{R}^2} \hat{\varphi}(a,-b) \left(\int_{\mathbb{R}} \xi(t) \overline{\eta(s_k p^i \left(\frac{t}{\rho_i^k}\right) + s_k b)} e^{-2\pi i a \cdot (t-s)} dt \right) \\
&\qquad \eta(s_k p^i \left(\frac{s}{\rho_i^k}\right) + s_k b) db da \\
&= \quad s_k \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\varphi}^2(s-t,-b) \xi(t) \overline{\eta(s_k p^i \left(\frac{t}{\rho_i^k}\right) + s_k b)} \eta(s_k p^i \left(\frac{s}{\rho_i^k}\right) + s_k b) dt db \\
&= \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\varphi}^2(s-t, -\frac{b}{s_k} + p^i \left(\frac{t}{\rho_i^k}\right)) \overline{\eta(b)} \\
&\qquad \eta(s_k \left(p^i \left(\frac{s}{\rho_i^k}\right) - p^i \left(\frac{t}{\rho_i^k}\right)\right) + b) \xi(t) dt db.
\end{aligned} \tag{3.18}$$

Since η has L_2-norm 1, using (3.17) and (3.18) we get

$$
\begin{aligned}
&(U(t_i^k)\circ\pi_{\ell_k}(f)\circ U(-t_i^k)\circ M_{s_k}-\nu(F(f)(0))(i,k)\circ M_{s_k})(\xi)(s)\\
=\;&\int_{-s_k}^{s_k}(\int_{\mathbb{R}}\hat f^2(s-t,(t+t_i^k)\cdot\ell_k))-\hat f^2(s-t,-\frac{b}{s_k}+p^i\left(\frac{t}{\rho_i^k}\right),0\dots)\\
&\overline{\eta(b)}\eta(s_k\left(p^i(\frac{s}{\rho_i^k})-p^i\left(\frac{t}{\rho_i^k}\right)\right)+b)db)\xi(t)dt\\
&+\int_{-s_k}^{s_k}(\int_{\mathbb{R}}\hat f^2(s-t,(t+t_i^k)\cdot\ell_k))\overline{\eta(b)}\\
&(\eta(b)-\eta(s_k\left(p^i\left(\frac{s}{\rho_i^k}\right)-p^i\left(\frac{t}{\rho_i^k}\right)\right)+b))db)\xi(t)dt.
\end{aligned}\tag{3.19}
$$

Proposition 3.11. *Let $\varphi\in C^*(\mathbb{R}^2)$, $i\in C$ and $k\in\mathbb{N}$. Then*

1. *the operator $\nu(\varphi)(i,k)$ is compact and $\|\nu(\varphi)(i,k)\|_{\text{op}}\le\|\varphi\|_{C^*(\mathbb{R}^2)}$;*
2. *we have that $\nu(\varphi)(i,k)^*=\nu(\varphi^*)(i,k)$;*
3. *furthermore*
$$\lim_{k\to\infty}\|\nu(\varphi)(i,k)\circ(\mathbb{I}-M_{s_k\rho_i^k})\|_{\text{op}}=0$$
and hence
$$\lim_{k\to\infty}\|(\mathbb{I}-M_{s_k\rho_i^k})\circ\nu(\varphi)(i,k)\circ M_{s_k\rho_i^k}\|_{\text{op}}=0.$$

Proof. 1.) It suffices to prove this for $\varphi\in\mathcal{D}(\mathbb{R}^2)$. We have that

$$
\begin{aligned}
\|\nu(\varphi)(i,k)\xi\|_2^2 &= \int_{\mathbb{R}}|\int_{\mathbb{R}}\int_{\mathbb{R}}\hat\varphi^2(s-t,-\frac{b}{s_k})\xi(t)\overline{\eta(s_kp^i(\frac{t}{\rho_i^k})+b)}dt\eta(s_kp^i(\frac{s}{\rho_i^k})+b)db|^2ds\\
&= \int_{\mathbb{R}}|\int_{\mathbb{R}}(\hat\varphi^2(-,-\frac{b}{s_k})*(\xi\overline{\eta_{k,b}}))(s)\eta(s_kp^i(\frac{s}{\rho_i^k})+b)db|^2ds\\
&\quad\text{(where } \eta_{k,b}(t):=\eta(s_kp^i(\frac{t}{\rho_i^k})+b), t\in\mathbb{R})\\
&\le \int_{\mathbb{R}^2}|(\hat\varphi^2(-,-\frac{b}{s_k})*(\xi\overline{\eta_{k,b}}))(s)|^2dbds\\
&\le \|\varphi\|^2_{C^*(\mathbb{R}^2)}\int_{\mathbb{R}}\|\xi\eta_{k,b}\|_2^2db\\
&= \|\varphi\|^2_{C^*(\mathbb{R}^2)}\int_{\mathbb{R}^2}|\xi(t)|^2|\eta(s_kp^i(\frac{t}{\rho_i^k})+b)|^2dbdt\\
&= \|\varphi\|^2_{C^*(\mathbb{R}^2)}\|\xi\|_2^2.
\end{aligned}
$$

Furthermore, since $\nu(\varphi)(i,k)$ is an integral of rank one operators, $\nu(\varphi)(i,k)$ must be compact. Hence for every $\varphi\in C^*(\mathbb{R}^2)$, $\nu(\varphi)(i,k)$ is a compact operator bounded by $\|\varphi\|_{C^*(\mathbb{R}^2)}$.
2.) Let $\varphi\in\mathcal{S}(\mathbb{R}^2)$. Then $\overline{\hat\varphi}=\hat{\varphi^*}$ and so

$$
\begin{aligned}
\nu(\varphi)(i,k)^* &= (s_k\int_{\mathbb{R}^2}\hat\varphi(u)P_{i,k,u}du)^*=s_k\int_{\mathbb{R}^2}\overline{\hat\varphi(u)}P_{i,k,u}du\\
&= s_k\int_{\mathbb{R}^2}\hat{\varphi^*}(u)P_{i,k,u}du=\nu(\varphi^*)(i,k).
\end{aligned}
$$

