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Stochastic Optimization of Structural Topology

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ABSTRACT

This work addresses the problem of hierarchical decision-making under uncertainty and, more specificaly, its applications in mechanical engineering. The motivation for this study comes from the need to build cost-effective structures that are robust under varying conditions; failure to take uncertainty into account may lead either to very expensive or very inefficient designs.

We are primarily concerned with the robust design (including the possibility of modifying the topology) of truss-like structures in unilateral contact and/or including members that are able to sustain only tensile forces. The problem is formulated and studied as a stochastic bilevel programming problem, which allows us to take both the randomness and the hierarchical nature of design optimization problems into account. Three particular questions are studied: (i) the *approximation* of the non-differentiable topology optimization problems with a sequence of simpler, differentiable, sizing optimization problems; (ii) the *robustness* of the optimal designs with respect to errors in the modelling of uncertainty; and (iii) the *discretization* of the infinite-dimensional stochastic structural optimization problems, or approximation with a sequence of finite-dimensional optimization problems.

Within (i), we generalize the known approximation results to the stochastic setting, but the main contribution is a new approximation method for stochastic stress constrained weight minimization problem based on the idea of penalty functions. With respect to (ii), we show the robustness of the optimal solutions to the stochastic compliance minimization problem, and we propose a relaxation of the stress constrained weight minimization problem which in contrast to the original formulation possesses robust solutions. In (iii) we prove the convergence of the discretizations for the approximate problems constructed in (i) under rather general assumptions about the random variables, defining the problem.

CONTENTS

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INTRODUCTION AND OVERVIEW

Bilevel programming

Hierarchical decision-making problems are encountered in a wide variety of domains in the engineering and experimental natural sciences, and in regional planning, management, and economics. These problems are all defined by the presence of two or more objectives with a prescribed order of priority or information. In many applications it is sufficient to consider a sub-class of these problems having two levels, or objectives. We refer to the upper level as the objective having the highest priority and/or information level; it is defined in terms of an optimization with respect to one set of variables. The lower-level problem, which in the most general case is described by a variational inequality, is then a supplementary problem parameterized by the upper-level variables. These models are known as generalized bilevel programming problems, or mathematical programs with equilibrium constraints (MPEC); see, for example, Luo et al. [LPR96].

Structural optimization problems have an inherent bilevel form. The upper level objective function measures some performance of the structure, such as its weight or stiffness. This objective function is optimized by selecting design parameters, which may express the shape of the structure, the choice of material or the amount of material being used. Further, the structure may be subject to limits on the amount of available material, and to behavioural constraints, such as bounds on the displacements, stresses and contact forces. The lower level problem describes the behaviour of the structure given the choices of the design variables, possible contact conditions with foundations or boundaries, and the external forces acting on it. The behaviour is for linear elastic structures given by the equilibrium law of minimal potential energy, which determines the values of the state variables (nodal displacements) at the lower level. Equivalently, the equilibrium law can be expressed as a (dual) principle of the minimum of complementary energy, determining the stresses and contact forces.

In applications relating to Stackelberg game theory, economics, and decision analysis, a number of the problem inputs will often be subject to uncertainty. This is true in particular with respect to costs, demands, and system capacities, which are subject to fluctuations and/or are difficult to measure. In hierarchical models of engineering design and physical phenomena, external conditions and measurement or manufacturing errors introduce uncertainty into the problems. In both of these cases, the uncertainty can be included explicitly by generalizing some of the problem parameters to random variables. However, this generalization complexifies the model significantly; resolution strategies will in many cases require some approximation methods to solve the resulting stochastic programs.

In the simplest case, the expected values of the random variables could be substituted for their stochastic counterparts and a deterministic model then solved. However, in a nonlinear problem subject to constraints, the effect of this simplification can be quite costly. Indeed, not only will the optimal cost of the expected value solution not necessarily represent the average of the possible optimal costs, but the solution may not even be feasible with respect to the realized values of the random variables.

To take into account explicitly the variability of the random inputs, as well as the possible infeasibility, we consider a stochastic programming extension of the mathematical programming problem with equilibrium constraints.

Let $(\Omega, \mathfrak{S}, \mathrm{P})$ be a complete probability space. The stochastic MPEC is:

$$\begin{aligned} &\min\ \mathrm{E}_\omega[f(x,\boldsymbol{\xi}(\omega),\omega)] := \int_\Omega f(x,\boldsymbol{\xi}(\omega),\omega)\,\mathrm{P}(d\omega) \\ &\text{s.t.} \quad \begin{cases} \big(x,\boldsymbol{\xi}(\omega)\big) \in \mathcal{Z}(\omega), & \text{P-a.s.} \\ \boldsymbol{\xi}(\omega) \in \mathcal{S}(x,\omega), & \text{P-a.s.} \end{cases} \end{aligned} \qquad [\mathrm{SMPEC} - \Omega]$$

where $\boldsymbol{\xi} : \Omega \to \mathbb{R}^m$ is a random element in $(\Omega, \mathfrak{S}, \mathrm{P})$, $\mathcal{Z} : \Omega \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ is a point-to-set mapping representing the upper-level constraints, and $\mathcal{S} : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^m$ is a set of solutions to a lower-level parametric variational inequality problem:

$$\mathcal{S}(x,\omega) := \{\, \xi \in \mathbb{R}^n \mid -T(x,\xi,\omega) \in N_{\mathcal{Y}(x,\omega)}(\xi) \,\}.$$

The lower-level problem is defined by the mapping $T : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ and a feasible set mapping $\mathcal{Y} : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^m$ having closed convex images, and $N_{\mathcal{Y}(x,\omega)} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ denotes the normal cone mapping to the set $\mathcal{Y}(x,\omega)$.

A special case of [SMPEC-Ω] is bilevel programming, which is obtained when the lower-level variational inequality problem reduces to the optimality conditions for an optimization problem, that is, when $T(x,\xi,\cdot) = \nabla_\xi t(x,\xi,\cdot)$ for some function $t : \mathcal{Z} \times \Omega \to \mathbb{R}$. Usually, bilevel programming is formulated in terms of the corresponding optimization problem, thus leading to the formulation

$$\begin{aligned} &\min\ \mathrm{E}_\omega[f(x,\boldsymbol{\xi}(\omega),\omega)] := \int_\Omega f(x,\boldsymbol{\xi}(\omega),\omega)\,\mathrm{P}(d\omega) \\ &\text{s.t.} \quad \begin{cases} \big(x,\boldsymbol{\xi}(\omega)\big) \in \mathcal{Z}(\omega), & \text{P-a.s.} \\ \boldsymbol{\xi}(\omega) \in \operatorname{argmin}_{z\in\mathcal{Y}(x,\omega)} t(x,z,\omega), & \text{P-a.s.} \end{cases} \end{aligned} \qquad [\mathrm{BP}-\Omega]$$

With the proper identification of mappings f, $\mathcal{Z}$, $\mathcal{Y}$, and t, most stochastic structural optimization problems can be formulated as [BP-Ω].

The problem [SMPEC-Ω] is treated in detail in Paper 1. Even in the non-stochastic case the analysis of the problem is quite an intricate one. The feasible set of the problem is not in general closed or connected, and standard assumptions made in nonlinear programming (constraint qualifications) are necessarily violated. Not much more than the existence of solutions to this problem can be established without further supposing, e.g., that the set $\mathcal{S}(x,\omega)$ is a singleton for all x and almost all ω. However, in the case of structural optimization problems such an assumption is necessarily satisfied, which allows us to develop approximation results for such problems.

Structural optimization

Structural optimization is a scientific discipline that is concerned with the assemblage of materials to carry prescribed loads as efficiently as possible. It has long been recognized that when determining the design of a mechanical structure it is vital to take into account the uncertain character of some of the parameters that will determine the ultimate design. Traditionally however, engineering models in shape, sizing and (more generally) topology optimization often ignore the presence of uncertainty in the data, such as random properties of the material used and conditions that will affect the structure once it has been built, such as varying weather conditions and external forces acting on it. This may result in the construction of designs that are unstable under varying conditions. There are three main approaches to the modelling of uncertainty in the area of structural optimization: probabilistic (stochastic), worst-case, and empirical multi-load approaches. (There exist a few minor alternatives, such as fuzzy sets based models and combinations of the approaches listed.)

The stochastic programming based models offer the richest modelling capabilities and at the same time require considerable effort for their numerical solution. There exists a subclass of stochastic programming problems (so called two-stage problems, or problems with recourse), which is very well studied from both theoretical and numerical points of view.

Therefore, one direction of research in the area concentrates on simplified probabilistic structural optimization models, which are approximable by stochastic programming problems with recourse (for example, see [Mar97]).

On the other hand, owing to the anticipated fact that the "real" probability model is never known, and the reported high sensitivity of solutions to stochastic structural optimization problems with respect to small changes in probability measure (e.g. [BHE90, pp. 20–22]), many probability-free worst-case ("pessimistic") models of uncertainty have been developed as an alternative to probabilistic ones [BHE90, BTN97]. Such an approach does yield stable structures but does not take into account the probability of occurrence of the different scenarios, thereby often resulting in unnecessarily costly designs. Furthermore, such models are unable to capture the essential properties of the underlying uncertain reality (such as the correlation between two events), which results in a very simplified prototype of reality. Nevertheless, the question of the stability of the optimal designs with respect to the errors in the modelling of uncertainty is not studied at all, which makes it even harder to estimate the quality of designs obtained.

Another approach is the multi-load design ([SvG68], [Ben95, p. 8]). In such an approach, an engineer picks up a few loading scenarios, empirically assigns them some "weights" and then solves, essentially, the stochastic optimization problem with a discrete probability measure, where each scenario has a probability proportional to its "weight".

In the Papers 2–4 of the present thesis, we advocate the stochastic approach to robust structural topology optimization by

Paper 2
- formulating the problems of robust topological design of trusses as stochastic bilevel programming problems;
- extending the classic topology optimization results to the stochastic case;
- analyzing the continuity of optimal designs with respect to changes in the probability measure;

Paper 3
- proposing approximation schemes suitable for stochastic topology optimization problems;
- interpreting stochastic optimal designs as limits of multi-load designs as the number of load cases goes to infinity;

Paper 4
- presenting an alternative formulation of the stochastic stress constrained weight minimization problem whose optimal solutions are continuous with respect to changes in probability measure;
- extending the results of Papers 2–3 to this reformulation.

In contrast with existing formulations of stochastic structural optimization problems, which focus on the optimization of statistical reliability properties of the structure, such as the probability of the failure, the primal focus in this study is the maximization of the structural performance while keeping the design reasonably robust.

Example: stochastic topology optimization of a truss

By a *truss* we mean a structure consisting of a finite number (denoted by m) of bars [Mic04]. The *design* of a truss can be determined by assigning the volume $x_i \geq 0$ of structural material to the bar i, $x_i = 0$ meaning that the bar is removed from the structure. A design x and a force vector $f(\omega)$ uniquely determine the distribution of stresses $\sigma_i(\omega)$ in structural members, if the structure can carry the given load. To prevent the damage of bars it is often desirable to constrain the admissible values of stresses for the bars, which are present in the structure (i.e., $x_i > 0$). Therefore, we can formulate the general stochastic topology optimization problem for a truss as follows:

$$\max_x \text{ the expected mechanical performance of the truss}$$

$$\text{s.t.} \begin{cases} \text{the structure determined by the design } x \text{ can carry} \\ \quad \text{the loads } \{f(\omega)\}, \text{ with probability one,} & (*) \\ \text{stress constraints are satisfied for all bars present} \\ \quad \text{in the structure (i.e., } x_i > 0\text{), with probability one.} & (**) \end{cases}$$

It can be immediately noticed that by allowing topological modifications (i.e., allowing the values $x_i = 0$ for some i) we make the problem inaccessible for the standard nonlinear programming techniques, because the number of constraints $(**)$ changes with the design.

ε-perturbation

> Although this may seem a paradox, all exact science is dominated by the idea of approximation.
>
> Russell, Bertrand (1872-1970)
> *in W. H. Auden and L. Kronenberger (eds.)*
> *The Viking Book of Aphorisms, New York: Viking Press, 1966*

Let us take a closer look at the constraint $(*)$ to fit the general stochastic topology optimization problem into the framework [BP-Ω]. This constraint is equivalent to the existence of the optimal solutions to the the principle of minimum complementary energy (in our case it is the (x, ω)-

parametric minimization problem):

$$(\mathcal{C})_x(\omega)\begin{cases}\min_s \mathcal{E}(x,s,\omega) := \dfrac{1}{2}\displaystyle\sum_{i=1}^m \dfrac{s_i^2}{E(\omega)x_i} \\ \text{s.t.} \begin{cases}\displaystyle\sum_{i\in\mathcal{I}(x)} B_i^T(\omega)s_i = f(\omega),\end{cases}\end{cases}$$

where we introduced artificial variables $s_i = x_i\sigma_i$ and an index set of the present members in the structure $\mathcal{I}(x) = \{\, i = 1,\dots,m \mid x_i > 0\,\}$; $E(\omega)$ is the Young's modulus for the structure material and $B_i(\omega)$ is the kinematic transformation matrix for the bar i. Not only the energy functional $\mathcal{E}$ is not differentiable at the points where the topology of the truss changes, but it is not even upper semicontinuous at such points!

Therefore, it seems natural to introduce a small but positive bound ε and require that $x_i \geq \varepsilon$. This approach is called ε-perturbation, and depending on the structural optimization problem under consideration the sequence of optimal solutions to perturbed problems may or may not converge to an optimal solution of the limiting problem as ε goes to zero (cf. [ChG97, Ach98, Pet01] for the discussion of ε-perturbation in the deterministic case). The construction of approximations of such a type for stochastic structural optimization problems is discussed in Papers 2–4.

