An Analytical Approximation to the Block Error Rate in Nakagami-m Non-Selective Block Fading Channels

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Abstract— **With few exceptions, an analytical closed-form expression for the block error rate (BLER) is lacking in block fading channels. Thus, the BLER is often obtained by numerical methods, such as Monte-Carlo simulations, resulting in additional computational complexity. In this paper, we propose a single-parameter analytical approximation for the BLER in flat, block-fading Nakagami-m channels, which significantly reduces the computational overhead. The low computational cost of the approximation makes it feasible to include the BLER in the objective function of larger optimization problems.**
Keywords: Fading channels, Block error rate, Approximation methods

I. Introduction

In this work, we consider the problem of finding a low-complexity analytical approximation for the BLER in Nakagami-m block fading channels [1], [2]. The choice of Nakagami-m ($m \geq 0.5$) is motivated by the fact that it has been shown to be a good fit to a wide variety of empirical data [3], [4]. Note that although the analytical solution to the BLER in Rayleigh and Nakagami-m block fading channels has been found in few special cases [5]–[8], there is a lack of analytical expressions for a wide range of modulation, coding, decoding methods, and channel models.

In practice, numerical methods such as Monte-Carlo simulations are used to obtain the BLER. Nonetheless, these simulations are computationally intensive especially at higher SNRs. To reduce the simulation complexity, Rayleigh fading channels are often modelled as a finite-state Markov chain with a single bit error probability being assigned to each state [9]. So, based on these models, less complex methods for simulating the BLER in Rayleigh fading channels have been proposed [10]–[12]. In case high accuracy is desirable, however, as many as 100 states might be required as shown in [10]. Also note that the complexity is specially important when BLER is incorporated into larger problems such as link level performance optimization where it becomes highly advantageous to have an analytical formula for the BLER [13].

The analytical approximation proposed here is obtained by employing a threshold model which assigns 0 or 1 to the instantaneous BLER given the SNR level. Similar methods have been used in [14]–[16] to study the BLER in slow Rayleigh fading channels and have been shown to be accurate. In this work, we examine the applicability of the threshold method to Nakagami-m block fading channels. We also study the effect of parameters such as m, block size, and modulation on the accuracy of the approximation. Furthermore, we offer a different definition for the SNR threshold and propose two simple heuristics to estimate that threshold.

The rest of this paper is organized as follows. Section II contains the system model and the BLER approximation obtained using the threshold model. Two different methods of finding the SNR threshold are proposed in Section III. In Section IV, the accuracy of these approximations are assessed by comparing it to simulation results. The conclusion is presented in Section V.

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II. BLER approximation

The block fading channels are flat fading channels in which the effect of fading on each block of bits is only a constant gain. Therefore, for each block, the channel can be modelled as an AWGN channel with the SNR properly adjusted by the current channel gain.

Similar to [5], [6], [10], we limit ourselves to linear block codes with a hard-decision decoder. The instantaneous BLER in fading channels, $P_a(\gamma)$, for a block of n coded bits capable of correcting t coded-bit errors with a bounded distance decoder [17] is given by

$$P_a(\gamma) = 1 - \sum_{i=0}^{t} \binom{n}{i} p(\gamma)^i [1 - p(\gamma)]^{n-i} \qquad (1)$$

where γ is the SNR per information bit during the block and $p(\gamma)$ is the coded-bit error probability. It is worth noting that (1) is not applicable to modulation formats which result in correlated bit errors such as differential PSK [5]. The above equation is applicable to non-binary modulation only if different bit positions have the same bit error rate which is not true, for instance, in case of 16-QAM [18]. Nonetheless, similar expressions are possible to derive. In case of Gray-coded 16-QAM, it can be shown that $P_a(\gamma)$ is given by

$$P_a(\gamma) = 1 - \sum_{l=0}^{t} \binom{n/2}{l} p_1(\gamma)^l [1 - p_1(\gamma)]^{n/2-l} \times \sum_{k=0}^{t-l} \binom{n/2}{k} p_2(\gamma)^k [1 - p_2(\gamma)]^{n/2-k} \quad (2)$$

where $p_1(\gamma)$ and $p_2(\gamma)$ are the two unique bit error probabilities for different bit positions and t is assumed to be less than $n/2$.

The BLER in fading channels, $P_f(\overline{\gamma})$, can be obtained by averaging the instantaneous BLER over the pdf of γ. That is,

$$P_f(\overline{\gamma}) = \int_0^\infty P_a(\gamma) f_\gamma(\gamma; \overline{\gamma}) d\gamma \quad (3)$$

where $\overline{\gamma}$ is the expected value of γ and $f_\gamma(\gamma; \overline{\gamma})$ is the probability distribution function of γ in a block fading channel for a given $\overline{\gamma}$. The difficulties in solving equation (3) is evident, for instance, in the case of 16-QAM. In addition, for large number of modulation, coding, and decoding methods, the exact analytical expression for $P_a(\gamma)$ is not known [19].

