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Four-Dimensional Optimized Constellations for Coherent Optical Transmission Systems

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Abstract *We investigate power-efficient modulation formats in the four-dimensional signal space that is used by most coherent transmission systems. The sensitivity and spectral efficiency trade-off is discussed, with and without forward error correction.*

Introduction

An electromagnetic wave has four degrees of freedom (two quadratures in two polarization components), and contemporary coherent transmisson systems modulate data in all four[1,2,3,4]. Even if one might use complex two-vectors (e.g., Jones vectors) to describe the data, communication-centric concepts like minimum distance, neighboring points, and maximum likelihood are, with some exceptions[5,6], rarely used in the analysis of optical transmission systems, which is why we find it worthwhile to adopt such an approach. For example, we recently[7,8] used four-dimensional (4-d) sphere-packing optimization to find power-efficient modulation formats, which are formats having better sensitivities than, e.g., binary phase-shift keying (BPSK) or its 4-d analogy, dual-polarization quadrature phase-shift keying (DP-QPSK). Such formats can be of importance in linear systems limited by additive noise, as well as in nonlinearly limited system where the power must be reduced to avoid nonlinear distortions. In this paper, we describe the system model in more detail, including a previously unpublished account for how Jones matrices are related to 4-d rotations. Then we will review some results on power-efficient 4-d modulation formats and describe how they relate to sphere packings. After describing a few promising formats and their performance in terms of sensitivity, with and without forward error correction (FEC), we end by summarizing the main results and providing some conclusions.

Four-dimensional description

We here briefly describe the system model for coherent fiber optical systems. As mentioned in the introduction, the electromagnetic field has four degrees of freedom, which span a 4-d signal space. The electric field of the optical wave can be written as the complex 2-component vector

$$\vec{E} = \begin{pmatrix} E_{x,r} + iE_{x,i} \\ E_{y,r} + iE_{y,i} \end{pmatrix} = \begin{pmatrix} |E_x| \exp(i\varphi_x) \\ |E_y| \exp(i\varphi_y) \end{pmatrix}, \qquad (1)$$

where subscripts x and y denote the polarization components and subscripts r and i denote the real and imaginary parts, respectively, of the field. The coordinate directions x and y are orthogonal to the propagation direction z. The transmission of this field through a linear optical system can be described in the frequency domain by a 2x2 complex transfer matrix J, as $\vec{E}_{\text{out}} = J\vec{E}_{\text{in}}$. In polarization optics, J is known as the *Jones matrix* of the system. In the absence of polarization-dependent losses (PDL), which is a reasonable assumption for most system components, J is a unitary matrix, i.e., its inverse J^{-1} equals its conjugate transpose J^{+}. Any unitary 2x2 matrix may be parametrized[9] by the matrix exponential as $J = \exp(ig_1)\exp(iH)$, where g_1 is an arbitrary real phase angle and

$$H = \begin{pmatrix} h_1 & h_2 - ih_3 \\ h_2 + ih_3 & -h_1 \end{pmatrix} \qquad (2)$$

for arbitrary real numbers h_1, h_2, h_3. Using $h = \sqrt{h_1^2 + h_2^2 + h_3^2}$ we may derive the useful formula $J = \exp(ig_1)[I\cos(h) + iH\sin(h)/h]$, where I is the 2x2 identity matrix.

Alternatively, we can express the signal as a real-valued 4-d vector

$$\vec{s} = \begin{pmatrix} E_{x,r} \\ E_{x,i} \\ E_{y,r} \\ E_{y,i} \end{pmatrix}, \qquad (3)$$

which is transformed by a 4x4 real matrix T according to $\vec{s}_{\text{out}} = T\vec{s}_{\text{in}}$. The absence of PDL corresponds in this case to T being orthogonal, i.e., $T^{-1} = T^{t}$, so that T is a 4-d rotation matrix. Such rotations have in general 6 degrees of freedom, which can be expressed by using the matrix exponential $T = \exp(M)$, where M equals the skew-symmetric matrix

$$\begin{pmatrix} 0 & -g_1 - h_1 & g_3 + h_3 & g_2 - h_2 \\ g_1 + h_1 & 0 & g_2 + h_2 & -g_3 + h_3 \\ -g_3 - h_3 & -g_2 - h_2 & 0 & -g_1 + h_1 \\ -g_2 + h_2 & g_3 - h_3 & g_1 - h_1 & 0 \end{pmatrix}. \tag{4}$$