3.) Take now $\varphi \in \mathcal{S}(\mathbb{R}^2)$, such that $\hat{\varphi}$ has a compact support. We denote by $[-s_k\rho_i^k, s_k\rho_i^k]^c$ the set $\mathbb{R} \setminus [-s_k\rho_i^k, s_k\rho_i^k]$. By (3.18) for any $\xi \in L^2(\mathbb{R}), s \in \mathbb{R}$ we have

$$\begin{aligned} &\nu(\varphi)(i,k) \circ (\mathbb{I} - M_{s_k\rho_i^k})(\xi)(s) \\ = &\int_{[-s_k\rho_i^k, s_k\rho_i^k]^c} \int_{\mathbb{R}} \hat{\varphi}^2(s-t, -\frac{b}{s_k} + p^i(\frac{t}{\rho_i^k}))\overline{\eta(b)}\eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b) db \xi(t) dt = 0 \end{aligned}$$

since for k large enough $\hat{\varphi}^2(s-t, -\frac{b}{s_k} + p^i(\frac{t}{\rho_i^k})) = 0$ for any $t \in [-s_k\rho_i^k, s_k\rho_i^k]^c$, $b \in \text{supp}(\eta)$, $s \in \mathbb{R}$. Hence $\nu(\varphi(i,k)) \circ (\mathbb{I} - M_{s_k\rho_i^k}) = 0$ for k large enough. Since the mapping ν is continuous, it follows that $\lim_{k\to\infty} \|\nu(\varphi)(i,k)(\mathbb{I} - M_{s_k\rho_i^k})\|_{\text{op}} = 0$ for all $\varphi \in C^*(\mathbb{R}^2)$ and every $i \in C$. Hence also

$$\begin{aligned} &\lim_{k\to\infty} \|(\mathbb{I} - M_{s_k\rho_i^k}) \circ \nu(\varphi)(i,k) \circ M_{s_k\rho_i^k}\|_{\text{op}} \\ = &\lim_{k\to\infty} \|(M_{s_k\rho_i^k} \circ \nu(\varphi^*)(i,k) \circ (\mathbb{I} - M_{s_k\rho_i^k})\|_{\text{op}} \\ \leq &\lim_{k\to\infty} \|\nu(\varphi^*)(i,k) \circ (\mathbb{I} - M_{s_k\rho_i^k})\|_{\text{op}} = 0. \end{aligned}$$

□

Definition 3.12. Let Let $A = (A(\ell) \in \mathcal{K}(L^2(\mathbb{R})), \ell \in S_N^{gen}, A(0) \in C^*(\mathbb{R}^2))$ be a field of bounded operators. We say that A satisfies the *generic condition* if for every properly converging sequence with perfect data $(\pi_{\ell_k})_k \subset \hat{G}_N$ and for every limit point $\pi_{\ell^i}, i \in D$, and for every adapted real sequence $(s_k)_k$

$$\lim_{k\to\infty} \|U(t_i^k) \circ A(\ell_k) \circ U(-t_i^k) \circ M_{s_k} - A(\ell^i) \circ M_{s_k}\|_{\text{op}} = 0. \tag{3.20}$$

A satisfies the *character condition* if for every properly converging sequence with perfect data $(\pi_{\ell_k})_k$, $\ell_k \in S_N^{gen}$ and for every limit point $\pi_{\ell^i}, i \in C$, and for every adapted real sequence $(s_k)_k$

$$\lim_{k\to\infty} \|U(t_i^k) \circ A(\ell_k) \circ U(-t_i^k) \circ M_{s_k\rho_i^k} - \nu(A(0))(i,k) \circ M_{s_k\rho_i^k}\|_{\text{op}} = 0.$$

A satisfies the *infinity* condition, if for any properly converging sequence (π_{ℓ_k}), $\ell_k \in S_N^{gen}$, with perfect data we have that

$$\lim_{k\to\infty} \|A(\ell_k) \circ M_{T_k}\|_{\text{op}} = 0,$$

where $T_k = \mathbb{R} \setminus \left(\bigcup_{i=1}^m [t_i^k - s_k\rho_i^k, t_i^k + s_k\rho_i^k]\right)$, and that for every sequence $(\ell_k)_k \subset S_N^{gen}$, for which the sequence of orbits $G_N \cdot \ell_k$ goes to infinity we also have

$$\lim_{k\to\infty} A(\ell_k) = 0.$$

We can now define the operator field C^*-algebra D_N^*, which will be the image of the Fourier transform of $C^*(G_N)$.

Definition 3.13. Let D_N^* be the space of all bounded operator fields $A = (A(\ell)) \in \mathcal{K}(L^2(\mathbb{R})), \ell \in S_N^{gen}, A(0) \in C^*(\mathbb{R}^2)$, such that A and the adjoint field A^* satisfy the generic, the character and the infinity conditions. Let for $A \in D_N^*$

$$\|A\|_\infty := \sup\{\|A(\ell)\|_{\text{op}}, \|A(0)\|_{C^*(\mathbb{R}^2)} : \ell \in S_N^{gen}\}.$$

It is clear that D_N^* is a Banach space for the norm $\|\cdot\|_\infty$, since the generic, the character and the infinity conditions are stable for the sum, for scalar multiplication and limits of sequences of operator fields.