To find an analytical approximation for $P_f(\overline{\gamma})$, $P_a(\gamma)$ is replaced with a simple approximation based on the threshold model. This model approximates $P_a(\gamma)$ by either 0 or 1 using a simple SNR threshold as

$$\widehat{P}_a(\gamma; \Theta) = \begin{cases} 1, & \text{if} \quad \gamma \leq \Theta \\ 0, & \text{if} \quad \gamma > \Theta \end{cases} \quad (4)$$

where the approximation $\widehat{P}_a(\gamma; \Theta)$ of $P_a(\gamma)$ is parameterized by SNR threshold, Θ. The proposed approximation of BLER can be derived by replacing $P_a(\gamma)$ in (3) with $\widehat{P}_a(\gamma; \Theta)$ as

$$\widehat{P}_f(\overline{\gamma}; \Theta) = \int_0^\infty \widehat{P}_a(\gamma; \Theta) f_\gamma(\gamma; \overline{\gamma}) d\gamma = \int_0^\Theta f_\gamma(\gamma; \overline{\gamma}) d\gamma = F_\gamma(\Theta; \overline{\gamma}) \quad (5)$$

where $F_\gamma(\gamma; \overline{\gamma})$ is the cumulative distribution function (CDF) of γ for a given $\overline{\gamma}$. In a Nakagami-m fading channel, γ is distributed according to the Gamma distribution with the CDF given by

$$F_\gamma(\gamma; \overline{\gamma}) = \frac{\Gamma(m, m\gamma/\overline{\gamma})}{\Gamma(m)}, \quad m \geq 0.5 \quad (6)$$

where $\Gamma(m)$ is the Gamma function, and $\Gamma(m, x)$ is the lower part incomplete Gamma function given by

$$\Gamma(m, x) = \int_0^x t^{m-1} e^{-t} dt \quad (7)$$

The only parameter needed for this approximation is Θ. Thus if Θ is known, an analytical approximation for $P_f(\overline{\gamma})$ can still be obtained, even if no analytical formula for $P_a(\gamma)$ is known. In the next section, methods for finding Θ are discussed.

III. How to find Θ

In previous works, several definitions for Θ have been presented. For instance, in [14], [15], Θ is defined as the inverse of the fading margin, while in work by Rodrigues et al. [16], it is defined as the iterative decoder convergence threshold. Note, however, that the latter definition, is only applicable to turbo-coded blocks. In general, the optimal value of Θ in a given range of $\overline{\gamma}$, e.g., $\overline{\gamma}_s \leq \overline{\gamma} \leq \overline{\gamma}_e$, can be found by minimizing an error measure. An example of such an error measure is

$$\varepsilon(\Theta) = \max_{\overline{\gamma}_s \leq \overline{\gamma} \leq \overline{\gamma}_e} \frac{|P_f(\overline{\gamma}) - \widehat{P}_f(\overline{\gamma}; \Theta)|}{P_f(\overline{\gamma})}, \quad (8)$$

which measures the maximum absolute relative deviation of the approximation from the correct value within a range of $\overline{\gamma}$ between $\overline{\gamma}_s$ and $\overline{\gamma}_e$. This error is, clearly, a function of Θ. Therefore, the optimal Θ is given by

$$\Theta_{\text{opt}} = \min_\Theta \varepsilon(\Theta) = \min_\Theta \max_{\overline{\gamma}_s \leq \overline{\gamma} \leq \overline{\gamma}_e} \frac{|P_f(\overline{\gamma}) - \widehat{P}_f(\overline{\gamma}; \Theta)|}{P_f(\overline{\gamma})} \quad (9)$$

As (9) demonstrates, finding Θ_{opt} is not trivial and can be achieved only if $P_f(\overline{\gamma})$ is known which renders $\widehat{P}_f(\overline{\gamma}; \Theta)$ redundant. Therefore, for the proposed approximation method to be useful, a simple method of finding Θ is required.

We propose two heuristic, sub-optimal methods to adjust Θ. The first method is fully analytical and simple but less accurate in some cases. The second method is more complex as it requires a single simulation to adjust Θ. It can also be extended to make use of multiple Θs to obtain a higher accuracy.

