# Solutions of the classical Yang-Baxter equation and Lagrangian subalgebras

IULIA POP

# Contents

# Introduction and overview of the thesis

At an early stage, the Yang-Baxter equation (YBE) appeared in several different works in literature and sometimes its solutions even preceded the equation. One can trace three streams of ideas from which YBE has emerged: the Bethe Ansatz, commuting transfer matrices in statistical mechanics and factorizable S matrices in field theory. In the pioneering works of the theoretical physicists C. N. Yang and R. J. Baxter, the YBE appears in the following form:

$$R^{\alpha\alpha'}_{\gamma\gamma'}(u-v)R^{\gamma\alpha''}_{\beta\gamma''}(u)R^{\gamma'\gamma''}_{\beta'\beta''}(v) = R^{\alpha'\alpha''}_{\gamma'\gamma''}(v)R^{\alpha\gamma''}_{\gamma\beta''}(u)R^{\gamma\gamma'}_{\beta\beta'}(u-v)$$

for a family of functions $R^{\alpha\beta}_{\gamma\delta}(u)$ on a complex variable $u$ and depending on four indices $\alpha$, $\beta$, $\gamma$, $\delta$ that range from 1 to a natural number $N$.

One of the first occurences of YBE can be found in the study of a one-dimensional quantum mechanical many-body problem with $\delta$ function interaction. By building the Bethe-type wavefunction, J. B. McGuire, F. A. Berezin, V. M. Finkelberg and others discovered that the $N$ -particle $S$ -matrix factorized in the product of two-particle ones. C. N. Yang treated the case of arbitrary statistics of particles by introducing the nested Bethe Ansatz. The YBE appears here as the consistency condition for the factorization. In statistical mechanics, R. J. Baxter investigated the role of YBE and its application for the eight vertex model on an arbitrary irregular lattice. He found that the partition function was left unchanged upon parallel displacement of the lines forming the lattice, a property which was called $Z$ invariance.

The topics concerning YBE began to be studied thoroughly in the 80's also by mathematicians like A. A. Belavin, V. G. Drinfeld, P. P. Kulish, E. K. Sklyanin, L. D. Faddeev and others. This study was motivated by the multitude of applications that YBE has in different areas of mathematics and physics: quantum theory, integrable systems, inverse scattering problems, group theory, algebraic geometry and statistical physics.

The classical Yang-Baxter equation (CYBE) was firstly introduced by E. K. Sklyanin. Compared to YBE, CYBE represents an important and simplified case since it can be formulated in the language of Lie algebras. The form of CYBE is the one given in Definition 1.1.1. One of the directions of study in this domain is the classification of solutions in the case of a simple complex Lie algebra. In [1], A. A. Belavin and V. G. Drinfeld investigate the nondegenerate solutions of CYBE for a finite-dimensional, simple, complex

Lie algebra. The authors prove that the poles of a nondegenerate solution form a discrete subgroup of the additive group of complex numbers. Moreover, with respect to the rank of this subgroup, one gives a classification of nondegenerate solutions: elliptic, trigonometric and rational. Concerning the first class, the authors reduce the problem of finding nondegenerate elliptic solutions to the one of describing triples $(\mathbf{g}, A_1, A_2)$, where $\mathbf{g}$ is a simple Lie algebra, $A_1$ and $A_2$ are commuting automorphisms of $\mathbf{g}$ of finite order, not having common fixed nonzero vectors. Moreover, they prove that if such triples exist then there is an isomorphism $\mathbf{g} \cong sl(n)$. Belavin and Drinfeld also succeeded in classifying the trigonometric solutions using the data from the Dynkin diagram. Regarding rational solutions, in [1] there are given several examples associated with Frobenius subalgebras of $\mathbf{g}$ and some arguments in favour of the idea that there are too many rational solutions to try to list them. However, in [13], A. Stolin reduces the problem of listing "nontrivial" rational solutions of CYBE to the classification of quasi-Frobenius subalgebras of $\mathbf{g}$ which, in turn, are related to the so-called maximal orders in the loop algebra corresponding to the extended Dynkin diagram.

In the first chapter of the thesis, we will be interested in the study and computation of the trigonometric solutions of CYBE for $sl(2)$ and $sl(3)$. In the sections 1.1-1.3, we make a survey of the main results concerning the general properties of the nondegenerate solutions of CYBE. We remind the method of Belavin and Drinfeld to determine the trigonometric solutions. This method uses the important notions of Coxeter automorphism, Dynkin diagram, simple weight and admissible triple. Section 1.4 is focused on the determination of the trigonometric solutions correponding to the case $sl(3)$. Firstly, we obtain the two Coxeter automorphisms, then the corresponding Dynkin diagrams and admissible triples, and finally the form of the solutions, given by (1.4.8), (1.4.12), (1.4.14) and (1.4.23).

In the next sections, we give another approach to trigonometric solutions, based on the determination of certain Lagrangian subalgebras. This approach is inspired by the method used in [13] for listing rational solutions. We will deal with a slightly different type of solutions that we will call "quasi-trigonometric". These solutions have the form $X(u,v) = \frac{u}{v-u}t + p(u,v)$, where $t$ denotes the quadratic Casimir element and $p(u,v)$ is some polynomial. Firstly, we describe the general form of the Lagrangian subspaces of $V_+ \oplus V_-$, where $V$ is a finite-dimensional linear space together with a nondegenerate symmetric bilinear form denoted by $<,>_+$, $V_+ = (V, <,>_+)$, $V_- = (V, -<,>_+)$ and the direct sum has a bilinear form induced by $<,>_+$: $<(a,b),(c,d)> = <a,c>_+ - <b,d>_+$. Then, given a simple, complex, finite-dimensional Lie algebra $\mathbf{g}$, the next step is to describe the Lagrangian subalgebras of $\mathbf{g}_+ \oplus \mathbf{g}_-$ with respect to the bilinear form induced by the Killing form of the Lie algebra $\mathbf{g}$. As it will be shown, this implies to determine all subalgebras $S$ of $\mathbf{g}$ that satisfy the condition $S^\perp \subseteq S$. After reminding the notion of parabolic subalgebra, this problem is solved in the cases $\mathbf{g} = sl(2)$ and $sl(3)$. Taking into consideration that the purpose is to construct some quasi-trigonometric solutions for $sl(2)$ and $sl(3)$, one has to determine a certain type of Lagrangian subalgebras. More precisely, in Sections 1.8 and 1.9, we find the Lagrangian subalgebras $\overline{W}$ that verify the property $\overline{W} \cap \overline{\Delta} = 0$, where

$$\overline{\Delta} = \left\{ \left( \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & -a \end{pmatrix} \right); a, b, c \in \mathbb{C} \right\},$$

when $\mathbf{g} = sl(2)$ and respectively

$$\overline{\Delta} = \left\{ \left( \begin{pmatrix} A & b \\ 0 & -TrA \end{pmatrix}, \begin{pmatrix} A & b' \\ 0 & -TrA \end{pmatrix} \right); A \in gl(2), b, b' \in \mathbb{C}^2 \right\},$$

when $\mathbf{g} = sl(3)$.

In Section 1.10, we give an overview of the basic results concerning Lie bialgebras and classical $r$-matrices which will be used for the computation of the quasi-trigonometric solutions. In Section 1.11, we explain the relationship between the quasi-trigonometric solutions and the Lagrangian subalgebras that are complementary to $\overline{\Delta}$. We consider the Lie algebra $\mathbf{g}[u]$ and a certain quasi-trigonometric solution of CYBE, $X_0(u,v) = \frac{u}{v-u}t + r$, where $t$ denotes the quadratic Casimir element and $r$ is the classical Drinfeld-Jimbo r-matrix. Then $X_0$ induces a 1-cocycle $\delta$: $\mathbf{g}[u] \to \mathbf{g}[u] \wedge \mathbf{g}[v]$, $\delta(a(u)) = [X_0(u,v), a(u) \otimes 1 + 1 \otimes a(v)]$, which defines a Lie bialgebra structure on $\mathbf{g}[u]$. The main result that enables us to make a correspondence between quasi-trigonometric solutions and Lagrangian subalgebras is Theorem 1.11.2. We prove that the classical double of $\mathbf{g}[u]$, induced by the 1-cocycle $\delta$, is isomorphic to the direct sum $\mathbf{g}((u^{-1})) \oplus \mathbf{g}$. In this way, one can find quasi-trigonometric solutions of CYBE by looking at the Lagrangian subalgebras of $\mathbf{g}((u^{-1})) \oplus \mathbf{g}$ which are complementary to $\Delta = \{(a(u), a(0)); a(u) \in \mathbf{g}[u]\}$. As it will be shown, in the situation $\mathbf{g} = sl(2)$ and $sl(3)$ to determine "nontrivial" quasi-trigonometric solutions (i.e. nonequivalent to $X_0 + const.$) implies exactly to find the Lagrangian subalgebras that are complementary to $\overline{\Delta}$. This has already been accomplished in Sections 1.8 and 1.9. It seems that there exists a correspondence between "nontrivial" quasi-trigonometric solutions and Lagrangian subalgebras of $\mathbf{g} \oplus \mathbf{g}$ that are complementary to $\overline{\Delta}$ (up to so-called gauge equivalence). In further work we will study the nature of this correspondence.

The second chapter of the thesis concerns the computation of constant solutions of the modified classical Yang-Baxter equation(mCYBE) for a certain class of Lie algebras that are not simple. We remind that, given a finite-dimensional complex Lie algebra $\Lambda$, a solution of mCYBE is a tensor $s \in \Lambda \otimes \Lambda$ which satisfies the following conditions:

$$[a, s^{12} + s^{21}] = 0$$

for any $a \in \Lambda$ and

$$[s^{12},s^{13}]+[s^{12},s^{23}]+[s^{13},s^{23}]=0.$$

Our goal is to construct solutions of mCYBE for parabolic subalgebras of a simple complex Lie algebra. We consider a finite-dimensional complex, simple Lie algebra $\mathbf{g}$, with root system $R$ and a simple set of roots $\Delta$, with respect to a fixed Cartan subalgebra $\mathbf{h}$. For any $\alpha \in R$, $\mathbf{g}^{\alpha}$ denotes the corresponding root space. If we consider the Killing form $K$ on $\mathbf{g}$, for any nonzero element $e_\alpha$ of $\mathbf{g}^{\alpha}$ there exists an element $e_{-\alpha}$ of $\mathbf{g}^{-\alpha}$ such that $K(e_\alpha, e_{-\alpha}) = 1$. With these notations, we remind that one of the classical Drinfeld-Jimbo r-matrices is the following:

$$r=\sum_{\alpha>0} e_\alpha \wedge e_{-\alpha}.$$

Now let $P_\alpha$ be the parabolic subalgebra of $\mathbf{g}$ corresponding to a simple positive root $\alpha$. $P_\alpha$ is the Lie algebra generated by the root vectors corresponding to the simple roots and their opposite except $-\alpha$. Obviously, it is spanned by all the linear spaces $\mathbf{g}^{\beta}$ for $\beta > 0$ and $\mathbf{g}^{-\beta}$ for $\beta \neq \alpha$ , $\beta > 0$. We consider the map $\delta_r : \mathbf{g} \to \mathbf{g} \otimes \mathbf{g}$ defined by $\delta_r(a) = [a \otimes 1 + 1 \otimes a, r]$. In the first section, we will prove that $\delta_r$ provides a Lie bialgebra structure for $P_\alpha$. In this way, there exists a classical double $D(P_\alpha)$ induced by the 1-cocycle $\delta_r$. In the next sections, we give a more precise description of $D(P_\alpha)$. We start with an example, $\mathbf{g} = sl(3)$, and we prove that the classical double of the parabolic subalgebra $P = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$ is isomorphic to $sl(3) \oplus gl(2)$. This gives us an idea about what to expect in the general case. In the third section, we show that if $\mathbf{g}$ is a complex, finite-dimensional, simple Lie algebra and $P_\alpha$ is the parabolic subalgebra corresponding to a simple positive root $\alpha$, then $D(P_\alpha)$ is isomorphic to $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$. Here $\mathbf{Red}(P_\alpha)$ denotes the reductive part of $P_\alpha$. Finally, this description of the classical double will be used to construct solutions of mCYBE in $P_\alpha$.

Acknowledgement. I am grateful to my supervisor, Alexander Stolin, who suggested the topic of the thesis and guided me carefully along the way. I would also like to thank my colleagues at the Department of Mathematics for their support. Last, but not least, I am grateful to my family and all my friends for their love.

CHAPTER 1

# On trigonometric solutions of CYBE

## 1.1. Preliminaries on the classical Yang-Baxter equation

Let $\mathbf{g}$ be a simple finite-dimensional Lie algebra over the field $\mathbb{C}$ of complex numbers. Consider an associative algebra $A$ with unit containing $\mathbf{g}$ (for example, the universal enveloping algebra). Let $\varphi_{13}, \varphi_{12}, \varphi_{23} : \mathbf{g} \otimes \mathbf{g} \to A \otimes A \otimes A$ be the linear maps respectively defined by $\varphi_{13}(a \otimes b) = a \otimes 1 \otimes b$, $\varphi_{12}(a \otimes b) = a \otimes b \otimes 1$, $\varphi_{23}(a \otimes b) = 1 \otimes a \otimes b$. For a function $X : \mathbb{C} \times \mathbb{C} \to \mathbf{g} \otimes \mathbf{g}$, we consider $X^{ij}$: $\mathbb{C} \times \mathbb{C} \to A \otimes A \otimes A$ defined by $X^{ij}(u_i, u_j) = \varphi_{ij}(X(u_i, u_j))$. Let $[,]$ be the usual Lie algebra bracket on the associative and unital algebra $A \otimes A \otimes A$.

DEFINITION 1.1.1. *The classical Yang-Baxter equation (CYBE)* for the simple Lie algebra $\mathbf{g}$ is the functional equation

$$[X^{12}(u_1, u_2), X^{13}(u_1, u_3)] + [X^{12}(u_1, u_2), X^{23}(u_2, u_3)]+ \tag{1.1.1}$$

$$+[X^{13}(u_1, u_3), X^{23}(u_2, u_3)] = 0,$$

with respect to the function $X : \mathbb{C} \times \mathbb{C} \to \mathbf{g} \otimes \mathbf{g}$.

REMARK 1.1.2. If $X(u_1, u_2)$ is a solution of CYBE and $\varphi(u)$ is a function with values in $Aut(\mathbf{g})$, then $\widetilde{X}(u_1, u_2) = (\varphi(u_1) \otimes \varphi(u_2))(X(u_1, u_2))$ is also a solution.

DEFINITION 1.1.3. The solutions $X$ and $\widetilde{X}$ are called *equivalent* if $\widetilde{X}$ is obtained from $X$ as in the previous remark.

A second method to obtain a new solution can be given if one starts with an invariant solution.

DEFINITION 1.1.4. The function $X(u_1, u_2)$ is said to be *invariant* with respect to $\varphi \in Aut(\mathbf{g})$ if $(\varphi \otimes \varphi)(X(u_1, u_2)) = X(u_1, u_2)$. The set of all such $\varphi$ is a group called *the invariance group of the function* $X(u_1, u_2)$. The function $X(u_1, u_2)$ is said to be *invariant* with respect to $h \in \mathbf{g}$ if $[h \otimes 1 + 1 \otimes h, X(u_1, u_2)] = 0$ (i.e., it is invariant with respect to $e^{t\mathbf{ad}h}$ for any $t \in \mathbb{C}$).

REMARK 1.1.5. If $X(u_1, u_2)$ is a solution of CYBE, invariant with respect to a subalgebra $\mathbf{h} \subset \mathbf{g}$ , and a tensor $r \in \mathbf{h} \otimes \mathbf{h}$ satisfies

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0, \tag{1.1.2}$$

$$r^{21} = -r^{12}, \tag{1.1.3}$$

then $\widetilde{X}(u_1, u_2) = X(u_1, u_2) + r$ is also a solution of CYBE. If the algebra $\mathbf{h}$ is Abelian, then (1.1.2) holds automatically.

Usually one considers solutions with additional conditions:

(1) the so-called *unitary* solution if $X^{12}(u_1, u_2) = -X^{21}(u_2, u_1)$.

(2) the function $X(u_1, u_2)$ depends only on $u_1 - u_2$. In this case, $X(u_1, u_2)$ is denoted by $X(u_1 - u_2)$ and (1.1.1) can be written as

$$[X^{12}(u), X^{13}(u+v)] + [X^{12}(u), X^{23}(v)] + [X^{13}(u+v), X^{23}(v)] = 0 \tag{1.1.4}$$

and $X(u_1, u_2)$ is unitary iff $X^{12}(u) = -X^{21}(-u)$.

REMARK 1.1.6. The second condition is not restrictive. In [2] there was shown the following result:

Let $U$ be a domain in $\mathbb{C}$. If $X(u_1, u_2)$ is a meromorphic solution of CYBE, defined on $U \times U$, such that the determinant of the coordinates is not identically zero, then there exist a domain $V$ in $\mathbb{C}$, a holomorphic map $\varphi : V \to Aut(\mathbf{g})$ and a nonconstant function $f : V \to U$ such that $(\varphi(v_1) \otimes \varphi(v_2))X(f(v_1), f(v_2))$ depends only on $v_1 - v_2$.

In [1] there was proved the equivalence of the following three conditions, under the assumption that $X(u)$ was a meromorphic solution, defined on some disk with center at zero:

A) The determinant of the matrix formed by the coordinates of the tensor $X(u)$ is not identically equal to zero;

B) The function $X(u)$ has at least one pole and there does not exist a Lie subalgebra $\mathbf{g}' \subset \mathbf{g}$ such that $X(u) \in \mathbf{g}' \otimes \mathbf{g}'$ for any $u$;

C) The function $X(u)$ has for $u = 0$ a pole of the first order with residue of the form $ct$, where $c \in \mathbb{C}$ and $t = \sum I_\mu \otimes I_\mu$ , for $\{I_\mu\}$ an orthonormal basis with respect to the Killing form on $\mathbf{g}$ (we will use the notation $t = I_\mu \otimes I_\mu$, meaning that we understand summation over identical indices).

DEFINITION 1.1.7. A solution $X(u)$ of (1.1.4), in the class of meromorphic functions defined on some disk $U \subset \mathbb{C}$ with center at zero, and satisfying one of the (equivalent) conditions from above is called *nondegenerate.*

## 1.2. Properties of nondegenerate solutions

Let $X(u)$ be a nondegenerate solution of (1.1.4) defined on a disk $U \subset \mathbb{C}$ with center at zero. We may assume that

$$\lim_{u \to 0} uX(u) = t. \tag{1.2.1}$$

In the next proposition we collect some results given in [1]:

PROPOSITION 1.2.1. *The following properties hold:*

*1) $X(u)$ satisfies the unitary condition $X^{12}(u) = -X^{21}(-u)$.*

*2) $X(u)$ can be extended to an entire meromorphic function.*

*3) Let $\Gamma$ be the set of poles of the function $X(u)$. Consider $\gamma \in \Gamma$. There exists $A_\gamma \in Aut(\mathbf{g})$ such that $X(u + \gamma) = (A_\gamma \otimes 1)X(u)$.*

More things can be said about the set of poles $\Gamma$, as it is shown by the following statements from [1]:

PROPOSITION 1.2.2. *1)* $\Gamma$ *is a discrete subgroup of* $\mathbb{C}$*;*

*2)* $A_{\gamma_1+\gamma_2} = A_{\gamma_1}A_{\gamma_2}$*, for any* $\gamma_1, \gamma_2 \in \Gamma$*;*

*3)* $X(u+\gamma) = (1 \otimes A_\gamma^{-1})X(u)$*, for any* $u \in \mathbb{C}$*,* $\gamma \in \Gamma$*;*

*4)* $(A_\gamma \otimes A_\gamma)X(u) = X(u)$*;*

*5) Suppose that* $rank\Gamma = 2$*. Then there is no nonzero* $x \in \mathbf{g}$ *such that* $A_\gamma(x) = x$ *for any* $\gamma \in \Gamma$*. There exists a subgroup of finite index* $\Gamma' \subset \Gamma$ *such that* $A_\gamma = 1$ *for any* $\gamma \in \Gamma'$*.*

The proof of all these properties is based on some results concerning quasi-Abelian functions. We remind the following

DEFINITION 1.2.3. A meromorphic function $\varphi$ on a complex linear space $L$ of dimension $n$ is called *quasi-Abelian* if there exist a system of coordinates $z_1, ..., z_n$ in the space $L$ and $p, q, r \in \mathbb{N}$, $p+q+r = n$ and $\gamma_1, ..., \gamma_{2r} \in \mathbb{C}^n$ such that the following statements hold:

1) for $z_{p+q+1}, ..., z_n$ fixed, $\varphi(z_1, ..., z_n)$ is a rational function of $z_1, ..., z_p$, $e^{z_{p+1}}, ..., e^{z_{p+q}}$;

2) $\gamma_i$ are periods for $\varphi$;

3) the vectors $\overline{\gamma_i} \in \mathbb{C}^{\mathbf{r}}$ formed by the last $r$ coordinates of the vectors $\gamma_i$ are linearly independent over $\mathbb{R}$.