Theorem 3.14. *Let $a \in C^*(G_N)$ and let A be the operator field defined by $A = F(a)$ as in Definition 3.6. Then A satisfies the generic, the character and the infinity conditions.*

Proof. For the infinity condition, it suffices to remark that for any $f \in L^1_c(G_N)$, and k large enough, we have that $\hat{f}^2(s-t, t\cdot\ell_k) = 0$ for every $s \in \mathbb{R}$, $t \in T_k$ and so $\pi_{\ell_k}(f) \circ M_{T_k} = 0$. If $G_N \cdot \ell_k$ goes to infinity in the orbit space, then $\mathbb{R}\cdot\ell_k$ is outside any given compact subset $K \subset \mathfrak{g}_N^*$ and so $\hat{f}^2(s-t, t\cdot\ell_k) = 0, s, t \in \mathbb{R}$ and hence $\pi_{\ell_k}(f) = 0$ for k large enough. Using the density argument, we see that the infinity condition is satisfied for every element in the Fourier transform of $C^*(G_N)$.
For the generic condition, let $(\ell_k)_k$ be a properly converging sequence in S_N with perfect data. Take $i \in D$. Then for an adapted sequence $(s_k)_k$, $f \in L^1_c(G_N)$ and $\xi \in L^2(\mathbb{R}), s \in \mathbb{R}$, we have that

$$\begin{aligned} &(U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k) \circ M_{s_k} - \pi_{\ell^i}(f) \circ M_{s_k})\xi(s) \\ (3.21) \qquad &= \int_{-s_k}^{s_k} \left(\hat{f}^2(s-t, (t+t_i^k)\cdot\ell_k)) - \hat{f}^2(s-t, t\cdot\ell^i))\right)\xi(t)dt. \end{aligned}$$

Let p_k and q_i be the polynomials corresponding to ℓ_k and ℓ^i respectively, i.e., $p_k(t) = \hat{\ell}_k(t)$ and $q_i(t) = \hat{\ell}^i(t)$. Since $\lim_{k\to\infty} \frac{s_k}{|t_i^k - a_j^k|} \to 0$, $j \in J(i)$, there exists $R > 0$ such that $(s-t, (t+t_i^k)\cdot\ell_k) = (s-t, p_k(t+t_i^k), -p_k'(t+t_i^k), \ldots)$ is out of the support of $\hat{f}^2$ if $t \in [-s_k, s_k]$ and $|t| > R$. In fact if $t \in [-s_k, s_k]$ we have

$$\begin{aligned} |p_k(t+t_i^k)| &= |c_k \prod_{j=1}^d (t+t_i^k - a_j^k)| = |c_k \prod_{j\in J(i)} |t_i^k - a_j^k| \prod_{j\in J(i)} |\frac{t}{t_i^k - a_j^k} + 1| \prod_{j\notin J(i)} |t + t_i^k - a_j^k| \\ &\geq |b_i| \prod_{j\in J(i)} |1 - \frac{s_k}{|t_i^k - a_j^k|}| \prod_{j\notin J(i)} |t + t_i^k - a_j^k|, \end{aligned}$$

where b_i is the leading coefficient of the polynomial q_i, giving the statement. Thus by (3.21)

$$\begin{aligned} &(U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k) \circ M_{s_k} - \pi_{\ell^i}(f) \circ M_{s_k})\xi(s) \\ = &\int_{-R}^{R} \left(\hat{f}^2(s-t, (t+t_i^k)\cdot\ell_k)) - \hat{f}^2(s-t, t\cdot\ell^i))\right)\xi(t)dt \end{aligned}$$

for k large enough. It is clear now that $U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k) \circ M_{s_k} - \pi_{\ell^i}(f) \circ M_{s_k}$ converges to 0 with respect to the Hilbert-Schmidt norm and hence in the operator norm.
Let $a \in C^*(G_N)$. Then for any $\varepsilon > 0$ there exists $f \in L^1_c(G_N)$ such that $||\pi(f) - \pi(a)||_{op} \leq ||f - a||_{C^*(G_N)} < \varepsilon$ for any representation π of $C^*(G_N)$. Thus for $A(\ell) = \pi_\ell(a)$, $\ell \in S_N^{gen}$ we have

$$\begin{aligned} &\|U(t_i^k) \circ A(\ell_k) \circ U(-t_i^k) \circ M_{s_k} - A(\ell^i) \circ M_{s_k}\|_{\mathrm{op}} = \|U(t_i^k) \circ (A(\ell_k) - \pi_{l_k}(f)) \circ U(-t_i^k)\|_{\mathrm{op}} \\ &+ \|U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k) \circ M_{s_k} - \pi_{\ell^i}(f) \circ M_{s_k}\|_{\mathrm{op}} + \|(\pi_{\ell^i}(f) - A(\ell^i))\|_{\mathrm{op}} \to 0, \end{aligned}$$

and hence A satisfies the generic condition.

Choose now $i \in C$. By (3.19), for $k \in \mathbb{N}, s \in \mathbb{R}, \xi \in L^2(\mathbb{R}), f \in L^1_c(G_N)$

$$\begin{aligned}
& U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k) \circ M_{s_k \rho_i^k} - \nu(F(f(0)))(i,k) \circ M_{s_k \rho_i^k}(\xi)(s) \\
= & \int_{-s_k \rho_i^k}^{s_k \rho_i^k} (\int_{\mathbb{R}} \hat{f}^2(s-t, (t+t_i^k)\cdot \ell_k)) - \hat{f}^2(s-t, -\frac{b}{s_k} + p^i(\frac{t}{\rho_i^k}), 0\ldots, 0) \\
& \overline{\eta(b)} \eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b) db) \xi(t) dt + \\
& \int_{-s_k}^{s_k} (\int_{\mathbb{R}} \hat{f}^2(s-t, (t+t_i^k)\cdot \ell_k)) \overline{\eta(b)} (\eta(b) - \eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b)) db) \xi(t) dt.
\end{aligned}$$