A. Method A: Analytical

By applying the threshold model, $P_a(\gamma)$ is approximated by 0 or 1. It is therefore reasonable to approximate $P_a(\gamma)$ with 1 for the range of γ when $P_a(\gamma) > 0.5$. For this to happen, Θ is a solution to the equation

$$P_a(\Theta) = 0.5 \quad (10)$$

Note that this method ignores the distribution of γ when computing Θ.

B. Method B: Simulation

This method is based on the assumption that $\widehat{P}_f(\overline{\gamma}; \Theta)$ and $P_f(\overline{\gamma})$ have similar shapes. Thus, it is reasonable to expect a satisfactory overall accuracy, if Θ is adjusted such that $P_f(\overline{\gamma})$ and $\widehat{P}_f(\overline{\gamma}; \Theta)$ are forced to have at least one common point. Hence, Θ is found as a solution to

$$\widehat{P}_f(\overline{\gamma}_c; \Theta) = P_f(\overline{\gamma}_c) \quad (11)$$

where $P_f(\overline{\gamma}_c)$ is obtained by simulation at the common point, $\overline{\gamma}_c$. Choosing $\overline{\gamma}_c$ optimally is, again, a complex optimization problem which depends on many parameters. Heuristically, however, one can simply choose $\overline{\gamma}_c$ in the middle of the range of interest, $\overline{\gamma}_c = (\overline{\gamma}_s + \overline{\gamma}_e)/2$.

The accuracy can be increased by forcing more common points between $\widehat{P}_f(\overline{\gamma}; \Theta)$ and $P_f(\overline{\gamma})$. This can be done by dividing the interval of interest into multiple smaller sub-intervals. Then, in each sub-interval, a single common point is required to adjust Θ similar to (11). For instance, if the interval between $\overline{\gamma}_s$ and $\overline{\gamma}_e$ is divided into N equal sized sub-intervals, the BLER estimates in each interval is given by

$$\widehat{P}_f(\overline{\gamma};\Theta_i) = \frac{\Gamma(m, m\Theta_i/\overline{\gamma})}{\Gamma(m)}, \quad \overline{\gamma}_s + (i-1)\Delta \le \overline{\gamma} < \overline{\gamma}_s + i\,\Delta \tag{12}$$

where $\Delta = (\overline{\gamma}_e - \overline{\gamma}_s)/N$, $i = 1, 2, ..., N$, and Θ_i is found as a solution to

$$\widehat{P}_f(\overline{\gamma}_i;\Theta_i) = P_f(\overline{\gamma}_i), \quad \overline{\gamma}_i = \overline{\gamma}_s + \frac{2i-1}{2}\Delta \tag{13}$$

More sub-intervals result in a higher accuracy at the price of higher complexity.

IV. Simulation results

In this section, the accuracy of our proposed approximation is examined by comparing $\widehat{P}_f(\overline{\gamma};\Theta)$ with $P_f(\overline{\gamma})$. In absence of an analytical solution, $P_f(\overline{\gamma})$ is obtained by extensive Monte-Carlo simulations. At each $\overline{\gamma}$, (3) is numerically solved by averaging over many randomly generated $P_a(\gamma)$. Simulations were performed for $\overline{\gamma}$ from 0 to 40 dB with 1 dB spacing. In the case of non-coherent FSK modulation over a Rayleigh fading channel, a known analytical formula is used [10]

$$P_f(\overline{\gamma}) = 1 - \sum_{l=0}^{t} \binom{n}{l} 2^{-n} \frac{2}{2+l\overline{\gamma}} \left[1 + \sum_{i=1}^{n-l} \prod_{j=1}^{i} \frac{n-l+1-j}{l+j+2/\overline{\gamma}}\right] \tag{14}$$

It is practically infeasible to verify the accuracy of $\widehat{P}_f(\overline{\gamma};\Theta)$ for all the possible scenarios, as a prohibitively large number of combinations can be generated by varying n, t, m, and modulation. Therefore, the scenarios in Table I are chosen such that a range of diverse cases are covered. For instance, non-coherent FSK was chosen as an example of binary modulation while 16-QAM is an example of non-binary modulation. Two block sizes of 16 and 1024 bits with both high and low code rates are considered. For a given t and n, the number of information bits per block, k, (and the code rate k/n) is found by assuming that code satisfies the Hamming bound with equality. The Hamming bound [19] is

$$2^{n-k} \le \sum_{i=0}^{t} \binom{n}{i} \tag{15}$$

Each case in Table I is simulated in different fading conditions by varying m from 1 to 7. Since the Rayleigh fading channel is a very common model for fading channels, its results are presented in more detail in Fig. 1 and Fig. 2.