REMARK 1.2.4. 1) This is a generalization of the notion of Abelian function. We recall that a function $\varphi$ on $\mathbb{C}^n$ is called *Abelian* if it has $2n$ periods, linearly independent over $\mathbb{R}$. It corresponds to the case $p = q = 0$, $r = n$.

2) For $n = 1$, there are three types of quasi-Abelian functions: elliptic ($p = q = 0$, $r = 1$); rational ($p = 1$, $q = r = 0$) and rational on $e^z$ ($p = r = 0$, $q = 1$).

By a Weierstrass-type theorem which involves quasi-Abelian functions, [[1], Th.2.1], one proves

PROPOSITION 1.2.5. *Let* $X(u)$ *be a nondegenerate solution of (1.1.4). Then there exist a natural number* $n$*, a vector* $a \in \mathbb{C}^n$ *and a quasi-Abelian function* $\overline{X} : \mathbb{C}^n \to \mathbf{g} \otimes \mathbf{g}$ *satisfying (1.1.4) such that* $X(u) = \overline{X}(ua)$*.*

Consequently, one obtains the following classification theorem for nondegenerate solutions of CYBE:

THEOREM 1.2.6. *1) If* $rank\Gamma = 2$ *then* $X(u)$ *is an* ***elliptic*** *solution, i.e. a meromorphic double periodic function;*

*2) If* $rank\Gamma = 1$ *then* $X(u)$ *is equivalent to a solution* $\widetilde{X}(u) = f(e^{ku})$*, where* $f$ *is a rational function. Such a solution is called* ***trigonometric****;*

*3) If* $\Gamma = 0$ *then* $X(u)$ *is equivalent to a* ***rational*** *solution.*

Concerning the first class, in [1] one reduces the problem of finding nondegenerate elliptic solutions to the one of describing triples $(\mathbf{g}, A_1, A_2)$, where $\mathbf{g}$ is a simple Lie algebra, $A_1$ and $A_2$ are commuting automorphisms of $\mathbf{g}$ of finite order, not having common fixed nonzero vectors. Moreover, the authors prove that if such triples exist then there is an isomorphism $\mathbf{g} \cong sl(n)$. Belavin and Drinfeld also succeed in classifying the trigonometric solutions using the data from the Dynkin diagram. We will remind their method in the following section. Regarding rational solutions, in [1] there are given several examples associated with Frobenius subalgebras of $\mathbf{g}$ and some arguments in favour of the idea that there are too many rational solutions to try to list them. However, this problem was solved by A. Stolin in [13].

## 1.3. Trigonometric solutions

According to [1], in the description of the trigonometric solutions of CYBE, the notions of Coxeter automorphism and simple weight are very important. Let $\mathbf{g}$ be a finite-dimensional, simple, complex Lie algebra. We denote by $Aut^0\mathbf{g}$ the connected component of the identity of the Lie group $Aut\mathbf{g}$. The elements of $Aut^0\mathbf{g}$ are called *inner automorphisms.* In other words, $Aut^0\mathbf{g}$ is the subgroup generated by $\{exp(\mathbf{ad}h); h \in \mathbf{g}\}$. It is known that $Aut\mathbf{g}/Aut^0\mathbf{g} \cong Aut\Delta$, where $\Delta$ is the Dynkin diagram of $\mathbf{g}$. The order of the group $Aut\Delta$ can be only 1, 2 or 6 (the last possibility is obtained for $\mathbf{g} = \mathcal{O}(8)$). If $\sigma \in Aut\Delta$, let $K_\sigma$ be the corresponding coset in $Aut\mathbf{g}/Aut^0\mathbf{g}$. We will now remind the notion of Coxeter automorphism:

DEFINITION 1.3.1. The automorphism $A \in K_\sigma$ is called a *Coxeter automorphism* if the following statements hold:

a) the algebra $\mathbf{g}^A = \{x \in \mathbf{g}; Ax = x\}$ is Abelian;

b) $A$ has the smallest order among the automorphisms $A' \in K_\sigma$ such that $\mathbf{g}^{A'}$ is Abelian.

According to [7], for any pair $(\mathbf{g}, \sigma)$, there exists a Coxeter automorphism $C \in K_\sigma$ which is unique up to conjugation by inner automorphisms. The order of the automorphism $C$ is called the *Coxeter number* of the pair $(\mathbf{g}, \sigma)$. We denote this number by $h$. From [1], a method to construct a Coxeter automorphism is to choose a system of Weyl generators $\{X_i, Y_i, H_i\}$, where $i$ runs through the set of vertices of $\Delta$ and let $C \in Aut\mathbf{g}$ such that $C(H_i) = H_{\sigma(i)}$, $C(X_i) = e^{2\pi i/h}X_{\sigma(i)}$ and $C(Y_i) = e^{-2\pi i/h}Y_{\sigma(i)}$. One verifies that $C$ is a Coxeter automorphism.

It is useful to remind the form of the Coxeter automorphisms in the case of simple Lie algebras, given in [1]. The following notations are made: $C$ is a Coxeter automorphism corresponding to the pair $(\mathbf{g}, \sigma)$, $m$ is the order of $\sigma$, $h$ is the Coxeter number of the pair $(\mathbf{g}, \sigma)$, $\omega = e^{2\pi i/h}$ and $S$ is the matrix with 1 on the auxiliary diagonal and 0 elsewhere. The Coxeter automorphisms of the classical simple Lie algebras are the following:

(1) $\mathbf{g} = sl(n)$, $m = 1 \Rightarrow h = n$, $C(X) = TXT^{-1}$, $T = diag(1, ..., \omega^{n-1})$;
(2) $\mathbf{g} = sl(2n+1)$, $m = 2 \Rightarrow h = 4n+2$, $C(X) = -TX^tT^{-1}$, $T = S \cdot diag(1, \omega, ..., \omega^{2n})$;
(3) $\mathbf{g} = sl(2n)$, $m = 2 \Rightarrow h = 4n-2$, $C(X) = -TX^tT^{-1}$, $T = S \cdot diag(1, \omega, ..., \omega^{2n-2})$;
(4) $\mathbf{g} = sp(2n) \Rightarrow h = 2n$, $C(X) = TXT^{-1}$, $T = diag(1, \omega, ..., \omega^{2n-1})$;
(5) $\mathbf{g} = o(2n+1) \Rightarrow h = 2n$, $C(X) = TXT^{-1}$, $T = diag(1, ..., \omega^{2n-1}, 1)$;
(6) $\mathbf{g} = o(2n)$, $m = 1 \Rightarrow h = 2n-2$, $C(X) = TXT^{-1}$, $T = diag(1, ..., \omega^{2n-3}, 1)$;

(7) $\mathbf{g} = o(2n)$, $m = 2 \Rightarrow h = 2n$, $C(X) = TXT^{-1}$, $T = (t_{ij})$ given by $t_{11} = \omega$, $t_{22} = \omega^2$,..., $t_{n-1n-1} = \omega^{n-1}$, $t_{n+2n+2} = \omega^{n+1}$,..., $t_{2n2n} = \omega^{2n-1}$, $t_{nn+1} = t_{n+1n} = 1$ and the rest are zero.

The second notion that has to be reminded in order to study the trigonometric solutions of CYBE is *simple weight*. Let $\sigma \in Aut\Delta$ be fixed and consider a Coxeter automorphism $C \in K_\sigma$. Let $\mathbf{h} = \{x \in \mathbf{g}; Cx = x\}$ be Abelian subalgebra of $\mathbf{g}$. Put $\omega = e^{2\pi i/h}$. If one considers the subspaces $\mathbf{g}_j = \{x \in \mathbf{g}; Cx = \omega^j x\}$ for $j = 0, ..., h-1$, then $\mathbf{g} = \oplus_{j\in\mathbb{Z}/h\mathbb{Z}}\mathbf{g}_j$. For any $\alpha \in \mathbf{h}^*$, consider $\mathbf{g}_j^\alpha = \{x \in \mathbf{g}_j; [a, x] = \alpha(a)x$, for any $a \in \mathbf{h}\}$. From [7], it results that the direct sum of these subspaces is $\mathbf{g}_j$ and their dimension is $\leq 1$ for $\alpha \neq 0$. The elements of the set $\Gamma = \{\alpha \in \mathbf{h}^*;\ \mathbf{g}_1^\alpha \neq 0\}$ are called *simple weights*. Also, $0 \notin \Gamma$ and $\dim \mathbf{g}_1^\alpha = 1$ for any $\alpha \in \Gamma$.

Having defined the simple weights, the next step is to construct the so-called *Dynkin diagram* of the pair $(\mathbf{g}, C)$. Let us denote by ( , ) the bilinear form on $\mathbf{h}^*$ induced by the Killing form $K$ of $\mathbf{g}$. It is known that the restriction of the Killing form to $\mathbf{h}$ is nondegenerate and therefore it induces an isomorphism between $\mathbf{h}$ and $\mathbf{h}^*$. The *Dynkin diagram* is a graph whose vertices are in 1-1 correspondence with the simple weights and the nature of the connection of two vertices $A$ and $B$, corresponding to the simple weights $\alpha$ and $\beta$, is determined by the following rules:

(1) the number of lines that connect $A$ and $B$ is $\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$;

(2) if $(\alpha, \alpha)(\beta, \beta) > 1$, the arrows of these lines are pointed towards $B$.

Using these notions, one constructs a particular solution of (1.1.4). Let $t = I_\mu \otimes I_\mu$, where $\{I_\mu\}$ is an orthonormal basis with respect to the Killing form. One verifies that $(C \otimes C)t = t$ and this implies that $t \in \oplus_{j\in\mathbb{Z}/h\mathbb{Z}}(\mathbf{g}_j \otimes \mathbf{g}_{-j})$. Let $t_j$ be the projection of $t$ on $\mathbf{g}_j \otimes \mathbf{g}_{-j}$. Denote

$$\xi(\lambda) = \frac{t_0}{2} + \frac{1}{\lambda^h - 1}\sum_{j=0}^{h-1} t_j \lambda^j; \tag{1.3.1}$$

$$X(u) = \xi(e^{u/h}). \tag{1.3.2}$$

We will remind the following result due to [1]:

PROPOSITION 1.3.2. *The function $X(u)$, defined by (1.3.1) and (1.3.2), is a solution of (1.1.4) with set of poles $2\pi i\mathbb{Z}$ and the residue at zero equals $t$.*

This result gives a particular solution of (1.1.4), but we are interested in describing the general form of a nondegenerate trigonometric solution. Let $X(u)$ be a nondegenerate trigonometric solution of (1.1.4). Because the rank of the set of poles of $X(u)$ equals 1, we may assume that the set of poles is $2\pi i\mathbb{Z}$. Let $A \in Aut\mathbf{g}$ such that $X(u + 2\pi i) = (A \otimes 1)X(u)$ (such an automorphism is given by Prop.1.2.1). Denote by $\sigma$ the automorphism of $\Delta$ induced by $A$. One says that the solution $X(u)$ *corresponds to* $\sigma$.

If $X(u)$ is replaced by an equivalent solution, then $A$ is replaced by $T_1AT_2^{-1}$, where $T_1$ and $T_2$ are in the same connected component of $Aut\mathbf{g}$. Thus, the class of conjugation of $\sigma$ remains the same.

We will describe, according to [1], the general form of the trigonometric solutions corresponding to $\sigma \in Aut\Delta$. Let us consider a Coxeter automorphism $C$ corresponding to a pair $(\mathbf{g}, \sigma)$ and let $\Gamma$ be the set of simple weights.

DEFINITION 1.3.3. A triple $(\Gamma_1, \Gamma_2, \tau)$, where $\Gamma_1, \Gamma_2 \subset \Gamma$ and $\tau : \Gamma_1 \to \Gamma_2$ is a bijective map, is called *admissible* if the following properties hold:

a) for any $\alpha, \beta \in \Gamma_1$, $(\tau(\alpha), \tau(\beta)) = (\alpha, \beta)$;

b) for any $\alpha \in \Gamma_1$, there exists $k \in \mathbb{N}^*$such that $\tau(\alpha), ..., \tau^{k-1}(\alpha) \in \Gamma_1$ and $\tau^k(\alpha) \notin \Gamma_1$.

Let $(\Gamma_1, \Gamma_2, \tau)$ be an admissible triple. One considers the following system:

$$r^{12} + r^{21} = t_0; \tag{1.3.3}$$

$$(\tau(\alpha) \otimes 1)(r) + (1 \otimes \alpha)(r) = 0, \alpha \in \Gamma_1, \tag{1.3.4}$$

where $r \in \mathbf{h} \otimes \mathbf{h}$ and the following notations are made: if $r = \sum h_i \otimes k_i$ and $\alpha \in \mathbf{h}^*$, then $(\alpha \otimes 1)(r) = \sum \alpha(h_i)k_i$ and $(1 \otimes \alpha)(r) = \sum \alpha(k_i)h_i$. According to [1], the previous system is consistent. The skew-symmetric tensors which belong to $\mathbf{h}_0 \otimes \mathbf{h}_0$, where $\mathbf{h}_0 = \{a \in \mathbf{h}; \alpha(a) = \tau(\alpha(a))$, for any $\alpha \in \Gamma_1\}$, are the only solutions of the corresponding homogeneous system.

One denotes by $\mathbf{a}_i$ the subalgebra of $\mathbf{g}$ generated by the subspaces $\mathbf{g}_1^\alpha$, $\alpha \in \Gamma_i$ , where $i \in \{1, 2\}$. Because $\mathbf{g} = \oplus_{j,\alpha}\mathbf{g}_j^\alpha$, the algebra $\mathbf{a}_i$ is the sum of some subspaces $\mathbf{g}_j^\alpha$. From this it results that there is a unique projector $P : \mathbf{g} \to \mathbf{a}_1$ such that $P(\mathbf{g}_j^\alpha) = 0$ if $\mathbf{g}_j^\alpha$ is not included in $\mathbf{a}_1$. For $\alpha \in \Gamma_1$, we fix an isomorphism of linear spaces $\mathbf{g}_1^\alpha \cong \mathbf{g}_1^{\tau(\alpha)}$(both have dimension 1). This isomorphism can be extended to a Lie algebras isomorphism $\theta : \mathbf{a}_1 \to \mathbf{a}_2$. Define then a linear operator $\widetilde{\theta}$: $\mathbf{g} \to \mathbf{g}$ by $\widetilde{\theta}(x) = \theta(P(x))$. This operator is nilpotent and one considers $\psi = \widetilde{\theta}/(1 - \widetilde{\theta}) = \widetilde{\theta} + \widetilde{\theta}^2 + ....$ With these notations the following general theorem can be proved:

THEOREM 1.3.4. *[[1], Th.6.1] 1) Let $r \in \mathbf{h} \otimes \mathbf{h}$ satisfy the system of equations (1.3.3) and (1.3.4). Then the function*

$$X(u) = r + \frac{1}{e^u - 1}\sum_{j=0}^{h-1} e^{ju/h}t_j - \sum_{j=1}^{h-1} e^{ju/h}(\psi \otimes 1)t_j + \sum_{j=1}^{h-1} e^{-ju/h}(1 \otimes \psi)t_{-j} \tag{1.3.5}$$

*is a solution of (1.1.4) with set of poles $2\pi i\mathbb{Z}$ and residue $t$ at zero. In addition, $X(u + 2\pi i) = (C \otimes 1)X(u)$.*

*2) Any trigonometric solution of (1.1.4) with set of poles $2\pi i\mathbb{Z}$ and residue $t$ at zero, corresponding to the automorphism $\sigma \in Aut\Delta$, is equivalent with a solution of the form (1.3.5).*

REMARK 1.3.5. 1) The particular solution from Prop.1.3.2 corresponds to the case $\Gamma_1 = \Gamma_2 = \emptyset$, $r = \frac{t_0}{2}$.

2) It can be shown that the solution (1.3.5) is $\mathbf{h}_0$-invariant. Hence, adding to this solution any skew-symmetric tensor from $\mathbf{h}_0 \otimes \mathbf{h}_0$, one obtains a new solution of (1.1.4). Thus one can get by this method all solutions corresponding to a fixed triple $(\Gamma_1, \Gamma_2, \tau)$, starting from one solution. It is easy to show that $\theta$, $\psi$ and hence $X(u)$ depend on the choice of the isomorphisms $\mathbf{g}_1^{\alpha} \cong \mathbf{g}_1^{\tau(\alpha)}$, $\alpha \in \Gamma_1$ and the change of them leads to the replacement of $X(u)$ by $(e^{\mathbf{ad}a} \otimes e^{\mathbf{ad}a})X(u)$, $a \in \mathbf{h}$. Therefore one obtains an equivalent solution. From the previous theorem, it results that, up to the methods of propagation of solutions and such trivial transformations as multiplication by a number and replacement of $u$ by $cu$, the number of nondegenerate trigonometric solutions of (1.1.4) is finite.

3) One can show that if the solutions $X(u)$ and $\widetilde{X}(u)$ of the form (1.3.5) are equivalent, then $\widetilde{X}(u) = (\varphi \otimes \varphi)X(u)$, $\varphi \in G$, where $G$ is the set of automorphisms of $\mathbf{g}$ that commute with $C$.

4) From 3), it follows that:

i) if the solutions $X(u)$ and $\widetilde{X}(u)$ of the form (1.3.5), corresponding to the triple $(\Gamma_1, \Gamma_2, \tau)$ and $(\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \widetilde{\tau})$ are equivalent, then $(\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \widetilde{\tau})$ is obtained by applying to $(\Gamma_1, \Gamma_2, \tau)$ some automorphism of the Dynkin diagram of the pair $(\mathbf{g}, C)$;

ii) if $(\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \widetilde{\tau})$ is obtained by applying to $(\Gamma_1, \Gamma_2, \tau)$ an automorphism of the Dynkin diagram of the pair $(\mathbf{g}, C)$, then a solution corresponding to $(\Gamma_1, \Gamma_2, \tau)$ is equivalent to some solution corresponding to $(\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \widetilde{\tau})$.