In order to show that

$$(3.22) \qquad \|U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k) \circ M_{s_k \rho_i^k} - \nu(F(f(0)))(i,k) \circ M_{s_k \rho_i^k}\| \to 0, \ \ k \to \infty,$$

consider

$$\begin{aligned}
q(k,i)(s,b) \quad = \quad & \int_{-s_k \rho_i^k}^{s_k \rho_i^k} (\hat{f}^2(s-t, (t+t_i^k)\cdot \ell_k)) - \hat{f}^2(s-t, -\frac{b}{s_k} + p^i(\frac{t}{\rho_i^k}), 0\ldots, 0)) \\
& \eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b)) \xi(t) dt = u(k,i) + v(k,i),
\end{aligned}$$

where

$$\begin{aligned}
u(k,i)(s,b) \quad = \quad & \int_{-s_k \rho_i^k}^{s_k \rho_i^k} (\hat{f}^2(s-t, p_k(t+t_i^k), -p_k'(t+t_i^k), \ldots) - \hat{f}^2(s-t, p_k(t+t_i^k), 0, \ldots)) \\
& \eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b)) \xi(t) dt, \\
v(k,i)(s,b) \quad = \quad & \int_{-s_k \rho_i^k}^{s_k \rho_i^k} (\hat{f}^2(s-t, p_k(t+t_i^k), 0, \ldots) - \hat{f}^2(s-t, -\frac{b}{s_k} + p^i(\frac{t}{\rho_i^k}), 0\ldots)) \\
& \eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b)) \xi(t) dt.
\end{aligned}$$

and let

$$w(k,i)(s) = \int_{\mathbb{R}} \int_{-s_k}^{s_k} (\int_{\mathbb{R}} \hat{f}^2(s-t, (t+t_i^k)\cdot \ell_k)) \overline{\eta(b)} (\eta(b) - \eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b)) db) \xi(t) dt.$$

Our aim is to prove that for $p(s,b) = 1_{\mathbb{R}\times \mathrm{supp}(\eta)}(s,b)$

$$(3.23) \qquad \|u(k,i)p\|_2 \le \omega_k ||\xi||_2, \|v(k,i)p\|_2 \le \delta_k ||\xi||_2 \text{ and } \|w(k,i)\|_2 \le r_k ||\xi||_2$$

with $\omega_k, \delta_k, r_k \to 0$ as $k \to \infty$. This will imply

$$\int_{\mathbb{R}} |\int_{\mathbb{R}} (q(k,i)(s,b)) \overline{\eta(b)} db|^2 ds \le \|q(k,i)\|_2^2 \|\eta\|_2 \le (\omega_k + \delta_k)^2 \|\xi\|_2^2$$

which together with $\|w(k,i)\|_2 \le r_k ||\xi||_2$ will give (3.22).

To see this we note first that since $\frac{s_k\rho_i^k}{|a_k^j-t_i^k|}\to 0$ if $j\notin L(i)$, we have that for $|t|\leq s_k\rho_i^k$

$$\begin{aligned}|p_k(t+t_i^k)| &= |c_k\prod_{j=1}^d(t+t_i^k-a_j^k)| = |c_k\prod_j|t_i^k-a_j^k|\prod_{j\notin L(i)}|\frac{t}{t_i^k-a_j^k}+1|\prod_{j\in L(i)}|\frac{t}{t_i^k-a_j^k}+1)| \\ &\geq \sigma\prod_{j\notin L(i)}|1-\frac{s_k\rho_i^k}{|t_i^k-a_j^k|}|\prod_{j\in L(i)}|\frac{|t|}{\rho_i^k}\frac{\rho_i^k}{|t_i^k-a_j^k|}-1|\end{aligned}$$

for some $\sigma>0$. Thus for large k there exists $R>0$ such that $\hat{f}^2(s-t,p_k(t+t_i^k),-p_k'(t+t_i^k),\ldots)=0$ and $\hat{f}^2(s-t,p_k(t+t_i^k),0,\ldots)=0$ if $|t|<s_k\rho_i^k$ and $|t|>R\rho_i^k$. Hence the integration over the interval $[-s_k,s_k]$ can be replaced by the integration over $[-R\rho_i^k,R\rho_i^k]$ in the expression for $u(k,i)$, $v(k,i)p$ and $w(k,i)$. Since $f\in L_c^1(G_N)$ we have that

$$|\hat{f}^2(s-t,p_k(t+t_i^k),0,\ldots)-\hat{f}^2(s-t,p^i(\frac{t}{\rho_i^k})-\frac{b}{s_k},0,\ldots)|\leq C|p_k(t+t_i^k)-p^i(\frac{t}{\rho_i^k})+\frac{b}{s_k}|\frac{1}{1+|t-s|^m}$$

