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A Novel Rate Allocation Method for Multilevel Coded Modulation

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***Abstract*—We present a new rate allocation scheme for multilevel coded modulation based on the minimization of the total block error rate (BLER). The proposed method uses affine code components and hard decision multistage decoding. Exhaustive search for the rate allocation which minimizes the total BLER justifies the near-optimum performance of the introduced method in moderate to high SNRs. Compared to previous approaches this new rate allocation scheme can improve the performance of the system by 1 dB at BLER = 10^{-6} for 16-QAM with Ungerboeck set partitioning. Interestingly, our results indicate that the optimum rate allocation is a function of the SNR. Finally, the performance of some specific codes are evaluated by simulation and union bounds to verify the theoretical results.**

I. Introduction

The early work by Imai and Hirakawa [1] introduced a special class of coded modulation called multilevel coded modulation (MLCM), which has attracted a lot of interest in the design of power and bandwidth efficient data transmission systems. According to [2] and [3], the MLCM scheme can be modeled as several parallel binary multiple access layers. Each layer has its own signal constellation with different minimum Euclidean space distance (MESD) [4], [5] and average number of neighbor symbols (neighboring coefficient) [6], [7] of the constellation symbols at the distance of the MESD. Obviously, the MESD and the neighboring coefficient of each layer are determined by the constellation symbols labeling or the set partitioning method.

Due to the large neighboring coefficients degradation in the early layers (see Section II) and the suboptimality of multistage decoding (MSD), despite its asymptotic optimality, MLCM systems have not been as widely adopted in practice [8] as trellis coded modulation [9]. However, the low complexity of MSD is an interesting feature of this technique making it suitable for very high rate data transmission systems such as optical fiber communication.

Since the early design approach of MLCM based on the balanced distance design rule (BDR) [1] does not take into account the effects of neighboring coefficients in the different layers, it shows poor performance in finite block error rate (BLER). Many efforts have been devoted to combating the high error rates due to the large neighboring coefficients of layers with smaller MESD, e.g., iterative decoding [10] and unequal error protection [5]. In [7], the optimum geometrical structure of MLCM for finite bit error rate (BER) was introduced to equalize the BER of different layers and mitigate the large neighboring coefficient degradation in the early decoding stages.

In this paper, a general design method of MLCM systems for hard decision MSD with affine code components, a family of block codes having linear relation between their Hamming distances and their code rates (see section III-B), is proposed by minimizing the total BLER of the system. RS and BCH codes [10] are special cases of affine codes. Moreover, it is shown that the rate allocation in different layers of the MLCM system should depend on the SNR and the neighboring coefficients of its layers.

Although the rate design rules based on the random coding principles such as capacity and cut-off rate design rules [2] are also taking into account the neighboring coefficient, they introduce fixed rate allocation for moderate to high SNRs. In other words, subject to a given total code rate, there is solely one choice for the rate allocation while the proposed method introduces a near-optimum rate allocation as a function of the SNR. In addition, the proposed method has a simpler implementation structure than the coding exponent rule [2] with no limitation on the length of block code.

The advantage of our proposed method to [7] is on the generality of the design, which holds for a large category of codes, and being based on an analytical design procedure. The results of the proposed method are verified by an exhaustive search and for finite BLER (around 10^{-6}), it justifies a near-optimum performance of the new method.

II. System Model

We consider a two dimensional constellation $\mathcal{C}$ with cardinality 2^L to construct an MLCM system consisting of L component binary block codes with the same block length n but different code rates R_i and Hamming distances δ_i for layer i. A two dimensional set partitioning algorithm ($\mathcal{M}$ according to Fig. 1) maps L encoded bits at each time instant to a two dimensional symbol.

In the block diagram of Fig. 1, the DEMUX unit splits the input bit vector U of length k bits into L different

Research supported by the Swedish Research Council under grant 2007-6223.