EXAMPLE 1.3.6. We will apply the general procedure in order to compute the trigonometric solutions for $sl(2)$. If we consider $\mathbf{g} = sl(2)$, the Dynkin diagram of $\mathbf{g}$ has only one automorphism, the identity. It results that the Coxeter number of the pair $(\mathbf{g}, id)$ equals 2 and the Coxeter automorphism is given by : $C(X) = TXT^{-1}$, where $T = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$ and $\omega = e^{\pi i} = -1$. Let $\mathbf{h} = \{X \in sl(2); C(X) = X\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}; a \in \mathbb{C} \right\}$. Consider $\mathbf{g}_1 = \{X \in sl(2); C(X) = -X\} = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}; x, y \in \mathbb{C} \right\}$ which is 2-dimensional. Let us take the canonical basis in $sl(2)$ : $\{e_{12}, e_{21}, e_{11} - e_{22}\}$ and define $\alpha_1 \in \mathbf{h}^*$ by $\alpha_1(e_{11} - e_{22}) = -2$ and put $\alpha_2 = -\alpha_1$. It is obvious that $\mathbf{g}_1^{\alpha_1} = \mathbb{C}e_{21}$ and $\mathbf{g}_1^{\alpha_2} = \mathbb{C}e_{12}$. The set of simple weights is $\Gamma = \{\alpha_1, \alpha_2\}$.

There exist two admissible triples $(\Gamma_1, \Gamma_2, \tau)$ :

a) $\Gamma_1 = \Gamma_2 = \emptyset$;

b) $\Gamma_1 = \{\alpha_1\}$, $\Gamma_2 = \{\alpha_2\}$, $\tau(\alpha_1) = \alpha_2$ (which coincides with the case $\Gamma_1 = \{\alpha_2\}$, $\Gamma_2 = \{\alpha_1\}$, $\tau(\alpha_2) = \alpha_1$).

The next step is to find $t$, $t_0$ and $t_1$. In order to do this, we construct an orthonormal basis in $sl(2)$ with respect to the Killing form. Let us take $I_1 = e_{12} + \frac{1}{2}e_{21}$, $I_2 = ie_{12} - \frac{i}{2}e_{21}$ and $I_3 = \frac{\sqrt{2}}{2}(e_{11} - e_{22})$ which is orthonormal.

We obtain the following:

$$t = \frac{1}{2}(e_{11} - e_{22}) \otimes (e_{11} - e_{22}) + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}; \tag{1.3.6}$$

$$t_0 = \frac{1}{2}(e_{11} - e_{22}) \otimes (e_{11} - e_{22}) \in \mathbf{h} \otimes \mathbf{h}; \tag{1.3.7}$$

$$t_1 = e_{12} \otimes e_{21} + e_{21} \otimes e_{12} \in \mathbf{g}_1 \otimes \mathbf{g}_1. \tag{1.3.8}$$

Consider the first situation:

a) $\Gamma_1 = \Gamma_2 = \emptyset$. The system (1.3.3), (1.3.4) becomes the following equation:

$$r^{12} + r^{21} = \frac{1}{2}(e_{11} - e_{22}) \otimes (e_{11} - e_{22}), r \in \mathbf{h} \otimes \mathbf{h}. \tag{1.3.9}$$

The only solution of (1.3.9) is $r = \frac{t_0}{2}$. In this case, the trigonometric solution is given by (1.3.1) and (1.3.2):

$$X_1(u) = \frac{e^u + 1}{4(e^u - 1)}(e_{11} - e_{22}) \otimes (e_{11} - e_{22}) + \frac{e_{12} \otimes e_{21} + e_{21} \otimes e_{12}}{e^{u/2} - e^{-u/2}}. \tag{1.3.10}$$

b) $\Gamma_1 = \{\alpha_1\}$, $\Gamma_2 = \{\alpha_2\}$, $\tau(\alpha_1) = \alpha_2$. The system (1.3.3), (1.3.4) is formed by (1.3.9) and the following equation:

$$(\alpha_2 \otimes 1)(r) + (1 \otimes \alpha_1)(r) = 0. \tag{1.3.11}$$

Again, the only solution of this system is $r = \frac{t_0}{2}$. According to the notations that have been introduced in this section, we have that $\mathbf{a}_1 = \mathbb{C}e_{21}$ and $\mathbf{a}_2 = \mathbb{C}e_{12}$. Because different isomorphisms between $\mathbf{a}_1$ and $\mathbf{a}_2$ lead to equivalent solutions, we may choose $\theta$ such that $\theta(e_{21}) = e_{12}$. Considering the projector $P : sl(2) \to \mathbb{C}e_{21}$ and $\widetilde{\theta} = \theta P$, it results that $\widetilde{\theta}(e_{12}) = 0$, $\widetilde{\theta}(e_{21}) = e_{12}$ and $\widetilde{\theta}(e_{11} - e_{22}) = 0$. Thus, $\widetilde{\theta}^2 = 0$ and $\psi = \widetilde{\theta}$. In conclusion, by (1.3.5), the trigonometric solution corresponding to $(\Gamma_1, \Gamma_2, \tau)$ is:

$$X_2(u) = X_1(u) + (e^{-u/2} - e^{u/2})(e_{12} \otimes e_{12}). \tag{1.3.12}$$

## 1.4. Trigonometric solutions of CYBE for $sl(3)$

The aim of this section is to construct all nonequivalent trigonometric solutions for the simple Lie algebra $sl(3)$, using the method of Belavin and Drinfeld presented in the previous section. Let us consider $\mathbf{g} = sl(3)$. In this situation, the Dynkin diagram of $\mathbf{g}$ has two automorphisms. According to the classification given in the third section, the corresponding Coxeter automorphisms are:

(1) for $m = 1 \Rightarrow h = 3$, $\omega = e^{2\pi i/3}$ and $C_1(X) = TXT^{-1}$, where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix};$$

(2) for $m = 2 \Rightarrow h = 6$, $\omega = e^{\pi i/3}$ and $C_2(X) = -TX^tT^{-1}$, where $T = \begin{pmatrix} 0 & 0 & e^{2\pi i/3} \\ 0 & e^{\pi i/3} & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

In (1), $\mathbf{h} = \{X \in sl(3);\ C_1(X) = X\} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix}; a, b \in \mathbb{C} \right\}$;

$\mathbf{g}_1 = \{X \in sl(3);\ C_1(X) = e^{2\pi i/3}X\} = \left\{ \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix}; a, b, c \in \mathbb{C} \right\}$ and

also $\mathbf{g}_{-1} = \{X \in sl(3); C_1(X) = e^{-2\pi i/3}X\} = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}; a, b, c \in \mathbb{C} \right\}$.

We make the remark that $\mathbf{g}_2 = \mathbf{g}_{-1}$and $\mathbf{g}_{-2} = \mathbf{g}_1$. The set of simple weights $\Gamma$ and the form of the subspaces $\mathbf{g}_1^{\alpha}$ can be easily found. Let us define $\alpha_i \in \mathbf{h}^*$, $i = 1, 2, 3$ by $\alpha_1(A) = b - a$; $\alpha_2(A) = -a - 2b$ and respectively $\alpha_3(A) = 2a + b$, for any $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \in \mathbf{h}$. It is obtained that $\Gamma = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathbf{g}_1^{\alpha_1} = \mathbb{C}e_{21}$, $\mathbf{g}_1^{\alpha_2} = \mathbb{C}e_{32}$, $\mathbf{g}_1^{\alpha_3} = \mathbb{C}e_{13}$.

We now determine $t, t_0, t_1, t_2$, with the notations introduced in the previous section. An orthonormal basis for $sl(3)$, with respect to the Killing form, is the following: $I_1 = \frac{\sqrt{2}}{2}(e_{12} + e_{21})$, $I_2 = \frac{i\sqrt{2}}{2}(e_{12} - e_{21})$, $I_3 = \frac{\sqrt{2}}{2}(e_{13} + e_{31})$, $I_4 = \frac{i\sqrt{2}}{2}(e_{13} - e_{31})$, $I_5 = \frac{\sqrt{2}}{2}(e_{23} + e_{32})$, $I_6 = \frac{i\sqrt{2}}{2}(e_{23} - e_{32})$, $I_7 = e_{11} - e_{33}$, $I_8 = e_{22} - e_{33}$. It results that

$$t = \frac{1}{3}\sum_{i<j}(e_{ii} - e_{jj}) \otimes (e_{ii} - e_{jj}) + \sum_{i \neq j} e_{ij} \otimes e_{ji}; \tag{1.4.1}$$

$$t_0 = \frac{1}{3}\sum_{i<j}(e_{ii} - e_{jj}) \otimes (e_{ii} - e_{jj}) \in \mathbf{h} \otimes \mathbf{h}; \tag{1.4.2}$$

$$t_1 = e_{21} \otimes e_{12} + e_{13} \otimes e_{31} + e_{32} \otimes e_{23} \in \mathbf{g}_1 \otimes \mathbf{g}_{-1}; \tag{1.4.3}$$

$$t_2 = e_{12} \otimes e_{21} + e_{31} \otimes e_{13} + e_{23} \otimes e_{32} \in \mathbf{g}_2 \otimes \mathbf{g}_{-2}. \tag{1.4.4}$$

Because the Dynkin diagram of the pair $(\mathbf{g}, C_1)$ is a triangle with vertices in $\alpha_i$, there exist three admissible triples $(\Gamma_1, \Gamma_2, \tau)$: i) $\Gamma_1 = \Gamma_2 = \emptyset$; ii) $\Gamma_1 = \{\alpha_1\}$, $\Gamma_2 = \{\alpha_2\}$ and $\tau(\alpha_1) = \alpha_2$; iii) $\Gamma_1 = \{\alpha_1, \alpha_2\}$, $\Gamma_2 = \{\alpha_2, \alpha_3\}$ and $\tau(\alpha_1) = \alpha_2$, $\tau(\alpha_2) = \alpha_3$.

In the situation i), we have to find $r \in \mathbf{h} \otimes \mathbf{h}$ such that

$$r^{12} + r^{21} = t_0. \tag{1.4.5}$$

Let us denote $E = e_{11} - e_{33}$, $F = e_{22} - e_{33}$ the canonical basis in $\mathbf{h}$. A simple computation gives the following:

$$r = aE \otimes F + bF \otimes E + \frac{1}{3}E \otimes E + \frac{1}{3}F \otimes F, \tag{1.4.6}$$

with $a + b = -\frac{1}{3}$. The trigonometric solution corresponding to i) is:

$$X_1(u) = r + \frac{1}{e^u - 1}\sum_{j=0}^{2} e^{ju/3}t_j. \tag{1.4.7}$$

By replacing $r, t_0, t_1, t_2$ we obtain:

$$X_1(u) = \sum_{i,j=1}^{3} \rho_{ij} e_{ii} \otimes e_{jj} + Y(u), \tag{1.4.8}$$

where

$$Y(u) = \frac{1}{e^u - 1}[\frac{1}{3}\sum_{i<j}(e_{ii}-e_{jj})\otimes(e_{ii}-e_{jj}) + e^{u/3}\sum_{i-j\equiv 1(\mathbf{mod} 3)} e_{ij}\otimes e_{ji} + \tag{1.4.9}$$

$$+ e^{2u/3}\sum_{i-j\equiv 2(\mathbf{mod} 3)} e_{ij}\otimes e_{ji}]$$

and $(\rho_{ij}) = \begin{pmatrix} \frac{1}{3} & a & b \\ b & \frac{1}{3} & a \\ a & b & \frac{1}{3} \end{pmatrix}$, $a + b = -\frac{1}{3}$.

In the situation ii), the system (1.3.3), (1.3.4) becomes (1.4.5) and the following:

$$(\alpha_2 \otimes 1)(r) + (1 \otimes \alpha_1)(r) = 0. \tag{1.4.10}$$

We know already that the general solution of (1.4.5) is (1.4.6). It results that $a = 0$ and thus

$$r = -\frac{1}{3}F \otimes E + \frac{1}{3}E \otimes E + \frac{1}{3}F \otimes F = \frac{1}{3}\sum_{i,j=1}^{3} r_{ij} e_{ii} \otimes e_{jj}, \tag{1.4.11}$$

where $(r_{ij}) = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$.

Let us consider now the subalgebra generated by $\mathbf{g}_1^{\alpha_1}$, $\mathbf{a}_1 = \mathbb{C}e_{21}$ and the subalgebra generated by $\mathbf{g}_1^{\alpha_2}$, $\mathbf{a}_2 = \mathbb{C}e_{32}$. Take the Lie algebras isomorphism $\theta : \mathbf{a}_1 \to \mathbf{a}_2$ given by $\theta(e_{21}) = e_{32}$. Since other choice of $\theta$ would lead to an equivalent solution, we may take $\theta$ in this way. Take the projector $P$: $\mathbf{g} \to \mathbf{a}_1$ and $\widetilde{\theta}$: $\mathbf{g} \to \mathbf{g}$ defined by $\widetilde{\theta} = \theta P$. It results that $\widetilde{\theta}(e_{21}) = e_{32}$ and the images of the other elements of the canonical basis in $\mathbf{g}$ are zero. It follows that $\widetilde{\theta}^2 = 0$ and thus $\psi = \widetilde{\theta}$. By (1.3.5), the following trigonometric solution is obtained:

$$(1.4.12)\quad X_2(u) = \frac{1}{3}\sum_{i,j=1}^{3} r_{ij} e_{ii} \otimes e_{jj} + Y(u) - e^{u/3} e_{32} \otimes e_{12} + e^{-u/3} e_{12} \otimes e_{32},$$

where $Y(u)$ is given by (1.4.9) and $(r_{ij})$ by (1.4.11).

Finally, we consider iii). The system (1.3.3), (1.3.4) is equivalent to (1.4.5), (1.4.10) and

$$(1.4.13)\qquad (\alpha_3 \otimes 1)(r) + (1 \otimes \alpha_2)(r) = 0,$$

which has the solution given by (1.4.11).

Let $\mathbf{a}_1$ be the subalgebra generated by $\mathbf{g}_1^{\alpha_1}$ and $\mathbf{g}_1^{\alpha_2}$ ; $\mathbf{a}_2$ the subalgebra spanned by $\mathbf{g}_1^{\alpha_2}$ and $\mathbf{g}_1^{\alpha_3}$. It results that $\mathbf{a}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & b & 0 \end{pmatrix} ; a, b, c \in \mathbb{C} \right\}$ and $\mathbf{a}_2 = \left\{ \begin{pmatrix} 0 & c & b \\ 0 & 0 & 0 \\ 0 & a & 0 \end{pmatrix} ; a, b, c \in \mathbb{C} \right\}$. We choose the Lie isomorphism $\theta$: $\mathbf{a}_1 \to \mathbf{a}_2$ given by $\theta \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & b & 0 \end{pmatrix} = \begin{pmatrix} 0 & c & b \\ 0 & 0 & 0 \\ 0 & a & 0 \end{pmatrix}$. The map $\widetilde{\theta}$ obtained from $\theta$ verifies $\widetilde{\theta}^3 = 0$ and we obtain that $\psi \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & -a_{11} - a_{22} \end{pmatrix} = \begin{pmatrix} 0 & a_{31} & a_{31} + a_{21} \\ 0 & 0 & 0 \\ 0 & a_{21} & 0 \end{pmatrix}$. The trigonometric solution corresponding to our triple is:

$$(1.4.14)\qquad X_3(u) = X_2(u) - e^{u/3} e_{13} \otimes (e_{12} + e_{23}) - e^{2u/3} e_{12} \otimes e_{13} +$$
$$+ e^{-u/3} (e_{12} + e_{23}) \otimes e_{13} + e^{-2u/3} e_{13} \otimes e_{12}.$$

Now we will do the computation of the trigonometric solutions corresponding to $(\mathbf{g}, C_2)$. In this situation, $\mathbf{g}_0 = \{X \in sl(3);\ C_2(X) = X\} = \mathbb{C}(e_{11} - e_{33})$. Analogously,

$\mathbf{g}_1 = \{X \in sl(3);\ C_2(X) = e^{\pi i/3} X\} = \mathbb{C}(e_{12} - e_{23}) \oplus \mathbb{C}e_{31}$;
$\mathbf{g}_2 = \{X \in sl(3);\ C_2(X) = e^{2\pi i/3} X\} = \mathbb{C}(e_{21} + e_{32})$;
$\mathbf{g}_3 = \{X \in sl(3);\ C_2(X) = -X\} = \mathbb{C}(e_{11} - 2e_{22} + e_{33})$;
$\mathbf{g}_4 = \{X \in sl(3);\ C_2(X) = -e^{\pi i/3} X\} = \mathbb{C}(e_{12} + e_{23})$;
$\mathbf{g}_5 = \{X \in sl(3);\ C_2(X) = -e^{2\pi i/3} X\} = \mathbb{C}(e_{21} - e_{32}) \oplus \mathbb{C}e_{13}$.

We remind that $t$ is given by (1.4.1) and it follows that

$$(1.4.15)\qquad t_0 = \frac{1}{2}(e_{11} - e_{33}) \otimes (e_{11} - e_{33}) \in \mathbf{g}_0 \otimes \mathbf{g}_0;$$

$$t_1 = \frac{1}{2}(e_{12} - e_{23}) \otimes (e_{21} - e_{32}) + e_{31} \otimes e_{13} \in \mathbf{g}_1 \otimes \mathbf{g}_5; \tag{1.4.16}$$

$$t_2 = \frac{1}{2}(e_{21} + e_{32}) \otimes (e_{12} + e_{23}) \in \mathbf{g}_2 \otimes \mathbf{g}_4; \tag{1.4.17}$$

$$t_3 = \frac{1}{6}(e_{11} - 2e_{22} + e_{33}) \otimes (e_{11} - 2e_{22} + e_{33}) \in \mathbf{g}_3 \otimes \mathbf{g}_3; \tag{1.4.18}$$

$$t_4 = \frac{1}{2}(e_{12} + e_{23}) \otimes (e_{21} + e_{32}) \in \mathbf{g}_4 \otimes \mathbf{g}_2; \tag{1.4.19}$$

$$t_5 = \frac{1}{2}(e_{21} - e_{32}) \otimes (e_{12} - e_{23}) + e_{13} \otimes e_{31} \in \mathbf{g}_5 \otimes \mathbf{g}_1. \tag{1.4.20}$$

The set of simple weights is $\Gamma = \{\beta_1, \beta_2\}$, where $\beta_i \in \mathbf{g}_0^*$ are defined by $\beta_1(e_{11} - e_{33}) = 1$ and respectively $\beta_2(e_{11} - e_{33}) = -2$. Also $\mathbf{g}_1^{\beta_1} = \mathbb{C}(e_{12} - e_{23})$, $\mathbf{g}_1^{\beta_2} = \mathbb{C}e_{31}$. Because $(\beta_1, \beta_1) \neq (\beta_2, \beta_2)$, there exists a unique admissible triple $\Gamma_1 = \Gamma_2 = \emptyset$. We consider the equation

$$r^{12} + r^{21} = t_0, \tag{1.4.21}$$

with $r \in \mathbf{g}_0 \otimes \mathbf{g}_0$ and $t_0$ given by (1.4.15). The only solution is

$$r = \frac{1}{4}(e_{11} - e_{33}) \otimes (e_{11} - e_{33}). \tag{1.4.22}$$

The trigonometric solution is therefore the following:

$$X_4(u) = \frac{1}{4}(e_{11} - e_{33}) \otimes (e_{11} - e_{33}) + \frac{1}{e^u - 1} \sum_{j=0}^{5} e^{ju/6} t_j, \tag{1.4.23}$$

with $t_j$ given by (1.4.15)-(1.4.20).

REMARK 1.4.1. The solution $X_4(u)$ is the classical $r$-matrix for the Zhiber-Shabat equation, [16].

Finally, we can conclude that the only nonequivalent trigonometric solutions of CYBE for $sl(3)$ are $X_i(u)$, $i \in \{1, 2, 3, 4\}$, given by (1.4.8), (1.4.12), (1.4.14) and (1.4.23). The nonequivalence of these solutions folows from Remark 1.3.5.