for some constant $C>0$ and $m\in\mathbb{N}$, $m\geq 2$. This gives

$$\begin{aligned}
\|v(k,i)p\|_2^2 &= C\int_{\mathbb{R}^2}\left|\int_{-R\rho_i^k}^{R\rho_i^k}\eta(s_k(p^i(\frac{s}{\rho_i^k})-p^i(\frac{t}{\rho_i^k})+b)\right.\\
&\quad\left.(p_k(t+t_i^k)-p^i(\frac{t}{\rho_i^k})+\frac{b}{s_k})\frac{\xi(t)}{1+|t-s|^m}dt\right|^2dsdb\\
&\leq\frac{3C}{s_k^2}\int_{\mathbb{R}^2}\int_{-R\rho_i^k}^{R\rho_i^k}\left|\eta(s_k(p^i(\frac{s}{\rho_i^k})-p^i(\frac{t}{\rho_i^k})+b)\right.\\
&\quad\left.(b+s_k(p^i(\frac{s}{\rho_i^k})-p^i(\frac{t}{\rho_i^k}))\frac{\xi(t)}{1+|t-s|^m}dt\right|^2dsdb\\
&+3C\int_{\mathbb{R}^2}\int_{-R\rho_i^k}^{R\rho_i^k}\left|\eta(s_k(p^i(\frac{s}{\rho_i^k})-p^i(\frac{t}{\rho_i^k}))+b)(p_k(t+t_i^k)-p^i(\frac{t}{\rho_i^k}))\frac{\xi(t)}{1+|t-s|^m}dt\right|^2dsdb\\
&+3C\int_{\mathbb{R}^2}\int_{-R\rho_i^k}^{R\rho_i^k}\left|\xi(t)\eta(s_k(p^i(\frac{s}{\rho_i^k})-p^i(\frac{t}{\rho_i^k}))+b)(p^i(\frac{t}{\rho_i^k})-p^i(\frac{s}{\rho_i^k}))\frac{1}{1+|t-s|^m}dt\right|^2dsdb\\
&\leq\frac{C_1}{s_k^2}\int_{\mathbb{R}^2}\int_{-R\rho_i^k}^{R\rho_i^k}\left|\xi(t)\tilde{\eta}(s_k(p^i(\frac{s}{\rho_i^k})-p^i(\frac{t}{\rho_i^k}))+b)\right|^2\frac{1}{1+|t-s|^m}dtdsdb\\
&\quad(\text{where }\tilde{\eta}(b)=b\eta(b))\\
&+C_1\int_{\mathbb{R}^2}\int_{-R\rho_i^k}^{R\rho_i^k}\left|\xi(t)\eta(s_k(p^i(\frac{s}{\rho_i^k})-p^i(\frac{t}{\rho_i^k}))+b)(p_k(t+t_i^k)-p^i(\frac{t}{\rho_i^k}))\right|^2\\
&\quad\frac{1}{1+|t-s|^m}dtdsdb\\
&+\int_{\mathbb{R}^2}\int_{-R\rho_i^k}^{R\rho_i^k}\left|\xi(t)\eta(s_k(p^i(\frac{s}{\rho_i^k})-p^i(\frac{t}{\rho_i^k}))+b)(p^i(\frac{t}{\rho_i^k})-p^i(\frac{s}{\rho_i^k}))\right|^2\\
&\quad\frac{1}{1+|t-s|^m}dtdsdb
\end{aligned}$$

$$\leq \frac{C_2}{s_k^2}\|\tilde{\eta}\|_2^2\|\xi\|_2^2 + C_2\|\eta\|_2^2 \int_{-R}^{R} |\xi(t\rho_i^k)|^2 |p_k(t\rho_i^k + t_i^k) - p^i(t)|^2 \rho_i^k dt$$

$$+ \quad C_3\|\eta\|_2^2 \int_{\mathbb{R}} \int_{-R\rho_i^k}^{R\rho_i^k} |\xi(t)|^2 \left| p^i(\frac{t}{\rho_i^k}) - p^i(\frac{s}{\rho_i^k}) \right|^2 \frac{1}{1+|t-s|^m} dtds$$

As $p_k(t\rho_i^k + t_i^k) - p^i(t)$ converges to 0 uniformly on each compact,

$$\int_{-R}^{R} |\xi(t\rho_i^k)|^2 |p_k(t\rho_i^k + t_i^k) - p^i(t)|^2 \rho_i^k dt \leq r_k \|\xi\|_2^2$$

with $r_k \to 0$ as $k \to \infty$. Moreover, $p^i(\frac{t}{\rho_i^k}) - p^i(\frac{s}{\rho_i^k}) = \frac{t-s}{\rho_i^k} \sum_l \alpha_l(\frac{t}{\rho_i^k}) \beta_l(\frac{t-s}{\rho_i^k})$ for some finite number of polynomials α_l, β_l which do not depend on k. Thus

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{-R\rho_i^k}^{R\rho_i^k} |\xi(t)|^2 \left| p^i(\frac{t}{\rho_i^k}) - p^i(\frac{s}{\rho_i^k}) \right|^2 \frac{1}{1+|t-s|^m} dtds \\
= & \int_{\mathbb{R}} \int_{-R\rho_i^k}^{R\rho_i^k} |\xi(t)|^2 \left| \frac{t-s}{\rho_i^k} \sum_l \alpha_l(\frac{t}{\rho_i^k}) \beta_l(\frac{t-s}{\rho_i^k}) \right|^2 \frac{1}{1+|t-s|^m} dtds \\
\leq & \frac{C_4}{(\rho_i^k)^2} \int_{-R\rho_i^k}^{R\rho_i^k} |\xi(t)|^2 dt \leq \frac{C_4}{(\rho_i^k)^2} \|\xi\|_2^2
\end{aligned}$$

for a properly chosen m. It follows now that

$$||v(k,i)p||_2 \leq \delta_k \|\xi\|_2$$

for some $\delta_k \to 0$ as $k \to \infty$.
For $w(k,i)$ we have

$$\begin{aligned}
\|w(k,i)\|^2 &= \int_{\mathbb{R}} \left| \int_{-R\rho_i^k}^{R\rho_i^k} \int_R \hat{f}^2(s-t, (t+t_i^k)\cdot \ell_k) \overline{\eta(b)} \right. \\
& \quad \left. (\eta(b) - \eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b)) \xi(t) dbdt \right|^2 ds \\
&\leq C\|\eta\|_2^2 \int_{\mathbb{R}} \left| \int_{-R\rho_i^k}^{R\rho_i^k} |s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k}))| \frac{1}{1+|t-s|^m} |\xi(t)| dt \right|^2 ds \\
&\leq C\|\eta\|_2^2 \int_R \int_{-R\rho_i^k}^{R\rho_i^k} |s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k}))|^2 \frac{1}{1+|t-s|^m} |\xi(t)|^2 dtds
\end{aligned}$$

for some constant C. Then using the previous arguments we get

$$\|w(k,i)\|^2 \leq \frac{Ds_k^2}{(\rho_i^k)^2} \|\xi\|_2^2 \|\eta\|_2^2.$$

As $\frac{s_k}{\rho_i^k} \to 0$ we get the desired inequality for $w(k,i)$.