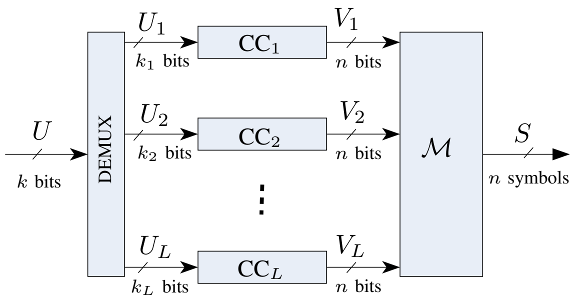

Figure 1. MLCM with L component codes CC_1, . . ., CC_L

vectors U_1,. . .,U_L of lengths k_1,. . .,k_L respectively, where $\sum_{l=1}^{L} k_l = k$. In the following, the component codes CC_1, . . ., CC_L encode these vectors to L row code vectors V_1, . . ., V_L of length n. Finally, the mapping function $\mathcal{M}$ receives the $L\times n$ binary coded block $\mathbb{V} = [V_1^T, \ldots, V_L^T]^T$ and maps each column of $\mathbb{V}$ to a two dimensional symbol. In summary, each input bits vector U with length k is mapped to a symbol vector S which contains n symbols from the constellation $\mathcal{C}$.

The mapping function from the columns of the matrix $\mathbb{V}$ to the two dimensional symbols is based on the Ungerboeck set partitioning algorithm [9]. It should be noted that the achieved results in this paper are independent of constellation labeling. We considered this labeling based on the Ungerboeck set partitioning merely due to the simplicity of deriving the uncoded BER of each layer. We denote the MESD and the neighboring coefficient of the layer i by d_i and A_i, respectively, for $i = 1, \ldots, L$. The neighboring coefficient, A_i [6], [7], determines the average number of neighbor symbols in the constellation of the layer i which have Euclidean distance d_i from a symbol in this constellation.

As an example, for the square 16-QAM, the squared MESDs of layers $1, \ldots, L$ are (d_1^2, d_2^2, d_3^2, d_4^2) = (d_1^2, $2d_1^2$, $4d_1^2$, $8d_1^2$), where $d_1^2 = \frac{2}{5}E_s$ and E_s is the average symbol energy, and the corresponding neighboring coefficients are (A_1, A_2, A_3, A_4) = $(3, 9/4, 2, 1)$.

Affine codes are defined as the family of (n, k) block codes with Hamming distance δ and code rate R for which $\delta = \alpha - \beta R$, where the constants α and β depend on the code length n but not on R or the number of information bits k, e.g., for the (255,k) RS codes $\delta = 256 - 255R$.

The channel model considered in this paper is a discrete-time memoryless AWGN channel with noise variance $N_0/2$ and hard decision MSD at the receiver.

III. Proposed rate allocation for MLCM systems

The neighboring coefficients significantly affect the BLER in the regime of moderate to high SNRs [5]. This is not accounted for by the BDR which is solely based on the MESD of the different layers. For practical SNRs, an optimal rate allocation is required to take into account the impact of neighboring coefficients.

Here, we propose a new rate allocation method based on the Lagrange multiplier minimization (LMM), for an MLCM system with affine code components. This approach is optimized for moderate to high SNR, or practical SNR. By using a union bound approximation for the BLER, which is quite tight for moderate to high SNR, we introduce an analytical approach to derive near-optimum code rates of different layers that will minimize the total BLER of the system.

A. General formulation

We can interpret the rate allocation of the MLCM system as an optimization problem with the constraint that the total code rate should be constant. In other words, one should find R_i, $i = 1, \ldots, L$ that will minimize the total BLER of the MLCM system by taking into account the error propagation among layers. If we suppose an MLCM scheme with a total code rate R, the BLER (a block contains k information bits) of the system is given by

$$P_{\mathrm{e}} = 1 - \prod_{l=1}^{L} (1 - P_l), \tag{1}$$

where P_l is the block error probability of layer l, which contains k_l information bits, conditioned on the fact that there is no error in layers $1, \ldots, l-1$ and $\sum_{l=1}^{L} k_l = k; R = \frac{k}{nL}$. Even though in the derivation of P_ls, no error is considered in the previous layers, P_{e} determines the BLER of the system with error propagation among layers. By some algebraic manipulations, we obtain

$$P_{\mathrm{e}} = \sum_{l=1}^{L} P_l - \sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\neq i}}^{L} P_i P_j + \ldots \tag{2}$$