## 1.5. Lagrangian subspaces of $V_+ \oplus V_-$

Our aim is to compute "quasi-trigonometric" solutions of CYBE for $sl(2)$ and $sl(3)$ using some special Lagrangian subalgebras. In this section, we will give a result concerning Lagrangian subspaces that will be used later. Let $V$ be a finite-dimensional linear space, together with a symmetric nondegenerate bilinear form denoted by $<,>_+$. Consider also the bilinear form given by

$$< a, b >_- = - < a, b >_+ . \tag{1.5.1}$$

Let us denote $V_+ = (V, <,>_+)$ and $V_- = (V, <,>_-)$. A nondegenerate bilinear form can be introduced on the linear space $V_+ \oplus V_-$ in the following way:

$$< (a,b),(c,d) >=< a,c >_+ + < b,d >_- . \tag{1.5.2}$$

We are interested in finding the linear subspaces $W \subset V_+ \oplus V_-$ such that $W = W^\perp$ with respect to the bilinear form on $V_+ \oplus V_-$ introduced in (1.5.2). Such a linear subspace is called *Lagrangian*. Their description is given in the following

THEOREM 1.5.1. *If $W$ is a Lagrangian subspace of $V_+ \oplus V_-$, then there exist two subspaces $W_+ \subseteq V_+$ and $W_- \subseteq V_-$ such that $W_+ \supseteq W_+^\perp$, $W_- \supseteq W_-^\perp$ and an isomorphism $\Phi : \frac{W_+}{W_+^\perp} \to \frac{W_-}{W_-^\perp}$ which preserves the bilinear form. Conversely, if $W_+$, $W_-$ and $\Phi$ are given satisfying the above properties, then*

$$W = (W_+^\perp \oplus W_-^\perp) + \{(\widetilde{x}, \Phi(\widetilde{x})); \widetilde{x} \in \frac{W_+}{W_+^\perp}\}$$

*is a Lagrangian subspace of $V_+ \oplus V_-$.*

REMARK 1.5.2. Because $W_+^\perp \subseteq W_+$, $\frac{W_+}{W_+^\perp}$ can be identified with the subspace of $W_+$ which is complementary to $W_+^\perp$.

PROOF. We will construct $W_+$, $W_-$ and $\Phi$. Consider $W$ a Lagrangian subspace of $V_+ \oplus V_-$. If $\pi_+$: $V_+ \oplus V_- \to V_+$ and $\pi_-$: $V_+ \oplus V_- \to V_-$ are the canonical projections, then let us take the linear subspaces $W_+ = \pi_+(W) \subseteq V_+$ and $W_- = \pi_-(W) \subseteq V_-$. These linear spaces verify the conditions

$$W_+^\perp \subseteq W_+, W_-^\perp \subseteq W_-, \tag{1.5.3}$$

where the orthogonals are considered with respect to $<,>_+$ and $<,>_-$. In order to check this, let us take $x \in W_+^\perp$. It results that $< x,y >_+ = 0$ for any $y \in W_+$. For all pairs $(y_+, y_-) \in W$, $< (x,0),(y_+,y_-) >=< x, y_+ >_+ = 0$. It follows that $(x,0) \in W^\perp$. Because $W$ is a Lagrangian subspace, $(x,0) \in W$ and thus $x \in W_+$. Therefore $W_+^\perp \subseteq W_+$. Similarly one gets the second property.

It is possible to consider now the linear spaces $\frac{W_+}{W_+^\perp}$ and $\frac{W_-}{W_-^\perp}$. The bilinear forms $<,>_+$ and $<,>_-$ induce two nondegenerate bilinear forms on these spaces:

$$< \widetilde{a}, \widetilde{b} >=< a,b >_+, \widetilde{a}, \widetilde{b} \in \frac{W_+}{W_+^\perp}; \tag{1.5.4}$$

$$< \widetilde{x}, \widetilde{y} >= - < x,y >_-, \widetilde{x}, \widetilde{y} \in \frac{W_-}{W_-^\perp}. \tag{1.5.5}$$

Obviously, $W \subseteq W_+ \oplus W_-$. This implies that $W^\perp \supseteq W_+^\perp \oplus W_-^\perp$. Let us consider

$$\widetilde{W} = \frac{W}{W_+^\perp \oplus W_-^\perp} \subseteq \frac{W_+ \oplus W_-}{W_+^\perp \oplus W_-^\perp} \cong \frac{W_+}{W_+^\perp} \oplus \frac{W_-}{W_-^\perp}. \tag{1.5.6}$$

It is easy to check that $\widetilde{W}$ is a Lagrangian subspace in $\frac{W_+}{W_+^\perp} \oplus \frac{W_-}{W_-^\perp}$. Let us prove now that $\frac{W_+}{W_+^\perp} \cong \frac{W_-}{W_-^\perp}$. Put $\dim \frac{W_+}{W_+^\perp} = n_1$ and $\dim \frac{W_-}{W_-^\perp} = n_2$. It is enough to show that $n_1 = n_2$. Assume for instance that $n_1 > n_2$. Then $\dim \widetilde{W} = \frac{n_1+n_2}{2} < n_1$ (here we used the nondegerency). Consider the projections $\widetilde{\pi}_+ \colon \widetilde{W} \to \frac{W_+}{W_+^\perp}$ and $\widetilde{\pi}_- \colon \widetilde{W} \to \frac{W_-}{W_-^\perp}$ . From our assumption, it results that $\widetilde{\pi}_+$ is not an epimorphism, which is false because for any $x \in W_+$, there exists $(x, y) \in W$ and thus $\widetilde{\pi}_+((x, y)+W_+^\perp \oplus W_-^\perp) = x+W_+^\perp$. Therefore $n_1 = n_2$ and there exists an isomorphism $\Phi : \frac{W_+}{W_+^\perp} \to \frac{W_-}{W_-^\perp}$, $\Phi = \widetilde{\pi}_- \widetilde{\pi}_+^{-1}$. We make the remark that both $\widetilde{\pi}_+$and $\widetilde{\pi}_-$ are isomorphisms. We finally notice that $\Phi$ preserves the bilinear form,

$$< \Phi(\widetilde{x}), \Phi(\widetilde{y}) >=< \widetilde{x}, \widetilde{y} >, \tag{1.5.7}$$

for any $\widetilde{x}, \widetilde{y} \in \frac{W_+}{W_+^\perp}$.

For the second part of the theorem, it is enough to show that $W \subseteq W^\perp$. Since $W_+^\perp \subseteq W_+$ , $W_-^\perp \subseteq W_-$ and $\Phi$ preserves the bilinear form, for any $u, v \in W_+^\perp$, $z, t \in W_-^\perp$and $\widetilde{x}, \widetilde{y} \in \frac{W_+}{W_+^\perp}$ we have :

(1.5.8)

$$< (u + \widetilde{x}, z + \Phi(\widetilde{x})), (v + \widetilde{y}, t + \Phi(\widetilde{y})) >=< \widetilde{x}, \widetilde{y} > - < \Phi(\widetilde{x}), \Phi(\widetilde{y}) >= 0.$$

Thus $W$ is a Lagrangian subspace and this ends the proof. □

## 1.6. Lagrangian subalgebras of $\mathbf{g}_+ \oplus \mathbf{g}_-$

Let us consider a simple, finite-dimensional, complex Lie algebra $\mathbf{g}$ and let $K$ be the Killing form on it. Let $\mathbf{g}_+ = (\mathbf{g}, K)$ and $\mathbf{g}_- = (\mathbf{g}, -K)$. In this section we will be interested in describing the Lie subalgebras $W \subset \mathbf{g}_+ \oplus \mathbf{g}_-$ which are Lagrangian subspaces with respect to the bilinear form on $\mathbf{g}_+ \oplus \mathbf{g}_-$ induced by $K$:

$$Q((a, b), (c, d)) = K(a, c) - K(b, d). \tag{1.6.1}$$

According to Theorem 1.5.1, a Lagrangian subspace $W$ is determined by two subspaces $W_+ \subseteq \mathbf{g}_+$ and $W_- \subseteq \mathbf{g}_-$, which satisfy the properties $W_+ \supseteq W_+^\perp$, $W_- \supseteq W_-^\perp$, and an isomorphism $\Phi : \frac{W_+}{W_+^\perp} \to \frac{W_-}{W_-^\perp}$ which preserves the bilinear form. Following the proof of the theorem, one can see that if $W$ is a Lie subalgebra, then $W_+$ and $W_-$ are also subalgebras and $\Phi$ must be a Lie algebras isomorphism too. We will now give a sufficient condition for a subalgebra $S$ of $\mathbf{g}$ to satisfy $S^\perp \subseteq S$. In order to do this, we have to remind the notion of *parabolic subalgebra.*

DEFINITION 1.6.1. A *parabolic subalgebra* is a subalgebra of a complex Lie algebra such that it contains a Borel subalgebra (i.e. a maximal solvable subalgebra).

REMARK 1.6.2. If $P$ is a parabolic subalgebra then $P^\perp \subseteq P$.

EXAMPLE 1.6.3. In the algebra $gl(n)$ of all $n \times n$- matrices over $\mathbb{C}$, the parabolic subalgebras are all the algebras of the form $p(\mu)$, where $\mu = (m_1, m_2, ..., m_s)$ is an arbitrary choice of natural numbers with the sum equal to $n$, and $p(\mu)$ consists of all block matrices which are upper-triangular and the diagonal blocks are $m_i \times m_i$- matrices, $i = 1, ..., s$.

Let $\mathbf{g}$ be the simple Lie algebra from the beginning. It is possible to describe all parabolic subalgebras of $\mathbf{g}$. Consider $R$ the root system of $\mathbf{g}$, with respect to a Cartan subalgebra $\mathbf{g}^0$, $\Delta$ the set of simple roots and $\{\mathbf{g}^\alpha\}_{\alpha \in R}$ the root spaces. For any $\Psi \subseteq \Delta$ , let us denote by $\Pi(\Psi)$ the set of all $x \in R$ such that if $x = \sum k_\alpha \alpha$, then for any $\alpha \in \Psi$, $k_\alpha \geq 0$. It can be proved that any parabolic subalgebra can be transformed by an inner automorphism in one of the subalgebras:

$$p_\Psi = \mathbf{g}^0 + \sum_{\alpha \in \Pi(\Psi)} \mathbf{g}^\alpha. \tag{1.6.2}$$

In this way, the number of classes modulo the conjugation of parabolic subalgebras of $\mathbf{g}$ is equal to $2^r$, where $r =| \Delta |$. If $\Psi_1 \subseteq \Psi_2$ then $p_{\Psi_1} \supseteq p_{\Psi_2}$. In particular, $p_\emptyset = \mathbf{g}$ and $p_\Delta$ is the minimal parabolic subalgebra of $\mathbf{g}$.

EXAMPLE 1.6.4. i) If $\mathbf{g} = sl(2)$ then $r = 1$ and there exist only two parabolic subalgebras (up to conjugation) : $p_\emptyset = \mathbf{g}$ and $p_\Delta = B_+ = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} ; a, b \in \mathbb{C} \right\}$.

ii) If $\mathbf{g} = sl(3)$, then $r = 2$ and there exist 4 parabolic subalgebras (up to conjugation). We have: $\Delta = \{\alpha_1, \alpha_2\}$ the set of simple roots, where $\alpha_1$: $\mathbf{g}^0 \to \mathbb{C}$ is defined by $\alpha_1 \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} = a - b$ and $\alpha_2 : \mathbf{g}^0 \to \mathbb{C}$ is given by $\alpha_2 \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} = a + 2b$. It follows that $\mathbf{g}^{\alpha_1} = \mathbb{C}e_{12}$, $\mathbf{g}^{\alpha_2} = \mathbb{C}e_{23}$ and $\mathbf{g}^{\alpha_1+\alpha_2} = \mathbb{C}e_{13}$. For $\Psi \subseteq \Delta$ we have the following possibilities:

a) $\Psi = \emptyset \Rightarrow p_\Psi = \mathbf{g}$ ;

b) $\Psi = \Delta \Rightarrow p_\Delta = B_+ = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & -a-d \end{pmatrix} ; a, b, c, d, e \in \mathbb{C} \right\}$;

c) $\Psi = \{\alpha_1\} \Rightarrow p_{\{\alpha_1\}} = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & -a-d \end{pmatrix} ; a, b, c, d, e, f \in \mathbb{C} \right\}$;

d) $\Psi = \{\alpha_2\} \Rightarrow p_{\{\alpha_2\}} = \left\{ \begin{pmatrix} a & b & c \\ f & d & e \\ 0 & 0 & -a-d \end{pmatrix} ; a,b,c,d,e,f \in \mathbb{C} \right\}.$

### 1.7. Lagrangian subalgebras of $\mathbf{g}_+ \oplus \mathbf{g}_-$, for $\mathbf{g} = sl(2), sl(3)$

As we have seen in the previous section, the determination of the Lagrangian subalgebras implies to find the subalgebras $S$ of $\mathbf{g}$ satisfying $S^\perp \subseteq S$. In this section, we will do this for $\mathbf{g} = sl(2)$ and $sl(3)$.

(1) $\mathbf{g} = sl(2)$. The only subalgebras $S$ that verify the condition $S^\perp \subseteq S$ are the parabolic ones (up to conjugation): $p_\emptyset = sl(2)$ and $p_\Delta = B_+$.
(2) $\mathbf{g} = sl(3)$. We distinguish two situations:

i) Assume that $S$ is a subalgebra containing $B_+$(therefore $S$ is parabolic). It follows that $S$ is (up to conjugation) one of the subalgebras given in Example 1.6.4 from the previous section :

$$sl(3), B_+, p_{\{\alpha_1\}}, p_{\{\alpha_2\}}. \tag{1.7.1}$$

ii) Assume that $S$ does not contain $B_+$ and $S^\perp \subseteq S$ . After a suitable conjugation, we may suppose that $N_+ \subseteq S$, where $N_+$ denotes the subset of $B_+$ with zero entries on the diagonal. Moreover, the inclusion is strict. Indeed, if one supposes that $S = N_+$, it follows that $N_+^\perp \subseteq S$ and therefore $B_+ \subseteq S$, which is a contradiction. If, for example, $e_{21} \in S$, we may assume that $S$ is generated by $e_{11} - e_{22}$ , $e_{12}$, $e_{13}$, $e_{21}$, $e_{23}$. Particularly, $A = e_{11} + e_{22} - 2e_{33} \in S^\perp$ and it follows that $B_+ \subseteq S$ , contradiction. Thus $e_{21} \notin S$. Similarly, $e_{31} \notin S$ and $e_{32} \notin S$. In this way one obtains that $N_+ \subset S \subset B_+$. Therefore any matrix of $S$ has the form $\begin{pmatrix} \lambda_1 t & a & b \\ 0 & \lambda_2 t & c \\ 0 & 0 & \lambda_3 t \end{pmatrix}$, where $t, a, b, c \in \mathbb{C}$ and $\lambda_1, \lambda_2, \lambda_3$ will be determined by the condition $S^\perp \subseteq S$. This implies that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$. We have also $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Thus,

$$S = \left\{ \begin{pmatrix} \lambda_1 t & a & b \\ 0 & \lambda_2 t & c \\ 0 & 0 & \lambda_3 t \end{pmatrix} ; t, a, b, c \in \mathbb{C} \right\}, \tag{1.7.2}$$

where $\lambda_i$ verify the conditions $\lambda_1 + \lambda_2 + \lambda_3 = 0$, $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$ and $\lambda_i \neq 0$. It follows that $\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 = 0$ and thus $\lambda_2 = \lambda_1 \epsilon$ ; $\lambda_3 = \epsilon^2 \lambda_1$, where $\epsilon^2 + \epsilon + 1 = 0$. We may assume $\lambda_1 = 1$ and one obtains

$$S = \left\{ \begin{pmatrix} t & a & b \\ 0 & \epsilon t & c \\ 0 & 0 & \epsilon^2 t \end{pmatrix} ; t, a, b, c \in \mathbb{C} \right\}. \tag{1.7.3}$$

Our aim is to compute "quasi-trigonometric" solutions of CYBE for $\mathbf{g} = sl(2)$ and $sl(3)$. These solutions correspond to certain types of Lagrangian subalgebras of $\mathbf{g}_+ \oplus \mathbf{g}_-$. Therefore, we are not interested in the determination of all Lagrangian subalgebras for $sl(2)$ and $sl(3)$, but only in finding the special ones that satisfy certain properties which will be presented in the next sections.

### 1.8. Special Lagrangian subalgebras for the case $\mathbf{g} = sl(2)$

Let us denote by $\mathbf{h}$ the usual Cartan subalgebra of $sl(2)$, $B_+$ the set of upper-triangular matrices and $B_-$ the set of lower-triangular matrices of $sl(2)$. We will find all Lagrangian subalgebras $\overline{W} \subset \mathbf{g}_+ \oplus \mathbf{g}_-$ such that $\overline{W} \cap \overline{\Delta} = 0$, where

$$\overline{\Delta} = \left\{ \left( \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & -a \end{pmatrix} \right); a, b, c \in \mathbb{C} \right\}. \tag{1.8.1}$$

REMARK 1.8.1. $\overline{\Delta}$ is exactly the Lagrangian subalgebra obtained for $W_+ = W_- = B_+$ and $\Phi = id_{\mathbf{h}}$ (here $id_{\mathbf{h}}$ denotes the identity automorphism of $\mathbf{h}$).

According to Section 1.7, a Lagrangian subalgebra $\overline{W}$ is determined by a triple $(\overline{W}_+, \overline{W}_-, \Phi)$, where $\overline{W}_+$ and $\overline{W}_-$ are conjugated to $sl(2)$ or $B_+$and $\Phi : \frac{W_+}{W_+^\perp} \to \frac{W_-}{W_-^\perp}$ is an isomorphism preserving the bilinear form. We will consider the following situations:

1) $\overline{W}_+ = \overline{W}_- = B_-$. An isomorphism $\Phi$ of the Cartan subalgebra $\mathbf{h}$ that preserves the Killing form is $id_{\mathbf{h}}$ or $-id_{\mathbf{h}}$.

i) if $\Phi = id_{\mathbf{h}}$, then

$$\overline{W} = \left\{ \left( \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix}, \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix} \right); a, b, c \in \mathbb{C} \right\} \Rightarrow \overline{W} \cap \overline{\Delta} \neq 0. \tag{1.8.2}$$

ii) if $\Phi = -id_{\mathbf{h}}$, then

$$\overline{W} = \left\{ \left( \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix}, \begin{pmatrix} -a & 0 \\ c & a \end{pmatrix} \right); a, b, c \in \mathbb{C} \right\} \Rightarrow \overline{W} \cap \overline{\Delta} = 0. \tag{1.8.3}$$

2) $\overline{W}_+ = \overline{W}_- = sl(2)$. Obviously $\overline{W} = \{(X, \Phi(X)); X \in sl(2)\}$, where $\Phi \in Aut(sl(2))$ which preserves the Killing form. But any automorphism of $sl(2)$ is inner, so there exists $S \in GL(2)$ such that $\Phi(X) = SXS^{-1}$for any $X \in sl(2)$. In order that $\overline{W} \cap \overline{\Delta} = 0$, we will find $S$ such that the equation

$$S \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} S^{-1} = \begin{pmatrix} a & c \\ 0 & -a \end{pmatrix} \tag{1.8.4}$$

has only the solution $a = b = c = 0$. Let us consider $S = (s_{ij}) \in GL(2)$. (1.8.4) is equivalent to a linear homogeneous system which has only the solution $a = b = c = 0$ iff $s_{21} \neq 0$. Thus

$$\overline{W} = \{(X, SXS^{-1}); X \in sl(2)\}, \tag{1.8.5}$$

where $S = (s_{ij}) \in GL(2)$, $s_{21} \neq 0$, is a Lagrangian subalgebra verifying $\overline{W} \cap \overline{\Delta} = 0$.

Simple computations show that other situations will not lead to a Lagrangian subalgebra that is complementary to $\overline{\Delta}$.