To prove the inequality for $u(k,i)$ we have as in the previous case that

$$|\hat{f}^2(s-t, p_k(t+t_i^k), -p_k'(t+t_i^k), \ldots) - \hat{f}^2(s-t, p_k(t+t_i^k), 0, \ldots)|$$
$$\leq C(\sum_{n=1}^{N-2} |p_k^{(n)}(t+t_i^k)|^2)^{1/2} \frac{1}{1+|t-s|^m}$$

for some constant $C > 0$ and $m \in \mathbb{N}$, $m \geq 2$; here $p_k^{(n)}$ denotes the n-th derivative of p_k. For $n = 1$ we have

$$|p_k'(t+t_i^k)| = |c_k \prod_j (t_i^k - a_k^j)| \sum_l \frac{1}{(t_i^k - a_k^l)} \prod_{j \neq l} (\frac{t}{t_i^k - a_k^j} + 1)| \leq \sigma \frac{1}{\rho_i^k} \left(\frac{|t|}{\rho_i^k} + 1 \right)^{d-1}$$

for some constant $\sigma > 0$. Similar inequalities hold for higher order derivatives $p_k^{(n)}(t + t_i^k)$ which show that

$$\begin{aligned}
& \|u(k,i)\|_2^2 = \\
= \; & \int_{\mathbb{R}^2} | \int_{-R\rho_i^k}^{R\rho_i^k} (\hat{f}^2(s-t, p_k(t+t_i^k), -p_k'(t+t_i^k), \ldots) - \hat{f}^2(s-t, p_k(t+t_i^k), 0, \ldots)) \\
& \eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b))\xi(t)dt|^2 dbds \\
\leq \; & \int_{\mathbb{R}^2} | \int_{-R\rho_i^k}^{R\rho_i^k} C(\sum_{n=1}^{N-2} |p_k^{(n)}(t+t_i^k)|^2)^{1/2} \frac{1}{1+|t-s|^m} \eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b))\xi(t)dt|^2 dbds \\
\leq \; & \frac{1}{\rho_i^k} \int_{\mathbb{R}^2} | \int_{-R\rho_i^k}^{R\rho_i^k} |p\left(\frac{|t|}{\rho_i^k}\right)| \frac{1}{1+|t-s|^m} \eta(s_k(p^i(\frac{s}{\rho_i^k}) - p^i(\frac{t}{\rho_i^k})) + b))\xi(t)dt|^2 dbds \\
\leq \; & \frac{C'}{\rho_i^k} \|\eta\|_2^2 \|\xi\|_2^2
\end{aligned}$$

for a polynomial p. Thus we get the required inequality for $u(k,i)$ and hence

$$\lim_{k\to\infty} \|U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k) \circ M_{s_k\rho_i^k} - \nu(F(f(0)))(i,k) \circ M_{s_k\rho_i^k}\|_{\mathrm{op}} = 0.$$

To show now that the character condition holds for the fields $A \in \widehat{C^*(G_N)}$ we use again the density of $L_c^1(G_N)$ in $C^*(G_N)$.

□

Corollary 3.15. *Let $(\pi_{\ell_k})_k$ be a properly converging sequence in $\widehat{G}_N$ with perfect data $((t_i^k)_k, (\rho_i^k), (s_i^k))$. Let $i \in C$. Then for every $\varphi, \psi \in C^*(\mathbb{R}^2)$ we have that*

$$\lim_{k\to\infty} \|\nu(\varphi)(i,k) \circ \nu(\psi)(i,k) - \nu(\varphi\psi)(i,k)\|_{\mathrm{op}} = 0.$$

Proof. Indeed, if we take first φ, ψ in $\mathcal{S}(\mathbb{R}^2)$, then we can choose $f, g \in \mathcal{S}(G_N)$, such that

$\rho(f) = \varphi, \rho(g) = \psi$ and so, by Proposition 3.11 and Theorem 3.14,

$$
\begin{aligned}
& \|\nu(i,k)(\varphi) \circ \nu(i,k)(\psi) - \nu(i,k)(\varphi\psi)\|_{\rm op} \\
\leq \ & \|(\nu(i,k)(\varphi) \circ \nu(i,k)(\psi) - \nu(i,k)(\varphi\psi)) \circ M_{s_k\rho_i^k}\|_{\rm op} \\
+ \ & \|(\nu(i,k)(\varphi) \circ \nu(i,k)(\psi) - \nu(i,k)(\varphi\psi)) \circ (\mathbb{I} - M_{s_k\rho_i^k})\|_{\rm op} \\
\leq \ & \|(\nu(i,k)(\varphi) \circ \nu(i,k)(\psi) \\
& -(U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k)) \circ (U(t_i^k) \circ \pi_{\ell_k}(g) \circ U(-t_i^k))) \circ M_{s_k\rho_i^k}\|_{\rm op} \\
+ \ & \|(U(t_i^k) \circ \pi_{\ell_k}(f*g) \circ U(-t_i^k) - \nu(i,k)(\varphi\psi)) \circ M_{s_k\rho_i^k}\|_{\rm op} \\
+ \ & \|(\nu(i,k)(\varphi) \circ \nu(i,k)(\psi) - \nu(i,k)(\varphi\psi)) \circ (\mathbb{I} - M_{s_k\rho_i^k})\|_{\rm op} \\
\leq \ & \|(\nu(i,k)(\varphi) - (U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k)) \circ (\mathbb{I} - M_{s_k\rho_i^k}) \circ \nu(i,k)(\psi) \circ M_{s_k\rho_i^k}\|_{\rm op} \\
+ \ & \|(\nu(i,k)(\varphi) - (U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k)) \circ M_{s_k\rho_i^k} \circ \nu(i,k)(\psi) \circ M_{s_k\rho_i^k}\|_{\rm op} \\
+ \ & \|((U(t_i^k) \circ \pi_{\ell_k}(f) \circ U(-t_i^k)) \circ (\nu(i,k)(\psi) - (U(t_i^k) \circ \pi_{\ell_k}(g) \circ U(-t_i^k)) \circ M_{s_k\rho_i^k}\|_{\rm op} \\
+ \ & \|(U(t_i^k) \circ \pi_{\ell_k}(f*g) \circ U(-t_i^k) - \nu(i,k)(\varphi\psi)) \circ M_{s_k\rho_i^k}\|_{\rm op} \\
+ \ & \|(\nu(i,k)(\varphi) \circ \nu(i,k)(\psi) - \nu(i,k)(\varphi\psi)) \circ (\mathbb{I} - M_{s_k\rho_i^k})\|_{\rm op} \to 0 \text{ as } k \to \infty.
\end{aligned}
$$