Distribution sensitivity

> Bridges would not be safer if only people who knew the proper definition of a real number were allowed to design them.
>
> Mermin, N. David
> *"Topological Theory of Defects"*
> *in Review of Modern Physics, v. 51 no. 3, July 1979*

We expect a robust optimal design to be insensitive to modelling errors, and, in particular, to the errors in the modelling of uncertainty. Formally, consider a sequence of probability measures $\{\mathrm{P}_k\}$ weakly converging to a limit P. Let $\{x_k^*\}$ be the sequence of optimal designs corresponding to $\{\mathrm{P}_k\}$. Do the limit points of this sequence solve the limiting stochastic topology optimization problem, corresponding to the probability measure P?

The answer is: not in general. For a subclass of the problems considered in the thesis, compliance minimization problems, which do not include any *behavioural* constraints ($*$), the answer is affirmative provided the sequence $\{\mathrm{P}_k\}$ *locally* converges to P. For another subclass, stress constrained weight minimization problems, the answer is negative even in the case of local convergence (cf. Paper 2). Therefore, for the latter problems we propose a reformulation based on a relaxation of the constraints $(*)$, which allows large violations of such constraints but provides a mechanism to control the probability of such violations (Paper 4). The relaxation, in contrast with the original formulation, possesses optimal designs that are robust with respect to local changes in probability measure.

Discretization

> A theory has only the alternative of being right or wrong.
> A model has a third possibility: it may be right, but irrelevant.
>
> EIGEN, MANFRED
> *in Jagdish Mehra (ed.), The Physicist's Conception of Nature, 1973*

In the case when the number of points in the set Ω is infinite, we are faced with the task of solving an infinite-dimensional optimization problem. For the purpose of numerical computations, it is important to construct finite-dimensional approximate problems. The most popular approach to solving stochastic optimization problems is based on the idea of approximating the probability measure with a sequence of discrete probability measures with finite support, resulting in a desired sequence of finite-dimensional optimization problems.

Unfortunately, in our case the straightforward implementation of such strategy is impossible, owing to the lack of standard constraint qualifications by the feasible set of the problem. Therefore, we approximate the given topology optimization problem by a sequence of simpler sizing optimization problems (ε-perturbations), and then discretize the latter problems (Papers 3–4).

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Paper 1

A NOTE ON EXISTENCE OF SOLUTIONS TO STOCHASTIC MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS

Anton Evgrafov* and Michael Patriksson*

Abstract

We generalize Stochastic Mathematical Programs with Equilibrium Constraints (SMPEC) introduced by Patriksson and Wynter [*Operations Research Letters*, 25:159–167, 1999] to allow joint upper-level constraints, and to change continuity assumptions w.r.t. uncertainty parameter assumed before by measurability assumptions. For this problem, we prove the measurability of a lower-level mapping and the existence of solutions. We also discuss algorithmic aspects of the problem, in particular the construction of an inexact penalty function for the SMPEC problem, and touch a question of distribution sensitivity. Applications to structural topology optimization and other fields are mentioned.

Key words: Bilevel programming; Equilibrium constraints; Stochastic programming; Existence of solutions; Stochastic Stackelberg game

1.1 Introduction

The preparation of this note was prompted by a mistake in the proof of the existence of solutions to stochastic mathematical programs with equilibrium constraints (SMPEC) [PaW99, Corollary 2.5]. We also generalize the framework presented there to include more general constraints and probabilistic settings.

SMPEC represents a model for hierarchical decision-making under uncertainty. It generalizes the deterministic MPEC, or generalized

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bilevel programming problems [LPR96] by explicitly incorporating possible uncertainties in the problem data to obtain robust solutions. For a discussion of possible applications of the model see [PaW99]; applications to structural optimization are discussed in [CPW01, PaP00]. A special form of SMPEC was formulated in [LCN87] in a framework of stochastic Stackelberg games. Thus the model has applications in economics as well.

Let $(\Omega, \mathfrak{S}, \mathrm{P})$ be a complete probability space. The stochastic MPEC is:

$$\begin{aligned} &\min\ \mathrm{E}_\omega[f(x, \boldsymbol{\xi}(\omega), \omega)] := \int_\Omega f(x, \boldsymbol{\xi}(\omega), \omega)\, \mathrm{P}(d\omega) \\ &\text{s.t.} \quad \begin{cases} \big(x, \boldsymbol{\xi}(\omega)\big) \in \mathcal{Z}(\omega), & \text{P-a.s.} \\ \boldsymbol{\xi}(\omega) \in \mathcal{S}(x, \omega), & \text{P-a.s.} \end{cases} \end{aligned} \qquad [\mathrm{SMPEC} - \Omega]$$

where $\boldsymbol{\xi} : \Omega \to \mathbb{R}^m$ is a random element in $(\Omega, \mathfrak{S}, \mathrm{P})$, $\mathcal{Z} : \Omega \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ is a point-to-set mapping representing the upper-level constraints, and $\mathcal{S} : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^m$ is a set of solutions to a lower-level parametric variational inequality problem:

$$\mathcal{S}(x, \omega) := \{\, \xi \in \mathbb{R}^n \mid -T(x, \xi, \omega) \in N_{\mathcal{Y}(x,\omega)}(\xi) \,\}. \tag{1.1}$$

The lower-level problem is defined by the mapping $T : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ and a feasible set mapping $\mathcal{Y} : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^m$ having closed convex images, and $N_{\mathcal{Y}(x,\omega)} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ denotes the normal cone mapping to the set $\mathcal{Y}(x, \omega)$.

The outline of the paper is as follows. In section 1.2 the question of feasibility is addressed. The main result is the measurability of the solution set to a variational inequality problem mapping, which is a generalization of the measurability of the marginal mapping for optimization problems (cf. Lemma III.39 [CaV77], Theorem 8.2.11 [AuF90]). In section 1.3, the existence of solutions to $[\mathrm{SMPEC} - \Omega]$ is proved, generalizing Corollary 2.5 [PaW99]. In section 1.4 as an example we apply the existence result to a structural optimization problem. Section 1.5 discusses penalization procedures, generalizing Theorem 9.2.2 of [BSS93] to $[\mathrm{SMPEC} - \Omega]$ and outlining one possible approach to solve SMPEC.

1.2 Feasibility

The crucial part of the proof of the existence of solutions to a deterministic MPEC is the closedness of the feasible set [LPR96]. The typical situation with SMPEC is that for almost any ω the closedness of an "ω-slice" $\mathcal{F}_\omega = \mathcal{Z}(\omega) \cap \mathrm{gr}[x \to \mathcal{S}(x, \omega)]$ of the feasible set could be established using the existing results.

Consider now $x \in \mathbb{R}^n$. Suppose that for almost any ω we obtain a point $(x, \xi(\omega)) \in \mathcal{F}_\omega$. The objective function can be evaluated at $(x, \xi(\cdot))$ *only* if the function $\xi(\omega)$ is $\mathfrak{S}$-measurable. Thus the question arises, whether we can guarantee the existence of some $\mathfrak{S}$-measurable function $\boldsymbol{\xi}$ such that for almost any ω the following two conditions hold: $(x, \boldsymbol{\xi}(\omega)) \in \mathcal{F}_\omega$ (feasible solution) and $f(x, \boldsymbol{\xi}(\omega), \omega) \le f(x, \xi(\omega), \omega)$ ("non-worse" solution).

Our approach to the problem is as follows. We will use the measurability in ω for fixed x of $\mathcal{S}(x, \omega)$ and $\mathcal{Z}_x(\omega) := \{ \xi \in \mathbb{R}^N \mid (x, \xi) \in \mathcal{Z}(\omega) \}$ (cf. [Him75, Section 2], [CaV77, Chapter III] or [AuF90, Chapter 8] for definition of measurability of set-valued mappings). After that, we can apply the theorem about the measurability of marginal mappings (cf. Lemma III.39 [CaV77] or Theorem 8.2.11 [AuF90]) to give an affirmative answer to the posed question.

We simply assume measurability in ω of $\mathcal{Z}_x(\omega)$ and $\mathcal{Y}(x, \omega)$ for any $x \in \mathbb{R}^n$. A sufficient condition is, e.g. Theorem 8.2.9 [AuF90], cited here for convenience.

Theorem 1.2.1 (Inverse image [AuF90, Theorem 8.2.9]). *Consider a complete σ-finite measure space $(\Omega, \mathfrak{S}, \mathrm{P})$, complete separable metric spaces X, Y, measurable set-valued maps $F : \Omega \rightrightarrows X$, $G : \Omega \rightrightarrows Y$ with closed images. Let $g : \Omega \times X \to Y$ be a Carathéodory map. Then, the set-valued map H, defined by $H(\omega) = \{ x \in F(\omega) \mid g(\omega, x) \in G(\omega) \}$ is measurable.*

Remark 1.2.1.1. If the mappings $\mathcal{Z}_x(\omega)$, $\mathcal{Y}(x, \omega)$ are defined by inequalities of the type $\{ \xi \in \mathbb{R}^m \mid g_x(\xi, \omega) \le 0 \}$, where g_x is a Carathéodory mapping, then they are measurable.

The next proposition asserts the measurability of the mapping $\mathcal{S}(x, \cdot)$.

Proposition 1.2.2 (Measurability of $\mathcal{S}(x, \cdot)$). *Suppose that the mapping $\mathcal{Y}$ is measurable in ω for any fixed x and has closed convex images for any x and almost any ω. Let the mapping T be continuous in y and measurable in ω (i.e. Carathéodory) for any x. Then, the mapping $\mathcal{S}$ is measurable in ω for any x.*

Proof. Fix x and consider the mapping $\widetilde{\mathcal{S}} : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^m$ given by the normal equation:

$$\widetilde{\mathcal{S}}(x, \omega) := \{ \nu \in \mathbb{R}^m \mid T(x, \Pi_{\mathcal{Y}(x,\omega)}(\nu), \omega) + \nu - \Pi_{\mathcal{Y}(x,\omega)}(\nu) = 0 \},$$

where $\Pi_{\mathcal{Y}(x,\omega)} : \mathbb{R}^m \to \mathbb{R}^m$ denotes the Euclidean projection operator onto the closed convex set $\mathcal{Y}(x, \omega)$. By Corollary 8.2.13 [AuF90], the mapping $\Pi_{\mathcal{Y}(x,\omega)}(\nu)$ is measurable for any ν. Since T is Carathéodory in the variables $\xi \times \omega$ and the Euclidean projection is continuous, the resulting mapping $T(x, \Pi_{\mathcal{Y}(x,\omega)}(\nu), \omega)$ is Carathéodory in variables $\nu \times \omega$. Thus we can apply Theorem 1.2.1 to conclude the measurability of $\widetilde{\mathcal{S}}$ for any ν.

Recalling that $\mathcal{S}(x,\omega) = \Pi_{\mathcal{Y}(x,\omega)}(\widetilde{\mathcal{S}}(x,\omega))$ by Proposition 1.3.3 [LPR96], we can apply Theorem 8.2.7 [AuF90] about direct image to get measurability of a mapping $\operatorname{cl}\mathcal{S}(x,\cdot)$ for any x. Since T is continuous in ξ and $\mathcal{Y}$ has closed images, the mapping $\mathcal{S}$ has closed images and we are through. #

1.3 Existence of solutions

Let $\mathcal{X}$ denote the projection of the feasible set of the upper-level problem on the space of x variables: $\mathcal{X} := \{\, x \in \mathbb{R}^n \mid \exists\,\xi(\omega) : (x,\xi(\omega)) \in \mathcal{Z}(\omega)$ for almost any $\omega \,\}$. Let also denote by $\mathcal{F}(x,\omega)$ the "x-slice" of the feasible set of [SMPEC $-\,\Omega$]: $\mathcal{F}(x,\omega) := \mathcal{Z}_x(\omega)\cap\mathcal{S}(x,\omega)$. We will say that the function $f : \mathbb{R}^n \times \mathbb{R}^m \times \Omega$ is uniformly weakly coercive w.r.t. to x and the set $\mathcal{X}$ if the set $\{x \in \mathcal{X} \mid f(x,\xi,\omega) \le c\}$ is bounded for any $c \in \mathbb{R}$.

The approach to the existence proof is close in spirit to that of [RoW98, Theorem 14.60] about the interchangeability of integration and optimization. The difficulty is that we have "coupling" variables x which do not allow us to use the pointwise minimization in a straightforward way.

The next theorem generalizes [PaW99, Corollary 2.5] in the following ways: we allow joint upper-level constraints $\mathcal{Z}$, do not require any continuity of the involved mappings with respect to ω, and consider an arbitrary probability measure on a complete probability space.

Theorem 1.3.1 (Existence of solutions). *Suppose that the following assumptions are fulfilled:*

(i) *the mappings $\mathcal{Z}_x(\cdot)$ and $\mathcal{S}(x,\cdot)$ are measurable for any x,*

(ii) *the set $\mathcal{Z}(\omega)$ and the mapping $x \to \mathcal{S}(x,\omega)$ are closed for almost all $\omega \in \Omega$,*

(iii) *the mapping $f(x,\xi,\omega)$ is continuous in (x,ξ), measurable in ω, uniformly weakly coercive w.r.t. x and the set $\mathcal{X}$, and bounded from below by an $(\mathfrak{S},\mathrm{P})$-integrable function,*

(iv) *for any $x \in \mathcal{X}$ there is a neighborhood $U_x \ni x$ such that the set $\cup_{\tilde{x}\in U_x\cap\mathcal{X}} Z_{\tilde{x}}(\omega)$ is bounded for almost any ω,*

(v) *the set $\mathcal{F}(x_0,\omega)$ is nonempty for some $x_0 \in \mathcal{X}$ and almost any ω.*

Then, there exists at least one optimal solution $\left(\bar{x},\bar{\boldsymbol{\xi}}\right)$ to a problem ([SMPEC $-\ \Omega$]).