By writing M as $M = M_L + M_R$, where M_L depends only on g_1, g_2, g_3 and M_R on h_1, h_2, h_3, one may show that

$$T = \exp(M_L)\exp(M_R) = T_L T_R = T_R T_L, \tag{5}$$

which means that each rotation T can be decomposed into two commuting subgroups with three degrees of freedom each. These are usually called *left-* and *right-isoclinic* rotations[10], which explains their indices. Polar decomposition formulas exist also here, so that, e.g., $T_R = I\cos(h) + M_R \sin(h)/h$, where I denotes the 4x4 identity matrix. A similar formula exists for T_L.

Now, relating this back to the complex polarization transformations $\vec{E}_{\text{out}} = J\vec{E}_{\text{in}}$, we see that those are given by the right-isoclinic rotations T_R. The left-isoclinic ones correspond to transformations of the form $\vec{E}_{\text{out}} = J\vec{E}^*_{\text{in}}$, which are of less relevance in optical transmission systems, apart from complex multiplication with a phase angle, which is modeled by the parameter g_1 alone. Thus, an arbitrary Jones matrix $J = \exp(ig_1)\exp(iH)$ with H described by (2) is equivalent to the 4-d transfer matrix $T = \exp(M)$, with M given by (4) using $g_2 = g_3 = 0$. The transmitted optical signal power is $P = \|\vec{s}\|^2 = \|\vec{E}\|^2 = E^2_{x,r} + E^2_{x,i} + E^2_{y,r} + E^2_{y,i}$.

We believe this is the first time that Jones matrices are explicitly related to a subset of 4-d rotation matrices, although 4-d transfer matrices have been described[5] earlier. Finally, we emphasize that the four-dimensional vector $\vec{s}$ should not be confused with the Stokes vector description of polarization states, which is defined in a completely different way and proportional to the intensity rather than being linear in the field.

System model

In general, the vector (3) varies continuously with time. For the purpose of digital communications, $\vec{s}(t)$ is designed to transmit a sequence of information symbols $(\vec{s}_0, \vec{s}_1, \vec{s}_2, ...)$, one every T_s seconds. The symbol $\vec{s}_n$ is taken from a finite set, or *constellation,* $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_M\}$ of N-dimensional vectors. In this article, we consider $N = 2$ or 4. Assuming that all constellation vectors are equally likely, $\log_2 M$ information bits are transmitted every T_s seconds, yielding an information bit rate of $R_B = (\log_2 M)/T_s$ bits/s.

In the continuous system model, the symbol sequence $(\vec{s}_0, \vec{s}_1, \vec{s}_2, ...)$ is modulated into a continuous waveform. The waveform is subject to additive white Gaussian noise (AWGN) with double-sided spectral density $N_0/2$, which in most cases of relevance stems from amplified spontaneous emission noise from optical amplifiers. The receiver then demodulates and samples the waveform to a set of *received vectors* $(\vec{r}_0, \vec{r}_1, \vec{r}_2, ...)$. It can be shown that under some conditions on the modulator and demodulator[7], $\vec{r}_n = \vec{s}_n + \vec{z}_n$, where $\vec{s}_n$ are the transmitted symbols and $\vec{z}_n$ are independent, Gaussian random vectors with variance $N_0/2$ in each dimension. This equation is a *discrete-time channel model,* which includes modulation, transmission, and demodulation. The average of $\|\vec{s}_n\|^2$ equals the *average energy per symbol*

$$E_s = \frac{1}{M}\sum_{k=1}^{M} \|\vec{c}_k\|^2 = PT_s, \tag{6}$$

while the average of $\|\vec{z}_n\|^2$ equals $NN_0/2$.