### 1.9. Special Lagrangian subalgebras for the case $\mathbf{g} = sl(3)$

In this section, we consider $\mathbf{g} = sl(3)$. Let $\mathbf{h}$ be the usual Cartan subalgebra of $sl(3)$, $B_+$ the set of upper-triangular matrices and $B_-$ the set of lower-triangular matrices. We are now interested in finding Lagrangian subalgebras $\overline{W}$ such that $\overline{W} \cap \overline{\Delta} = 0$, where

$$\overline{\Delta} = \left\{ \left( \begin{pmatrix} A & b \\ 0 & -TrA \end{pmatrix}, \begin{pmatrix} A & b' \\ 0 & -TrA \end{pmatrix} \right); A \in gl(2), b, b' \in \mathbb{C}^2 \right\}. \tag{1.9.1}$$

REMARK 1.9.1. $\overline{\Delta}$ is a Lagrangian subalgebra.

According to the results in Section 1.7, for the determination of a Lagrangian subalgebra, one has several choices. We will analyse the following cases:

1) $\overline{W}_+ = B_+, \overline{W}_- = B_- \Rightarrow \frac{\overline{W}_+}{\overline{W}_+^\perp} = \frac{B_+}{N_+} \cong \mathbf{h}$ and $\frac{\overline{W}_-}{\overline{W}_-^\perp} = \frac{B_-}{N_-} \cong \mathbf{h}$. Consider $\Phi \in Aut(\mathbf{h})$ which preserves the Killing form. We take the canonical basis in $\mathbf{h}$ formed by $E = e_{11} - e_{33}$ , $F = e_{22} - e_{33}$ and set $\Phi(E) = a_1 E + a_2 F$ , $\Phi(F) = b_1 E + b_2 F$. Obviously, $\Phi$ preserves the Killing form iff the folowing conditions hold:

$$< \Phi(E), \Phi(F) >=< E, F >; \tag{1.9.2}$$

$$< \Phi(E), \Phi(E) >=< E, E >; \tag{1.9.3}$$

$$< \Phi(F), \Phi(F) >=< F, F > . \tag{1.9.4}$$

These are equivalent to the following:

$$2a_1 b_1 + a_1 b_2 + a_2 b_1 + 2a_2 b_2 = 1; \tag{1.9.5}$$

$$a_1^2 + a_2^2 + a_1 a_2 = 1; \tag{1.9.6}$$

$$b_1^2 + b_2^2 + b_1 b_2 = 1. \tag{1.9.7}$$

The corresponding Lagrangian subalgebra is
(1.9.8)

$$\overline{W} = \left\{ \left( \begin{pmatrix} x & * & * \\ 0 & y & * \\ 0 & 0 & -x-y \end{pmatrix}, \begin{pmatrix} a_1 x + b_1 y & 0 & 0 \\ * & a_2 x + b_2 y & 0 \\ * & * & -(...) \end{pmatrix} \right) \right\}.$$

Obviously, $(e_{13}, 0) \in \overline{W} \cap \overline{\Delta}$ and thus $\overline{W} \cap \overline{\Delta} \neq 0$.

2) $\overline{W}_+ = \overline{W}_- = B_-$. Similar computations show that $(e_{21}, e_{21}) \in \overline{W} \cap \overline{\Delta}$.

3) $\overline{W}_+ = \overline{W}_- = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\}$ ; it results that $\overline{W}_+^\perp = \overline{W}_-^\perp = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix}$ and thus $\frac{\overline{W}_+}{\overline{W}_+^\perp} = \frac{\overline{W}_-}{\overline{W}_-^\perp} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$. Any automorphism

$\Phi$ of $\frac{W_+}{W_+^\perp}$ which preserves the Killing form is given by $f \in Aut(gl(2))$ that verifies the identity:

$$Tr(f(A)f(B)) + Tr(f(A))Tr(f(B)) = Tr(AB) + TrATrB. \tag{1.9.9}$$

The corresponding Lagrangian subalgebra is

$$\overline{W} = \left\{ \left( \begin{pmatrix} A & 0 \\ * & -TrA \end{pmatrix}, \begin{pmatrix} f(A) & 0 \\ * & -Trf(A) \end{pmatrix} \right) \right\}, \tag{1.9.10}$$

where $f \in Aut(gl(2))$ satisfies (1.9.9). Because any such $f$ has at least one nonzero fixed point, it follows that $\overline{W} \cap \overline{\Delta} \neq 0$.

4) $\overline{W}_+ = \overline{W}_- = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}$, $\overline{W}_+^\perp = \overline{W}_-^\perp = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \right\}$

and $\frac{\overline{W}_+}{\overline{W}_+^\perp} = \frac{\overline{W}_-}{\overline{W}_-^\perp} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \Rightarrow \overline{W}$ is given by

$$\overline{W} = \left\{ \left( \begin{pmatrix} -TrA & 0 \\ * & A \end{pmatrix}, \begin{pmatrix} -Trf(A) & 0 \\ * & f(A) \end{pmatrix} \right) \right\}, \tag{1.9.11}$$

where $f \in Aut(gl(2))$ satisfies (1.9.9). It is easily seen that $(e_{21}, e_{21}) \in \overline{W} \cap \overline{\Delta}$ for any $f$.

5) $\overline{W}_+ = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\}$, $\overline{W}_- = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}$. In this case, $\frac{\overline{W}_+}{\overline{W}_+^\perp} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$ and $\frac{\overline{W}_-}{\overline{W}_-^\perp} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$. A Lagrangian subalgebra is given as usual by $\overline{W}_+$, $\overline{W}_-$ and an isomorphism $\Phi : \frac{\overline{W}_+}{\overline{W}_+^\perp} \to \frac{\overline{W}_-}{\overline{W}_-^\perp}$ which preserves the bilinear form. We will construct $\Phi$ such that the corresponding Lagrangian subalgebra $\overline{W}$ satisfies the condition $\overline{W} \cap \overline{\Delta} = 0$. Let us write every matrix from $\frac{\overline{W}_+}{\overline{W}_+^\perp}$ in the following way: $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & -a-d \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & a+d & 0 \\ 0 & 0 & -a-d \end{pmatrix}$. We may suppose that

$$\Phi \begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & c & -a \end{pmatrix} \tag{1.9.12}$$

for all $a$, $b$, $c \in \mathbb{C}$. We are allowed to do this because if we take instead an inner automorphism of $sl(2)$, this will lead to a Lagrangian subalgebra which is "gauge equivalent" to the one that we obtain now. Let us put $\Phi(e_{22} - e_{33}) = \alpha(e_{11} - e_{33}) + \beta(e_{22} - e_{33})$, where $\alpha$ and $\beta$ will be found such that $\overline{W}$ is complementary to $\overline{\Delta}$. First of all, because $\Phi$ preserves the bilinear form, it follows (*) $\alpha^2+\beta^2+\alpha\beta=1$. The Lagrangian subalgebra corresponding to $\Phi$ has the following form:

(1.9.13)
$$\overline{W}=\left\{\left(\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ * & * & -a-d \end{pmatrix}, \begin{pmatrix} \alpha(a+d) & 0 & 0 \\ * & (1+\beta)a+\beta d & b \\ * & c & -(...) \end{pmatrix}\right)\right\}.$$

The condition $\overline{W}\cap\overline{\Delta}=0$ is equivalent to the fact that the linear system $a=\alpha(a+d)$, $d=a+\beta(a+d)$ has only trivial solution $a=d=0$. In other words, $2\alpha+\beta\neq 1$. From the relation (*), it results that $(\alpha,\beta)\neq(0,1)$ and $(1,-1)$.

In conclusion, the Lagrangian subalgebra given by (1.9.13), where $\alpha^2+\beta^2+\alpha\beta=1$ and $(\alpha,\beta)\neq(0,1)$; $(1,-1)$, has the property $\overline{W}\cap\overline{\Delta}=0$.

6) $\overline{W}_+=\left\{\begin{pmatrix} \lambda_1 t & * & * \\ 0 & \lambda_2 t & * \\ 0 & 0 & \lambda_3 t \end{pmatrix}\right\}$, $\overline{W}_-=\left\{\begin{pmatrix} \mu_1 u & * & * \\ 0 & \mu_2 u & * \\ 0 & 0 & \mu_3 u \end{pmatrix}\right\}$, as in (1.7.2). In this case, $\overline{W}_+=\overline{W}_+^{\perp}$, $\overline{W}_-=\overline{W}_-^{\perp}$ and thus $\overline{W}=\overline{W}_+\oplus\overline{W}_-$. It is easy to see that $(e_{13},0)\in\overline{W}\cap\overline{\Delta}$ and therefore $\overline{W}\cap\overline{\Delta}\neq 0$.

7) $W_+=\left\{\begin{pmatrix} \lambda_1 t & * & * \\ 0 & \lambda_2 t & * \\ 0 & 0 & \lambda_3 t \end{pmatrix}\right\}$, $W_-=\left\{\begin{pmatrix} \mu_1 u & 0 & 0 \\ * & \mu_2 u & 0 \\ * & * & \mu_3 u \end{pmatrix}\right\}$ and again $\overline{W}=\overline{W}_+\oplus\overline{W}_-$. For example, $(e_{13},0)\in\overline{W}\cap\overline{\Delta}$.

8) $\overline{W}_+=\overline{W}_-=sl(3)\Rightarrow\overline{W}=\{(X,\Phi(X));X\in sl(3)\}$, $\Phi\in Aut(sl(3))$. The condition $\overline{W}\cap\overline{\Delta}=0$ is equivalent to the fact that the equation

$$\Phi\begin{pmatrix} A & b \\ 0 & -TrA \end{pmatrix}=\begin{pmatrix} A & b' \\ 0 & -TrA \end{pmatrix} \tag{1.9.14}$$

admits only the solution $A=0$, $b=b'=0$. We will determine $\Phi\in Aut(sl(3))$ that satisfy this property. We make the remark that if $\Phi\in Aut(sl(3))$ verifies the above condition, then $\Phi^{-1}$ also does.

Let us denote by $Aut^0(sl(3))$ the set of inner automorphisms of $sl(3)$. One knows that $Aut(sl(3))/Aut^0(sl(3))\cong Aut\Delta$ , i.e. the set of automorphisms of the Dynkin diagram of $sl(3)$. Because there are only two automorphisms of the Dynkin diagram, it results that any $\Phi\in Aut(sl(3))$ has one of the forms:

i) $\Phi(X)=SXS^{-1}$, for any $X\in sl(3)$;

ii) $\Phi(X)=-SX^tS^{-1}$, for any $X\in sl(3)$, where $S\in GL(3)$.

In the first situation, i), if $S = (s_{ij})$, the linear system represented by (1.9.14) admits only the trivial solution iff the following condition holds:
(1.9.15)

$$rank \begin{pmatrix} 0 & -s_{21} & s_{12} & 0 & 0 & 0 & -s_{31} & 0 \\ -s_{12} & s_{11}-s_{22} & 0 & s_{12} & 0 & 0 & -s_{32} & 0 \\ -2s_{13} & -s_{23} & 0 & -s_{13} & s_{11} & s_{12} & -s_{33} & 0 \\ s_{21} & 0 & s_{22}-s_{11} & -s_{21} & 0 & 0 & 0 & -s_{31} \\ 0 & s_{21} & -s_{12} & 0 & 0 & 0 & 0 & -s_{32} \\ -s_{23} & 0 & -s_{13} & -2s_{23} & s_{21} & s_{22} & 0 & -s_{33} \\ 2s_{31} & 0 & s_{32} & s_{31} & 0 & 0 & 0 & 0 \\ s_{32} & s_{31} & 0 & 2s_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{31} & s_{32} & 0 & 0 \end{pmatrix} = 8.$$

For example, (1.9.15) holds in one of the following situations:

(1) $s_{31} = 0,\ s_{32} \neq 0,\ s_{21} \neq 0,\ s_{11}s_{22} - s_{12}s_{21} \neq 0$ (and $detS \neq 0$);
(2) $s_{32} = 0,\ s_{31} \neq 0,\ s_{12} \neq 0,\ s_{11}s_{22} - s_{12}s_{21} \neq 0$ (and $detS \neq 0$);
(3) $s_{31}s_{32} \neq 0,\ s_{11}s_{22}-s_{12}s_{21} \neq 0,\ s_{31}s_{32}(s_{11}-s_{22})-s_{12}s_{31}^2+s_{21}s_{32}^2 \neq 0$.

But there are also other possibilities. A concrete example is $S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and also $S^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ verifies (1.9.15).

In the second situation, ii), the matrix of the linear system representing (1.9.14) is the following:
(1.9.16)

$$\begin{pmatrix} -2s_{11} & -s_{12}-s_{21} & 0 & 0 & -s_{13} & 0 & -s_{31} & 0 \\ -s_{12} & -s_{22} & -s_{11} & -s_{12} & 0 & -s_{13} & -s_{32} & 0 \\ 0 & -s_{23} & 0 & s_{13} & 0 & 0 & -s_{33} & 0 \\ -s_{21} & -s_{22} & -s_{11} & -s_{21} & -s_{23} & 0 & 0 & -s_{31} \\ 0 & 0 & -s_{21}-s_{12} & -2s_{22} & 0 & -s_{23} & 0 & -s_{32} \\ s_{23} & 0 & -s_{13} & 0 & 0 & 0 & 0 & -s_{33} \\ 0 & -s_{32} & 0 & s_{31} & -s_{33} & 0 & 0 & 0 \\ s_{32} & 0 & -s_{31} & 0 & 0 & -s_{33} & 0 & 0 \\ s_{33} & 0 & 0 & s_{33} & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we can prove that the rank of this matrix is at most 7. Let us consider the following situations:

a) $s_{33} = 0$. One can compute the only minor of order 8, without a null line, by using Laplace's rule, and obtain that this minor also vanishes.

b) $s_{33} \neq 0$. By suitable elementary transformations applied to the matrix from (1.9.16), we get

(1.9.17)

$$\begin{pmatrix} 0 & 0 & -s_{13} & 0 & -s_{31} & -s_{12}-s_{21} & 0 & 2s_{11} \\ 0 & -s_{13} & s_{23} & s_{31} & -s_{32} & 0 & 0 & 0 \\ 0 & -s_{23} & 0 & -s_{32} & 0 & 0 & -s_{12}-s_{21} & -2s_{22} \\ 0 & 0 & -s_{23} & -s_{31} & 0 & -s_{22} & -s_{11} & 0 \\ 0 & 0 & 0 & 0 & -s_{33} & -s_{23} & 0 & s_{13} \\ 0 & 0 & 0 & -s_{33} & 0 & 0 & -s_{13} & -s_{23} \\ 0 & 0 & -s_{33} & 0 & 0 & -s_{32} & 0 & s_{31} \\ 0 & -s_{33} & 0 & 0 & 0 & 0 & -s_{31} & -s_{32} \\ s_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and one can check that the columns of the equivalent matrix are linearly dependent and therefore the rank of the matrix cannot be 8. We conclude that, in case ii), $\overline{W} \cap \overline{\Delta} \neq 0$.

One can check easily that other choices of the triples $(\overline{W}_+, \overline{W}_-, \Phi)$ do not lead to a Lagrangian subalgebra that satisfies the condition $\overline{W} \cap \overline{\Delta} = 0$.

## 1.10. Lie bialgebras and classical $r$-matrices

In this section we will give an overview of the basic results concerning Lie bialgebras and classical $r$-matrices. These will be used in the last section for the computation of the "quasi-trigonometric" solutions.

Firstly, let us remind the notions of *Lie bialgebra* and *classical double* associated to it. Let $\Lambda$ be a complex Lie algebra and $\Lambda^*$ its dual vector space. Suppose that $\Lambda^*$has also a fixed Lie algebra structure. Let $\varphi : \Lambda^* \otimes \Lambda^* \to \Lambda^*$be defined by the rule: $\varphi(l_1 \otimes l_2) = [l_1, l_2]$. Then $\varphi^*\colon \Lambda \to \Lambda \otimes \Lambda$ satisfies the following conditions:

(1) $Im\varphi^* \subseteq \Lambda \wedge \Lambda$;
(2) $Per(\varphi^* \otimes 1)\varphi^*(a) = 0$ for any $a \in \Lambda$, where $Per(a \otimes b \otimes c) = a \otimes b \otimes c + c \otimes a \otimes b + b \otimes c \otimes a$ (co-Jacobi identity).

The following result was given in [4]:

THEOREM 1.10.1. *([4],Th.2) The following conditions are equivalent:*

*1) The map $\varphi^*\colon \Lambda \to \Lambda \otimes \Lambda$ is a 1-cocycle, being understood that $\Lambda$ acts on $\Lambda \otimes \Lambda$ by means of the adjoint representation, i.e. for any $a, b \in \Lambda$, $\varphi([a,b]) = a.\varphi(b) - b.\varphi(a)$, where $a.(b \otimes c) = [a \otimes 1 + 1 \otimes a,\ b \otimes c] = [a,b] \otimes c + b \otimes [a,c]$.*

*2) There is a Lie algebra structure on $\Lambda \oplus \Lambda^*$ inducing the given Lie algebra structures on $\Lambda$ and $\Lambda^*$which is such that the bilinear form $Q$ given by the formula*

$$Q((x_1, l_1), (x_2 + l_2)) = l_1(x_2) + l_2(x_1) \tag{1.10.1}$$

*for any $x_1, x_2 \in \Lambda$ and $l_1, l_2 \in \Lambda^*$, is invariant with respect to the adjoint representation of $\Lambda \oplus \Lambda^*$. Moreover, such a Lie algebra structure on $\Lambda \oplus \Lambda^*$is unique if it exists.*

DEFINITION 1.10.2. The Lie algebra structure on $\Lambda \oplus \Lambda^*$satisfying the second condition is called *the classical double of* $\Lambda$ and it is denoted by $D(\Lambda, \varphi)$ or simply $D(\Lambda)$. The pair $(\Lambda, \varphi)$ is called a *Lie bialgebra.*

REMARK 1.10.3. $D(\Lambda)$ is not generally speaking unique for a given $\Lambda$ because there are many different Lie algebra structures on $\Lambda^*$and some satisfy the conditions of the previous theorem.

DEFINITION 1.10.4. Two Lie bialgebra structures $(\Lambda, \varphi_1)$ and $(\Lambda, \varphi_2)$ on $\Lambda$ are called *equivalent* if there exists a Lie algebra isomorphism $f : D(\Lambda, \varphi_1) \to D(\Lambda, \varphi_2)$, which preserves the canonical forms $Q_i$ on $D(\Lambda, \varphi_i)$, and such that $fj_1 = j_2$, where $j_1$ and respectively $j_2$ are the embeddings of $\Lambda$ in the two doubles.

We recall a result due to A. Stolin and E. Karolinsky:

THEOREM 1.10.5. *Two Lie bialgebra structures* $(\Lambda, \varphi_1)$ *and* $(\Lambda, \varphi_2)$ *on* $\Lambda$ *are equivalent if and only if* $\varphi_2 = \varphi_1 + ds$*, where* $s \in \Lambda \wedge \Lambda$ *and* $< s, s >= Alt(\varphi \otimes id)(s)$*, where* $Alt(x) = x^{123} + x^{231} + x^{312}$ *for* $x \in \Lambda^{\otimes 3}$.

