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A Low-Complexity Semi-Analytical Approximation to the Block Error Rate in Nakagami-m Block Fading Channels

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Abstract— **There are few analytical formulas that can be used for calculating the block error rate (BLER) in block fading channels. Thus, an estimate of the BLER is often obtained using numerical methods. One such method is the threshold method which assigns 0 or 1 to the instantaneous BLER given the signal to noise ratio (SNR) level. It has been shown that utilizing such a method results in an accurate approximation of the BLER in Nakagami-m block fading channels for a wide range of m.**

In this work, we consider a recently proposed simple method of obtaining the threshold and study the effect of adopting different physical layer and channel parameters on that threshold. We show that, while the value of this threshold depends on the modulation, coding, and block size, it is almost unaffected by the m parameter of Nakagami-m channels for a wide range of practical values. In addition, for a given modulation and coding method, the threshold is shown to be a simple function of block size. As a result, the computational complexity required to obtain the threshold can be significantly reduced.

I. Introduction

Block fading channels are flat fading channels in which several symbols (a block of symbols) are exposed to the same channel gain [1]. This channel model is applicable, for instance, in systems based on orthogonal frequency division multiplexing (OFDM) within a time-frequency grid in which the channel is constant. The size of this time-frequency grid mainly depends on the mobility of the nodes. This channel model is also relevant in systems with slow time-frequency hopping such as Bluetooth or GSM. Furthermore, the fading channel in many wireless sensor networks (WSNs), in which the network consists of low mobility sensor nodes equipped with a narrow band radio, can be accurately modeled as a block fading channel.

This work is focused on a family of block fading channels in which the variation of the channel gain between different blocks in the time scale of interest can be accurately modeled by the Nakagami-m probability distribution function (pdf). The Nakagami-m ($m \geq 0.5$) is chosen because it has been shown to be a good fit to a wide variety of empirical data [2], [3]. By varying the m parameter, various fading conditions can be generated.

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The analytical solutions for the BLER in Nakagami-m block fading channels are, to the best of our knowledge, limited to non-coherent FSK modulation and hard-decision block codes with bounded distance decoders [4]–[8]. Therefore, for a wide range of physical layer parameters, numerical methods such as Monte-Carlo simulation or numerical integration are used to obtain the BLER. One such numerical method is the threshold method [9]–[11], which assigns 0 or 1 to the instantaneous BLER depending on the instantaneous SNR. This approximation method, explained in details in Sec. II, results in a simple analytical formula for the BLER estimate in many channel models. The validity of this method has been examined in Nakagami-m channels [12], [13] and has been shown to provide accurate results for wide range of practical values of m.

In this work, the definition of the threshold presented in [12], [13] is adopted and the effect of several physical layer parameters and the Nakagami parameter on this threshold is studied. For instance, it is shown that the threshold remains almost unchanged as m varies. Thus, the threshold obtained from a simulation in Rayleigh block fading channels can be utilized to approximate the BLER in other Nakagami-m channels. In addition, a simple relation between the block size and the threshold is derived. A combination of these results can be used to significantly reduce the computational complexity. Furthermore, the sensitivity of this model to channel variations is also studied by simulation.

The rest of this paper is organized as follows. In Sec. II, the approximation method is explained and an analytical formula is derived for Nakagami-m channels. A heuristic method to find the SNR threshold is discussed in Sec. III. In Sec. IV, the accuracy of the proposed method is examined by extensive simulations. The procedure for obtaining the BLER approximation is summarized in Sec. V, and conclusions are presented in Sec. VI.

II. BLER approximation

The block diagram of the system considered in this work is shown in Fig. 1. A block of k information bits, $\mathbf{X}$, is encoded with a rate R convolutional code. The resulting $n = k/R$

coded bits are interleaved using a random block interleaver and modulated by an M-ary binary reflected Gray-coded QAM-modulator. The resulting block of q symbols is transmitted through a Nakagami-m block fading channel. A block of received information bits, $\widehat{\mathbf{X}}$, is generated at the receiver by demodulating, deinterleaving and decoding the received symbols. A block is declared in error when $\mathbf{X} \neq \widehat{\mathbf{X}}$.