In the AWGN model, the received vector $\vec{r}_n$ has an isotropic distribution around $\vec{s}_n$ in an N-dimensional space, and for a maximum-likelihood receiver, the symbol decision is based on which signal in the constellation set is closest (in the Euclidian sense) to the received vector.

A simple and useful approximation to the symbol error rate (SER) is the *union bound* which can be expressed as [11, p. 191]

$$SER \leq \frac{1}{M}\sum_{k=1}^{M}\sum_{\substack{j=1\\ j\neq k}}^{M} \frac{1}{2}\,\text{erfc}\left(\frac{d_{kj}}{2\sqrt{N_0}}\right), \tag{7}$$

where erfc denotes the complementary error function and $d_{kj} = \|\vec{c}_k - \vec{c}_j\|$ is the Euclidean distance between the symbols. In the limit of high SNR, this expression can be further simplified to

$$SER \approx \frac{M_{\min}}{M}\,\text{erfc}\Big(\sqrt{\gamma \frac{E_b}{N_0}}\Big), \tag{8}$$

where $E_b/N_0 = E_s/(N_0 \log_2 M)$ is the common measure of signal-to-noise ratio (SNR) and $M_{\min}$ is the number of symbol pairs whose distance equals the *minimum distance* $d_{\min} = \min_{j\neq k}\{d_{kj}\}$ of the constellation. The asymptotic power efficiency is $\gamma = d^2_{\min}/4E_b$ and gives a measure of how well a given constellation trades $d_{\min}$ for $E_b = E_s/\log_2 M$. This is a purely geometrical property of the constellation, and a judicious selection of signaling levels $\vec{c}_k$ that minimize the average energy per symbol E_s without decreasing $d_{\min}$ is crucial for a modulation format to perform well. Such a selection is equivalent to the problem of packing M N-dimensional spheres with diameters $d_{\min}$ and centers at $\vec{c}_k$, so that E_s (which is equal to the average second moment of $\vec{c}_k$) is minimized. Another interpretation of γ is as the *sensitivity gain* over BPSK to transmit the same data rate, since γ = 0 dB for BPSK, QPSK, and DP-QPSK.

Most common modulation formats have a penalty with respect to BPSK; for example, for M-ary PSK and M-ary quadrature amplitude modulation (QAM) we have [11, pp. 226, 234]

$$\gamma_{\text{M-PSK}} = \sin^2(\pi/M)\log_2 M, \tag{9}$$

$$\gamma_{\text{M-QAM}} = \frac{3\log_2 M}{2(M-1)}, \tag{10}$$

where (10) is valid for M being a power of 4. We can show from these expressions that both M-PSK and M-QAM have efficiencies $\gamma \leq 0$ dB for all values of M (with the notable exception of 3-PSK, which will be discussed in the next section).

Two- and four-dimensional sphere packings

By dynamically simulating how clusters of equal-radius, hard spheres with random initial positions relax under suitably chosen attractive forces, one can find the constellation that hypothetically minimizes E_s for each given value of M and N. We denote such constellations with $\mathcal{C}_{N,M}$, and they are of fundamental interest as they give the best possible sensitivity for N-dimensional M-ary transmission in the high-SNR limit.

The first investigation on the trade-offs between dimensionality N, constellation size M, and SER was done[12] by Shannon. His objective, which was capacity-approaching coded systems, is slightly different from ours, which mainly focuses on uncoded, low-dimensional systems.

For uncoded transmission, the planar case (N = 2) was investigated[13] for some selected values of $M \leq 16$ and later[14] for all $M \leq 500$ (whereof only the constellations for $M \leq 100$ are conjectured to be optimal). The best constellations are empirically found to be hexagonal packings. For N = 3 and 4, on the other hand, the best found constellations[15,16] have very different structures depending on M. For the most regular of these constellations, exact SER expressions can be derived[7,17]. Some of these are plotted together with union bounds in Fig. 1, which indicates that the union bounds are very accurate for $SER < 10^{-3}$.