In these figures, $P_f(\overline{\gamma})$ obtained from simulations are compared with $\widehat{P}_f(\overline{\gamma};\Theta)$ obtained from both method A and B with $\overline{\gamma}_c = 20$ dB. These results demonstrate a close agreement between $P_f(\overline{\gamma})$ and $\widehat{P}_f(\overline{\gamma};\Theta)$ for both method A and B. Therefore, considering the results in [14]–[16], it is expected

TABLE I
Scenarios

	n	t	code rate	Modulation
case 1	16	0	1	Non-coherent FSK
case 2	16	0	1	16-QAM
case 3	16	3	0.41	Non-coherent FSK
case 4	16	3	0.41	16-QAM
case 5	1024	0	1	Non-coherent FSK
case 6	1024	0	1	16-QAM
case 7	1024	115	0.50	Non-coherent FSK
case 8	1024	115	0.50	16-QAM

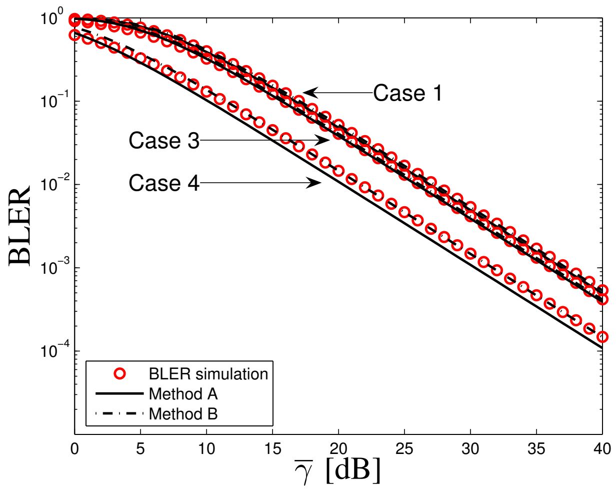

Fig. 1. $P_f(\overline{\gamma})$ and $\widehat{P}_f(\overline{\gamma};\Theta)$ in Rayleigh fading channel ($m = 1$) for case 1, 3, and 4.

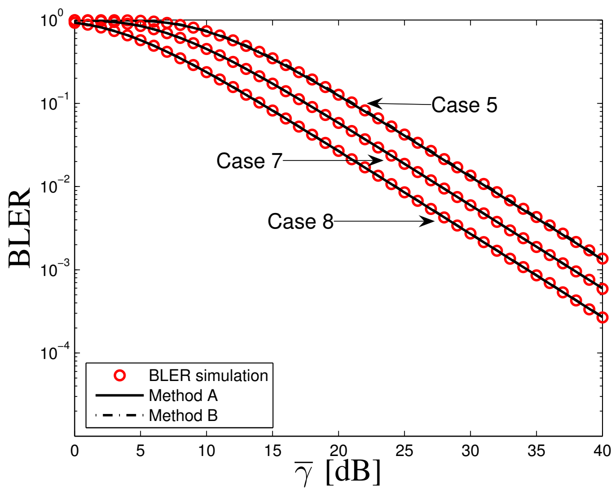

Fig. 2. $P_f(\overline{\gamma})$ and. $\widehat{P}_f(\overline{\gamma};\Theta)$ in Rayleigh fading channel ($m = 1$) for case 5, 7, and 8.

that the BLER approximation based on the threshold method will prove to be highly accurate for a wide range of block sizes, error correction capabilities, and modulations in Rayleigh block fading channels.

A careful examination of these figures also reveals that the accuracy of $\widehat{P}_f(\overline{\gamma};\Theta)$ is varying between different cases. This can be explained by considering how accurately $\widehat{P}_a(\gamma;\Theta)$ approximates $P_a(\gamma)$. As shown in Fig. 3, $\widehat{P}_a(\gamma;\Theta)$ is a better approximation of $P_a(\gamma)$ in case 5 compared to case 1 resulting in better accuracy for $\widehat{P}_f(\overline{\gamma};\Theta)$. While not shown here, similar trends can be observed between other simulated scenarios.

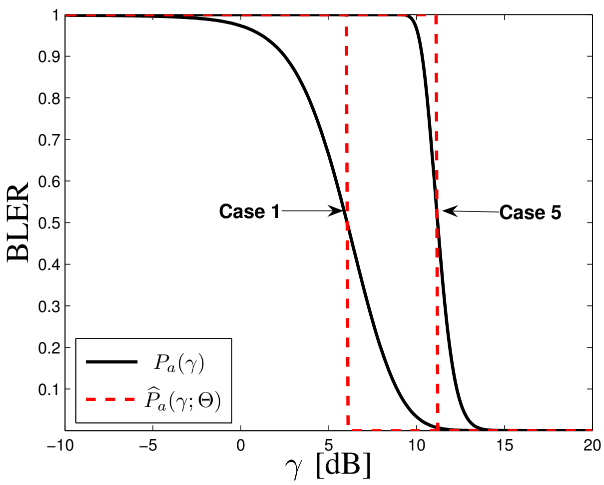

Fig. 3. $P_a(\gamma)$ and $\widehat{P}_a(\gamma;\Theta)$ for case 1 and 5.