The Lie bialgebras are closely related to the so-called *classical* $r$*-matrices.* We will remind the following proposition from [9]:

PROPOSITION 1.10.6. *([9], Prop.4.1.1) Let* $\Lambda$ *be a Lie algebra and* $r \in \Lambda \wedge \Lambda$*. Consider the linear map* $\varphi : \Lambda \to \Lambda \otimes \Lambda$ *given by*

$$\varphi(a) = [a \otimes 1 + 1 \otimes a, r], \tag{1.10.2}$$

*for any* $a \in \Lambda$*. Then* $(\Lambda, \varphi)$ *is a Lie bialgebra if and only if the tensor*

$$< r, r >= [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \tag{1.10.3}$$

*is* ***ad****-invariant. Here we use the usual notation:* $r^{12} = r \otimes 1$*,* $r^{23} = 1 \otimes r$ *and* $r^{13} = \sum_k a_k \otimes 1 \otimes b_k$ *if* $r = \sum_k a_k \otimes b_k$*. The commutators on the right-hand side are taken in* $(U\Lambda)^{\otimes 3}$*, where* $U\Lambda$ *denotes the universal enveloping algebra.*

DEFINITION 1.10.7. A tensor $r$ that satisfies the condition $< r, r >$ is **ad**-invariant is called *classical r-matrix*. The equation (1.10.3) is called *the modified classical Yang-Baxter equation (mCYBE).*

Let us recall the following result:

PROPOSITION 1.10.8. *([9], Prop.2.3.6) Let* $\Lambda$ *be a Lie bialgebra and* $D(\Lambda)$ *the classical double associated to it. Suppose that* $\{I_\alpha\}$ *is a basis in* $\Lambda$ *and* $\{I^\alpha\}$ *is the dual basis of* $\Lambda^*$*. Let us identify* $I_\alpha$ *with* $(I_\alpha, 0) \in D(\Lambda)$ *and* $I^\alpha$ *with* $(0, I^\alpha) \in D(\Lambda)$*. Then the cobracket in* $D(\Lambda)$ *is given by (1.10.2) for any* $a \in D(\Lambda)$*, where*

$$r = \sum_\alpha I_\alpha \otimes I^\alpha \in D(\Lambda) \otimes D(\Lambda). \tag{1.10.4}$$

### 1.11. Quasi-trigonometric solutions

In this section we will give some suggestions about the construction of "quasi-trigonometric" solutions of CYBE for $sl(2)$ and $sl(3)$ using the Lagrangian subalgebras $\overline{W}$ that satisfy the condition $\overline{W} \cap \overline{\Delta} = 0$.

Let us now consider $\mathbb{C}[[u^{-1}]]$ the ring of formal power series in $u^{-1}$ and $\mathbb{C}((u^{-1}))$ the field of its quotients. For an arbitrary simple, complex, finite-dimensional Lie algebra $\mathbf{g}$, we set $\mathbf{g}[u] = \mathbf{g} \otimes \mathbb{C}[u]$, $\mathbf{g}[[u^{-1}]] = \mathbf{g} \otimes \mathbb{C}[[u^{-1}]]$ and $\mathbf{g}((u^{-1})) = \mathbf{g} \otimes \mathbb{C}((u^{-1}))$.

Let us take $\{I_\mu\}$ an orthonormal basis in $\mathbf{g}$ with respect to the Killing form $K$ and $t = I_\mu \otimes I_\mu$ (with summation over identical indices). For any root $\alpha$ let us choose $e_\alpha$ such that $K(e_\alpha, e_{-\alpha}) = 1$. Also, we take $h_\alpha = [e_\alpha, e_{-\alpha}]$. The classical Drinfeld-Jimbo r-matrix defined by

$$r = \frac{1}{2}\sum_{\alpha>0} e_\alpha \wedge e_{-\alpha} + \frac{t}{2} \tag{1.11.1}$$

satisfies the following equations:

$$r + r^{21} = t, \tag{1.11.2}$$

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0. \tag{1.11.3}$$

Let us consider the function $X_0$ defined by $X_0(u,v) = \frac{u}{v-u}t + r$, for any $u, v \in \mathbb{C}$. Obviously, $X_0(u,v) + X_0^{21}(v,u) = 0$. Let us check that $X_0(u,v)$ verifies CYBE. Firstly, because $K([I_i, I_j], I_k) = K(I_i, [I_j, I_k])$ for any $i, j, k$ and $\{I_i\}_{i=1,..,n}$ is orthonormal with respect to the Killing form $K$, it results that $[t^{12}, t^{13}] = -[t^{12}, t^{23}] = [t^{13}, t^{23}]$. Secondly, it is known that $[t, a \otimes 1 + 1 \otimes a] = 0$ for any $a \in \mathbf{g}$. This implies that $[r^{12} + r^{13}, t^{23}] = 0$, $[r^{13} + r^{23}, t^{12}] = 0$ and $[r^{21} + r^{23}, t^{13}] = 0$. Therefore $X_0(u,v)$ satisfies CYBE.

REMARK 1.11.1. If $u = e^\lambda$, $v = e^\mu$, then $\widetilde{X_0}(\lambda, \mu) = X_0(e^\lambda, e^\mu) = \frac{e^{\lambda-\mu}}{1-e^{\lambda-\mu}}t + r$ is a solution of CYBE which depends only on $\lambda - \mu$. In addition, $\widetilde{X_0}$ is trigonometric.

We will say that a solution is *quasi-trigonometric* if it has the form $X(u,v) = \frac{u}{v-u}t + p(u,v)$, where $p(u,v)$ is some polynomial. Two solutions $X_1$ and $X_2$ will be called *gauge equivalent* if there exists $\varphi(u) \in Aut(\mathbf{g}[u])$ such that $X_2(u,v) = (\varphi(u) \otimes \varphi(v))(X_1(u,v))$.

Let us define the map $\delta: \mathbf{g}[u] \to \mathbf{g}[u] \wedge \mathbf{g}[v]$ by $\delta(a(u)) = [X_0(u,v), a(u) \otimes 1 + 1 \otimes a(v)]$. Because $[\frac{t}{u-v}, a(u) \otimes 1 + 1 \otimes a(v)] = -[t, 1 \otimes \frac{a(v)-a(u)}{v-u}] \in \mathbf{g}[u] \wedge \mathbf{g}[v]$, $\delta(a(u))$ is indeed an element of $\mathbf{g}[u] \wedge \mathbf{g}[v]$. The map $\delta$ is a 1-cocycle and therefore it defines a Lie bialgebra structure on the Lie algebra $\mathbf{g}[u]$. The following theorem gives a description of the double $D(\mathbf{g}[u])$ of this Lie bialgebra as the direct sum $\mathbf{g}((u^{-1})) \oplus \mathbf{g}$, where we have considered the following invariant bilinear form

$$Q((f(u),a),(g(u),b)) = K(f(u)g(u))_0 - K(a,b). \tag{1.11.4}$$

Here the index zero means that we have considered the free term in the series expansion.

THEOREM 1.11.2. *Suppose that $D(\mathbf{g}[u])$ is the classical double of $\mathbf{g}[u]$ induced by the 1-cocycle $\delta$. Then $D(\mathbf{g}[u])$ is isomorphic to $\mathbf{g}((u^{-1}))\oplus\mathbf{g}$.*

PROOF. The algebra $\mathbf{g}[u]$ is embedded in $\mathbf{g}((u^{-1}))\oplus\mathbf{g}$ in a natural way : the image of $a(u)$ is $(a(u),a(0))$. Let us consider the set $\Delta = \{(a(u),a(0)); a(u)\in\mathbf{g}[u]\}$. In order to prove the theorem, we will construct a Lagrangian subalgebra $W_0$ of $\mathbf{g}((u^{-1}))\oplus\mathbf{g}$ (with respect to the bilinear form $Q$) that is complementary to $\Delta$. A basis of $\Delta$ is given by the following elements: $(I_iu^k,0)$ for $i=1,..,n$ and any $k>0$, $(e_\alpha,e_\alpha)$ for any root $\alpha$, and $(h_\beta,h_\beta)$ for any simple root $\beta$. Let us consider the linear space $W_0$ spanned by the elements: $(I_iu^{-k},0)$ for $i=1,..,n$ and any $k>0$, $(e_{-\alpha},0)$, $(0,-e_\alpha)$ for any positive root $\alpha$, and $(h_\beta,-h_\beta)$ for any simple root $\beta$. It can be easily seen that $W_0$ is a Lagrangian subalgebra such that $W_0\oplus\Delta=\mathbf{g}((u^{-1}))\oplus\mathbf{g}$.

Let us choose two dual bases in $\Delta$ and respectively $W_0$. We have only to replace $(h_\beta,-h_\beta)$ in order to make them dual to $(h_\gamma,h_\gamma)$. Let us consider $\widetilde{h}_\beta=\sum a_{\beta\lambda}h_\lambda$ and impose the condition $Q((\widetilde{h}_\beta,-\widetilde{h}_\beta),(h_\gamma,h_\gamma))=1$ if $\beta=\gamma$ and 0 otherwise. It follows that $\sum a_{\beta\lambda}K(h_\lambda,h_\gamma)=\frac{1}{2}$ if $\beta=\gamma$ and 0 otherwise. Because the Cartan matrix $(K(h_\lambda,h_\gamma))$ is nondegenerate, the unknown $a_{\beta\lambda}$ can be found precisely.

We consider now the tensor built from these dual bases

$$r_0(u,v)=\sum_{k>0}(I_iu^k,0)\otimes(I_iv^{-k},0)+\sum_{\alpha>0}((e_\alpha,e_\alpha)\otimes(e_{-\alpha},0)+$$

$$+\sum_{\alpha>0}(e_{-\alpha},e_{-\alpha})\otimes(0,-e_\alpha)+\sum_\beta a_{\beta\lambda}(h_\beta,h_\beta)\otimes(h_\lambda,-h_\lambda).$$

If $p$ is the canonical projection of $\mathbf{g}((u^{-1}))\oplus\mathbf{g}$ on $\mathbf{g}[u]$ then $(p\otimes p)(r_0(u,v))=X_0(u,v)$ and therefore $D(\mathbf{g}[u])$ can be identified with $\mathbf{g}((u^{-1}))\oplus\mathbf{g}$. □

REMARK 1.11.3. This result enables us to reduce the problem of listing the quasi-trigonometric solutions of CYBE to the one of finding the Lagrangian subalgebras $W$ of $\mathbf{g}((u^{-1}))\oplus\mathbf{g}$ that are complementary to $\Delta$ and such that $W\supseteq u^{-N}\mathbf{g}[[u^{-1}]]$ for some $N>0$. If $W$ is a Lagrangian subalgebra that satisfies these conditions, let us consider two dual bases $\{E_i(u)\}$ in $\Delta$ and respectively $\{F^i(v)\}$ in $W$. The tensor $r(u,v)=\sum_i E_i(u)\otimes F^i(v)$ satisfies CYBE and $r^{12}(u,v)+r^{21}(v,u)$ is $\mathbf{ad}$-invariant. The tensor $X(u,v)=(p\otimes p)(\sum_i E_i(u)\otimes F^i(v))$ has the form $X(u,v)=X_0(u,v)+p(u,v)$, where $p(u,v)$ is a polynomial.

Let us restrict our interest to $\mathbf{g}=sl(n)$ and make the following

REMARK 1.11.4. If $T\in GL(n,\mathbb{C}[u])$, $X_0(u,v)$ and $\widetilde{X}(u,v)=\mathbf{Ad}(T(u)\otimes T(v))(X_0(u,v))$ are equivalent solutions. Here $T(u)$ acts on $sl(n,\mathbb{C}[u])$ by conjugation. According to Theorem 1.10.5, it results that $X_0(u,v)$ and $\widetilde{X}(u,v)$ induce the same classical double $D(\mathbf{g}[u])$ of $\mathbf{g}[u]$. A simple calculus shows that $\widetilde{X}(u,v) = X_0(u,v) + \widetilde{p}(u,v)$, where $\widetilde{p}$ is a polynomial.

Now we will give the correspondence between the quasi-trigonometric solutions and the Lagrangian subalgebras that we have determined in Sections 1.8 and 1.9. We consider the following two situations:

A. Suppose that $\mathbf{g} = sl(2)$. From [13], it results that any Lagrangian subalgebra $W$ of $\mathbf{g}((u^{-1})) \oplus \mathbf{g}$ which is complementary to $\Delta$ and such that $W \supseteq u^{-N} sl(2, \mathbb{C}[[u^{-1}]])$ can be embedded (up to gauge equivalence) in $L = diag(1,u) sl(2, \mathbb{C}[[u^{-1}]]) diag(1, u^{-1}) \oplus sl(2)$ or in $L' = sl(2, \mathbb{C}[[u^{-1}]]) \oplus sl(2)$. Here "gauge equivalence" between two Lagrangian subalgebras $W_1$ and $W_2$ means that there exists $T \in GL(2, \mathbb{C}[u])$ such that $(\mathbf{Ad}T(u), \mathbf{Ad}T(0))W_1 = W_2$. Let us prove that $\frac{L}{L^\perp} \cong sl(2) \oplus sl(2)$ (the orthogonal is considered with respect to the bilinear form $Q$). Firstly, let us make the remark that $sl(2, \mathbb{C}[[u^{-1}]])^\perp = u^{-1} sl(2, \mathbb{C}[[u^{-1}]])$ where the orthogonal is in $sl(2, \mathbb{C}((u^{-1})))$ with respect to the bilinear form $< f(u), g(u) >= Tr(f(u)g(u))_0$. It follows that

$$\frac{L}{L^\perp} \cong \frac{diag(1,u) sl(2, \mathbb{C}[[u^{-1}]]) diag(1, u^{-1})}{diag(u^{-1}, 1) sl(2, \mathbb{C}[[u^{-1}]]) diag(1, u^{-1})} \oplus sl(2). \tag{1.11.5}$$

One identifies the class of the matrix $\begin{pmatrix} a & bu^{-1} \\ cu & -a \end{pmatrix}$ with $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ and therefore the quotient that appears on the right-hand side is isomorphic to $sl(2)$.

On the other hand, a straightforward computation shows that

$$L \cap \Delta = \left\{ \left( \begin{pmatrix} a & 0 \\ b + cu & -a \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} \right), a, b, c \in \mathbb{C} \right\}. \tag{1.11.6}$$

The image of this set via the canonical projection $\pi : L \to \frac{L}{L^\perp} \cong sl(2) \oplus sl(2)$ is $\overline{\Delta} = \left\{ \left( \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} \right); a, b, c \in \mathbb{C} \right\}$. If $W$ is a Lagrangian subalgebra of $D(\mathbf{g}[u])$ which is complementary to the diagonal, then $\pi(W)$ is a Lagrangian subalgebra of $sl(2) \oplus sl(2)$ complementary to $\overline{\Delta}$. We remind that in Section 1.8 we have determined the two Lagrangian subalgebras $\overline{W}$ in $sl(2) \oplus sl(2)$ which verify the condition $\overline{W} \cap \overline{\Delta} = 0$ ($\overline{\Delta}$ is conjugated to the one written in Section 1.8). These Lagrangian subalgebras will lead to two quasi-trigonometric solutions (equivalent or not).

Let us consider the case when $W$ is embedded in $L'$. It is obvious that $\frac{L'}{L'^\perp} \cong sl(2) \oplus sl(2)$. If $\pi'$ denotes the canonical projection of $L'$ onto $sl(2) \oplus sl(2)$, then $\pi'(W)$ is a Lagrangian subalgebra of $sl(2) \oplus sl(2)$ complementary to $\{(x,x);\ x \in sl(2)\}$. It is known that there is only one (up to equivalence) such Lagrangian subalgebra, which is related to the unique solution (up to equivalence) of mCYBE in $sl(2)$ (see Theorem 4.2.2. in [9]). Therefore $W$ leads to one of the two trigonometric solutions obtained in Section 1.3.

B. Suppose that $\mathbf{g} = sl(3)$. Again it follows from [13] that any Lagrangian subalgebra $W$ of $\mathbf{g}((u^{-1})) \oplus \mathbf{g}$, complementary to $\Delta$, and such that $W \supseteq u^{-N} sl(3, \mathbb{C}[[u^{-1}]])$ can be embedded (up to gauge equivalence) in $L = diag(1, u, u) sl(3, \mathbb{C}[[u^{-1}]]) diag(1, u^{-1}, u^{-1}) \oplus sl(3)$ or in $L' = sl(3, \mathbb{C}[[u^{-1}]]) \oplus sl(3)$. Since $sl(3, \mathbb{C}[[u^{-1}]])^{\perp} = u^{-1} sl(3, \mathbb{C}[[u^{-1}]])$, the orthogonal being considered with respect to the form $< f(u), g(u) >= Tr(f(u)g(u))_0$, we have that

$$\frac{L}{L^{\perp}} \cong \frac{diag(1, u, u) sl(3, \mathbb{C}[[u^{-1}]]) diag(1, u^{-1}, u^{-1})}{diag(u^{-1}, 1, 1) sl(3, \mathbb{C}[[u^{-1}]]) diag(1, u^{-1}, u^{-1})} \oplus sl(3). \tag{1.11.7}$$

One identifies the class of $\begin{pmatrix} a & bu^{-1} & cu^{-1} \\ du & e & f \\ gu & h & -a-e \end{pmatrix}$ with $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix}$ and therefore one obtains that $\frac{L}{L^{\perp}} \cong sl(3) \oplus sl(3)$.

A simple calculus shows that

$$L \cap \Delta = \left\{ \left( \begin{pmatrix} a & 0 & 0 \\ b_0 u + b_1 & d & e \\ c_0 u + c_1 & f & -a-d \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ b_1 & d & e \\ c_1 & f & -a-d \end{pmatrix} \right) \right\}. \tag{1.11.8}$$

The image of this set via the map $\pi : L \to \frac{L}{L^{\perp}} \cong sl(3) \oplus sl(3)$ is

$$\overline{\Delta} = \left\{ \left( \begin{pmatrix} a & 0 & 0 \\ b_0 & d & e \\ c_0 & f & -a-d \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ b_1 & d & e \\ c_1 & f & -a-d \end{pmatrix} \right) \right\}. \tag{1.11.9}$$

Then $\pi(W)$ is a Lagrangian subalgebra of $sl(3) \oplus sl(3)$ complementary to $\overline{\Delta}$. In Section 1.9, we have found two Lagrangian subalgebras $\overline{W}$ which have this property ($\overline{\Delta}$ written in (1.11.9) is conjugated to the one from Section 1.9). These will lead to two "nontrivial" quasi-trigonometric solutions. Consequently, it seems to exist a 1-1 correspondence between "nontrivial" quasi-trigonometric solutions and Lagrangian subalgebras $\overline{W}$ such that $\overline{W} \cap \overline{\Delta} = 0$ (up to gauge equivalence). A more detailed analysis will be done in future work.

We consider now the case when the Lagrangian subalgebra $W$ is embedded in $L'$. Obviously, $\frac{L'}{L'^{\perp}} \cong sl(3) \oplus sl(3)$. One can easily check that $\pi'(\Delta) = \{(x, x);\ x \in sl(3)\}$, where $\pi'$ denotes the canonical projection of $L'$ onto $sl(3) \oplus sl(3)$. Therefore $\pi'(W)$ is a Lagrangian subalgebra of $sl(3) \oplus sl(3)$ complementary to $\{(x, x);\ x \in sl(3)\}$. The Lagrangian subalgebras in $sl(3) \oplus sl(3)$ which are complementary to the diagonal are in 1-1 correspondence with solutions of mCYBE (see Theorem 4.2.2. in [9] and [5]). These Lagrangian subalgebras will provide two trigonometric solutions that have been obtained in Section 1.4, corresponding to trivial triples.

CHAPTER 2

# On the classical double of parabolic subalgebras

## 2.1. Basic Lie bialgebra structure for a parabolic subalgebra $P_\alpha$

In this chapter we will be interested in the computation of constant solutions of the modified classical Yang-Baxter equation(mCYBE) for a certain class of Lie algebras that are not simple. We remind that, given a finite-dimensional complex Lie algebra $\Lambda$, a solution of mCYBE is a tensor $s \in \Lambda \otimes \Lambda$ which satisfies the following conditions:

$$[a, s^{12} + s^{21}] = 0 \tag{2.1.1}$$

for any $a \in \Lambda$ and

$$[s^{12}, s^{13}] + [s^{12}, s^{23}] + [s^{13}, s^{23}] = 0. \tag{2.1.2}$$

Our goal is to construct solutions of mCYBE for parabolic subalgebras of a simple complex Lie algebra. Let $\mathbf{g}$ be a finite-dimensional complex, simple Lie algebra, with root system $R$ and a simple set of roots $\Delta$, with respect to a fixed Cartan subalgebra $\mathbf{h}$. For any $\alpha \in R$, let us denote by $\mathbf{g}^\alpha$ the corresponding root space. If we consider the Killing form $K$ on $\mathbf{g}$, for any nonzero element $e_\alpha$ of $\mathbf{g}^\alpha$ there exists an element $e_{-\alpha}$ of $\mathbf{g}^{-\alpha}$ such that $K(e_\alpha, e_{-\alpha}) = 1$. Indeed, assuming the contrary, it would follow that $K(e_\alpha, e_\beta) = 0$ for any $e_\beta \in \mathbf{g}^\beta$, no matter what $\beta$, and then the natural bilinear form $K$ would be singular. With these notations, let us remind that one of the classical Drinfeld-Jimbo r-matrices is the following:

$$r = \sum_{\alpha > 0} e_\alpha \wedge e_{-\alpha}. \tag{2.1.3}$$

It is well-known that $r$ satisfies the following conditions:

$$r^{12} + r^{21} = 0; \tag{2.1.4}$$

and the tensor $< r, r >$defined by

$$< r, r >= [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \tag{2.1.5}$$

is **ad**-invariant.