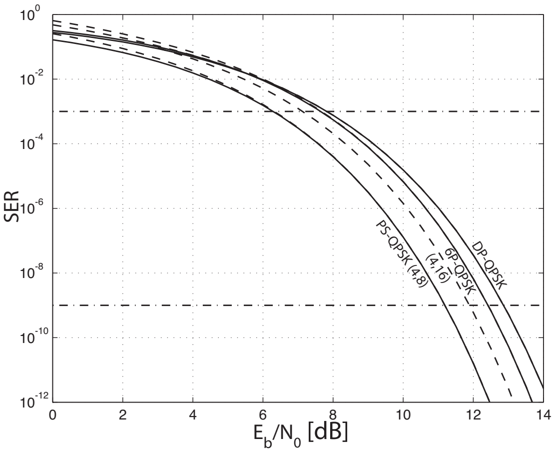

Fig. 1: SER vs. SNR for DP-QPSK, 6P-QPSK, PS-QPSK, and $\mathcal{C}_{4,16}$. Union bounds are shown as dashed lines and exact expressions as solid lines. No exact SER expression is known for $\mathcal{C}_{4,16}$ (denoted (4,16)).

The asymptotic efficiencies γ for these formats have been discussed[7] earlier, so here we will instead present absolute sensitivities for given symbol error rates, obtained by using the union bound (7). The sensitivities vs. spectral efficiencies for $\mathcal{C}_{2,M}$ and $\mathcal{C}_{4,M}$ are plotted in Fig. 2, for SER equal to 10^{-3} and 10^{-9} (cf. the dash-dotted lines in Fig. 1). We define the spectral efficiency, *SE*, as the number of transmitted bits per symbol and polarization, where each polarization represents a dimension pair, i.e.,

$$SE = \frac{\log_2 M}{N/2} \text{ [bits/(symbol} \cdot \text{polarization)]}. \tag{11}$$

With this definition, BPSK, QPSK, and DP-QPSK all have the same spectral efficiency of 2 bits per symbol and polarization, and they coincide at the point (2,4) in Fig. 2. Thus, QPSK is the 2-d, 4-point constellation requiring the least SNR to achieve a given, low, SER. The fact that DP-QPSK (and more generally any N-dimensional cubic constellation) lies at the same point in the chart is because it represents essentially two parallel, independent QPSK channels. The most power-efficient 2-d format is 3-PSK[19], being $3\log_2(3)/4$ = 0.75 dB better than QPSK asymptotically. In four dimensions, the most efficient format is (in the low-SER limit) $\mathcal{C}_{4,8}$, which is 3/2=1.76 dB better than QPSK.

In the following section, we describe some of the best known 4-d constellations $\mathcal{C}_{4,M}$. A more detailed account is to appear[18], including, for the first time, coordinate representations for many of the constellations, as well as an extension to constellations optimized with respect to maximum, rather than average symbol energy.