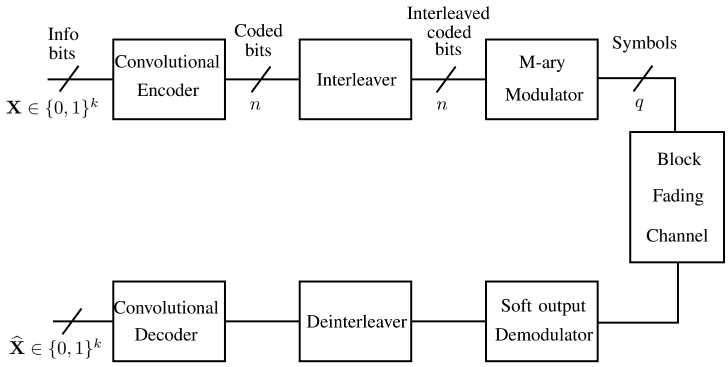

Fig. 1. Block diagram of the studied system.

In block fading channels, the SNR is constant during q consecutive transmitted symbols. Therefore, for each block, the channel can be modeled as an AWGN channel with the SNR properly adjusted by the current channel gain. Let E_b represent the average received energy per information bit and $N_0/2$ represent the noise spectral density. For notational convenience, let the received average (information bit) SNR, E_b/N_0 be denoted by $\overline{\gamma}$. Now, the instantaneous SNR per information bit (which is also fixed during a block) is denoted by γ is given by

$$\gamma = \kappa^2 \overline{\gamma} \tag{1}$$

where κ is a unit-power Nakagami-m distributed fading gain (channel gain). As a result, γ is distributed according to the Gamma distribution with the probability distribution function given by

$$f_\gamma(\gamma; \overline{\gamma}, m) = \frac{1}{\Gamma(m)} \left(\frac{m}{\overline{\gamma}}\right)^m \gamma^{(m-1)} \exp(-m\gamma/\overline{\gamma}) \tag{2}$$

where $\Gamma(m)$ is the Gamma function. Consequently, if the BLER on the AWGN channel is represented by $P_a(\gamma)$, the BLER on the Nakagami-m block fading channel is given by

$$P_f(\overline{\gamma}; m) = \Pr\{\mathbf{X} \neq \widehat{\mathbf{X}}\} = \int_0^\infty P_a(x) f_\gamma(x; \overline{\gamma}, m) dx. \tag{3}$$

Note that (3) is valid regardless of how quickly the fading varies between blocks as long as the distribution of γ (during the time scale of interest) approaches the Gamma distribution.

As mentioned in the Introduction, scarcely any analytical solution to (3) exists for a wide range of physical layer parameters in Nakagami-m block fading channels. This is partly due to the fact that there are few exact analytical expressions for $P_a(\gamma)$ for a large number of modulation, coding, and decoding methods [14]. More often, only an analytical upper bound on $P_a(\gamma)$ is known. For instance, in the case of soft decoding of a convolutional code and BPSK modulation, instantaneous BLER is given by

$$P_a(\gamma) \leq \sum_{d=d_f}^{\infty} a_d Q(\sqrt{2Rd\gamma}) \tag{4}$$

where $Q(x) = (\sqrt{2\pi})^{-1} \int_x^\infty \exp(-z^2/2) dz$, d_f is the free distance of the code, and a_d are code specific weights [15]. Obviously due to lack of exact expression for $P_a(\gamma)$ in such scenarios, solving (3) analytically is not feasible.

In addition, (3) becomes mathematically intractable for many combinations of physical layer parameters and channel models even when the exact expression for $P_a(\gamma)$ exists. Thus, numerical methods are often the only possible solution to obtain the $P_f(\overline{\gamma})$.

In this work, the threshold method is chosen because as we will see, it results in a simple analytical formula. Applying the threshold method, $P_a(\gamma)$ is replaced with the simple approximation given by

$$\widehat{P}_a(\gamma; \Theta) = \begin{cases} 1, & \text{if} \quad \gamma \leq \Theta \\ 0, & \text{if} \quad \gamma > \Theta \end{cases} \tag{5}$$

where Θ is the SNR threshold and $\widehat{P}_a(\gamma; \Theta)$ is the approximation of $P_a(\gamma)$. Thus, the approximation of BLER is given by