The accuracy of $\widehat{P}_f(\overline{\gamma};\Theta)$ in Nakagami-m fading channels is further examined in case 1 and 5. The results presented in Figs. 4 and 5 show that as expected, $\widehat{P}_f(\overline{\gamma};\Theta)$ is more accurate in case 5 compared to case 1. In addition, it can be observed that an increase in m has negative effect on the accuracy of $\widehat{P}_f(\overline{\gamma};\Theta)$. This is also expected considering that the Nakagami-m fading channel tends to the AWGN channel as m goes to infinity (i.e., fading variance tends to zero). In the AWGN channel, the proposed approximation breaks down as there is no SNR variation. Nevertheless, the BLER approximation is shown to be valid for the wide range of practical values of m.

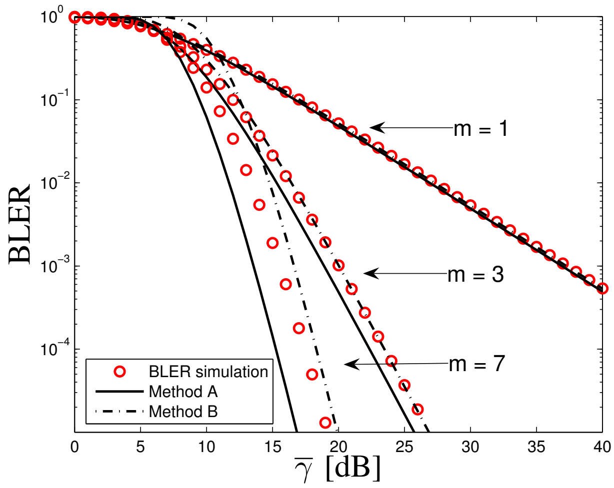

Fig. 4. $P_f(\overline{\gamma})$ and $\widehat{P}_f(\overline{\gamma};\Theta)$ for case 1 in Nakagami-m fading channel.

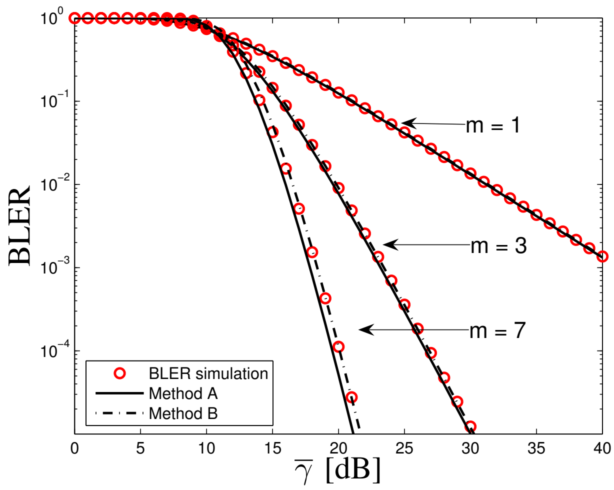

Fig. 5. $P_f(\overline{\gamma})$ and $\widehat{P}_f(\overline{\gamma};\Theta)$ for case 5 in Nakagami-m fading channel.

V. Conclusion

We have derived an analytical formula for the approximation of the BLER in Nakagami-m block fading channels by applying the threshold model to the instantaneous BLER. The proposed formula has a single parameter, Θ, which represents the effect of code, decoder, block length, and modulation on the BLER. In addition, this analytical approximation can be obtained even if no analytical formula for the instantaneous BLER is known.

The difficulties involved in obtaining the optimal Θ are discussed and two heuristic methods for finding Θ are proposed which at most require a single simulation. The outcome is an analytical approximation for the BLER with substantially lower computational complexity compared to many other numerical methods.

The accuracy of the BLER approximation is examined in several scenarios with different values of block lengths, error correction capabilities, and modulation in different fading conditions. Our results demonstrate a close agreement between the BLER approximation and simulation results in Rayleigh block fading channels. The accuracy of the BLER approximation reduces with an increase in m for Nakagami channels, but our simulations indicate that accuracy is satisfactory for a wide range of practical values of m.

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