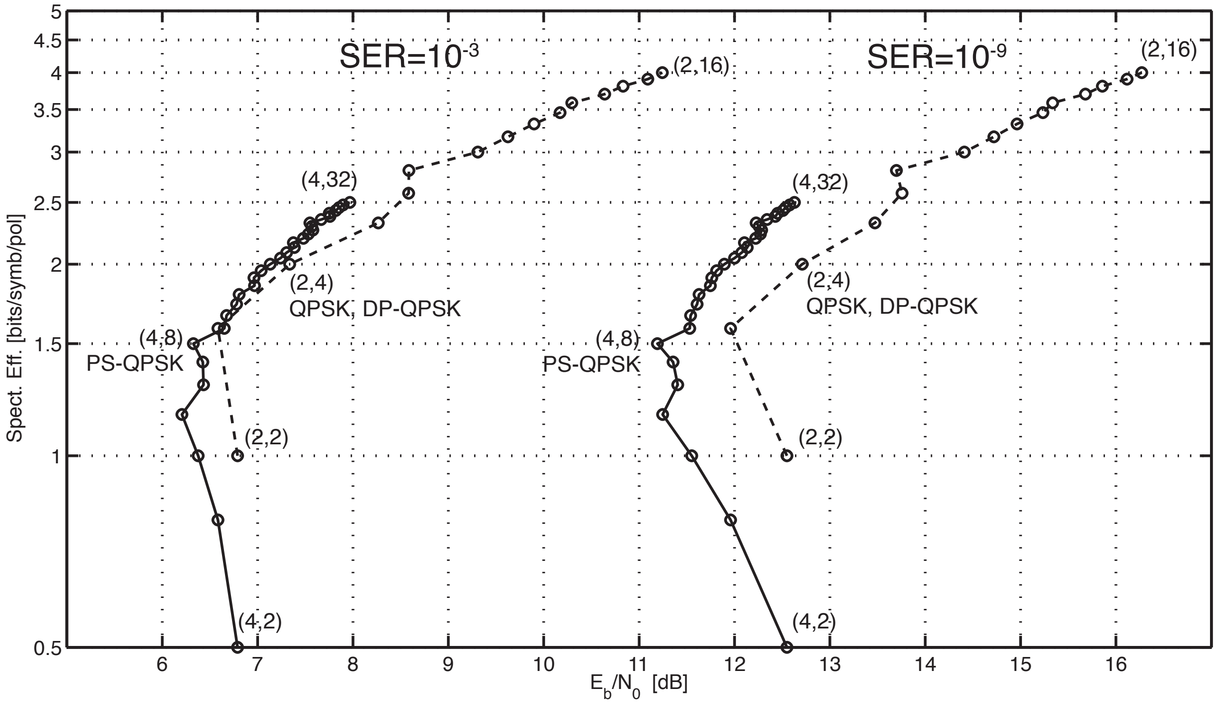

Fig. 2: Chart over the sensitivity vs. spectral efficiency of the best known M-ary, N-dimensional constellations at an SER of 10^{-3} (left curves) and 10^{-9} (right curves). The boundary cases and some selected formats are indicated with (N, M), and the lines connecting the points (solid for N = 4, dashed for N = 2) are included as a guide to the eye.

Specific 4-d formats

The $\mathcal{C}_{4,5}$ constellation is the simplex[7,20,21,22], which is quite power efficient; 1.62 dB better than QPSK, but its 5 levels make the bit-to-symbol mapping difficult. The $\mathcal{C}_{4,8}$ constellation is known as the *cross-polytope*, illustrated in Fig. 3(a). This modulation format has been referred to as polarization-switched QPSK (PS-QPSK)[8], since it essentially transmits QPSK in either of two polarizations but not in both simultaneously. It has 1.76 dB improved sensitivity over QPSK. The best 10-level constellation, $\mathcal{C}_{4,10}$, is the *rectified simplex*, formed by the midpoints between adjacent vertices in the 4-d simplex. It has an improvement over QPSK of 1.41 dB and was found[23] already in 1963.

The common DP-QPSK format is a 4-d cube with 16 levels (see Fig. 3(b)), but as we will see this is far from the best way of packing 16 4-d spheres. The union of the 4-d cube and the cross-polytope, scaled to the same E_s, is a regular polytope with 24 vertices known as the *24-cell*, visualized in Fig. 3(c). This modulation format has been studied in a general communications context[21,22] and for optical systems[7,24], including a bit-to-symbol mapping. It can be seen as QPSK transmitted in 6 different polarization states, and we refer to it as 6P-QPSK. These 24 points, plus the origin, form the $\mathcal{C}_{4,25}$ constellation.

The 4-d cube is not the optimum way of packing 16 spheres in four dimensions. The best known constellation $\mathcal{C}_{4,16}$ is an intriguing structure that can be described as a single point, a 3-d octahedron, a 3-d cube, and another single point layered along one coordinate, say, x_1. Its coordinate representation is

$$\begin{aligned}\mathcal{C}_{4,16} = \{&(a+\sqrt{2},0,0,0),(a,\pm\sqrt{2},0,0),\\ &(a,0,\pm\sqrt{2},0),(a,0,0,\pm\sqrt{2}),\\ &(a-c,\pm1,\pm1,\pm1),(a-c-1,0,0,0)\}\end{aligned} \tag{12}$$

with all combinations of signs, where $a = (1-\sqrt{2}+9c)/16$ and $c = \sqrt{2\sqrt{2}-1}$. Two projections of this structure are shown in Fig. 4. It has a 1.11 dB asymptotic gain over DP-QPSK.