Now let $P_\alpha$ be the parabolic subalgebra of $\mathbf{g}$ corresponding to a simple positive root $\alpha$. $P_\alpha$ is the Lie algebra generated by the root vectors corresponding to the simple roots and their opposite except $-\alpha$. Thus, it is spanned by all the linear spaces $\mathbf{g}^{\beta}$ for $\beta > 0$ and $\mathbf{g}^{-\beta}$ for $\beta \neq \alpha$ , $\beta > 0$. We consider the map $\delta_r : \mathbf{g} \to \mathbf{g} \otimes \mathbf{g}$ defined by $\delta_r(a) = [a \otimes 1 + 1 \otimes a, r]$. In this section, we prove that $\delta_r$ provides a Lie bialgebra structure for $P_\alpha$. Therefore we will have a classical double $D(P_\alpha)$ induced by the 1-cocycle $\delta_r$.

PROPOSITION 2.1.1. *The map $\delta_r$ endows the parabolic subalgebra $P_\alpha$ with a Lie bialgebra structure.*

PROOF. Because $\delta_r$ is a 1-cocycle that satisfies the co-Jacoby identity, it is enough to show that $\delta_r(P_\alpha) \subseteq P_\alpha \wedge P_\alpha$. For all $\gamma \in R$, choose $e_\gamma \in \mathbf{g}^\gamma$ such that $K(e_\gamma, e_{-\gamma}) = 1$. It is known that $\{e_\gamma\}_{\gamma>0}$ is a basis for $n^+ = \sum_{\gamma>0} \mathbf{g}^\gamma$ and $\{e_{-\gamma}\}_{\gamma>0}$ is a basis for $n^- = \sum_{\gamma>0} \mathbf{g}^{-\gamma}$. Therefore it is enough to show that $\delta_r(e_\gamma) \in P_\alpha \wedge P_\alpha$ for any $\gamma > 0$ and $\gamma < 0$ which does not contain $-\alpha$. Define $N_{\alpha,\beta}$ by the formula $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ for $\alpha$,$\beta$,$\alpha + \beta \in R$. If $\alpha$,$\beta \in R$ and $\alpha + \beta \notin R$ we set $N_{\alpha,\beta} = 0$. The number $N_{\alpha,-\alpha}$ remains indefinite. Consider $\gamma > 0$. We have:
(2.1.6)
$$\delta_r(e_\gamma) = [e_\gamma \otimes 1 + 1 \otimes e_\gamma, \sum_{\beta>0} e_\beta \wedge e_{-\beta}] = \sum_{\beta>0} ([e_\gamma, e_\beta] \wedge e_{-\beta} + e_\beta \wedge [e_\gamma, e_{-\beta}]).$$

We make the following remarks: If $\beta$ does not contain $\alpha$ then $[e_\gamma, e_\beta] \wedge e_{-\beta} = N_{\gamma,\beta} e_{\gamma+\beta} \wedge e_{-\beta}$ or it equals zero or belongs to $\mathbf{h} \wedge P_\alpha$, in any case is an element from $P_\alpha \wedge P_\alpha$. If $\gamma - \beta$ is not a root or a positive one or is a negative root but does not contain $-\alpha$, then $e_\beta \wedge [e_\gamma, e_{-\beta}] = N_{\gamma,-\beta} e_\beta \wedge e_{\gamma-\beta} \in P_\alpha \wedge P_\alpha$ or it equals zero or belongs to $P_\alpha \otimes \mathbf{h}$. Thus the terms that remain to be considered are $N_{\gamma,\beta} e_{\gamma+\beta} \wedge e_{-\beta}$ when $\beta = \alpha + \omega$, $\omega > 0$ and $N_{\gamma,-\beta} e_\beta \wedge e_{\gamma-\beta}$ when $\gamma - \beta = -\alpha - \delta$, $\delta > 0$. Let $A_{\alpha,\gamma} = \{\omega > 0 : \alpha + \omega, \alpha + \omega + \gamma \in R\}$. Consequently,

$$(2.1.7) \quad \delta_r(e_\gamma) - \sum_{\omega \in A_{\alpha,\gamma}} (N_{\gamma,\alpha+\omega} + N_{\gamma,-\alpha-\gamma-\omega}) e_{\gamma+\alpha+\omega} \wedge e_{-\alpha-\omega} \in P_\alpha \wedge P_\alpha.$$

Similarly, for $\gamma > 0$ which does not contain $\alpha$,
(2.1.8)
$$\delta_r(e_{-\gamma}) = [e_{-\gamma} \otimes 1 + 1 \otimes e_{-\gamma}, \sum_{\beta>0} e_\beta \wedge e_{-\beta}] = \sum_{\beta>0} ([e_{-\gamma}, e_\beta] \wedge e_{-\beta} + e_\beta \wedge [e_{-\gamma}, e_{-\beta}])$$

and the only "problematic" terms are $N_{-\gamma,\beta} e_{-\gamma+\beta} \wedge e_{-\beta}$ for $\beta = \alpha + \omega$, $\omega > 0$ and $N_{-\gamma,-\beta} e_\beta \wedge e_{-\gamma-\beta}$ for $\beta + \gamma = \alpha + \delta$, $\delta > 0$. Therefore
(2.1.9)
$$\delta_r(e_{-\gamma}) - \sum_{\omega \in A_{\alpha,-\gamma}} (N_{-\gamma,\alpha+\omega} + N_{-\gamma,-\alpha+\gamma-\omega}) e_{-\gamma+\alpha+\omega} \wedge e_{-\alpha-\omega} \in P_\alpha \wedge P_\alpha.$$

The following lemma will prove that the two sums that appear in (2.1.7) and respectively (2.1.9) cancel and thus $\delta_r(e_\gamma)$ for $\gamma > 0$ and $\delta_r(e_{-\gamma})$ for $\gamma > 0$ not containing $\alpha$ are elements of $P_\alpha \wedge P_\alpha$. □

LEMMA 2.1.2. *Suppose that $\alpha$, $\beta$, $\gamma \in R$ satisfy the relation $\alpha + \beta + \gamma = 0$. Then $N_{\alpha,\gamma} + N_{\beta,\gamma} = 0$.*

PROOF. We have the following:

$$N_{\alpha,\gamma} = N_{\alpha,\gamma}K(e_{-\beta}, e_\beta) = K(N_{\alpha,\gamma}e_{\alpha+\gamma}, e_\beta) = K([e_\alpha, e_\gamma], e_\beta) =$$

$$= K(e_\alpha, [e_\gamma, e_\beta]) = K(e_\alpha, N_{\gamma,\beta}e_{-\alpha}) = N_{\gamma,\beta} = -N_{\beta,\gamma}.$$

□

In the next sections we will try to give a description for the classical double of the parabolic subalgebra $P_\alpha$, for different simple complex Lie algebras $\mathbf{g}$.

## 2.2. Example: $\mathbf{g} = sl(3)$

Consider $\mathbf{g} = sl(3)$ and the parabolic subalgebra $P = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$ corresponding to the root $\alpha$ defined by $\alpha \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} = a + 2b$. We put $r = \frac{1}{2}(e_{12} \wedge e_{21} + e_{23} \wedge e_{32} + e_{13} \wedge e_{31})$ and take the Lie bialgebra structure on $P$ induced by $\delta_r$. We intend to prove the following result:

THEOREM 2.2.1. *The classical double $D(P)$ is isomorphic to $sl(3) \oplus gl(2)$ (as Lie algebras).*

Because they have the same dimension, the two linear spaces are obviously isomorphic. We will prove in several steps that the bracket is the same.

LEMMA 2.2.2. *Let $Q$ be the symmetric bilinear form on $sl(3) \oplus gl(2)$ defined by*

$$Q((A, B), (C, D)) = Tr(AC) - Tr(BD) - TrBTrD. \tag{2.2.1}$$

Then $Q$ is nondegenerate and invariant.

PROOF. On $sl(3)$ let us consider the symmetric bilinear form $Q_1$ defined by $Q_1(A, C) = Tr(AC)$. For $gl(2)$ let us take $Q_2$ given by $Q_2(B, D) = Tr(BD) + TrBTrD$. It is known that $Q_1$ is nondegenerate and invariant. It can be easily check that $Q_2$ has the same properties. We make now the remark that $Q((A, B), (C, D)) = Q_1(A, C) - Q_2(B, D)$ and the conclusion follows immediately. □

Consider now the map $d : P \to sl(3) \oplus gl(2)$ defined by $d(x) = (x, \pi(x))$, for any $x \in P$, where $\pi : P \to gl(2)$ is given by $\pi \begin{pmatrix} A & b \\ 0 & -TrA \end{pmatrix} = A \in gl(2)$. Let us make the following remark: if $P^\perp$ is the orthogonal of $P$ with respect to $Q_1$ and $\mathbf{Red}(P)$ denotes the reductive part of $P$, then $P^\perp =$

$\left\{\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}\right\}$ and $\mathbf{Red}(P) \cong \frac{P}{P^\perp} = \left\{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}\right\} \cong gl(2)$. The next step is to find a Lagrangian subalgebra $W \subset sl(3) \oplus gl(2)$ (with respect to the nondegenerate bilinear form $Q$ from the previous lemma) that satisfies $d(P) \cap W = 0$. Or, equivalently, the linear spaces $W$ and $d(P)$ are complementary in $sl(3) \oplus gl(2)$.

LEMMA 2.2.3. *There is at least one Lagrangian subalgebra $W \subset sl(3) \oplus gl(2)$ satisfying $d(P) \cap W = 0$, namely*

$$W = \left\{ \left( \begin{pmatrix} a & 0 & 0 \\ * & b & 0 \\ * & * & -a-b \end{pmatrix}, \begin{pmatrix} -a & * \\ 0 & -b \end{pmatrix} \right) \right\}.$$

PROOF. Consider the canonical projections $\pi_1 : sl(3) \oplus gl(2) \to sl(3)$ and $\pi_2 : sl(3) \oplus gl(2) \to gl(2)$. Suppose that $W$ is a Lagrangian subalgebra. It follows that $W_i = \pi_i(W)$ satisfies $W_i^\perp \subseteq W_i$ , $i = 1, 2$, where the orthogonal is considered with respect to $Q_i$ introduced in the previous lemma. Because $W_1^\perp \oplus W_2^\perp \subseteq W$, we can consider the linear space $\frac{W}{W_1^\perp \oplus W_2^\perp} \subset \frac{W_1}{W_1^\perp} \oplus \frac{W_2}{W_2^\perp}$ which is also Lagrangian. It follows from the construction that there exists an isomorphism $\Phi : \frac{W_1}{W_1^\perp} \to \frac{W_2}{W_2^\perp}$ which preserves the bilinear form induced by $Q$. Thus $W = W_1^\perp \oplus W_2^\perp + \left\{ (\widetilde{x}, \Phi(\widetilde{x})) : \widetilde{x} \in \frac{W_1}{W_1^\perp} \right\}$. Because $W_1 \supseteq W_1^\perp$, $W_1$ is one of the following (up to conjugation): $sl(3)$ , $B_+$ , $B_-$, $P$, $P' = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} t & a & b \\ 0 & \epsilon t & c \\ 0 & 0 & \epsilon^2 t \end{pmatrix} ; t, a, b, c \in \mathbb{C}\right\}$ , $\epsilon^2 + \epsilon + 1 = 0$. On the other hand, $W_2$ must satisfy the conditon $W_2 \supseteq W_2^\perp$ and be a subalgebra of $gl(2)$. We choose for example $W_2 = \widetilde{B_+} = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$. It follows that $W_2^\perp = \left\{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\right\}$ and thus $\frac{W_2}{W_2^\perp} = \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$. Taking into consideration that $\frac{W_1}{W_1^\perp}$ and $\frac{W_2}{W_2^\perp}$ must have the same dimension, it follows that the only interesting situations are:

a) $W_1 = B_+$, $W_2 = \widetilde{B_+} \Rightarrow \frac{W_1}{W_1^\perp} \cong \mathbf{h} \cong \frac{W_2}{W_2^\perp}$. In this case, it is easy to see that $d(P) \cap W \neq 0$ for any choice of $\Phi$.

b) $W_1 = B_-$, $W_2 = \widetilde{B_+}$. Consider the Lie isomorphism $\Phi : \frac{W_1}{W_1^\perp} \to \frac{W_2}{W_2^\perp}$ defined by $\Phi\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix}$. The corresponding Lagrangian subalgebra $W$ is $\left\{ \left( \begin{pmatrix} a & 0 & 0 \\ * & b & 0 \\ * & * & -a-b \end{pmatrix}, \begin{pmatrix} -a & * \\ 0 & -b \end{pmatrix} \right) \right\}$ and verifies $d(P) \cap W = 0$. □

Our aim is to prove that $W$ and $P^*$can be identified as Lie algebras. Because they have the same dimension, they are obviously isomorphic as linear spaces. Let us construct now two dual bases in $d(P)$ and respectively $W$. The canonical basis of $P$ is formed by $e_{12}$, $e_{13}$, $e_{21}$, $e_{23}$, $e_{11} - e_{33}$, $e_{22} - e_{33}$. Consider the following basis in $d(P)$: $e_1 = (e_{12}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$, $e_2 = (e_{13}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$, $e_3 = (e_{21}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$, $e_4 = (e_{23}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$, $e_5 = (e_{11} - e_{33}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})$, $e_6 = (e_{22} - e_{33}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$. Take the following basis in $W$: $h^1 = (e_{21}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$, $h^2 = (e_{31}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$, $h^3 = (0, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix})$, $h^4 = (e_{32}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$, $h^5 = (e_{11} - e_{33}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix})$, $h^6 = (e_{22} - e_{33}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix})$. Trivial computations show the following: $Q(e_i, h^i) = 1$ for $i = 1, 2, 3, 4$, $Q(e_5, h^5) = Q(e_6, h^6) = 4$, $Q(e_5, h^6) = Q(e_6, h^5) = 2$ and $Q(e_i, h^j) = 0$ if $i \neq j$ and $(i, j) \neq (5, 6), (6, 5)$. In other words, we have to modify the elements $h^5$ and $h^6$ in order to obtain a dual basis to the one chosen for $d(P)$. Take $f^5 = \alpha h^5 + \beta h^6$ and $f^6 = \gamma h^5 + \delta h^6$ and we impose the duality conditions. It follows immediately that $\alpha = \delta = \frac{1}{3}$ and $\beta = \gamma = -\frac{1}{6}$. Thus we have proved that the basis $\{h^1, h^2, h^3, h^4, f^5, f^6\}$ in $W$ is dual to $\{e_i\}$, $i = 1, 6$ of $d(P)$. For symmetry, let us also denote $h^i$ by $f^i$, $i = 1, 4$.

LEMMA 2.2.4. *The Lie algebras $W$ and $P^*$are isomorphic.*

PROOF. We firstly remind that $P^*$ has a Lie algebra structure $[,]_{P^*}$, induced by the 1-cocycle $\delta_r$. Because $W$ and $P^*$ are already identified as linear spaces, it is enough to show that the structure constants for two suitable bases are the same. For $W$ let us take the basis $\{f^i\}, i = 1, 6$ that we constructed above. Simple computations show the following: $[f^1, f^5] = \frac{1}{2} f^1$, $[f^2, f^5] = \frac{1}{2} f^2$, $[f^1, f^6] = -\frac{1}{2} f^1$, $[f^4, f^6] = \frac{1}{2} f^4$, $[f^3, f^5] = \frac{1}{2} f^3$, $[f^3, f^6] = -\frac{1}{2} f^3$. If we consider $c_k^{ij}$ given by $[f^i, f^j] = c_k^{ij} f^k$ then we have obtained: $c_1^{15} = \frac{1}{2}$, $c_1^{16} = -\frac{1}{2}$, $c_2^{25} = \frac{1}{2}$, $c_3^{35} = \frac{1}{2}$, $c_3^{36} = -\frac{1}{2}$, $c_4^{46} = \frac{1}{2}$ and the rest are zero (of course $c_k^{ij} = -c_k^{ji}$). On the other hand, $\delta_r(e_{12}) = \frac{1}{2} e_{12} \wedge (e_{11} - e_{22})$,$\delta_r(e_{13}) = \frac{1}{2} e_{13} \wedge (e_{11} - e_{33})$,$\delta_r(e_{21}) = \frac{1}{2}(e_{22} - e_{11}) \wedge e_{21}$,$\delta_r(e_{23}) = \frac{1}{2} e_{23} \wedge (e_{22} - e_{33})$,$\delta_r(e_{11} - e_{33}) = \delta_r(e_{22} - e_{33}) = 0$. We identify each $e_i$ to its first component and we consider $d_i^{jk}$ given by $\delta_r(e_i) = d_i^{jk} e_j \otimes e_k$ . It follows that $d_i^{jk} = c_i^{jk}$ for every $i, j, k$ . By considering $f^i, i = 1, 6$ as elements of $P^*$, we obtain

$$[f^j, f^k]_{P^*}(e_i) = \delta_r^*(f^j, f^k)(e_i) = (f^j \otimes f^k)(d_i^{uv} e_u \otimes e_v) = d_i^{jk}. \tag{2.2.2}$$

Thus $[f^j, f^k]_{P^*} = d_i^{jk} f^i = c_i^{jk} f^i = [f^j, f^k]$ and $W \cong P^*$. □

This lemma shows that the Lie algebra structure on the linear space $sl(3) \oplus gl(2)$ induces the Lie brackets on $P$ and $P^*$. This actually proves the theorem announced in the beginning, that $D(P)$ and $sl(3) \oplus gl(2)$ can be identified as Lie algebras.

As a consequence of the theorem, we will construct a solution of mCYBE in $P$.

COROLLARY 2.2.5. *Let* $s = e_{12} \otimes e_{21} + \frac{1}{3}(e_{11} - e_{33}) \otimes (e_{11} - e_{33}) + \frac{1}{3}(e_{22} - e_{33}) \otimes (e_{22} - e_{33}) - \frac{1}{6}(e_{11} - e_{33}) \otimes (e_{22} - e_{33}) - \frac{1}{6}(e_{22} - e_{33}) \otimes (e_{11} - e_{33})$. *Then* $s$ *is a solution of mCYBE in* $P$.