For moderate to high SNR, P_e can be approximated very well by using only first term of (2). Thus the problem is reduced to minimizing

$$P_{\mathrm{e}} \approx \sum_{l=1}^{L} P_l, \tag{3}$$

subject to the constraint $\frac{1}{L}\sum_{l=1}^{L} R_l = R$, where R is the total code rate of the MLCM system. This problem can be solved using Lagrange multipliers, with the Lagrange functional

$$J(R_1, \ldots, R_L, \lambda) = \sum_{l=1}^{L} P_l + \lambda(\sum_{l=1}^{L} R_l - RL). \tag{4}$$

We change the code rate constraint to a Hamming distance constraint, to exploit the P_l expression as a function of the Hamming distance δ_l of the component code l by the union bound. The BLER for layer $l = 1, \ldots, L$ of an MLCM system with hard decision MSD can be approximated based on the union bound

$$P_l \leq \sum_{i=t_l+1}^{n} \binom{n}{i} p_l^i (1 - p_l)^{n-i}, \tag{5}$$

where p_l is the uncoded bit error probability of layer l and t_l is the correcting capability of component code l.

Moreover, by approximating (5) with its dominant term $i = t_l + 1$ (see [11, p. 435]), we obtain

$$P_l \approx \exp\left(f(t_l+1) + n\ln(1-p_l) + (t_l+1)\ln\frac{p_l}{1-p_l}\right), \tag{6}$$

where $f(t) = \ln\binom{n}{t}$ and ln is the natural logarithm function. To be able to solve the Lagrange multiplier approach analytically, we approximate $f(t)$ with a quadratic function

$$f(t) \approx a_2 t^2 + a_1 t + a_0, \tag{7}$$

where a_2, a_1, a_0 for each n are fitted to $f(t)$ for $0 \le t \le \min\{T, \lfloor\frac{n-1}{2}\rfloor\}$ in a least squares sense, where T is the constraint on the sum of the correcting capabilities of the components codes.

For example, as Wachsmann et al. in [2] exploited the Gilbert-Varshamov bound with equality [11, p. 444] to compute the rate distribution of the BDR, we can use this well-known bound for linear block codes with equality to yield the Lagrange functional

$$J(R_1,\ldots,R_L,\lambda) = \sum_{l=1}^{L} P_l + \lambda\left(L(1-R) - \sum_{l=1}^{L} H_b(\delta_l/n)\right), \tag{8}$$

where H_b denotes the binary entropy function. However, we should emphasize that, due to the nonlinear function H_b, minimization of the Lagrange multiplier in (8) is analytically difficult.

B. Affine codes

In this section, we select the component codes from an affine code family. These codes, including RS and BCH codes, are far more useful in practice than Gilbert-Varshamov-achieving codes (of which none are known for large block lengths n), and we can use the linear property of these codes to optimize the Lagrange multiplier (4) analytically. By replacing the correcting capability t_l of the affine code in layer l with its code rate R_l, we obtain

$$t_l = \lfloor\frac{\delta_l - 1}{2}\rfloor \approx \frac{1}{2}(\alpha + \beta R_l - 1). \tag{9}$$

According to the definition of affine codes and using the MLCM model as described in section II, all the layers have the same α and β. By changing the constraint on the total code rate R into the sum of correcting capability of affine codes, we obtain

$$\sum_{l=1}^{L} t_l = T = \frac{L}{2}(\alpha + \beta R - 1). \tag{10}$$

Eventually, we have the following Lagrange functional

$$J(t_1,\ldots,t_L,\lambda) = \sum_{l=1}^{L} P_l + \lambda(\sum_{l=1}^{L} t_l - T). \tag{11}$$

Since according to (5)–(6), P_l is a convex function of t_l, we can assume real values for t_l and then apply the Lagrange multiplier approach to minimize the separable function $\sum_{l=1}^{L} P_l$ [12, p. 248]. After optimizing the Lagrange multiplier, we round the real values to the nearest integer to compute the correcting capabilities of the MLCM layers.