$$\begin{aligned} \widehat{P}_f(\overline{\gamma}; m, \Theta) &= \int_0^\infty \widehat{P}_a(x; \Theta) f_\gamma(x; \overline{\gamma}, m) dx \\ &= \int_0^\Theta f_\gamma(x; \overline{\gamma}, m) dx = F_\gamma(\Theta; \overline{\gamma}, m) \end{aligned} \tag{6}$$

where

$$F_\gamma(\gamma; \overline{\gamma}, m) = \frac{\Gamma(m, m\gamma/\overline{\gamma})}{\Gamma(m)} \tag{7}$$

is the cumulative distribution function (CDF). Here,

$$\Gamma(m, x) = \int_0^x t^{m-1} e^{-t} dt \tag{8}$$

is the lower part incomplete Gamma function. Hence, by combining (6) and (7), $\widehat{P}_f(\overline{\gamma}; m, \Theta)$ can be written as

$$\widehat{P}_f(\overline{\gamma}; m, \Theta) = \frac{\Gamma(m, m\Theta/\overline{\gamma})}{\Gamma(m)}. \tag{9}$$

A simpler expression for $\widehat{P}_f(\overline{\gamma}; m, \Theta)$ can be obtained for the special case of integer m where $\Gamma(m, x)$ can be simplified to

$$\Gamma(m, x) = \Gamma(m) \left[1 - e^{-x} \sum_{k=0}^{m-1} \frac{x^k}{k!}\right]. \tag{10}$$

In this case, we have

$$\widehat{P}_f(\overline{\gamma}; m, \Theta) = 1 - e^{-\frac{m\Theta}{\overline{\gamma}}} \sum_{k=0}^{m-1} \frac{\left(\frac{m\Theta}{\overline{\gamma}}\right)^k}{k!}. \tag{11}$$

In a given Nakagami-m channel, the only parameter needed to make use of the BLER approximations in (9) is Θ which may vary depending on the physical layer parameters. In the next section, a method for finding Θ is discussed.

III. How to find Θ

In previous works, several definitions for Θ have been presented. For instance, in [9], [10], Θ is defined as the inverse of the fading margin, while in work by Rodrigues et al. [11], it is defined as the iterative decoder convergence threshold. Note, however, that the latter definition is only applicable to turbo-coded systems.

In general, at an arbitrarily chosen $\overline{\gamma}$, Θ can be found such that $P_f(\overline{\gamma};m)$ and $\widehat{P}_f(\overline{\gamma};m,\Theta)$ are equal (i.e. zero error). However, in a given range of $\overline{\gamma}$, e.g., $\overline{\gamma}_s \le \overline{\gamma} \le \overline{\gamma}_e$, the optimal value of Θ depends on the definition of the error measure. An example of such an error measure is

$$\varepsilon(\Theta) = \max_{\overline{\gamma}_s \le \overline{\gamma} \le \overline{\gamma}_e} \frac{|P_f(\overline{\gamma};m) - \widehat{P}_f(\overline{\gamma};m,\Theta)|}{P_f(\overline{\gamma};m)}, \qquad (12)$$

which measures the maximum absolute relative deviation of the approximation from the correct value within a given range of $\overline{\gamma}$ between $\overline{\gamma}_s$ and $\overline{\gamma}_e$. The optimum Θ based on this error definition is given by

$$\Theta_{\text{opt}} = \min_{\Theta} \varepsilon(\Theta). \qquad (13)$$

Since for any meaningful error measure, the knowledge of $P_f(\overline{\gamma};m)$ is also required, the complexity of finding Θ_{opt} exceeds the benefits of using the threshold method. Therefore, a simple method of finding Θ is required. In this work, we adopt a heuristic sub-optimum method proposed in [13] which finds Θ by forcing $P_f(\overline{\gamma};m)$ and $\widehat{P}_f(\overline{\gamma};m,\Theta)$ to have a common point. Therefore, Θ is the solution to

$$\widehat{P}_f(\overline{\gamma}_c;m,\Theta) = P_f(\overline{\gamma}_c;m) \qquad (14)$$

where $P_f(\overline{\gamma}_c;m)$ is obtained by a simulation at $\overline{\gamma}_c$ corresponding to the common point. A range of different numerical methods can be used to find Θ as the solution to (14). However, if the simulation is performed in Rayleigh fading channels (Nakagami-1), Θ can be found by combining (11) and (14) as

$$\Theta = \overline{\gamma}_c \ln \frac{1}{1 - P_f(\overline{\gamma}_c;1)}. \qquad (15)$$