PROOF. We consider the projection $p : sl(3) \oplus gl(2) \to P$ defined by

$$p(\begin{pmatrix} A & b \\ b' & -TrA \end{pmatrix}, A') = \begin{pmatrix} A & b \\ 0 & -TrA \end{pmatrix}. \tag{2.2.3}$$

We remind that we have constructed two dual bases $\{e_i\}$ and $\{h^i\}$, $i = 1, 6$ in $d(P)$ and respectively $W$. We take

$$\widetilde{s} = \sum_{i=1}^{6} (p \otimes p)(e_i \otimes h^i). \tag{2.2.4}$$

A simple computation shows that $s = \widetilde{s}$. According to Prop.1.10.8 and because $D(P)$ is isomorphic to $sl(3) \oplus gl(2)$, the tensor $\sum e_i \otimes h^i$ is a solution of mCYBE in $D(P)$. Therefore $s$ satisfies mCYBE in $P$. □

## 2.3. The general case

In this section we consider a simple, complex, finite-dimensional Lie algebra $\mathbf{g}$ with root system $R$, with respect to a Cartan subalgebra $\mathbf{h}$. Denote by $\mathbf{g}^{\mu}$ , $\mu \in R$ the root spaces. For a simple positive root $\alpha$, we take the parabolic subalgebra corresponding to it, $P_\alpha$. We will give a description for $D(P_\alpha)$. The classical double is considered with respect to the 1-cocycle $\delta_r$, introduced in the first section, and we have replaced $r$ by $\frac{1}{2}r$. As suggested by the result obtained in the previous section, the classical double will be isomorphic to $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$. We remind that $\mathbf{Red}(P_\alpha)$ denotes the reductive part of $P_\alpha$ and it is isomorphic to $\frac{P_\alpha}{P_\alpha^\perp}$. This holds because $P_\alpha^\perp$coincides with the nil radical of $P_\alpha$(the maximal nilpotent ideal) and therefore $\frac{P_\alpha}{P_\alpha^\perp}$ is a reductive (but not semisimple) Lie algebra.

LEMMA 2.3.1. *Let* $\alpha$ *be a simple positive root and* $A_\alpha = \{\mu \in R : \mu$ *does not contain* $\alpha$ *or* $-\alpha\}$. *Then*

$$\mathbf{Red}(P_\alpha) = \mathbf{h} \oplus \sum_{\mu \in A_\alpha} \mathbf{g}^{\mu}. \tag{2.3.1}$$

PROOF. We know that $P_\alpha$ is generated by the root vectors corresponding to all simple roots except $-\alpha$. We determine $P_\alpha^\perp$(with respect to the Killing form $K$ on $\mathbf{g}$). Because $K(\mathbf{g}^\gamma, \mathbf{g}^\delta) = 0$ when $\gamma + \delta \neq 0$, it results that

$K(\mathbf{g}^\gamma, \mathbf{g}^\delta) = 0$ if $\delta > 0$ contains $\alpha$ and $\gamma$ does not contain $-\alpha$. Put $B_\alpha = \{\delta > 0 : \delta \text{ contains } \alpha\}$. Thus

$$P_\alpha^\perp = \sum_{\delta \in B_\alpha} \mathbf{g}^\delta \tag{2.3.2}$$

The conclusion follows now immediately. □

This lemma allows us to compute the dimension of $\mathbf{Red}(P_\alpha)$ and we get that the linear spaces $D(P_\alpha) = P_\alpha \oplus P_\alpha^*$ and $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$ have the same dimension. Thus they are isomorphic as vector spaces. Consider now the map $d : P_\alpha \to \mathbf{g} \oplus \mathbf{Red}(P_\alpha)$ defined by $d(x) = (x, \pi(x))$, where $\pi : P_\alpha \to \frac{P_\alpha}{P_\alpha^\perp}$ is the natural projection. $\mathbf{Red}(P_\alpha)$ is equipped with the Killing form of $\mathbf{g}$, which is nondegenerate on $\mathbf{Red}(P_\alpha)$. Therefore we can define a nondegenerate symmetric bilinear form $Q$ on $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$ by $Q((a,b),(c,d)) = K(a,c) - K(b,d)$. We are interested in finding a Lagrangian subalgebra $W$ of $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$ which satisfies the additional condition $W \cap d(P_\alpha) = 0$. We give its construction in the next lemma:

LEMMA 2.3.2. *There exists a Lagrangian subalgebra $W$ in $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$ (with respect to the bilinear form $Q$) such that $W \cap d(P_\alpha) = 0$.*

PROOF. Suppose that for all $\gamma \in R$ we have chosen $e_\gamma \in \mathbf{g}^\gamma$ such that $K(e_\gamma, e_{-\gamma}) = 1$. Put $[e_\gamma, e_{-\gamma}] = h_\gamma \in \mathbf{h}$. The canonical basis of $P_\alpha$ is formed by $h_\beta$ for all $\beta \in \Delta$ (set of simple roots) and $e_\gamma$ for $\gamma \in R$ which does not contain $-\alpha$. We make the following remarks: i) if $\gamma > 0$ contains $\alpha$ then $\pi(e_\gamma) = 0$; ii) if $\gamma > 0$ does not contain $\alpha$ or $\gamma < 0$ does not contain $-\alpha$, then $\pi(e_\gamma) = e_\gamma$; iii) for any $\beta \in \Delta$, $\pi(h_\beta) = h_\beta$. Consider $W$ the linear subspace of $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$ generated by the following elements: $F^\gamma$which denotes $(e_{-\gamma}, 0)$ if $\gamma > 0$ and $(0, -e_{-\gamma})$ for $\gamma < 0$, which does not contain $-\alpha$ , and $H^\beta = (h_\beta, -h_\beta)$ for all $\beta \in \Delta$. In fact, $W$ is a Lie subalgebra. We will prove that $W$ is a Lagrangian subalgebra. Firstly, $\dim W = \dim P_\alpha = \frac{1}{2}\dim(\mathbf{g} \oplus \mathbf{Red}(P_\alpha))$. It is enough to show that $W \subseteq W^\perp$. We have: $Q((e_{-\gamma}, 0), (e_{-\delta}, 0)) = K(e_{-\gamma}, e_{-\delta}) = 0$ for $\gamma, \delta > 0$; $Q((0, -e_{-\gamma}), (0, -e_{-\delta})) = -K(e_{-\gamma}, e_{-\delta}) = 0$ for $\gamma, \delta < 0$ which do not contain $-\alpha$; $Q((e_{-\gamma}, 0), (h_\beta, -h_\beta)) = K(e_{-\gamma}, h_\beta) = 0$ for $\gamma > 0$ and $\beta \in \Delta$; $Q((0, -e_{-\gamma}), (h_\beta, -h_\beta)) = -K(e_{-\gamma}, h_\beta) = 0$ and $Q((h_\beta, -h_\beta), (h_\theta, -h_\theta)) = 0$ when $\beta, \theta \in \Delta$. In conclusion, $W$ is a Lagrangian subalgebra.

On the other hand, the canonical basis in $d(P_\alpha)$ is formed by the elements: $E_\gamma$ representing $(e_\gamma, 0)$ for $\gamma > 0$ which contains $\alpha$; $(e_\gamma, e_\gamma)$ for $\gamma \in R$ which does not contain $\alpha$ or $-\alpha$; $G_\beta = (h_\beta, h_\beta)$ for all $\beta \in \Delta$. Now it follows immediately that $W \cap d(P_\alpha) = 0$. This ends the proof. □

In order to get to the main result of this section, we will build a basis in $W$ which is dual to the canonical basis of $d(P_\alpha)$ written in the previous lemma. Firstly we notice that $Q((e_\gamma, 0), (e_{-\delta}, 0)) = K(e_\gamma, e_{-\delta}) = 1$ if $\gamma = \delta$ and 0 otherwise; $Q((e_\gamma, e_\gamma), (e_{-\delta}, 0)) = K(e_\gamma, e_{-\delta})$ and $Q((e_\gamma, e_\gamma), (0, -e_{-\delta})) =$

$K(e_\gamma, e_{-\delta})$. We only have to change the elements $(h_\beta, -h_\beta), \beta \in \Delta$ in order to make them dual to $(h_\theta, h_\theta), \theta \in \Delta$. For any $\beta \in \Delta$ take

$$\widetilde{h_\beta} = \sum_{\lambda \in \Delta} a_{\beta\lambda} h_\lambda. \tag{2.3.3}$$

By imposing the condition (*) $Q((h_\theta, h_\theta), (\widetilde{h_\beta}, -\widetilde{h_\beta})) = 1$ if $\beta = \theta$ and 0 otherwise, we will determine $a_{\beta\lambda}$. We have

$$Q((h_\theta, h_\theta), (\widetilde{h_\beta}, \widetilde{h_\beta})) = 2 \sum_{\lambda \in \Delta} a_{\beta\lambda} K(h_\theta, h_\lambda). \tag{2.3.4}$$

Because the restriction of $K$ to the Cartan subalgebra $\mathbf{h}$ is nondegenerate, the matrix $(K(h_\theta, h_\lambda))_{\theta, \lambda \in \Delta}$ is invertible and thus the linear system equivalent to condition (*) has an unique solution for any fixed $\beta \in \Delta$. Let us denote $\widetilde{H^\beta} = (\widetilde{h_\beta}, -\widetilde{h_\beta})$ (which we determined above). We have proved the following result:

Lemma 2.3.3. *The systems $(E_\gamma, G_\beta)$ and $(F^\gamma, \widetilde{H^\beta})$ are dual bases in $d(P_\alpha)$ and respectively $W$.*

The last step that we need in order to prove that the Lie algebras $D(P_\alpha)$ and $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$ are isomorphic is:

Lemma 2.3.4. *The Lie algebras $W$ and $P_\alpha^*$ are isomorphic.*

Proof. We remind that $P_\alpha^*$ has a Lie algebra structure induced by the 1-cocycle $\delta_r$. The linear spaces $W$ and $P_\alpha^*$ have the same dimension, so they are isomorphic. Let us consider the dual basis $(E_\gamma, G_\beta)$ and $(F^\gamma, \widetilde{H^\beta})$ from the previous lemma. Take $\gamma > 0$. We have the following computations:

(2.3.5)
$$[F^\gamma, \widetilde{H^\beta}] = [(e_{-\gamma}, 0), (\sum_{\lambda \in \Delta} a_{\beta\lambda} h_\lambda, -\sum_{\lambda \in \Delta} a_{\beta\lambda} h_\lambda)] = \sum_{\lambda \in \Delta} a_{\beta\lambda}([e_{-\gamma}, h_\lambda], 0).$$

But $[e_{-\gamma}, h_\lambda] = \gamma(h_\lambda) e_{-\gamma} = K(h_\gamma, h_\lambda) e_{-\gamma}$ because of the following computations:

(2.3.6)
$$K(h_\lambda, h_\gamma) = K(h_\lambda, [e_\gamma, e_{-\gamma}]) = K([h_\lambda, e_\gamma], e_{-\gamma}) = \gamma(h_\lambda) K(e_\gamma, e_{-\gamma}) = \gamma(h_\lambda).$$

Thus we have obtained

$$[F^\gamma, \widetilde{H^\beta}] = (\sum_{\lambda \in \Delta} a_{\beta\lambda} K(h_\gamma, h_\lambda))(e_{-\gamma}, 0). \tag{2.3.7}$$

On the other hand, if we write $h_\gamma = \sum c_{\gamma\theta} h_\theta$ (after all $\theta \in \Delta$) and we take into consideration that the constants $a_{\beta\lambda}$ verify the conditions

$$\sum_{\lambda \in \Delta} a_{\beta\lambda} K(h_\beta, h_\lambda) = \frac{1}{2}; \tag{2.3.8}$$

$$\sum_{\lambda\in\Delta} a_{\beta\lambda}K(h_\theta,h_\lambda)=0,\theta\neq\beta, \tag{2.3.9}$$

we conclude that

$$[F^\gamma,\widetilde{H^\beta}]=(\sum_{\lambda,\theta\in\Delta} c_{\gamma\theta}a_{\beta\lambda}K(h_\theta,h_\lambda))F^\gamma=\frac{1}{2}c_{\gamma\beta}F^\gamma. \tag{2.3.10}$$

Analogously, when $\gamma>0$ does not contain $\alpha$, reminding that $F^{-\gamma}=(0,-e_\gamma)$, we obtain the following

$$[F^{-\gamma},\widetilde{H^\beta}]=\frac{1}{2}c_{\gamma\beta}F^{-\gamma}. \tag{2.3.11}$$

Now we have to analyse the bracket induced by $\delta_r$ on $P_\alpha^*$. By using the computations from the second section, it results that for any $\gamma>0$,

$$\delta_r(e_\gamma)=\frac{1}{2}e_\gamma\wedge h_\gamma=\frac{1}{2}\sum_{\theta\in\Delta}c_{\gamma\theta}e_\gamma\wedge h_\theta \tag{2.3.12}$$

and for any $\gamma>0$ which does not contain $\alpha$,

$$\delta_r(e_{-\gamma})=\frac{1}{2}e_{-\gamma}\wedge h_\gamma=\frac{1}{2}\sum_{\theta\in\Delta}c_{\gamma\theta}e_{-\gamma}\wedge h_\theta. \tag{2.3.13}$$

We can consider $F^\gamma$ and $\widetilde{H^\beta}$as elements of $P_\alpha^*$(because the linear spaces $W$ and $P_\alpha^*$are already identified). Because the Lie algebra structure on $P_\alpha^*$ is induced by the 1-cocycle $\delta_r$, we have the following computations

$$[F^\gamma,\widetilde{H^\beta}]_{P_\alpha^*}(e_\lambda)=\delta_r^*(F^\gamma,\widetilde{H^\beta})(e_\lambda)=(F^\gamma\otimes\widetilde{H^\beta})(\delta_r(e_\lambda)). \tag{2.3.14}$$

Suppose that $\gamma>0$. By construction, the bases $(E_\gamma,G_\beta)$ in $d(P_\alpha)$ and $(F^\gamma,\widetilde{H^\beta})$ in $W$ are dual. Thus, if $\lambda\neq\gamma$, then $[F^\gamma,\widetilde{H^\beta}]_{P_\alpha^*}(e_\lambda)=0$. Otherwise, $[F^\gamma,\widetilde{H^\beta}]_{P_\alpha^*}(e_\gamma)=\frac{1}{2}c_{\gamma\beta}$. In this way we see that

$$[F^\gamma,\widetilde{H^\beta}]_{P_\alpha^*}=\frac{1}{2}c_{\gamma\beta}F^\gamma. \tag{2.3.15}$$

If $\gamma>0$ and does not contain $\alpha$, we obtain analogously that

$$[F^{-\gamma},\widetilde{H^\beta}]_{P_\alpha^*}=\frac{1}{2}c_{\gamma\beta}F^{-\gamma}. \tag{2.3.16}$$

It is not necessary to consider brackets between other type of elements from the system $(F_\gamma,\widetilde{H^\beta})$ because they are zero.

In conclusion, we have proved that the Lie bracket is in fact the same and thus $W\cong P_\alpha^*$ as Lie algebras. □

We can now state the main result of this chapter. The proof is straightforward from the previous lemma.

THEOREM 2.3.5. *Let* $\mathbf{g}$ *be a complex, finite-dimensional and simple Lie algebra. Let* $P_\alpha$ *be the parabolic subalgebra corresponding to a simple positive root* $\alpha$. *Then the classical double* $D(P_\alpha)$, *considered with respect to the 1-cocycle* $\delta_r$, *is isomorphic to the Lie algebra* $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$.

As a consequence of the theorem, we will construct a solution for mCYBE in $P_\alpha$.

COROLLARY 2.3.6. *Let* $B_\alpha = \{\gamma > 0$ *which does not contain* $\alpha\}$ *and*

$$s = \sum_{\gamma \in B_\alpha} e_\gamma \otimes e_{-\gamma} + \sum_{\beta,\lambda \in \Delta} a_{\beta\lambda} h_\beta \otimes h_\lambda. \tag{2.3.17}$$

*Then the tensor* $s$ *is a solution of mCYBE in* $P_\alpha$.

PROOF. Put $C_\alpha = \{\gamma \in R$ such that $e_\gamma \in P_\alpha\}$. Let $p$ denote the projection of $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$ onto $P_\alpha$. We take the dual bases that we constructed before, $(E_\gamma, G_\beta)$ in $d(P_\alpha)$ and respectively $(F^\gamma, \widetilde{H^\beta})$ in $W$. Consider the following tensor:

$$\widetilde{s} = (p \otimes p)\Big(\sum_{\gamma \in C_\alpha} (E_\gamma \otimes F^\gamma) + \sum_{\beta \in \Delta} (G_\beta \otimes \widetilde{H^\beta})\Big). \tag{2.3.18}$$

A simple computation shows that $\widetilde{s} = s$. Because we have identified $D(P_\alpha)$ with the Lie algebra $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$, from Prop.1.10.8 it results that $\widetilde{s}$ constructed as above is a solution of mCYBE in $P_\alpha$. This ends the proof. □

REMARK 2.3.7. In principle, the theorem allows us to compute more solutions of mCYBE in $P_\alpha$. In the previous corollary, we have constructed only the solution corresponding to a certain Lagrangian subalgebra $W$. In general, it is known that there is a correspondence between the solutions of mCYBE and Lagrangian subalgebras in $D(P_\alpha)$ which are complementary to the diagonal (up to gauge equivalence) (see [5]). The classical double is now known. Therefore we have reduced the problem of computing solutions of mCYBE to that of finding Lagrangian subalgebras in $\mathbf{g} \oplus \mathbf{Red}(P_\alpha)$, with respect to the bilinear form $Q$. Once determined a Lagrangian subalgebra, we have to find a basis in it which is dual to $(E_\gamma, G_\beta)$ and then to construct the tensor $\widetilde{s}$ as in the proof of the corollary.

# Bibliography


[1] BELAVIN, A. A., DRINFELD, V. G., *Solutions of the classical Yang-Baxter equations for simple Lie algebras,* L. D. Landau Inst. of Theoretical Physics, Vol.**16**, No.3, 1-29 (1982).

[2] BELAVIN, A. A., DRINFELD, V. G., *On the classical Yang-Baxter equation for simple Lie algebras, Func. Analysis and Appl.,*Vol. **17**, No.3, 69-70 (1983).

[3] CUCULESCU, I., *Algebre Lie*, Ed. Academiei Romane, Bucharest (1963).

[4] DRINFELD, V. G., *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation*, Sov. Math. Dokl., **27** (1983).

[5] DRINFELD, V. G., *On Poisson homogeneous spaces of Poisson-Lie groups,* Theor. Math. Phys., **95** (1993), 226-227.

[6] JIMBO, M., *Yang-Baxter equation in integrable systems*, Advanced Series in Mathematical Physics, Vol. **10** (1989).

[7] KATS, V. G., *Automorphisms of finite order of simple Lie algebras,* Funk. Analysis Pril., T. **3**, Fasc. 3, 94-96 (1969).

[8] KATS, V. G., *Simple irreductible graded Lie algebras of finite height*, Izv.Akad. Nauk SSSR, SEr. Mat. **32**, No. 6, 1323-1367 (1968).

[9] KOROGODSKI, L. I., SOIBELMAN Y. S., *Algebras of Functions on Quantum Groups, Part I,* Mathematical Surveys and Monographs, Vol. **56**, (1998).

[10] KULISH, P. P., SKLYANIN, E. K., *Solutions of the Yang-Baxter equation*, Differential Geometry, Lie groups and Mechanics [in Russian], Zap. Nauch. Sem. Leningrad Otd. Mat. Inst., **95**, 129-160 (1980).

[11] MOROZOV, V. V., *Proof of the regularity theorem*, Uspehi. Mat. Nauk, **11**, No. 5 (71), 191-194 (1956).

[12] SERRE, J. P., *Lie Algebras and Lie Groups*, W.A. Benjamin (1966).

[13] STOLIN, A., *On rational solutions of the classical Yang-Baxter equation*, Doctoral Disertation, Dept. Mat. Univ. of Stockholm, (1991).

[14] STOLIN, A., *Some remarks on Lie bialgebra structures on simple complex Lie algebras*, Communications in Algebra, **27** (9), 4289-4302 (1999).

[15] VARADARAJAN, V. S., *Lie Groups, Lie Algebras and Their Representations*, Springer-Verlag (1984).

[16] ZHIBER, A. V., SHABAT, A. B., *Klein-Gordon equations with nontrivial group*, Dokl. Akad. Nauk SSSR, **247**, No. 5, 1103-1107 (1979).