Proof. Owing to the conditions (i), (v), and the Measurable Selection Theorem (e.g. [Aum69, Him75]) there exists a random element $\boldsymbol{\xi}(\omega) \in \mathcal{F}(x_0,\omega)$ for almost all ω, i.e., the problem is feasible. Consider an arbitrary minimizing sequence $\{(x_k,\boldsymbol{\xi}_k)\}$. Uniform weak coercivity in assumption (iii) implies that there must be a subsequence of the sequence with a converging x-component. Let us renumber the whole sequence, so that $\bar{x} := \lim_{k\to\infty} x_k$. Consider now a measurable function

$\widetilde{f}(\omega) := \liminf_{k\to\infty} f(x_k, \boldsymbol{\xi}_k(\omega), \omega)$. Using the lower boundedness of f (in assumption (iii)), we get $\mathrm{E}_\omega[\widetilde{f}(\omega)] \leq \lim_{k\to\infty} \mathrm{E}_\omega[f(x_k, \boldsymbol{\xi}_k(\omega), \omega)]$.

On the other hand, the uniform local boundedness assumption (iv) implies that for almost any ω there is an infinite sequence of indices $k(\omega)$ such that there exists $\widetilde{\xi}(\omega) := \lim_{k(\omega)\to\infty} \boldsymbol{\xi}_{k(\omega)}(\omega)$ and so that $\widetilde{f}(\omega) = \lim_{k(\omega)\to\infty} f(x_{k(\omega)}, \boldsymbol{\xi}_{k(\omega)}(\omega), \omega)$. The assumed closedness of the mappings $\mathcal{Z}$ and $\mathcal{S}$ (ii) implies that $\widetilde{\xi}(\omega) \in \mathcal{F}(\bar{x}, \omega)$ for almost any ω. Note that the continuity assumptions on f imply that $f(\bar{x}, \widetilde{\xi}(\omega), \omega) = \widetilde{f}(\omega)$ almost everywhere.

Consider now the ω-parametric optimization problem in the variables $\xi(\omega)$:

$$\begin{aligned} &\min f(\bar{x}, \xi(\omega), \omega) \\ &\text{s.t.} \quad \begin{cases} \big(\bar{x}, \xi(\omega)\big) \in \mathcal{Z}(\omega), & \text{P-a.s.} \\ \xi(\omega) \in \mathcal{S}(\bar{x}, \omega), & \text{P-a.s.} \end{cases} \end{aligned} \tag{1.2}$$

We know that the problem has a nonempty, closed and bounded feasible set for almost any ω, that also depends on ω in a measurable way. Thus we can apply Theorem 8.2.11 [AuF90] to obtain the existence of a measurable solution $\bar{\boldsymbol{\xi}}(\omega)$ such that $f(\bar{x}, \bar{\boldsymbol{\xi}}(\omega), \omega) \leq f(\bar{x}, \widetilde{\xi}(\omega), \omega)$ owing to the optimality of $\bar{\boldsymbol{\xi}}$ and the feasibility of $\widetilde{\xi}$ for the problem (1.2).

Thus we have found a feasible solution $(\bar{x}, \bar{\boldsymbol{\xi}}(\omega))$ with desirable properties. #

Remark 1.3.1.1. For examples of conditions implying the closedness of $x \to \mathcal{S}(x, \omega)$ (assumption (i)) we cite assumption (iii) in [PaW99] (which must hold for almost any ω in addition to the continuity of the mapping T in (x, y)):

(iii) The lower-level constraint set, $\mathcal{Y}(x)$, is of the form $\mathcal{Y}(x) := \{\xi \in \mathbb{R}^m \mid g_i(x, \xi) \leq 0, \quad i = 1, \ldots, k\}$, where each function $g_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m$ and convex in ξ for each x. Further, either $g_i(x, \cdot) = g_i(\cdot)$, $i = 1, \ldots, k$, that is, $\mathcal{Y}(x) = \mathcal{Y}$, or for each upper-level feasible x there is a $\xi \in \mathbb{R}^m$ such that $g_i(x, \xi) < 0$, $i = 1, \ldots, k$.

Another example is Corollary 3.1 in [PaP00], which works for a specific stochastic bilevel programming problem considered in the cited paper.

1.4 Application to stochastic structural optimization

In this section we apply Theorem 1.3.1 to show the existence of a truss with a minimal weight under stochastic loads and stress constraints. In the case of discrete measures with finite support the problem was extensively studied in [PaP00].

The problem formulation is:

$$\min_{(x,s(\cdot))} 1^T x$$
$$\text{s.t.} \begin{cases} 0 \le x \\ |s(\omega)| \le \sigma x, s(\omega) \text{ solves } (\mathcal{C})_x(\omega), \quad \text{P-a.s.} \end{cases} \tag{$\mathcal{W}$}$$

where the lower-level problem $(\mathcal{C})_x(\omega)$ is:

$$\min_s \mathcal{E}(x,s) := \frac{1}{2}\sum_{i=1}^n \frac{s_i^2}{Ex_i}$$
$$\text{s.t.} \sum_{i=1}^n B_i^T s_i = F(\omega). \tag{$(\mathcal{C})_x(\omega)$}$$

The upper-level (*design*) variable x_i represents a volume of material allocated at the bar i ($x_i = 0$ represents structural void), the lower-level (*state*) variable $s_i(\cdot)$ represents a forse in the bar i multiplied by the bar length, E is the Young's modulus of the structure material, σ is the maximal allowable stress, $F : \Omega \to \mathbb{R}^k$ is a stochastic load, B_i, $i = 1,\dots,n$ are the kinematic transformation matrices, and $\mathcal{E} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is an extended real-valued functional, representing the elastic energy of the structure. The problem $(\mathcal{C})_x(\omega)$ is the mechanical *principle of minimum of complementary energy*. Thus making the identifications $\mathcal{Z}(\omega) := \{\, (x,s) \in \mathbb{R}^n \times \mathbb{R}^m \mid 0 \le x, |s| \le \sigma x \,\}$ and $\mathcal{S}(x,\omega) := \{\, s \in \mathbb{R}^m \mid s \text{ solves } (\mathcal{C})_x(\omega) \,\}$ we can see that the problem $(\mathcal{W})$ perfectly fits in a framework of [SMPEC – Ω].

Proposition 1.4.1. *Let the load $F : \Omega \to \mathbb{R}^k$ be measurable. Suppose that the problem $(\mathcal{W})$ has a feasible point $(x, s(\omega))$ such that $\mathrm{P}(\mathcal{E}(x,s(\omega)) < \infty) = 1$. Then it posess at least one optimal solution.*

Proof. Obviously, assumptions (iii)–(v) of Theorem 1.3.1 are fulfilled. Furthermore, assumption (i) holds (it is an immediate consequence of Theorem 1.2.1 for $\mathcal{Z}_x(\omega)$ and [CaV77, Lemma III.39] for $\mathcal{S}(x,\cdot)$). The set $\mathcal{Z}(\omega)$ is closed for any ω. Thus it remains to show the closedness of $x \to \mathcal{S}(x,\omega)$ for any ω to verify assumption (ii) and conclude the existence of solutions.

The required property follows from [PaP00, Corollary 3.1] under additional assumption of boundedness of energy functional $\mathcal{E}(x,s)$. Furthermore, [PaP00, Theorem 4.3] implies that one can add *redundant* (such that no optimal solution can violate it) constraint $\mathcal{E}(x,s) \le \nu$ to the problem $(\mathcal{W})$. Since the function $\mathcal{E}$ is l.s.c. by [Roc70, p. 83], the set $\widetilde{\mathcal{Z}}(\omega) := \{\, (x,s) \in \mathcal{Z}(\omega) \mid \mathcal{E}(x,s) \le \nu \,\}$ is closed for any ω.

We finish the proof by application of Theorem 1.3.1. #

1.5 Inexact penalization

One-level problems have been studied much more than bilevel ones. Bilevel optimization algorithms are much less straightforward to develop owing to the non-convex nature of the problem and its absence of constraint qualifications for nonlinear programming [LPR96]. One approach is to move the equilibrium constraint as a penalty into the objective function. For examples of penalty functions leading to algorithmic solutions to MPEC, see [LPR96, Pan97, YZZ97, ScS99] and references therein. In particular, the exact penalties are of great importance, since they lead to exact solutions while they do not require the penalty parameter to tend to infinity [Bur91]. One cannot however expect to be able to construct an exact penalty for SMPEC problems, given an exact penalty for each ω, as the following simple example shows. The reason is again the presence of the "coupling" upper-level variables.

Example 1.5.1. Let $(\Omega, \mathfrak{S}, \mathrm{P}) = ([0,1], \bar{\mathcal{B}}([0,1]), \lambda)$, where λ is a Lebesgue measure on $[0,1]$, and $\bar{\mathcal{B}}([0,1])$ is a σ-algebra of Lebesgue measurable sets. Let $\mathcal{Z}(\omega) = [0,\omega] \times \{\, 0 \,\}$, $f(x,\xi,\omega) = (x-1/2)^2$, $\mathcal{Y}(x,\omega) = \{\, 0 \,\}$, $T(x,\xi,\omega) = 0$, For any $\omega \in [0,1]$ an exact penalty for the "fixed-ω" problem is, for example, $G(x,\xi,\omega) = \max\{\, x-\omega, 0 \,\}$.

Nevertheless, since

$$\int_0^1 [(x-1/2)^2 + \mu \max\{\, x-\omega, 0 \,\}]\, \lambda(d\omega) = (x-1/2)^2 + \mu \frac{x^2}{2},$$

the minimizing sequence is $x_\mu = 1/(\mu+2) \to 0$ as $\mu \to \infty$, and thus it does not reach the optimal (actually, the only feasible) point of the given SMPEC, $x^* = 0$, for any finite value of μ.

In the following theorem we show that, given a penalty function for almost any ω, we can construct an inexact penalty function for SMPEC. It generalizes Theorem 9.2.2 in [BSS93]. Note that we do not necessarily have compact sequences for the lower-level variables, so we do not necessarily have convergence for these variables. In the case of discrete measures supported by finite sets, the theorem reduces to [BSS93, Theorem 9.2.2].

We will write $\mathrm{val}([\mathcal{P}])$ for the optimal value of the optimization problem $[\mathcal{P}]$.

Theorem 1.5.2. *Suppose that the assumptions of Theorem 1.3.1 are satisfied, so that there is an optimal solution to* $[\mathrm{SMPEC} - \Omega]$. *Let also* $G(x,\xi,\omega)$ *be non-negative, continuous in* (x,ξ) *for almost any* ω, *and measurable in* ω *for any* $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m$ *function such that* $\mathcal{S}(x,\omega) = \{\, \xi \mid (x,\xi) \in \mathcal{Z}(\omega), G(x,\xi,\omega) = 0 \,\}$. *Then the penalized problem:*

$$\begin{aligned} \min\ & \mathrm{E}_\omega[f(x,\boldsymbol{\xi}(\omega),\omega) + \mu G(x,\boldsymbol{\xi}(\omega),\omega)] \\ \text{s.t.}\ & \big(x,\boldsymbol{\xi}(\omega)\big) \in \mathcal{Z}(\omega), \quad \mathrm{P}\text{-}a.s. \end{aligned} \qquad [\mathrm{SMPEC} - \Omega]_\mu$$

has an optimal solution for any $\mu \geq 0$ *and*

$$\sup_{\mu \geq 0} \mathrm{val}([\mathrm{SMPEC} - \Omega]_\mu) = \lim_{\mu \to \infty} \mathrm{val}([\mathrm{SMPEC} - \Omega]_\mu) = \mathrm{val}([\mathrm{SMPEC} - \Omega])$$

Furthermore, any limit point of the upper-level optimal solutions $\{x_\mu\}$ *to* $[\mathrm{SMPEC} - \Omega]_\mu$ *(and there is at least one) is an upper-level optimal solution to* $[\mathrm{SMPEC} - \Omega]$.

Proof. For any $\mu \geq 0$ the problem $[\mathrm{SMPEC}-\Omega]_\mu$ satisfies the assumptions of Theorem 1.3.1 (where we can put $\mathcal{S}_\mu(x,\omega) = \{\, \xi \in \mathbb{R}^m \mid (x,\xi) \in \mathcal{Z}(\omega) \,\}$), and thus possess a solution $(x_\mu, \boldsymbol{\xi}_\mu(\cdot))$.

Following the proof of Lemma 9.2.1 and Theorem 9.2.2 in [BSS93], we get:

$$\begin{aligned} \mathrm{val}([\mathrm{SMPEC} - \Omega]) &\geq \sup_{\mu \geq 0} \mathrm{val}([\mathrm{SMPEC} - \Omega]_\mu) \\ &= \lim_{\mu \to \infty} \mathrm{val}([\mathrm{SMPEC} - \Omega]_\mu) \\ &= \lim_{k \to \infty} \mathrm{E}_\omega[f(x_{\mu_k}, \boldsymbol{\xi}_{\mu_k}(\omega), \omega)] \end{aligned} \tag{1.3}$$

for some $\mu_k \to \infty$.