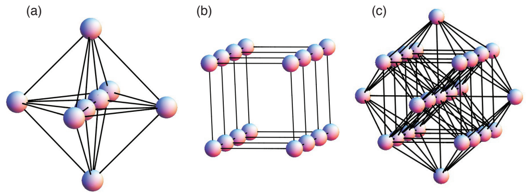

Fig. 3: Collinear projections of the crosspolytope $\mathcal{C}_{4,8}$ (a), the 4-d cube (b) and the extended 24-cell, $\mathcal{C}_{4,25}$ (c) showing how the latter is formed as a union by the two former, plus a point at the origin. The lines connect nearest neighbors and have equal length in 4-d space.

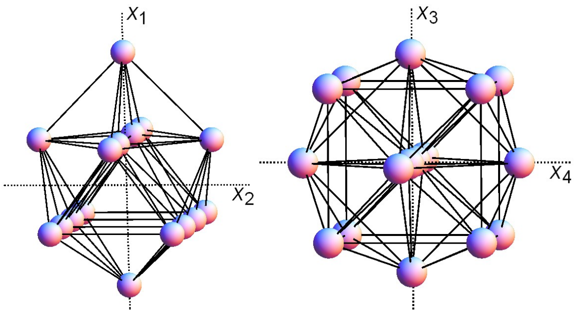

Fig. 4: The 4-d constellation $\mathcal{C}_{4,16}$ in two different projections, the left one (almost) perpendicularly to x_3 and x_4, showing the layered structure along x_1, and the right one (almost) perpendicularly to x_1 and x_2, showing the cubic/octahedral symmetry.

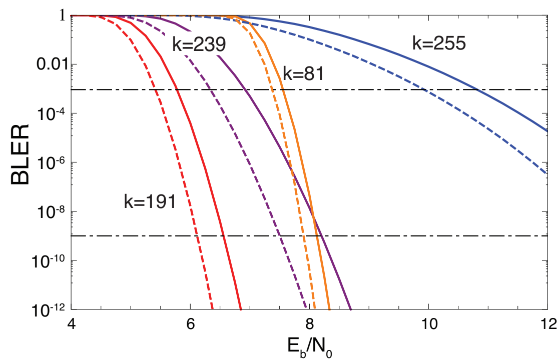

Fig. 5: Block error rate (BLER) for DP-QPSK (solid) and $\mathcal{C}_{4,16}$ (dashed) when using a (255, k) RS code.

Influence of error correction coding

Most future optical communication systems will use some kind of FEC coding together with multilevel modulation to reach both good spectral efficiency and good sensitivity. It is therefore of interest to compare the performance of the 4-d constellations not only in an uncoded system but also in the presence of FEC. As an instructive example, we compare $\mathcal{C}_{4,16}$ and DP-QPSK, which have the same number of points and hence the same spectral efficiency, in a system with Reed–Solomon (RS) coding. This family of codes (and in particular the (255,239) RS code) are often used in long-haul optical systems.

The coded system model is defined as follows. A sequence of information bits is partitioned into blocks of $8k$ bits, for any integer $1 \leq k \leq n$, where we choose $n = 255$. Each block is interpreted as k 256-ary symbols, which is fed into the encoder of an (n, k) RS code. The code rate is $R_C = k/n$. The output of the encoder consists of n 256-ary symbols, which are each mapped onto a pair of points of a 16-point constellation. After transmission over a noisy channel, a maximum-likelihood detector estimates most likely pair of points, i.e., the most likely 256-ary symbol. The RS decoder, which operates on blocks of n such estimates, is assumed to perform hard-decision bounded-distance decoding, in the sense that each block with $t = (n-k)/2$ or fewer symbol errors are corrected, while blocks with more than t errors remain incorrect.