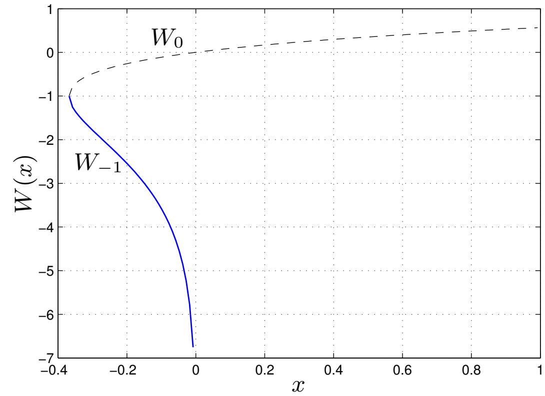

Figure 2. LambertW function.

Substituting (7) into (6) and (6) into (11), then differentiating J with respect to $t_l + 1$ and simplifying, we obtain the optimum code correcting abilities as

$$t_l = \sqrt{\frac{\mathrm{W}\left(\lambda^2 z_l\right)}{2a_2}} - \frac{\gamma_l}{2a_2} - 1 \quad ; \quad 0 \le l < L. \tag{12}$$

However, since t_l must be nonnegative, the Kuhn-Tucker conditions should also be taken into account in finding the optimum solution [13, p. 276], i.e., if t_l turns out to be negative, it should be reset to 0. In the following, by using equations (12) and (10), a simple equation for the multiplier λ is obtained as

$$\sum_{l=1}^{L}\left(\sqrt{\frac{\mathrm{W}\left(\lambda^2 z_l\right)}{2a_2}} - \frac{\gamma_l}{2a_2}\right) - T = 0, \tag{13}$$

where

$$z_l = -\frac{1}{2a_2}\exp\left(\frac{1}{2a_2}\gamma_l^2 - 2\left(a_0 + n\ln(1-p_l)\right)\right) \tag{14}$$

and

$$\gamma_l = a_1 + \ln\frac{p_l}{1-p_l}. \tag{15}$$

In (12) and (13), W is the LambertW function [14], which is defined to be the inverse of the function $h(W) = We^W$(where e^W is the natural exponential function). This function is a multivalued function with two possible real value branches, as shown in Fig. 2. W_0 is called the principal branch of the LambertW function and the other branch, satisfying $W \le -1$ and denoted by W_{-1}, is useful in our context.

Since the function $f(t)$ is a concave function, the second derivative of its approximation, a_2, is negative. In addition, it is readily seen that z_l is always negative and hence $W\left(\lambda^2 z_l\right)/(2a_2)$ will always be positive. Using an accurate approximation of W_{-1} [15], equation (13) can easily be

Table I
RS CODE CORRECTING CAPABILITIES AND CODE RATES FOR AN MLCM WITH HARD DECISION MSD, $\rho_b = 10\, dB$, TOTAL CODE RATE $R = 0.924$, AND 8-PSK CONSTELLATION.

Rate allocation method	$\{t_1, \ldots, t_3\}$	$\{R_1, \ldots, R_3\}$
BDR	$\{17, 9, 3\}$	$\{0.87, 0.93, 0.97\}$
CDR	$\{29, 0, 0\}$	$\{0.77, 1, 1\}$
LMM	$\{28, 1, 0\}$	$\{0.78, 0.99, 1\}$

Table II
RS CODE CORRECTING CAPABILITIES AND CODE RATES FOR AN MLCM WITH HARD DECISION MSD, $\rho_b = 10.5\, dB$, TOTAL CODE RATE $R = 0.929$, AND 16-QAM CONSTELLATION.

Rate allocation method	$\{t_1, \ldots, t_4\}$	$\{R_1, \ldots, R_4\}$
BDR	$\{19, 10, 5, 2\}$	$\{0.86, 0.92, 0.96, 0.98\}$
CDR	$\{33, 3, 0, 0\}$	$\{0.74, 0.98, 1, 1\}$
LMM	$\{31, 4, 1, 0\}$	$\{0.75, 0.97, 0.99, 1\}$

solved by standard numerical methods. Finally, we obtain the correcting capabilities of different layers by rounding the real values of the Lagrange multipliers optimization results in (12) to the nearest integer values.

It is seen in (12) that the rate allocation depends on the SNR and the neighboring coefficients. It is worth mentioning that even though the derivations of (12) and (13) are a bit complicated, these equations provide a straightforward way for computing the near-optimum rate allocation of the MLCM system.