The usual density condition shows that the statement holds for all φ, $\psi \in C^*(\mathbb{R}^2)$. □

Theorem 3.16. *The space D_N^* is a C^*-algebra, which is isomorphic with $C^*(G_N)$ for every $N \in \mathbb{N}, N \geq 3$.*

Proof. Let us first show that D_N^* is a C^*-algebra. We prove first that D_N^* is closed under multiplication. Let $A = (A(\ell), \ell \in S_N)$ and $B = (B(\ell), \ell \in S_N)$ satisfy the generic condition and let $(\pi_{\ell_k})_k \subset \hat{G}_N$ be a properly convergent sequence with perfect data such that for every limit point π_{l^i}, $i \in D$, and for every adapted real sequence $(s_k)_k$ the fields A, B satisfy (3.20). Then

$$
\begin{aligned}
& \|U(t_i^k) \circ A(\ell_k) \circ B(\ell_k) \circ U(-t_i^k) \circ M_{s_k} - A(\ell^i) \circ B(\ell^i) \circ M_{s_k}\|_{\rm op} \\
& \leq \|(U(t_i^k) \circ A(\ell_k) \circ U(-t_i^k) \circ M_{s_k} - A(\ell^i) \circ M_{s_k}) \circ U(t_i^k) \circ B(\ell_k) \circ U(-t_i^k) \circ M_{s_k}\|_{\rm op} \\
& +\|A(\ell^i) \circ M_{s_k} \circ (U(t_i^k) \circ B(\ell_k) \circ U(-t_i^k) \circ M_{s_k} - B(\ell^i) \circ M_{s_k})\|_{\rm op} \\
& +\|U(t_i^k) \circ A(\ell_k) \circ U(-t_i^k) \circ (\mathbb{I} - M_{s_k}) \circ (U(t_i^k) \circ B(\ell_k) \circ U(-t_i^k) \circ M_{s_k} - B(\ell^i) \circ M_{s_k})\|_{\rm op} \\
& +\|A(\ell^i) \circ (\mathbb{I} - M_{s_k}) \circ B(\ell^i) \circ M_{s_k}\|_{\rm op} \\
& +\|U(t_i^k) \circ A(\ell_k) \circ U(-t_i^k) \circ (\mathbb{I} - M_{s_k}) \circ B(\ell^i) \circ M_{s_k}\|_{\rm op}.
\end{aligned}
$$

Since $B(\ell^i)$ is compact and $\mathbb{I} - M_{s_k}$ converges to 0 strongly, $\|(\mathbb{I} - M_{s_k}) \circ B(\ell^i)\|_{\rm op} \to 0$ giving that the product $A(\ell) \circ B(\ell)$ satisfies the generic condition.
To see that the character condition is closed under multiplication we argue as before, but use $\|(\mathbb{I} - M_{s_k\rho_i^k}) \circ \nu(\varphi)(i,k) \circ M_{s_k\rho_i^k}\|_{\rm op} \to 0$ which is due to Propsition 3.11.
The infinity condition is clearly closed under multiplication of fields.
By Theorem 3.14, the Fourier transform F maps $C^*(G_N)$ into D_N^* Let us show that F is also onto. By the Stone-Weierstrass approximation theorem, we must only prove that the dual space of D_N^* is the same as the dual space of $C^*(G_N)$. We proceed by induction on N. If $N = 3$, then G_N is the Heisenberg group and the statement follows from Theorem 2.16. Let $\pi \in \widehat{D_N^*}$.

Let for $M = 3, \cdots N-1$, $R_M : D_N^* \to D_M^*$ be the restriction map, i.e. denote by $q_M : \mathfrak{g}_N \to \mathfrak{g}_N/\mathfrak{b}_{N-M} \simeq \mathfrak{g}_M$ the quotient map and by $q_M^t : \mathfrak{g}_M^* \simeq \mathfrak{b}_{N-M}^\perp \to \mathfrak{g}_N^*$ its transpose. Then for an operator field $A \in D_N^*$ we define the operator field $R_M(A)$ over S_M^{gen} by:

$$R_M(A)(\tilde{\ell}) := A(q_M^t(\tilde{\ell})), \tilde{\ell} \in S_M^{\text{gen}}.$$

It follows from the definition of D_N^* that the image of R_M is contained in D_M^*. Hence R_M is a homomorphism of C^*-algebras, whose kernel I_M is the ideal

$$I_M = \{A \in D_N^*, A(\ell) = 0 \text{ for all } \ell \in S_N \cap \mathfrak{b}_{N-M}^\perp\}.$$