By the uniform coercivity (assumption (iii) of Theorem 1.3.1) of f in x, and by the properties of G as a penalty function, the sequence $\{x_{\mu_k}\}$ is bounded. Switching to a subsequence if necessary, we may assume that $x_{\mu_k} \to \widetilde{x}$. Owing to the lower boundedness of f (assumption (iii) of Theorem 1.3.1) we have that $\lim_{k\to\infty} \mathrm{E}_\omega[f(x_{\mu_k}, y_{\mu_k}(\omega), \omega)] \geq \mathrm{E}_\omega[\liminf_{k\to\infty} f(x_{\mu_k}, y_{\mu_k}(\omega), \omega)]$. By the boundedness of the feasible set (assumption (iv) of Theorem 1.3.1) for almost any ω, there is a sequence $k(\omega)$ such that $\boldsymbol{\xi}_{\mu_{k(\omega)}}(\omega) \to \widetilde{\xi}(\omega)$ and $\liminf_{k\to\infty} f(x_{\mu_k}, \boldsymbol{\xi}_{\mu_k}(\omega), \omega) = \lim_{k(\omega)\to\infty} f(x_{\mu_{k(\omega)}}, \boldsymbol{\xi}_{\mu_{k(\omega)}}(\omega), \omega) \geq f(\widetilde{x}, \widetilde{\xi}(\omega), \omega)$, for P-almost any ω. Owing to the closedness (assumption (ii) of Theorem 1.3.1) of $\mathcal{Z}$, $(\widetilde{x}, \widetilde{\xi}(\omega)) \in \mathcal{Z}(\omega)$, P-a.s.

Following the proof of [BSS93, Theorem 9.2.2] we get:

$$0 = \lim_{k\to\infty} \mathrm{E}_\omega[G(x_{\mu_k}, \boldsymbol{\xi}_{\mu_k}(\omega), \omega)] \geq \mathrm{E}_\omega[\liminf_{k\to\infty} G(x_{\mu_k}, \boldsymbol{\xi}_{\mu_k}(\omega), \omega)],$$

and, by the continuity and non-negativity of G, $\liminf_{k\to\infty} G(x_{\mu_k}, \boldsymbol{\xi}_{\mu_k}(\omega), \omega) \geq G(\widetilde{x}, \widetilde{\xi}(\omega), \omega) \geq 0$ for P-almost any ω, thus showing that $\widetilde{\xi}(\omega) \in \mathcal{F}(x,\omega)$ for P-almost any ω. Considering the parametric optimization problem (1.2) we can find a measurable function $\widetilde{\boldsymbol{\xi}}(\cdot) \in \mathcal{F}(x,\cdot)$ such that $f(\widetilde{x}, \widetilde{\xi}(\omega), \omega) \geq f(\widetilde{x}, \widetilde{\boldsymbol{\xi}}(\omega), \omega)$ P-a.s., thus showing that

$$\sup_{\mu \geq 0} \mathrm{val}\,[\mathrm{SMPEC} - \Omega]_\mu \geq \mathrm{E}_\omega[f(\widetilde{x}, \widetilde{\boldsymbol{\xi}}(\omega), \omega)] \geq \mathrm{val}\,[\mathrm{SMPEC} - \Omega].$$

Together with (1.3) this proves the claim. #

1.6 Concluding remarks

The case of discrete measures was considered in [PaW99] and some algorithms were proposed. For a general SMPEC problem the discretization that is, the approximation of the probability measure by a sequence of discrete measures, seems to be the only way to solve it. The discretization procedure could be applied either to the original problem ([SMPEC $-\Omega$]) or to the penalized one ([SMPEC $-\Omega]_\mu$).

The question of convergence of the discretizations is related to the stability of optimization problems with respect to small changes in probability measure. The question of stability of bilevel programming problems is not so well investigated in the literature even in the deterministic case. For existing results we mention [LiM95, WWU96, JYW98].

Existing results about the stability of optimization problems with respect to changes in the probability measure usually presumes the existence of a constraint qualification [Lep90], which are by no means satisfied by SMPEC problems, or they are posed in the spaces of continuous functions [RoW87, Kal87, RöS91], which also is not the case for a general SMPEC. To apply latter results we need to assume the uniqueness of solutions to a lower-level problem and the continuity of solutions with respect to ω.

One can also view the lower-level problem as a variational inequality problem (VIP) in a Banach space X, under the additional assumptions that $\boldsymbol{\xi}(\cdot) \in X$ and $T(x, \boldsymbol{\xi}(\cdot), \cdot) \in X^*$, hoping to use sensitivity analysis results in this area [Din97, Wat97, Lev99, Din00]. The difficulty with such an identification is that the resulting VIP is not necessarily monotone even if the operator $T(x, \cdot, \omega)$ is monotone for almost any ω.

Despite all these difficulties it is possible to show the convergence of some discretization schemes under additional assumptions, for the specific cases of SMPEC discussed in [PaP00] in application to structural optimization in contact mechanics. Furthermore, assuming the continuity of the problem's data with respect to ω, it is possible to analyze a distribution sensitivity for such stochastic structural optimization models.

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Paper 2

STOCHASTIC STRUCTURAL TOPOLOGY OPTIMIZATION: EXISTENCE OF SOLUTIONS AND SENSITIVITY ANALYSES

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Abstract

We consider structural topology optimization problems including unilateral constraints arising from, for example, non-penetration conditions in contact mechanics or non-compression conditions for elastic ropes. To construct more realistic models and to hedge off possible failures or inefficient behaviour of optimal structures, we allow parameters (for example, loads) defining the problem to be stochastic. The resulting nonsmooth stochastic optimization problem is an instance of stochastic mathematical programs with equilibrium constraints (MPEC), or stochastic bilevel programs. The existence as well as the continuity of optimal solutions with respect to the lower bounds on the design variables are established. The question of continuity of optimal solutions with respect to small changes in probability measure is analysed. For a subclass of the problems considered the answer is affirmative, thus showing the robustness of optimal solutions.

Key words: Bilevel programming, stochastic programming, robust optimization, ε-perturbation, stress constraints

2.1 Introduction and notation

Does the introduction of multiple load cases into a topology optimization problem always lead to robust optimal designs? The large number of

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publications aiming to achieve robust solutions by optimizing for several (in some cases the continuum) load cases suggests that the answer should be positive.

The answer of course depends on the definition of "robustness" and the type of optimization problem under consideration. The reason for considering several load cases is to incorporate the *uncertain* nature of the loads into the model, while the desired property of a robust design is to change continuously as a model of reality (loading conditions, material properties, etc.) changes. To thoroughly answer the posed question it is necessary to measure the closeness of two models of (uncertain) reality.

In this paper we consider two of the most natural and classic structural topology optimization problems: the finding of a maximally stiff truss under a volume constraint, and the finding of a truss of minimal weight under stress constraints. The uncertainty due to several factors (such as loads unknown in advance, varying material properties, manufacturing errors, etc.) is taken into account. Capturing the uncertainty in the model through the use of probability theory allows us to construct general models, and through the associated probability measure, it is possible to interpret the "continuous change in the model of reality" as a continuous change in a topological space of measures.

To include a wide range of applications we allow mechanical structures to be *unilaterally* constrained, i.e., some parts of the structure might come into unilateral frictionless contact with rigid obstacles, while some other parts might sustain only tensile forces. Practical applications of unilateral contact include such machine elements as joints, hinges, press-fits, and examples of structures with tensile-only members include suspension bridges and cranes.

In addition to extending "classic" structural topology optimization results (existence of optimal designs, convergence of ε-perturbations) to the general stochastic setting, we analyse the continuity of optimal solutions with respect to changes in the probability measure. The results of this analysis give us explicit information about when the introduction of uncertainty into the structural topology optimization models indeed leads to robust optimal designs.

2.1.1 Historical overview

The study of the topology optimization of trusses dates back at least as early as the beginning of the previous century [Mic04]. Practice has shown that the idea of allowing truss topology to change leads to exceedingly efficient designs. Thus both the optimization and mechanical models were considerably generalized in many aspects by many authors during the last thirty years (see for example surveys [RKB95, Roz01]).

Unfortunately, designs obtained from a topology optimization procedure have a principal drawback. They may be very inefficient or can even fail when loading conditions slightly change. An attempt to maintain the efficiency of topology optimization while hedging off possible failures or inefficient behaviour has given rise to a field of *robust topology optimization*. Owing to the anticipated fact that the "real" probability model is never known, and the reported high sensitivity of solutions to stochastic structural optimization problems with respect to small changes in probability measure (e.g. [BHE90, pp. 20–22]), many probability-free worst-case ("pessimistic") models of uncertainty have been developed as an alternative to probabilistic ones. In such worst-case models uncertain parameters are assumed to vary in convex ("convex uncertainty models") or even in polyhedral ("polyhedral uncertainty models") sets. An efficient numerical approach to solve problems of this type is known [BTN97], which however has a considerable drawback: the algorithm can treat uncertainties with respect to loading conditions only. Furthermore, loads are restricted to lie in some small ellipsoid around the "primal loads", a condition that further reduces the generality of the algorithm.

Recently in [CPW01] and [PaP00] stochastic structural topology optimization problems have been formulated and analysed in the case of discrete probability spaces, with emphasis on sensitivity analysis leading to numerical methods. The sensitivity analysis conducted as one part of this paper is an extension of the results in [PaP00] to a more general probabilistic setting.

2.1.2 Mechanical equilibrium

In this subsection we introduce the notation and mechanical principles necessary to state the problems we are going to analyse.

Given positions of the nodes the *design* (and topology in particular) of a truss can be described by the following sets of *design* variables:

- $x_i \geq 0$, $i = 1, \ldots, m$, representing the volume of material, allocated to the bar i in the structure;
- $X_j \geq 0$, $j = 1, \ldots, r_2$, representing the volume of material, allocated to the cable j.

We introduce two index sets of the present (or active) members in the structure: $\mathcal{I}(x) = \{\, i = 1, \ldots, m \mid x_i > 0 \,\}$ and $\mathcal{J}(X) = \{\, j = 1, \ldots, r_2 \mid X_j > 0 \,\}$.

Let $(\Omega, \mathfrak{S}, \mathrm{P})$ be a complete probability space. Given a particular design the status of the linear elastic mechanical system is governed by the principle of minimum complementary energy $(\mathcal{C})_{(x,X)}(\omega)$ (in our case it is the (x, X, ω)-parametric minimization problem):

$$\begin{cases} \min_{(s,S,\lambda)} \mathcal{E}(x, X, s, S, \lambda, \omega) := \frac{1}{2} \sum_{i \in \mathcal{I}(x)} \frac{s_i^2}{E(\omega) x_i} + g_1^T(\omega)\lambda \\ \qquad\qquad + \sum_{j \in \mathcal{J}(X)} \Big(\frac{(L_j(\omega) S_j)^2}{2E_c(\omega) X_j} + (g_2(\omega))_j S_j \Big), \\ \text{s.t.} \begin{cases} C_1^T(\omega)\lambda + \sum_{i \in \mathcal{I}(x)} B_i^T(\omega) s_i + \sum_{j \in \mathcal{J}(X)} S_j \gamma_j(\omega) = f(\omega), \\ \lambda \geq 0, \\ S_{\mathcal{J}(X)} \geq 0, \end{cases} \end{cases}$$

where the functions in the problem have the following meaning from a mechanical point of view:

- $E(\omega)$ and $E_c(\omega)$ are Young's moduli for the structure and cable materials respectively;
- $B_i(\omega)$ is the kinematic transformation matrix for the bar i;
- $\gamma_j(\omega)$ is the unit direction vector of the cable j;
- $(g_2(\omega))_j$ is the initial slack of the cable j;
- $L_j(\omega)$ is the length of the cable j;
- $C_1(\omega)$ is the quasi-orthogonal kinematic transformation matrix for rigid obstacles;
- $g_1(\omega) \geq 0$ is the vector of the initial gaps;
- $f(\omega)$ is the vector of external forces.

For the problem to be tractable we assume that all functions listed above are $\mathfrak{S}$-measurable. We further assume that the matrix C_1 is quasi-orthogonal, that is, that $C_1 C_1^T = I$. That condition is fulfilled if at each node either there is at most one rigid support or multiple supports "act" in orthogonal directions to each other.

The variables in the problem $(\mathcal{C})_{(x,X)}(\omega)$ have the following interpretation:

- s_i is the tensile force in the bar times its length;
- S_j is the tensile force in the cable;
- λ is the vector of contact forces.

Note, that from the quasi-orthogonality of C_1 it follows that λ is uniquely determined by (s, S) and depends continuously on them:

$$\lambda = C_1(\omega)\Big(f(\omega) - \sum_{i \in \mathcal{I}(x)} B_i^T(\omega) s_i - \sum_{j \in \mathcal{J}(X)} S_j \gamma_j(\omega)\Big). \tag{2.1}$$

These facts will be often used without backward reference.

2.1.3 General stochastic minimum compliance problem

We are now ready to state the first problem considered in this paper — the general stochastic minimum compliance problem:

$$(\mathcal{P}_1)\begin{cases} \min_{(x,X,s(\cdot),S(\cdot))} c^f(x,X,s(\cdot),S(\cdot),\lambda(\cdot)) := \\ \qquad \int_\Omega \mathcal{E}(x,X,s(\omega),S(\omega),\lambda(\omega),\omega)\,\mathrm{P}(d\omega) \\ \text{s.t.} \begin{cases} \underline{x} \le x \le \overline{x}, & 1_m^T x \le v, \\ \underline{X} \le X \le \overline{X}, & 1_{r_2}^T X \le V, \\ (s(\omega),S(\omega),\lambda(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), \text{ P-a.s.,} \end{cases} \end{cases}$$

where v and V are the limits on the amount of cable and structure material correspondingly. In this problem we minimize the *average* value of compliance for multiple load cases.

In topology optimization we set lower bounds $\underline{x} = 0$ and $\underline{X} = 0$.