Choosing $\overline{\gamma}_c$ optimally is, again, a complex optimization problem that depends on several parameters. Heuristically, however, one can simply choose $\overline{\gamma}_c$ as the middle value in the range of interest, $\overline{\gamma}_c = (\overline{\gamma}_s + \overline{\gamma}_e)/2$. This is due to the fact that the approximation of BLER is expected to be more accurate around γ_c. In the next section, we study the effect of different physical layer and channel parameters on Θ.

IV. Simulation results

In this section, the effect of different physical layer parameters and channel models on the proposed approximation is examined by comparing the BLER approximates with results obtained from extensive simulations. To ensure the accuracy of simulations, 10^6 random blocks are generated at each $\overline{\gamma}$ point. Our simulated scenarios are presented in Table I. These scenarios are chosen such that a variety of modulations and error correction capabilities are examined. The convolutional codes chosen in this work have maximum free distance and good distance spectrum properties [15]. Following common practice, the generator polynomials are presented in octal format.

The simulations are performed in C++ with the help of the IT++ library [16] according to the block diagram in Fig. 1. At the encoder, enough zero tail bits are added to ensure that the trellis ends in the all-zero state. The interleaver is a random block interleaver with the size equal to block length, n. At the receiver, a soft-decision Viterbi decoder is used. The soft bits required by the Viterbi are generated at the receiver by the Log-MAP algorithm [14].

A. Θ *versus* m

To examine the effect of m on Θ, a series of simulations were performed in different Nakagami-m channels for each of the scenarios in Table I and for various block sizes. For each scenario and block size, Θ is obtained at $\overline{\gamma}_c = 10$ dB by solving (14). Our results indicate that there is little variation in the value of Θ as a function of m. As an example, the results for $n = 2400$ are presented in Fig. 2.

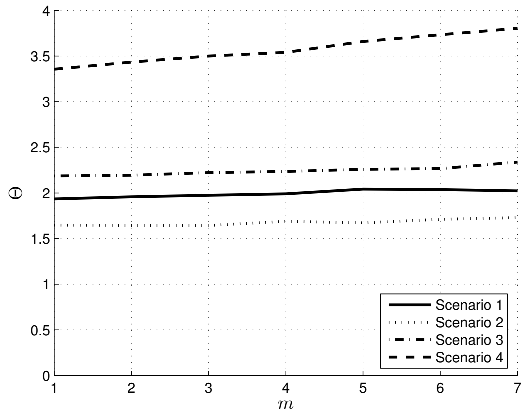

Fig. 2. Θ as a function of m for $n = 2400$.

B. Θ *versus* n

For each scenario, $K = 6$ different block sizes in the range of 400-12000 bits are simulated in block Rayleigh fading channels at $\overline{\gamma}_c = 10$ dB and Θ is numerically calculated using (15). A careful examination of a large set of results indicated that Θ may be modeled by a simple function of n. Several functions such as logarithmic and power series were tested, and a suitable candidate was found to be

$$\widehat{\Theta}(n) = a + n^b \qquad (16)$$

where $\widehat{\Theta}(n)$ is the approximation of Θ obtained from the model at block size n, and a and b are the model parameters. For each set of modulation, coding and decoding methods, a and b are obtained numerically by fitting $\widehat{\Theta}(n)$ to Θ obtained

TABLE I

SIMULATION SCENARIOS AND THE RESULTING a AND b WHERE $\Theta = a + n^b$

Scenario	Modulation	Generator polynomial	Free distance	R	a	b
1	BPSK	$(23, 35)$	7	$1/2$	-0.3839	0.1070
2	QPSK	$(133, 171)$	10	$1/2$	-0.4536	0.0935
3	16-QAM	$(133, 171, 165)$	15	$1/3$	-0.4148	0.1196
4	64-QAM	$(51, 55, 67, 77)$	18	$1/4$	-0.6307	0.1757

from simulations using the least square criteria, i.e.

$$(a, b) = \arg\min_{a,b} \sum_{k=1}^{K} \left[a + n_k^b - \Theta_k\right]^2 \tag{17}$$

where Θ_k is found from (15), and where $P_f(\overline{\gamma}_c; 1)$ is found through a simulation using the block length n_k. Fig. 3 shows Θ obtained from the simulations and $\widehat{\Theta}(n)$ from the model for all the scenarios presented in Table I. It is clear from this figure that the function defined in (16) successfully models the variations of Θ as function of n. The resulting a and b for each scenario are given in Table I.