Let $Q_M : C^*(G_N) \to C^*(G_M) \simeq C^*(G_N/B_{N-M})$ be the canonical projection. Then the kernel of this projection is the ideal $J_M := \{a \in C^*(G_N); \pi_\ell(a) = 0, \ell \in S_N \cap \mathfrak{b}_{N-M}^\perp\}$. Let us write F_M for the Fourier transform $C^*(G_M) \to D_M^*$. With these notations we have the formula

$$R_M(F_N(a)) = F_M(a \text{ modulo } J_M), a \in C^*(G_N). \tag{3.24}$$

Since by the induction hypothesis $\widehat{D_M^*} = F_M(\widehat{C^*(G_M)})$ for every $3 \le M \le N-1$ we see from (3.24) that $R_M(F_N(C^*(G_N))) = F_M(C^*(G_M)) = D_M^*$ and so the mapping R_M is surjective for such an M. Hence $D_N^*/I_M \simeq C^*(G_M)$. We have also $I_{N-1} \subseteq I_{N-2} \subseteq \ldots \subseteq I_3$.
If $\pi(I_{N-1}) = \{0\}$ then $\pi \in \widehat{G_N/B_1} \subset \widehat{G_N}$.
Suppose now that $\pi(I_{N-1}) \neq \{0\}$. Let us show that $I_{N-1} \simeq C_0(S_N^1, \mathcal{K}(L^2(\mathbb{R})))$. It is clear from the definition of D_N^* that $C_0(S_N^1, \mathcal{K}(L^2(\mathbb{R}))) \subset D_N^*$ and so is contained in I_{N-1}. It suffices to show now that $I_{N-1} \subset C_0(S_N^1, \mathcal{K}(L^2(\mathbb{R})))$. For that it is enough to see that for any element A in I_{N-1} and any sequence $(\ell_k)_k$ in S_N^1 for which either (π_{ℓ_k}) converges to infinity or to a representation π_ℓ with $\ell \notin S_N^1$, we have that $\lim_k \|A(\ell_k)\|_{\text{op}} = 0$. This follows from the infinity condition in the first case. In the second case no limit point of the sequence (π_{ℓ_k}) is in S_N^1 by Remark 3.7. It suffices to show then that $\lim_k \|A(\ell_k)\|_{\text{op}} = 0$ for every subsequence with perfect data (also indexed by k for simplicity of notation). We have with the notations of Definition 3.12 that for $k \in \mathbb{N}$

$$A(\ell_k) \quad = \quad A(\ell_k) \circ M_{S_k} + A(\ell_k) \circ M_{T_k}.$$

where $S_k = \cup_i (t_i^k - s_k\rho_i^k, t_i^k + s_k\rho_i^k)$, $T_k = \mathbb{R} \setminus S_k$.
Since $A(\ell) = 0$ for every π_ℓ in the limit set of the sequence $(\pi_{\ell_k})_k$, the generic and the character conditions say that

$$\lim_k \|U(t_i^k) \circ A(\ell_k) \circ U(t_i^k) \circ M_{s_k\rho_i^k}\|_{\text{op}} = 0.$$

Hence

$$\lim_k \|A(\ell_k) \circ M_{t_i^k, s_k\rho_i^k}\|_{\text{op}} = 0$$

and since also

$$\lim_k \|A(\ell_k) \circ M_{T_k}\|_{\text{op}} = 0$$

it follows that $\lim_k \|A(\ell_k)\|_{\text{op}} = 0$. Hence $I_{N-1} \subset C_0(S_N^1, \mathcal{K}(L^2(\mathbb{R})))$ and so $I_{N-1} = C_0(S_N^1, \mathcal{K}(L^2(\mathbb{R})))$. Finally $\pi_{|I_{N-1}}$ is evaluation in some point $\ell \in S_N^1$ and so $\pi \in \widehat{G_N}$. This finishes the proof of the theorem.

□

Acknowledgements. We would like to thank K. Juschenko for the reference [Gor]. The second author was supported by the Swedish Research Council.

References

[AKLSS] Archbold, R. J.; Kaniuth, E.; Ludwig, J.; Schlichting, G.; Somerset, D. W. B. Strength of convergence in duals of C^*-algebras and nilpotent Lie groups. Adv. Math. **158** (2001), no. 1, 26–65.

[ALS] Archbold, R. J.; Ludwig, J.; Schlichting, G. Limit sets and strengths of convergence for sequences in the duals of thread-like Lie groups. Math. Z. **255** (2007), no. 2, 245–282.

[CG] Corwin, L.J.; Greenleaf, F.P. Representations of nilpotent Lie groups and their applications. Part I. Basic theory and examples. Cambridge Studies in Advanced Mathematics, 18. Cambridge University Press, Cambridge, 1990. viii+269 pp.

[De] Delaroche, C. Extensions des C^*-algébres. (French) Bull. Soc. Math. France Mém., No. 29. Supplément au Bull. Soc. Math. France, Tome 100. Société Mathématique de France, Paris, 1972. 142 pp.

[Di] Dixmier, J. C^*-algebras. Translated from the French by Francis Jellett. North-Holland Mathematical Library, Vol. 15. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. xiii+492 pp.

[Gor] Gorbachev, N. V. C^*-algebra of the Heisenberg group. Uspekhi Mat. Nauk **35** (1980), no. 6(216), 157–158.

[Lee1] Lee, R-Y. On the C^* Algebras of Operator Fields Indiana University Mathematics Journal, **26** No. 2 (1977), 351-372.

[Lee2] Lee, R-Y. Full Algebras of Operator Fields Trivial Except at One Point. Indiana University Mathematics Journal, **25** No. 4 (1976), 303-314.

[LRS] Ludwig,J; Rosenbaum, G.; Samuel, J. The elements of bounded trace in the C^*-algebra of a nilpotent Lie group. Invent. Math. **83** (1985), no. 1, 167–190.

[L] Ludwig, J. On the behaviour of sequences in the dual of a nilpotent Lie group. Math. Ann. **287**, 239-257 (1990).

[R] Rosenberg, J. Homological invariants of extensions of C^*-algebras. Operator algebras and applications, Part 1 (Kingston, Ont., 1980), pp. 35–75, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, RI, 1982.

[W] Wegge-Olsen, N. E. K-theory and C^*-algebras. A friendly approach. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

Jean Ludwig, *Laboratoire LMAM, UMR 7122, Département de Mathématiques, Université Paul Verlaine Metz, Ile de Saulcy, F-57045 Metz cedex 1, France, ludwig@univ-metz.fr*

Lyudmila Turowska, *Department of Mathematics, Chalmers University of Technology and University of Gothenburg, SE-412 96 Göteborg, Sweden, turowska@chalmers.se*