2.1.4 Stochastic stress constrained minimum weight problem

The formal problem formulation is as follows:

$$(\mathcal{P}_2)\begin{cases} \min_{(x,X,s(\cdot),S(\cdot))} w(x,X) := \rho_1 1_m^T x + \rho_2 1_{r_2}^T X \\ \text{s.t.} \begin{cases} \underline{x} \le x \le \overline{x}, & & \\ \underline{X} \le X \le \overline{X}, & & \\ |s_i(\omega)| \le \overline{\sigma}_1 x_i, & i = 1,\dots,m, & \text{P-a.s.,} \\ L_j S_j(\omega) \le \overline{\sigma}_2 X_j, & j = 1,\dots,m, & \text{P-a.s.,} \\ (s(\omega),S(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), & & \text{P-a.s.,} \end{cases} \end{cases}$$

where $\overline{\sigma}_1$ and $\overline{\sigma}_2$ are the maximal allowable effective stresses in, and ρ_1 and ρ_2 the densities of, the structure and the cable materials respectively. In this problem we require stress constraints to hold for *almost all* load cases, or we allow them to be violated *with probability zero*.

In topology optimization we set lower bounds $\underline{x} = 0$ and $\underline{X} = 0$.

2.1.5 Outline

The outline of the remaining part of the paper is as follows. In Section 2.2 the existence of solutions to the stated problems is proved. Section 2.3 is dedicated to the analysis of the continuity of solutions with respect to changes in the lower bound on the design variables. The stability of solutions with respect to small changes in the probability measure is the topic of Section 2.4. Proofs of the auxiliary results can be found in the appendix.

2.2 Existence of solutions

In this section we show the existence of optimal designs for problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ under reasonable assumptions about the underlying mechanical model. The results depend on the closedness of the feasible set, which is typically the main issue when the existence of optimal solution to MPEC is in question [LPR96, Example 1.1.2]. Bilevel structural topology optimization problems, where the lower-level problem is $(\mathcal{C})_{(x,X)}(\omega)$, were extensively studied in [PaP00]. We cite three important results for the reader's convenience.

Proposition 2.2.1. *Fix* $\omega \in \Omega$.

(i) *[PaP00, Theorem 2.1] Suppose the feasible set of the problem* $(\mathcal{C})_{(x,X)}(\omega)$ *is nonempty for some nonnegative design* (x, X). *Then, there exists a unique optimal solution to the problem* $(\mathcal{C})_{(x,X)}(\omega)$.

(ii) *[PaP00, Theorem 3.1] Let* $\{(x_k, X_k)\}$ *be a nonnegative sequence of designs, converging to* (x, X). *Suppose, that* $\{(s_k, S_k, \lambda_k)\}$ *is the corresponding sequence of optimal solutions to* $(\mathcal{C})_{(x_k,X_k)}(\omega)$, *and assume that the sequence of energies is bounded, that is, that* $\mathcal{E}(x_k, X_k, s_k, S_k, \lambda_k, \omega) \leq c < \infty$ *for all* k. *Then, there exists a unique optimal solution* (s, S, λ) *to* $(\mathcal{C})_{(x,X)}(\omega)$, *and* $\lim_{k\to\infty}(s_k, S_k, \lambda_k) = (s, S, \lambda)$.

(iii) *[PaP00, Corollary 3.2] Let* (x, X) *be a nonnegative design for which there exists an optimal solution* (s, S, λ) *to the problem* $(\mathcal{C})_{(x,X)}(\omega)$. *Let* $\{(x_k, X_k)\}$ *be a sequence of nonnegative designs which converges to* (x, X), *and suppose that* $\{(s_k, S_k, \lambda_k)\}$ *is the corresponding sequence of optimal solutions to* $(\mathcal{C})_{(x_k,X_k)}(\omega)$. *Then,* $\lim_{k\to\infty}(s_k, S_k, \lambda_k) = (s, S, \lambda)$.

We note that if the problem $(\mathcal{C})_{(x,X)}(\omega)$ is feasible for some nonnegative design (x, X), then it is feasible for any design $(\widetilde{x}, \widetilde{X}) \geq (x, X)$. Furthermore, the feasible set of $(\mathcal{C})_{(\widetilde{x},\widetilde{X})}(\omega)$ includes that of $(\mathcal{C})_{(x,X)}(\omega)$.

To prove the existence of optimal designs we need an auxiliary result, which asserts the measurability of solutions to $(\mathcal{C})_{(x,X)}(\omega)$ as functions of ω. In particular, the measurability of the solutions together with the lower semi-continuity of the energy functional imply that we can integrate the energy unless it is "too large".

Corollary 2.2.2. *Suppose the measurability assumptions stated in Section 2.1 hold. Suppose further that for almost any* ω *the feasible set of the problem* $(\mathcal{C})_{(x,X)}(\omega)$ *is nonempty. Then there exists a unique (up to changes on sets of probability zero) triple of functions* $(s(\cdot), S(\cdot), \lambda(\cdot))$ *almost everywhere solving the parametric problem* $(\mathcal{C})_{(x,X)}(\cdot)$. *In addition, these functions are* $\mathfrak{S}$*-measurable.*

The following result is a generalization of [PaP00, Theorem 3.1 and Corollary 3.2] to a stochastic setting.

Proposition 2.2.3. *Let a sequence of nonnegative designs* $\{(x_k, X_k)\}$ *converge to* $(\bar{x}, \bar{X})$*. Suppose that* $(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))$ *solve* $(\mathcal{C})_{(x_k,X_k)}(\cdot)$*, and that the sequence of energy expectations is bounded:*

$$\int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)\,\mathrm{P}(d\omega) \le C < \infty.$$

Then there exists a solution $(\bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$ *to the problem* $(\mathcal{C})_{(\bar{x},\bar{X})}(\cdot)$*, and* $\{(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))\}$ *almost sure converges to* $(\bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$*.*

Theorems 2.2.4 and 2.2.5 show the existence of optimal solutions to problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$.

Theorem 2.2.4 (Existence of solutions to $(\mathcal{P}_1)$). *Suppose that for some feasible point* $(x_0, X_0, s(\cdot), S(\cdot))$ *in the problem* $(\mathcal{P}_1)$ *we have* $c^f(x, X, s(\cdot), S(\cdot), \lambda(\cdot)) < \infty$*. Then, there exists at least one optimal solution to* $(\mathcal{P}_1)$*.*

Proof. Consider an arbitrary minimizing sequence $\{(x_k, X_k, s_k(\cdot), S_k(\cdot))\}$ for the problem $(\mathcal{P}_1)$ together with the corresponding sequence of contact forces $\{\lambda_k(\cdot)\}$. Since the the feasible design space is compact, without any loss of generality we may assume that the sequence $\{(x_k, X_k)\}$ converges to a limit (x^*, X^*) satisfying the design constraints. Proposition 2.2.3 for such a sequence implies that the sequence of state variables $\{(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))\}$ almost sure converges to a limit $(s^*(\cdot), S^*(\cdot), \lambda^*(\cdot))$ solving $(\mathcal{C})_{(x^*,X^*)}(\cdot)$ thus showing the feasibility of the limit in $(\mathcal{P}_1)$. Furthermore, owing to the l.s.c. property of the energy functional for each ω and Fatou's Lemma, the following inequality holds:

$$0 \le c^f(x^*, X^*, s^*(\cdot), S^*(\cdot), \lambda^*(\cdot)) \le \liminf_{k\to\infty} c^f(x_k, X_k, s_k(\cdot), S_k(\cdot), \lambda_k(\cdot)),$$

whence $(x^*, X^*, s^*(\cdot), S^*(\cdot))$ is an optimal solution to $(\mathcal{P}_1)$. #

The next theorem generalizes Proposition 4.1 in [EvP01], which asserts the existence of solutions to the problem $(\mathcal{P}_2)$ for trusses without unilateral constraints.

Theorem 2.2.5 (Existence of solutions to $(\mathcal{P}_2)$). *Suppose that the following assumptions are satisfied:*

(i) *the feasible set of the problem* $(\mathcal{P}_2)$ *is nonempty;*

(ii) $\mathrm{P}(E(\cdot) \ge c) = \mathrm{P}(E_c(\cdot) \ge c) = 1$ *for some constant* $c > 0$*;*

(iii) *the functions* $L_j(\cdot)$, $g_1(\cdot)$, $g_2(\cdot)$, $C_1(\cdot)$, $B_i(\cdot)$ *and* $f(\cdot)$ *are essentially bounded.*

Then there exists at least one optimal solution to the problem $(\mathcal{P}_2)$.

Proof. Assumption (iii) together with the stress constraints and equation (2.1) implies the essential upper-boundedness of the term $g_1^T(\omega)\lambda(\omega)$ on the feasible set by some constant $C < \infty$. Following the proof of Theorem 4.3 in [PaP00] we can show the existence of the upper bound on energy for some strictly positive design $(\widehat{x}, \widehat{X}) \le (\overline{x}, \overline{X})$:

$$\begin{aligned}
&\mathcal{E}(x, X, s(\omega), S(\omega), \lambda(\omega), \omega) \le \frac{1}{2}\sum_{i=1}^{m} \frac{\widehat{x}_i(\overline{\sigma}_1)^2}{E(\omega)} \\
&+ \sum_{j=1}^{r_2} \max_{\underline{X}_j \le X_j \le \widehat{X}_j} \Big\{0, \Big(\frac{X_j(L_j(\omega)\overline{\sigma}_2)^2}{2E_c(\omega)} + (g_2(\omega))_j\overline{\sigma}_2 X_j\Big)\Big\} + C \\
&=: v(\omega) + C, \qquad \text{P-a.s.},
\end{aligned}$$

such that no optimal design can violate it. Assumptions (ii) and (iii) imply that $\mathrm{P}(v(\cdot) < \infty) = 1$. We then add a redundant constraint $\mathcal{E}(x, X, s(\omega), S(\omega), \lambda(\omega), \omega) \le v(\omega) + C$ to our problem and use Proposition 2.2.1(ii) to obtain the closedness of the feasible set for almost any ω. Now Theorem 3.1 in [EvP01] asserting the existence of solutions to SMPEC can be applied, and we are done. #

Remark 2.2.5.1. Assumption (ii) does not allow the cable and structure material to break with a positive probability, because in this case we usually cannot expect the existence of a mechanical equilibrium with probability 1. Assumption (iii) is satisfied by most mechanical models; the only questionable assumption is the boundedness of the loads $f(\cdot)$. Our interpretation of this assumption is that since we work in a framework of the linear elasticity we cannot consider unbounded loads.

2.3 Convergence of ε-perturbations

The so-called ε-perturbation of structural topology optimization problems, or approximation with a sequence of *sizing* optimization problems, has become a classic topic.

For compliance minimization problems a naive replacement of the lower design bounds $(\underline{x}, \underline{X}) = 0$ with a small positive value $\varepsilon > 0$ tending to zero (whence the name — ε-perturbation) is sufficient. Theorem 2.3.1 below is an extension of the corresponding result for discrete probability measures (Theorem 4.2 in [PaP00]).

The situation with the stress constrained weight minimization is far more complicated. Sved and Ginos [SvG68] observed that the problem may have singular solutions, which cannot be approximated by the simplistic approach outlined above. The properties of the feasible region were further investigated by Kirsch [Kir90], Cheng and Jiang [ChJ92], Rozvany and Birker [RoB94]. Cheng and Guo [ChG97] proposed a more sophisticated relaxation procedure, where not only lower bounds but also stress constraints were perturbed. They showed the convergence of optimal values of perturbed problems to the optimal value of the original problem, while Petersson [Pet01] showed the convergence of optimal solutions. The ε-relaxation was extended to continuum structures by Duysinx and Bendsøe [DuB98] and Duysinx and Sigmund [DuS98]. Patriksson and Petersson [PaP00] generalized the result for stochastic truss topology optimization problems including unilateral constraints and discrete probability measures. Theorem 2.3.3 below extends the latter result for general probability spaces.

Stolpe and Svanberg [StS01] demonstrated that singular topologies can occur in multi-load cases even if all other parameters (material properties, stress limits) are kept uniform among the structural members. This implies that in our case singular topologies are quite likely to occur.

2.3.1 ε-perturbation of $(\mathcal{P}_1)$

Consider the following ε-perturbation of the problem $(\mathcal{P}_1)$:

$$(\mathcal{P}_1^\varepsilon)\begin{cases} \min\limits_{(x,X,s(\cdot),S(\cdot))} c^f(x,X,s(\cdot),S(\cdot),\lambda(\cdot)) \\ \text{s.t.} \begin{cases} \varepsilon 1_m \le x \le \overline{x}, & 1_m^T x \le v, \\ \varepsilon 1_{r_2} \le X \le \overline{X}, & 1_{r_2}^T X \le V, \\ (s(\omega),S(\omega),\lambda(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), \text{ P-a.s.} \end{cases} \end{cases}$$

Theorem 2.3.1. *Suppose that for some* $\varepsilon_0 > 0$ *there is a solution* $(x_0,X_0,s_0(\cdot),S_0(\cdot),\lambda_0(\cdot))$ *that is feasible in* $(\mathcal{P}_1)$ *with* $(x_0,X_0)\ge\varepsilon_0 1_{m+r_2}$ *and* $c^f(x_0,X_0,s_0(\cdot),S_0(\cdot),\lambda_0(\cdot)) < \infty$. *For each* $\varepsilon_0 \ge \varepsilon > 0$, *let* $(x_\varepsilon^*,X_\varepsilon^*,s_\varepsilon^*(\cdot),S_\varepsilon^*(\cdot),\lambda_\varepsilon^*(\cdot))$ *denote an arbitrary optimal solution to* $(\mathcal{P}_1^\varepsilon)$. *Then any limit point of the sequence* $\{(x_\varepsilon^*,X_\varepsilon^*,s_\varepsilon^*(\cdot),S_\varepsilon^*(\cdot),\lambda_\varepsilon^*(\cdot))\}$ *(and there is at least one) is an optimal solution to* $(\mathcal{P}_1^0)=(\mathcal{P}_1)$.