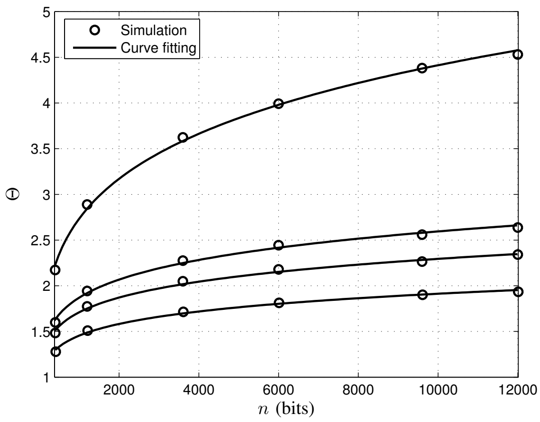

Fig. 3. Θ as a function of n for scenarios 1 to 4.

To further verify the accuracy of $\widehat{\Theta}(n)$, a series of simulations were performed at block sizes other than those used to obtain a and b and the results are compared with $\widehat{P}_f(\overline{\gamma}_c; m, \widehat{\Theta}(n))$. One such result for scenario 1 at $n = 7200$ is presented in Fig. 4, which confirms that $\widehat{\Theta}(n)$ obtained from the model in (16) can be used to determine the approximation of the BLER. Similar behavior is observed in other simulated scenarios not shown here due to lack of space.

C. Effect of mobility on the accuracy of approximation

In the block fading channel model, the fading gain is assumed to be constant during the block. In reality, however, the channel gain may vary depending on the relative speed between transmitter and receiver. Here, we study the sensitivity of the proposed approximation to variations of the fading gain during the block. Simulations in this section are limited to Rayleigh fading channels with correlation according to the Clark's channel model [17]. The correlation between the

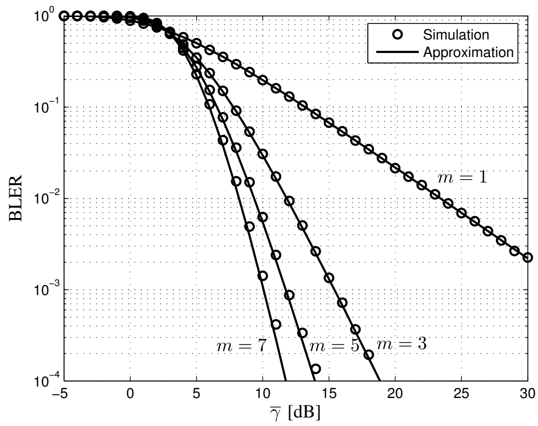

Fig. 4. $\widehat{P}_f(\overline{\gamma}; m, \widehat{\Theta})$ (solid line) and simulated BLER (circles) for scenario 1 in Nakagami-m channels ($m = 1, 3, 5, 7$) for $n = 7200$.

channel gains spaced T_s apart is

$$J_0(2\pi f_m T_s) \tag{18}$$

where J_0 is a Bessel function of the first kind and order zero, and f_m is the Doppler frequency. Since the variation of the channel over the block is of interest here, we define the normalized Doppler frequency as $f_m T_B$, where T_B is block transmission period. A series of simulation experiments were performed for various normalized Doppler frequencies and different scenarios and two examples of results are presented in Figs. 5-6. These results suggest that $\widehat{P}_f(\overline{\gamma}; m, \Theta)$ has acceptable accuracy for the normalized Doppler frequency of less than 0.1.

V. SUMMARY

This work aims to propose a low-complexity approximation method for finding the BLER in Nakagami-m block fading channels. The results presented in Sec. IV confirm that the model presented in Sec. II is valid for a number of different modulation and coding schemes and for a large number of different $\overline{\gamma}$, n and m. The BLER approximation can easily be obtained by inserting $\overline{\gamma}$, Θ, and m into (9).