Tables I and II show the results of the rate allocation based on the total BLER minimization for hard decision MSD by Lagrange multipliers compared with the BDR and capacity design rule (CDR) [2] results. The component codes are selected from t_i-error correcting RS component codes over GF(2^8), the Galois field with 2^8 elements.

IV. SIMULATION RESULTS

In this section, we compare the performance of the new proposed rate allocation LMM from equations (12) and (13) with the BDR [1] and the CDR. We ignored the coding exponent design rule [2] due to its high complexity, particularly when one has to consider the error propagation among the layers in moderate SNR. Since the derived code rates from the cut-off rate rule [2] are very similar to those derived from the CDR, we ignored this rate allocation method as well. Finally, the equal error probability rule [2] was also ignored, since there is no well presented algorithm for that scheme, rather than brute force search.

RS component codes over GF(2^8) are selected in the simulation of two systems with 8-PSK and 16-QAM signal constellations and Ungerboeck set partitioning. All the three methods compute the rate allocation subject to the total code rate RL constraint. The rate allocation is done only once for

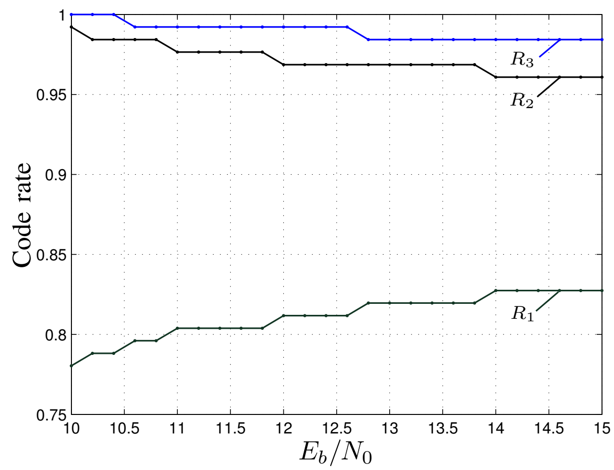

Figure 3. Rate allocation using LMM method for an MLCM scheme with 8-PSK and constraint total code rate of R = 0.924.

the BDR and the CDR, while in the LMM, it is performed for each SNR. For example at ρ_b =10.25 dB, the code rate vector {0.74, 0.98, 1, 1} minimizes the BLER of MLCM with 16-QAM constellation, while at ρ_b = 11 dB, it is changed to {0.77, 0.96, 0.99, 1}. Figures 3 and 4 show the rate allocation of two MLCM systems versus SNR$_b$. It is known that the BDR maximizes the MESD of MLCM scheme in asymptotic SNRs [4, p. 225], therefore the BDR will minimize the BLER in asymptotic SNRs. Since the LMM minimizes the BLER as well, the rate distribution of LMM will converge to the BDR rate allocation in asymptotically high SNRs, as seen in tables I, II and Fig. 3, 4.

The rate allocation based on the CDR is done by assuming each layer of the MLCM system to be a binary symmetric channel (BSC) with probability p_l which can be computed exactly for 8-PSK with three layers and 16-QAM in four layers. The capacity of each BSC channel is obtained by $C = 1 - H_b(p_l)$ using exact values of p_l. Figure 5 shows the performance comparison for the MLCM system with 8-PSK constellation and total code rate R = 0.924 and Fig. 6 for the MLCM system with 16-QAM constellation and R = 0.929.

As seen in Fig. 5 and 6, the performance of the LMM has 0.75 to 1 dB gain over previous methods for the MLCM system at a BLER around 10^{-6}. The approximation curves (dashed lines) in Fig. 5 and 6 are based on (6) using the derived rate allocation from each method. The solid curves in Fig. 5 and 6 show the exact BLER

$$P_l = \sum_{j=t_l+1}^{n} P(W_j) = \sum_{j=t_l+1}^{n} \sum_{i=j}^{8j} P(W_j \mid w_i) P(w_i), \quad (16)$$

where w_i and W_j denote the events corresponding to i binary bit errors and j nonbinary symbol errors in a received codeword, respectively. We derived the exact BLER by substituting equations (18) and (19) from [16] into (16).