Proof. According to Theorem 2.2.4 a solution to $(\mathcal{P}_1^\varepsilon)$ exists for each $\varepsilon_0 \ge \varepsilon \ge 0$. The sequence $\{(x_\varepsilon^*,X_\varepsilon^*,s_\varepsilon^*(\cdot),S_\varepsilon^*(\cdot),\lambda_\varepsilon^*(\cdot))\}$ is feasible to the original problem $(\mathcal{P}_1^0)$. Furthermore, the sequence $\{c^f(x_\varepsilon^*,X_\varepsilon^*,s_\varepsilon^*(\cdot),S_\varepsilon^*(\cdot),\lambda_\varepsilon^*(\cdot))\}$ is non-increasing. Applying Proposition 2.2.3 we can obtain a feasible solution $(\bar{x},\bar{X},\bar{s}(\cdot),\bar{S}(\cdot),\bar{\lambda}(\cdot))$ to $(\mathcal{P}_1)$ such that it is an a.s.-limit of $\{(x_\varepsilon^*,X_\varepsilon^*,s_\varepsilon^*(\cdot),S_\varepsilon^*(\cdot),\lambda_\varepsilon^*(\cdot))\}$.

On the other hand, for any feasible solution $(x,X,s(\cdot),S(\cdot),\lambda(\cdot))$ in $(\mathcal{P}_1)$ with $c^f(x,X,s(\cdot),S(\cdot),\lambda(\cdot)) < \infty$ there is a sequence $\{(x_\varepsilon,X_\varepsilon,s_\varepsilon(\cdot),S_\varepsilon(\cdot),\lambda_\varepsilon(\cdot))\}$ of feasible solutions to $(\mathcal{P}_1^\varepsilon)$ such that $(x_\varepsilon,X_\varepsilon)\to(x,X)$ (cf. Proposition 1.1.2 in [AuF90]). Proposition 2.2.1(iii) implies that the sequence $\{(s_\varepsilon(\cdot),S_\varepsilon(\cdot),\lambda_\varepsilon(\cdot))\}$ a.s. converges to $(s(\cdot),S(\cdot),\lambda(\cdot))$.

Finally,

$$\begin{aligned} c^f(\bar{x},\bar{X},\bar{s}(\cdot),\bar{S}(\cdot),\bar{\lambda}(\cdot)) &\le \liminf_{\varepsilon\to 0} c^f(x^*_\varepsilon,X^*_\varepsilon,s^*_\varepsilon(\cdot),S^*_\varepsilon(\cdot),\lambda^*_\varepsilon(\cdot)) \\ &\le \liminf_{\varepsilon\to 0} c^f(x_\varepsilon,X_\varepsilon,s_\varepsilon(\cdot),S_\varepsilon(\cdot),\lambda_\varepsilon(\cdot)) \\ &\le \lim_{\varepsilon\to 0} c^f(x_\varepsilon,X_\varepsilon,s(\cdot),S(\cdot),\lambda(\cdot)) \\ &= c^f(x,X,s(\cdot),S(\cdot),\lambda(\cdot)), \end{aligned} \tag{2.2}$$

where the inequalities are owing to the l.s.c.-property of $\mathcal{E}$ and Fatou's Lemma, the optimality of $(x^*_\varepsilon,X^*_\varepsilon,s^*_\varepsilon(\cdot),S^*_\varepsilon(\cdot),\lambda^*_\varepsilon(\cdot))$ in $(\mathcal{P}_1^\varepsilon)$, and the optimality of $(s_\varepsilon(\cdot),S_\varepsilon(\cdot),\lambda_\varepsilon(\cdot))$ in $(\mathcal{C})_{(x_\varepsilon,X_\varepsilon)}(\cdot)$ correspondingly. The equality follows from the continuity of c^f with respect to (x,X) (we can move variables (x,X) out of the integrals).

Since the feasible point $(x,X,s(\cdot),S(\cdot))$ was arbitrary, the inequality (2.2) shows the optimality of $(\bar{x},\bar{X},\bar{s}(\cdot),\bar{S}(\cdot))$ in $(\mathcal{P}_1^0)$. #

Remark 2.3.1.1. For the convergence of optimal values we note first that the sequence of energies $\{\mathcal{E}(x^*_\varepsilon,X^*_\varepsilon,s^*_\varepsilon(\cdot),S^*_\varepsilon(\cdot),\lambda_\varepsilon(\cdot))\}$ a.s. converges to $\mathcal{E}(\bar{x},\bar{X},\bar{s}(\cdot),\bar{S}(\cdot),\bar{\lambda}(\cdot))$ owing to Proposition 2.2.1, parts (ii) and (iii), applied for almost any ω. Thus to apply the Dominated Convergence Theorem it is only necessary to have an upper bound on the energies. Such upper bound exists, e.g., in the setting of section 2.4.

2.3.2 ε-perturbation of $(\mathcal{P}_2)$

In this subsection we restrict ourselves to a very important special case of a truss without unilateral constraints under stochastic loading. In this case, a mechanical equilibrium condition can be formulated as a system of linear equations, parametrized by x. To this end, we first define B as the $m\times n$ matrix created by stacking the matrices B_i on top of each other, and, $D(x)$ as the $m\times m$ diagonal matrix with elements x_iE on a diagonal. Then $s(\omega)$ is a state vector corresponding to a nonnegative design x under loading $f(\omega)$ if and only if for some vector $u(\omega)$ of Lagrange multipliers (nodal displacements) the following system is satisfied:

$$(\mathcal{Q})_x(\omega) \quad \begin{pmatrix} 0 & B^T \\ D(x)B & -I \end{pmatrix} \begin{pmatrix} u(\omega) \\ s(\omega) \end{pmatrix} = \begin{pmatrix} -f(\omega) \\ 0 \end{pmatrix}.$$

Consider now the following ε-perturbation of the problem $(\mathcal{P}_2)$:

$$(\mathcal{P}_2^\varepsilon) \begin{cases} \min\limits_{(x,s(\cdot))} w(x) \\ \text{s.t.} \begin{cases} o(\varepsilon)1_m \le x \le \overline{x} + o(\varepsilon)1_m, \\ |s_i(\omega)| \le \overline{\sigma}_1 x_i + \varepsilon, \qquad i=1,\dots,m, \quad \text{P-a.s.}, \\ s(\omega) \text{ solves } (\mathcal{C})_x(\omega), \qquad \text{P-a.s.}, \end{cases} \end{cases}$$

where from the function $o : \mathbb{R}_{++} \to \mathbb{R}_{++}$ we only require the properties that $\{o(\varepsilon)/\varepsilon\}$ converges to zero while $\{o(\varepsilon)/\varepsilon^2\}$ is bounded away from zero (e.g., $o(\varepsilon) = \varepsilon^2$ satisfies these requirements).

Before stating the theorem we need a lemma, which asserts the directionally Lipschitz dependence of state variables $s(\omega)$ on design x uniformly in ω.

Lemma 2.3.2. *Consider a truss without unilateral constraints under essentially bounded stochastic loading $f(\cdot)$. Let $x \geq 0$ be a design for which the problem $(\mathcal{C})_x(\omega)$ has a solution $s(\omega)$ for almost any ω. Let $\Psi > 0$ be arbitrary in $\mathbb{R}^m$ and for $\varepsilon > 0$ set*

$$x_\varepsilon = x + \varepsilon\Psi.$$

Denote by $s_\varepsilon(\omega)$ the corresponding optimal solution to $(\mathcal{C})_{x_\varepsilon}(\omega)$. Then, for some positive constant τ and almost any ω, the inequality

$$\|s_\varepsilon(\omega) - s(\omega)\| \leq \tau\varepsilon \tag{2.3}$$

holds for all $\varepsilon > 0$.

Theorem 2.3.3. *In addition to the the assumptions of Lemma 2.3.2, suppose that for some $\varepsilon_0 > 0$ there is a solution $(x_0, s_0(\cdot))$ that is feasible in $(\mathcal{P}_2)$ with $x_0 \geq o(\varepsilon_0)1_m$.*

For each $\varepsilon_0 \geq \varepsilon > 0$ let $(x^*_\varepsilon, s^*_\varepsilon(\cdot))$ denote an arbitrary optimal solution to $(\mathcal{P}^\varepsilon_2)$. Then any limit point of the sequence $\{x^*_\varepsilon\}$ (and there is at least one) is an optimal solution to $(\mathcal{P}^0_2) = (\mathcal{P}_2)$. The sequence of optimal values $\{w(x^*_\varepsilon, X^*_\varepsilon)\}$ converges to the optimal value of $(\mathcal{P}^0_2)$.

Furthermore, for any converging subsequence of the design variables $\{x^*_{\varepsilon_k}\} \to x^*$ the corresponding sequence of state variables $\{s^*_{\varepsilon_k}(\cdot)\}$ a.s. converges to the optimal state $s^*(\cdot)$, corresponding to x^*.

Proof. Using Theorem 2.2.5 we can verify the existence of optimal solutions to $(\mathcal{P}^\varepsilon_2)$ for each $\varepsilon_0 \geq \varepsilon \geq 0$. Now one can follow the proof of Theorem 4.4 in [PaP00] to conclude that the sequence $\{(x^*_\varepsilon, s^*_\varepsilon(\cdot))\}$ as well as the sequence of energies $\{\mathcal{E}(x^*_\varepsilon, s^*_\varepsilon(\cdot))\}$ is essentially bounded. Thus there exists a limit point x^* of the design variables. Then for almost any ω, there is a subsequence $\{\varepsilon_{k(\omega)}\}$ such that $\lim_{k(\omega)\to\infty}(x^*_{\varepsilon_{k(\omega)}}, s^*_{\varepsilon_{k(\omega)}}(\omega)) = (x^*, s^*(\omega))$. Proposition 2.2.1 (ii) implies that the limit is the solution to the problem $(\mathcal{C})_{x^*}(\omega)$. The continuity of other constraints implies that $(x^*, s^*(\cdot))$ is feasible in $(\mathcal{P}^0_2)$.

Let $(x, s(\cdot))$ be an arbitrary feasible solution to $(\mathcal{P}_2)$, set $x_\varepsilon = x + o(\varepsilon)1_m$ and let $s_\varepsilon(\cdot)$ solve $(\mathcal{C})_{x_\varepsilon}(\cdot)$. Owing to Lemma 2.3.2 the following estimation holds:

$$\begin{aligned}|s_\varepsilon(\cdot)| \leq \tau o(\varepsilon) + |s(\cdot)| &\leq \tau o(\varepsilon) + \overline{\sigma}_1[x + (o(\varepsilon) - o(\varepsilon))1_m] \\ &= (\tau - \overline{\sigma}_1)o(\varepsilon) + \overline{\sigma}_1 x_\varepsilon \leq \overline{\sigma}_1 x_\varepsilon + \varepsilon,\end{aligned}$$

for all ε small enough, where we have used the assumption that $\{o(\varepsilon)/\varepsilon\} \to 0$. Since clearly x_ε satisfies the design constraints, $(x_\varepsilon, s_\varepsilon(\cdot))$ is feasible in $(\mathcal{P}_2^\varepsilon)$. Hence, $w(x_\varepsilon^*) \le w(x_\varepsilon)$. Letting ε tend to zero in this inequality, we obtain that $w(x^*) \le w(x)$, whence we may conclude that $(x^*, s^*(\cdot))$ solves $(\mathcal{P}_2)$.

The continuity of the objective function implies the convergence of the optimal values. The boundedness of energies at the optimal solution clarifies the usage of Proposition 2.2.1 (iii), hence the convergence of state variables also follows. #

We illustrate Theorem 2.3.3 with a small numerical example.

Example 2.3.4 (4-bar truss). Consider the problem of minimizing the weight of the 4-bar structure shown in Figure 2.1. The stress limit for each bar is $\overline{\sigma} = 1$, and the Young's modulus is $E = 1$. Assume that the

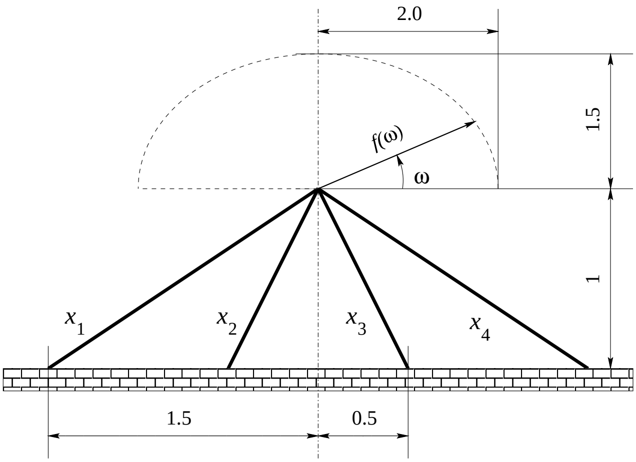

Fig. 2.1: The 4-bar truss problem.

upper design bounds are inactive, and that the force vector $f(\omega)$ equals $(2\cos(\omega), 1.5\sin(\omega))$, where $0 \le \omega \le \pi$. The probability measure is the uniform one on $[0, \pi]$ (it is easy to see that only the support of the probability measure is necessary to define stress constrained weight minimization; see also remarks in the end of Section 2.4). Since the initial structural topology as well as the loading conditions are symmetric, we can expect symmetric optimal solutions (i.e., $x_1^* = x_4^*$, $x_2^* = x_3^*$). Figure 2.2 shows the projection of the set of feasible designs onto the linear subset $\{ x \in \mathbb{R}^4 \mid x_1 = x_4, x_2 = x_3 \}$. Note that the feasible set is not *a finite union of polyhedra*, because we work with an infinite number of load cases (compare with the similar Problem 1 in [StS01]). Despite the large number of load cases, at the globally optimal solution,

$x^* = (0, 2.5, 2.5, 0)$, the structural topology was modified (i.e., bars 1 and 4 were removed).

There are three local minima, two of which (including the globally optimal solution) are singular. The nonsingular non-global, local minimum of the original problem is the global minimum for the "naively" perturbed problem for all small values of ε. Therefore, we cannot approximate the globally optimal solution by the "naive" ε-perturbation.