With this system model, it can be shown that the block error rate (BLER) is exactly

$$BLER = \sum_{i=t+1}^{n} \binom{n}{i} p^i (2-p)^i (1-p)^{2n-2i}, \qquad (13)$$

where p is the raw (uncoded) SER of the 16-point constellation, plotted in Fig. 1. The BLER of four RS codes shown is shown in Fig. 5, where $k = 255$ corresponds to uncoded transmission of 510 16-ary symbols. Evidently, RS coding provides large gains even at a high code rate, which is why the (255,239) RS code is so popular in practical optical communication systems. The sensitivity can be further improved by reducing the rate, because the Hamming distance of the code, which is equal to $n + 1 - k$, increases. However, if the rate gets below a certain threshold, the sensitivity increases again, as exemplified for $k = 81$. The reason is that for a given bit energy E_b, the average symbol energy $E_s = R_C E_b \log_2 M$ decreases with decreasing k. The sensitivity is optimized when the increasing Hamming distance and the decreasing symbol energy balance each other. At asymptotically low BLER (high SNR), this occurs at $R_C = 1/2$, but, as we shall see in the next figure, the threshold is about 0.75 in the regime of practical interest.

In Fig. 6, the SE is shown vs. the sensitivity at a BLER of 10^{-9}, which corresponds to the lower dash-dotted line in Fig. 5. The uncoded 4-d constellations from Fig. 2 are included for reference, altough it is somewhat unfair to compare SER (4 bits) with BLER ($8k$ information bits or 2040 coded bits). It is clearly seen that RS codes with $R_C \lesssim 0.75$ waste both power and bandwidth. Noteworthy is also that the sensitivity gain in going from DP-QPSK to $\mathcal{C}_{4,16}$ is less in a coded system than in an uncoded, and it decreases with the code rate.

A natural question is how much better we can perform with more complex codes, such as low-density parity-check (LDPC) codes or other so-called capacity-achieving codes. It is well known[6] that if there is no constraint on the block length or complexity, reliable communication is possible at an SE equal to the *mutual information* between the channel's input and output. The mutual information, constrained to DP-QPSK and $\mathcal{C}_{4,16}$ with equally

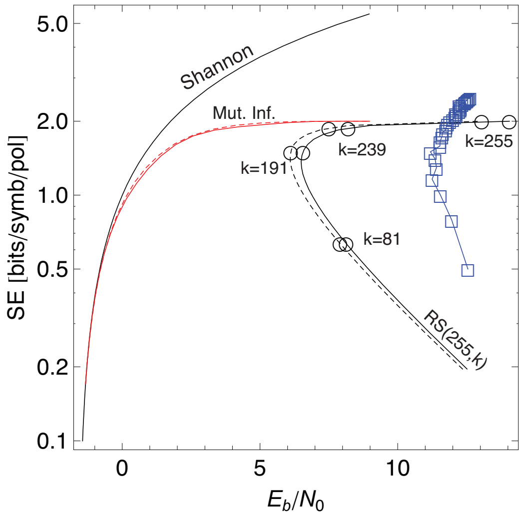

Fig. 6: The SE vs. sensitivity (at a BLER of 10^{-9}) for the DP-QPSK (solid) and $\mathcal{C}_{4,16}$ (dashed) formats with (255, k) RS codes for various values of k. Some specific values of k are shown as circles. The uncoded sensitivity (for $SER = 10^{-9}$) of the 4-d constellations, from Fig. 2, is shown as squares. The mutual information indicates the best possible performance with capacity-achieving codes.

likely input symbols, is also included in Fig. 6, as is Shannon's AWGN *channel capacity* $E_b/N_0 = (2^{SE}-1)/SE$, which is the supremum of the mutual information over all constellations and input probability distributions. We conclude that with capacity-achieving codes, the performance difference between the two constellations is negligible.

Summary and conclusions

We analyzed coherent optical transmission in a 4-d signal space and presented how the conventional Jones transfer matrices for optical systems can be viewed as a subset of 4-d rotations. Some 4-d modulation formats optimized for low sensitivity in the high-SNR limit were presented and compared. Sensitivity gains of up to 1.76 dB over independent BPSK transmission in each dimension can be obtained at high SNR, at the expense of higher modulation and demodulation complexity. In coded systems, the gains are in general less.

A future aspect that remains to be investigated is the nonlinear robustness of these power-efficient formats. Their reduced power requirements may lead to improved nonlinear tolerance. For example, cross-phase modulation between wavelength channels might be reduced by using these formats.

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