For each set of modulation, coding, and decoding methods, Θ has to be carefully adjusted. It was demonstrated in Sec. IV that Θ can be accurately modelled as function of a and b as defined in (16). These parameters can be obtained as follows:

1. Identify the range of possible block sizes in the system and choose a suitable set of block sizes for simulations.
2. Identify the range of $\overline{\gamma}$ of interest and choose a suitable $\overline{\gamma}_c$.

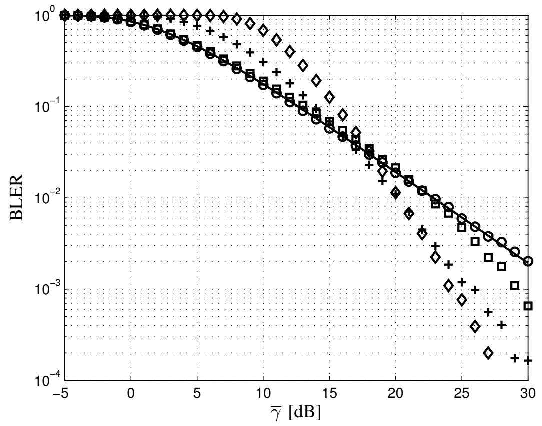

Fig. 5. The BLER for the first scenario and $n = 2400$ bits. $\widehat{P}_f(\overline{\gamma}; m, \Theta)$(solid line) and simulated BLER at different normalized Doppler frequencies 0.01 (circle), 0.1 (square), 1(plus), and 10 (diamond).

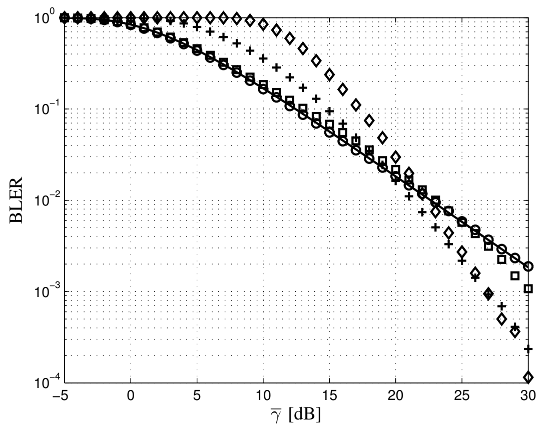

Fig. 6. The BLER for the second scenario and $n = 7200$ bits. $\widehat{P}_f(\overline{\gamma}; m, \Theta)$(solid line) and simulated BLER at different normalized Doppler frequencies 0.01 (circle), 0.1 (square), 1 (plus), and 10 (diamond).

3. Perform a single simulation for each of the selected block sizes at $\overline{\gamma}_c$ and in block Rayleigh fading channels.
4. Obtain Θ using (15) at each block size.
5. Numerically solve (17) to obtain a and b.

For example, in this work 6 simulations were performed at $\overline{\gamma}_c = 10$ dB for 6 different block sizes to obtain a and b.

The method proposed here leads to a significant reduction in the number of required simulations compared to the Monte-Carlo method, where one simulation is required for every combination of $\overline{\gamma}$, n, and m. In addition, for each set of modulation, coding and decoding methods, only two values (i.e., a and b) are needed to be stored instead of large set of numerical results.

VI. Conclusion

In this work, the approximation method based on a single parameter Θ, introduced in [11], is further studied for convolutional codes and Nakagami-m channels. It is shown that Θ is almost unaffected by the Nakagami parameter, m; therefore, Θ obtained from a simulation in the block Rayleigh fading channel is applicable to a wide range of practical Nakagami-m channels. More significantly, it is shown that Θ can be accurately modelled by a simple function of the block size, n. The combination of these results imply that with a few simulations, one can obtain an analytical approximation that is highly accurate for a wide range of block sizes in many Nakagami-m channels. In addition, for each set of modulation, coding, and decoding methods, only two values need to be stored, which also significantly simplifies the storing and handling of the numerical data.

The effect of channel variations during a block is also studied by generating a correlated Rayleigh fading channel. Our results indicate that the BLER estimate has an acceptable accuracy in channels with the normalized Doppler frequencies less than 0.1.

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