The "correct" ε-perturbation scheme allows us to recover the global optimal solution. Figure 2.3 shows the convergence of the optimal solutions to the ε-perturbed problems to the solution of the original problem, as ε decreases to zero (variables x_3 and x_4 are not shown, owing to the symmetry of the calculated optimal solutions). We have used the nested formulation (with eliminated state variables) and a finite difference approximation of the derivatives to solve the problem using an SQP algorithm.

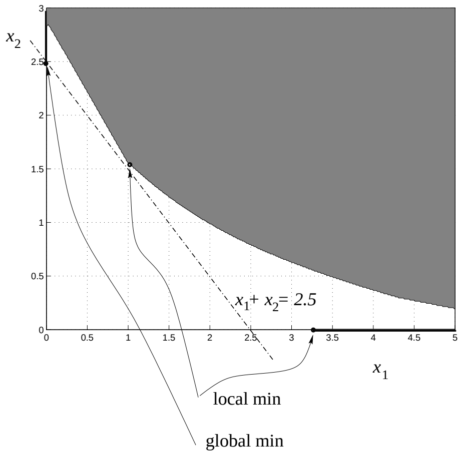

Fig. 2.2: The feasible design domain of the 4-bar truss problem.

2.4 Distribution sensitivity

The analysis of stability of optimal solutions with respect to small changes in probability measure is of great importance. From the computational point of view it allows one to replace the original stochastic problem by a sequence of simpler problems, involving approximations (discretizations) of the probability measure. From the practical point of view, it asserts that solutions to the problem obtained using statistical estimations of the unknown stochastic distribution are "close" to exact solu-

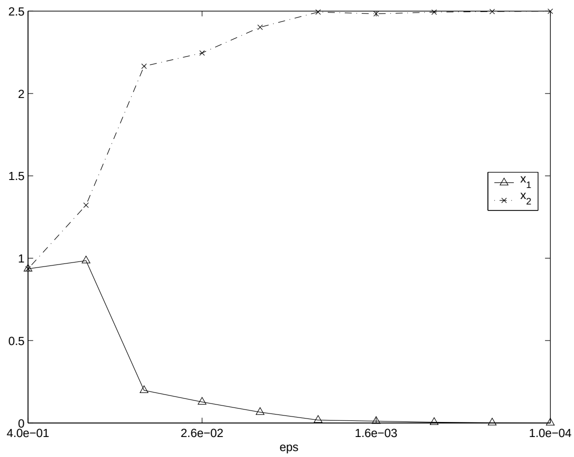

Fig. 2.3: Convergence of the ε-perturbations for the 4-bar truss problem.

tions. From the theoretical point of view, it confirms the robustness of the probabilistic approach with respect to possible errors in the probability distribution.

Throughout this section we assume that Ω is a compact metric space, $\mathfrak{S} = \mathcal{B}(\Omega)$ and the only sources of uncertainty are the loads $f(\cdot)$, gaps $g_1(\cdot)$ and slacks $g_2(\cdot)$, which in addition are assumed to be continuous functions. Although we do not necessarily work with a complete probability space in such a setting, under additional assumptions about the feasibility of the lower-level problem it is possible to omit the adverb "almost" from the discussion.

Corollary 2.4.1. *Given a nonnegative design* (x, X)*, suppose that the problem* $(\mathcal{C})_{(x,X)}(\omega)$ *is feasible for any* ω*. Then the solution* $(s(\omega), S(\omega), \lambda(\omega))$ *exists, is unique and continuous, and the optimal value function* $\mathcal{E}(x, X, s(\omega), S(\omega), \lambda(\omega), \omega)$ *is continuous.*

Proof. Both existence and uniqueness were announced in Corollary 2.2.2. Continuity then follows from [BGK$^+$83, Theorem 5.5.1]. #

In the remainder of the section we assume that the assumptions of Corollary 2.4.1 hold for any positive design (x, X). This assumption can easily be satisfied by choosing a "rich enough" ground structure, which can sustain loads for any ω.

Proposition 2.4.2. *Let the sequence* $\{(x_k, X_k)\}$ *of positive designs converge to a nonnegative design* $(\bar{x}, \bar{X})$*. Let* $(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))$ *be the continuous solution to* $(\mathcal{C})_{(x_k,X_k)}(\cdot)$*. Suppose that the sequence of energy estimates* $\{\int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)\, \mathrm{P}(d\omega)\}$ *is bounded. Then* *the sequence* $\{(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))\}$ *converges on a set of measure one to a limit* $(\bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$*, the solution to the problem* $(\mathcal{C})_{(\bar{x},\bar{X})}(\cdot)$*. Furthermore, the sequence of optimal values* $\{\mathcal{E}(x_k, X_k, s_k(\cdot), S_k(\cdot), \lambda_k(\cdot), \cdot)\}$ *uniformly converges on a set of measure one to* $\mathcal{E}(\bar{x}, \bar{X}, \bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot), \cdot)$.

2.4.1 *Stability of solutions to* $(\mathcal{P}_1)$

Before proceeding with analyzing the stability of solutions to the stochastic compliance minimization problem with respect to small changes in the probability measure, we note that in the settings of this section the problem $(\mathcal{P}_1)$ has a feasible solution $(x_0, X_0, s_0(\cdot), S_0(\cdot))$ such that $(x_0, X_0) > 0$ provided the bounds on the available material (v, V) are positive. We denote the optimal value of a problem $(\mathcal{P})$ by $\mathrm{val}(\mathcal{P})$.

Corollary 2.4.3. *The following equalities hold:*

$$\mathrm{val}(\mathcal{P}_1) = \inf_{\varepsilon>0} \mathrm{val}(\mathcal{P}_1^\varepsilon) = \lim_{\varepsilon\to 0} \mathrm{val}(\mathcal{P}_1^\varepsilon).$$

Proof. Follows immediately from Theorem 2.3.1 and Proposition 2.4.2. #

Consider a sequence of probability measures $\{\mathrm{P}_k\}$ defined on $\mathcal{B}(\Omega)$, together with a sequence of optimization problems:

$$(\mathcal{P}_1)^k \begin{cases} \displaystyle\min_{(x,X,s(\cdot),S(\cdot))} c_k^f(x, X, s(\cdot), S(\cdot)) := \\ \qquad\qquad \displaystyle\int_\Omega \mathcal{E}(x, X, s(\omega), S(\omega), \lambda(\omega), \omega)\, \mathrm{P}_k(d\omega) \\ \text{s.t.} \begin{cases} \underline{x} \le x \le \overline{x}, & 1_m^T x \le v, \\ \underline{X} \le X \le \overline{X}, & 1_{r_2}^T X \le V, \\ (s(\omega), S(\omega), \lambda(\omega)) \text{ solves } (\mathcal{C})_{(x,X)}(\omega), \ \mathrm{P}_k\text{-a.s.} \end{cases} \end{cases}$$

Lemma 2.4.4. *Suppose that the sequence of probability measures* $\{\mathrm{P}_k\}$ *weakly converges to* P*. Then* $\mathrm{val}(\mathcal{P}_1) \ge \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_1)^k$.

Proof. Consider an arbitrary sufficiently small $\varepsilon > 0$ such that there is an optimal solution $(x_\varepsilon, X_\varepsilon, s_\varepsilon(\cdot), S_\varepsilon(\cdot))$ to the problem $(\mathcal{P}_1^\varepsilon)$. Owing to Corollary 2.4.1 this point is feasible for all problems $(\mathcal{P}_1)^k$, so we get:

$$\begin{aligned} \mathrm{val}(\mathcal{P}_1^\varepsilon) &= c^f(x_\varepsilon, X_\varepsilon, s_\varepsilon(\cdot), S_\varepsilon(\cdot), \lambda_\varepsilon(\cdot)) \\ &= \lim_{k\to\infty} c_k^f(x_\varepsilon, X_\varepsilon, s_\varepsilon(\cdot), S_\varepsilon(\cdot), \lambda_\varepsilon(\cdot)) \ge \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_1)^k, \end{aligned}$$

and, owing to Corollary 2.4.3:

$$\mathrm{val}(\mathcal{P}_1) = \inf_{\varepsilon>0} \mathrm{val}(\mathcal{P}_1^\varepsilon) \ge \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_1)^k.$$

#

To prove the reverse inequality we assume additional regularity properties on the sequence $\{\mathrm{P}_k\}$. Namely, we suppose that each measure P_k has a density $p_k(\cdot)$ with respect to a Lebesgue measure on Ω and that the sequence $\{p_k(\cdot)\}$ converges to a density $p(\cdot)$ of P Lebesgue-almost everywhere. The existence of densities is not a very restrictive assumption from the theoretical point of view, and it is usually assumed in engineering applications of probability theory (for just a few examples, see [Lin00, Soi00, Vro00, MAM01]).

Theorem 2.4.5. *Let* $\{(x_k, X_k, s_k(\cdot), S_k(\cdot))\}$ *be a sequence of solutions to* $\{(\mathcal{P}_1)^k\}$. *Then any limit point (and there is at least one) of the sequence* $\{(x_k, X_k, s_k(\cdot), S_k(\cdot))\}$ *is a solution to the limiting problem* $(\mathcal{P}_1)$.

Proof. The sequence of design variables $\{(x_k, X_k)\}$ is bounded and has a limit point (x_0, X_0). Thus we may assume that the original sequence has converging design components.

Fatou's Lemma and Lemma 2.4.4 imply:

$$\begin{aligned}
&\int_\Omega \liminf_{k\to\infty} \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega) p(\omega)\, d\omega \\
&\leq \int_\Omega \liminf_{k\to\infty} [\mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega) p_k(\omega)]\, d\omega \\
&\leq \liminf_{k\to\infty} \int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega) p_k(\omega)\, d\omega \leq \mathrm{val}(\mathcal{P}_1) < \infty.
\end{aligned}$$

Thus we see that the P-probability of the set $\Omega_f = \{\omega \in \Omega \mid \liminf_{k\to\infty} \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega) < \infty\}$ is one. Using Proposition 2.2.1 (iii) we can verify the existence of a limiting state $(s_0(\cdot), S_0(\cdot), \lambda_0(\cdot))$ corresponding to the design (x_0, X_0), and the P-a.s. convergence of $(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))$ to this state. Using the lower semicontinuity of $\mathcal{E}$, this implies:

$$\begin{aligned}
\mathrm{val}(\mathcal{P}_1) &\leq \int_\Omega \mathcal{E}(x_0, X_0, s_0(\omega), S_0(\omega), \lambda_0(\omega), \omega) p(\omega)\, d\omega \\
&\leq \int_\Omega \liminf_{k\to\infty} \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega) p(\omega)\, d\omega \\
&\leq \liminf_{k\to\infty} \int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega) p_k(\omega)\, d\omega \\
&= \liminf_{k\to\infty} \mathrm{val}(\mathcal{P}_1)^k.
\end{aligned}$$

Together with estimation of $\limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_1)^k$ given by Lemma 2.4.4 this finishes the proof. #

2.4.2 *Example: Instability of solutions to* $(\mathcal{P}_2)$

In the proof of continuity of solutions to the general stochastic compliance minimization problem with respect to small changes in the probability measure we have used three properties of the problem:

- the boundedness of the objective function implies the convergence of the state variables;
- the optimal value of the problems $(\mathcal{P}_1^\varepsilon)$ monotonically decreases as ε goes to zero;
- any positive design defines a solution, which is feasible for all probability measures.

Unfortunately, the stress constrained weight minimization problem possesses neither of these properties. The following numerical example shows that solutions to the stochastic weight constrained minimization problem are not in general continuous with respect to changes in the probability measure.

Example 2.4.6 (One bar with a cable). Figure 2.4 shows a simple one-dimensional structure introduced and analysed in [PaP00] that consists of a bar suspended with one cable. Suppose that $\Omega = [-1, 2]$, P is the

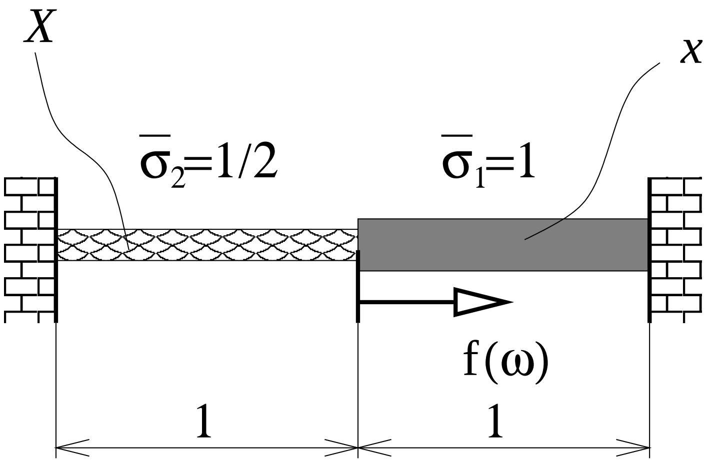

Fig. 2.4: The cable suspended one-bar truss.

uniform distribution on $[-1/2, 1]$, $f(\omega) = \omega$, $E = E_c = 1$, $\rho_1 = \rho_2 = 1$, $\overline{x} = 1$ and $\overline{X} = 2$. After eliminating the state variables one obtains the following optimization problem:

$$\begin{cases} \min_{(x,X)} x + X \\ \text{s.t.} \begin{cases} 0 \le x \le 1, & \\ 0 \le X \le 2, & \\ \dfrac{x\omega}{x+X} \le x, & \text{P-a.s.,} \\ -\omega \le x, & \text{P-a.s.,} \\ 0 \le \dfrac{X\omega}{x+X} \le X/2, & \text{P-a.s.} \end{cases} \end{cases}$$

Figure 2.5 shows the feasible domain for the design variables (x, X).