Finally, the optimality of the proposed method has been

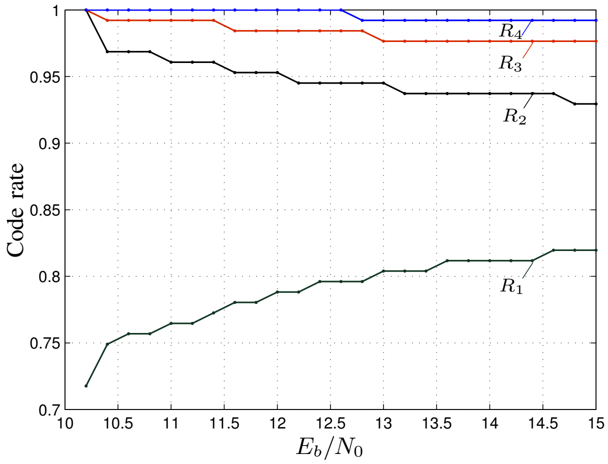

Figure 4. Rate allocation using LMM method for an MLCM scheme with 16-QAM and constraint total code rate of $R = 0.929$.

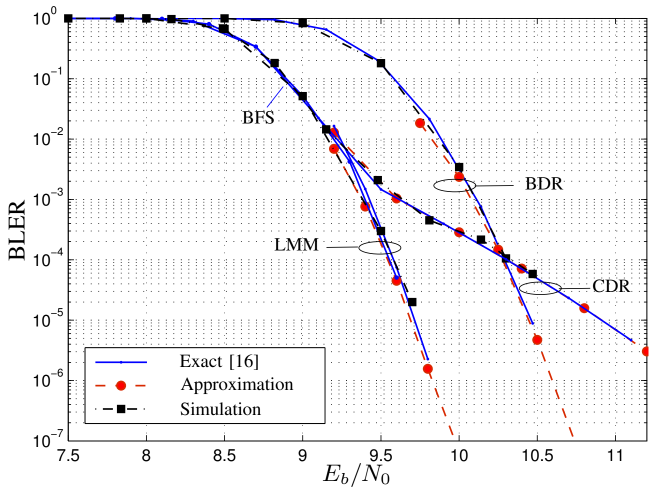

Figure 5. Performance comparison of an MLCM system for three different rate allocation methods with 8-PSK constellation.

checked by brute force search method (BFS) for the optimum rate allocation to attain minimum total BLER (1) of the MLCM system. As can be seen, the results are the same as with LMM for moderate to high SNR, at a much higher complexity. Even though the BLER curve of the BDR will theoretically converge to the LMM BLER curve at asymptotically high SNR, LMM shows a considerable performance improvement for practical SNRs.

V. Conclusion

A pragmatic rate allocation method was proposed for the design of MLCM with different constellations. The proposed method is simpler to implement than, e.g., the capacity design rule, while it shows up to 1 dB performance improvement in comparison to previous known methods for moderate to high SNRs. In the new method, the rate allocation accounts for neighboring coefficients, MESD, and SNR.

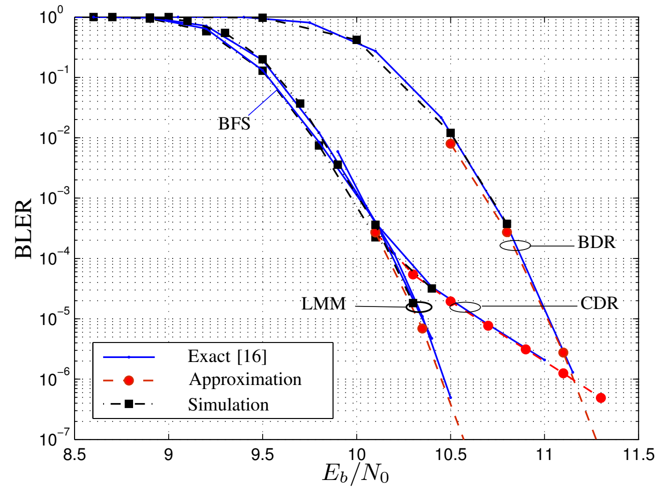

Figure 6. Performance comparison of an MLCM system for three different rate allocation methods with 16-QAM constellation.

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