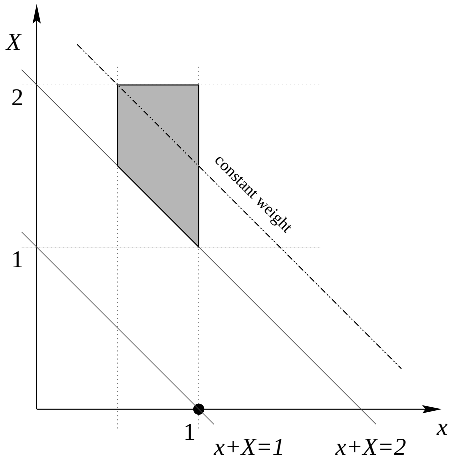

Fig. 2.5: The admissible design domain. The optimal solution is at the black circle.

Note that this domain consists of the union of a two-dimensional, convex domain and the isolated optimal point $(x^*, X^*) = (1, 0)$ with corresponding optimal weight $w^* = 1$.

Let P_k be a uniform distribution on $[-1/2, 1] \cup [2 - 1/k, 2]$. We note that the sequence $\{\mathrm{P}_k\}$ weakly converges to P and all the measures possess densities. On the other hand, for any vector (x_k, X_k) feasible to $(\mathcal{P})^k$ the inequality $X_k \ge 1$ holds, owing to the fact that $\mathrm{P}_k(f(\omega) > 1) > 0$. Thus the sequence of optimal solutions to $(\mathcal{P}_2)^k$ cannot converge to the optimal solution $(x^*, X^*) = (1, 0)$ of $(\mathcal{P}_2)$.

A few remarks are in order. The stress-constrained weight minimization problem depends only on the support of the probability measure, not on the measure itself. The assumption of fixed compact support of the approximating measures, being extremely restrictive from the theoretical point of view, is nevertheless satisfied in some applications (such as the topology optimization of machine parts, where all loading conditions can be prescribed).

In accordance with the previous remark, the stochastic weight minimization problem could be treated as a semi-infinite mathematical program. However, the presence of a probability measure could be exploited algorithmically. For example, when constructing penalty functions for the problem, one can impose larger penalties on more probable violations of constraints.

2.5 Concluding remarks and further research

In Section 2.4 we have shown that the introduction of uncertainty into structural topology optimization problems does not necessarily lead to robust solutions, if one understands robustness as insensitivity to modelling errors. It is possible to prove the robustness of the optimal solutions to stochastic compliance minimization under regularity assumptions on the approximating probability measures. The main reason for the instability of the optimal solutions to stochastic stress constrained weight minimization is that stress constraints are imposed in a too restrictive way — we require them to hold with probability one. The alternative is to allow small *average* violations of stress constraints, which, however, may result in rather large violations with small probabilities. This approach is one of the topics of current research.

One uncovered question in this paper is the possible construction of efficient numerical methods for stochastic topology optimization problems. The current research topics in this area include:

- the approximation of stochastic structural topology optimization problems with simpler finite-dimensional problems with discrete measures;
- alternative ε-perturbation approaches, removing stress constraints from the original problem.

This would allow us to apply existing algorithms for nonsmooth optimization (e.g. BT algorithm [OKZ98] or implicit programming algorithm [LPR96]) to stochastic topology optimization problems.

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2.A Proofs of the auxiliary results

Proof of Corollary 2.2.2. Both existence and uniqueness follow from Proposition 2.2.1(i) applied for almost any ω.

The feasible set of the problem is defined by inequality and equality constraints involving only measurable mappings, whence a point-to-set mapping $\omega \to \{\text{ feasible set of } (\mathcal{C})_{(x,X)}(\omega)\,\}$ is measurable for fixed (x, X), owing to Theorem 8.2.9 in [AuF90]. Each function s^2/x is l.s.c. with respect to s by [Roc70, p. 83], whence the objective function is measurable. Then we apply Lemma III.39 and its Application in [CaV77] to verify the existence of $\mathfrak{S}$-measurable solution. Since the solution is unique, we are done. #

Proof of Proposition 2.2.3. Let $\widetilde{\mathcal{E}}(\omega) := \liminf_{k\to\infty} \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)$, which is non-negative and $\mathfrak{S}$-measurable being a lower limit of functions satisfying these properties (where measurability is owing to Corollary 2.2.2). Let $k(\omega)$ be a sequence of indices such that $\widetilde{\mathcal{E}}(\omega) = \lim_{k(\omega)\to\infty} \mathcal{E}(x_{k(\omega)}, X_{k(\omega)}, s_{k(\omega)}(\omega), S_{k(\omega)}(\omega), \lambda_{k(\omega)}(\omega), \omega)$.

Using the non-negativity of $\mathcal{E}$, the assumed feasibility of the problem, and Fatou's Lemma, we get:

$$\begin{aligned} 0 &\le \int_\Omega \widetilde{\mathcal{E}}(\omega)\,\mathrm{P}(d\omega) \\ &\le \liminf_{k\to\infty} \int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)\,\mathrm{P}(d\omega) \le C < \infty. \end{aligned}$$

Thus $\mathrm{P}(\widetilde{\mathcal{E}}(\omega) < \infty) = 1$ holds, and for almost all $\omega \in \Omega$ the sequence $\{\mathcal{E}(x_{k(\omega)}, X_{k(\omega)}, s_{k(\omega)}(\omega), S_{k(\omega)}(\omega), \lambda_k(\omega), \omega)\}$ is bounded. Proposition 2.2.1(ii) for such ω implies that there exists $(\bar{s}(\omega), \bar{S}(\omega), \bar{\lambda}(\omega))$ solving the problem $(\mathcal{C})_{(\bar{x},\bar{X})}(\omega)$, and such that $(\bar{s}(\omega), \bar{S}(\omega), \bar{\lambda}(\omega)) = \lim_{k(\omega)\to\infty}(s_{k(\omega)}(\omega), S_{k(\omega)}(\omega), \lambda_{k(\omega)}(\omega))$. From Corollary 2.2.2 we know that $(\bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$ is measurable.

Now we can use the l.s.c. property of $\mathcal{E}$ for fixed ω (Lemma 3.2

in [PaP00]):

$$\begin{aligned}
0 &\le \mathcal{E}(\bar{x}, \bar{X}, \bar{s}(\omega), \bar{S}(\omega), \bar{\lambda}(\omega), \omega) \\
&\le \lim_{k(\omega)\to\infty} \mathcal{E}(x_{k(\omega)}, X_{k(\omega)}, s_{k(\omega)}(\omega), \lambda_{k(\omega)}(\omega), S_{k(\omega)}(\omega), \omega) \\
&= \liminf_{k\to\infty} \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega),
\end{aligned}$$

whence

$$\begin{aligned}
0 &\le \int_\Omega \mathcal{E}(\bar{x}, \bar{X}, \bar{s}(\omega), \bar{S}(\omega), \bar{\lambda}(\omega), \omega)\, \mathrm{P}(d\omega) \\
&\le \liminf_{k\to\infty} \int_\Omega \mathcal{E}(x_k, X_k, s_k(\omega), S_k(\omega), \lambda_k(\omega), \omega)\, \mathrm{P}(d\omega) \le C < \infty.
\end{aligned}$$

The latter inequality and Proposition 2.2.1(iii) imply the almost sure convergence of the sequence $\{(s_k(\cdot), S_k(\cdot), \lambda_k(\cdot))\}$ to $(\bar{s}(\cdot), \bar{S}(\cdot), \bar{\lambda}(\cdot))$. #

Proof of Lemma 2.3.2. For each $\varepsilon > 0$ and almost each ω the problem $(\mathcal{Q})_{x_\varepsilon}(\omega)$ has a unique solution $(u_\varepsilon(\omega), s_\varepsilon(\omega))$. Owing to the Hoffman error bound [Hof52], the inequality

$$\begin{aligned}
\|s_\varepsilon(\omega) - s(\omega)\| &\le \widehat{\tau} \left\| \begin{pmatrix} 0 & B^T \\ D(x)B & -I \end{pmatrix} \begin{pmatrix} u_\varepsilon(\omega) \\ s_\varepsilon(\omega) \end{pmatrix} - \begin{pmatrix} f(\omega) \\ 0 \end{pmatrix} \right\| \\
&= \widehat{\tau} \left\| \begin{pmatrix} 0 \\ \varepsilon D(\Psi) u_\varepsilon(\omega) \end{pmatrix} \right\|,
\end{aligned}$$

holds. Therefore, to finish the proof it is sufficient to show the uniform essential boundedness of $u_\varepsilon(\cdot)$ for $\varepsilon > 0$.

Owing to Theorem 3.2 in [Pet01] (see also Theorem 3.2 in [PaP00]) the inequality $\|u_\varepsilon(\omega)\|_\Psi \le \|u(\omega)\|_\Psi$ holds for almost each ω, where $u(\omega)$ is an arbitrary Lagrange multiplier for the problem $(\mathcal{C})_x(\omega)$, and $\|\cdot\|_\Psi$ is an elliptic norm associated with the positive definite matrix $\sum_{i=1}^m \Psi_i B_i^T E B_i$.

Let $\Omega_1 \in \mathfrak{S}$ be a set of probability one such that each problem $(\mathcal{C})_x(\omega)$ with $\omega \in \Omega_1$ has an optimal solution and the set $Q := \{\, f(\omega) \mid \omega \in \Omega_1 \,\}$ is bounded. The system $(\mathcal{Q})_x$ being a KKT system for a quadratic problem $(\mathcal{C})_x$ is solvable for each vector f in a closure of a convex hull of the set Q, owing to Theorem 5.5.1 in [BGK+83]. Furthermore, holding x fixed, the problem $(\mathcal{C})_x(\cdot)$ satisfies the constant rank constraint qualification (CRCQ), which, in turn, implies the sequentially bounded constraint qualification (SBCQ) (cf. Proposition 1.3.8 in [LPR96]). In particular we can infer the existence of a constant C, independent of ω, bounding the minimum norm Lagrange multipliers $\bar{u}(\omega)$. This finishes the proof. #

Proof of Proposition 2.4.2. The pointwise convergence of optimal solutions on a set Ω_1 of measure one holds owing to Proposition 2.2.3. On the other hand, for each (x_k, X_k) the optimal value function to $(\mathcal{C})_{(x_k, X_k)}(\cdot)$ is a quadratic function of $(f(\cdot), g_1(\cdot), g_2(\cdot))$ owing to Theorem 5.5.2 in [BGK+83]. Using a characterization of a solubility set of a parametric quadratic programming problem with parameters in the linear part of the objective function and in the right-hand sides of the constraints (Theorem 5.5.1 in [BGK+83]) we can infer the convergence of solutions to $(\mathcal{C})_{(x_k,X_k)}$ towards the solution of $(\mathcal{C})_{(\bar{x},\bar{X})}$ for all (f, g_1, g_2) in some convex bounded polyhedral set, a.s. containing $\{\, (f(\omega), g_1(\omega), g_2(\omega)) \mid \omega \in \Omega_1 \,\}$. The convergence of quadratic functions on a bounded convex polyhedron is uniform, and this finishes the proof. #

Paper 3

STOCHASTIC STRUCTURAL TOPOLOGY OPTIMIZATION: DISCRETIZATION AND PENALTY FUNCTION APPROACH

Anton Evgrafov* and Michael Patriksson*

Abstract

We consider structural topology optimization problems including unilateral constraints arising from, for example, non-penetration conditions in contact mechanics or non-compression conditions for elastic ropes. To construct more realistic models and to hedge off possible failures or inefficient behaviour of optimal structures, we allow parameters (for example, loads) defining the problem to be stochastic. The resulting nonsmooth stochastic optimization problem is an instance of stochastic mathematical programs with equilibrium constraints (MPEC), or stochastic bilevel programs. We propose a solution scheme based first on the approximation of the given topology optimization problem with a sequence of simpler sizing optimization problems, and second on approximating the probability measure in the latter problems. For stress constrained weight minimization problems an alternative to ε-perturbation based on a new penalty function is proposed.

Key words: Bilevel programming, stochastic programming, robust optimization, discretization, ε-perturbation, stress constraints

3.1 Introduction and notation

In this paper we propose a solution scheme for stochastic generalizations of two of the most natural and classic structural topology optimization problems: the finding of a maximally stiff truss under a volume constraint, and the finding of a truss of minimal weight under stress constraints. The reason for introducing stochasticity into the problem is that

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uncertainty due to several factors (such as loads unknown in advance, varying material properties, manufacturing errors, etc.) has to be taken into account to obtain robust optimal solutions. Capturing the uncertainty in the model through the use of probability theory allows us to construct arbitrarily general models.

To include a wide range of applications we allow mechanical structures to be *unilaterally* constrained, i.e., some parts of the structure might come into unilateral frictionless contact with rigid obstacles, while some other parts might sustain only tensile forces. Practical applications of unilateral contact include such machine elements as joints, hinges, press-fits, and examples of structures with tensile-only members include suspension bridges and cranes.

The resulting optimization problem is an instance of stochastic mathematical programming problems with equilibrium constraints (SMPEC), or stochastic bilevel programming problems [PaW99, EvP01]. In context of structural optimization, this class of problems was studied for discrete probability measures in [CPW01, PaP00], and for general probability measures in [EPP02].

In the case of discrete probability measures there is a strong connection with the intuitive multiple load approach ([SvG68], [Ben95, p. 8], [StS01]), which is widely used to obtain designs that are more robust with respect to unknown loading conditions. In such an approach, an engineer picks up a few loading scenarios, empirically assigns them some "weights" and then solves, essentially, the stochastic optimization problem with a discrete probability measure, where each scenario has a probability proportional to its "weight". By considering our problems in the framework of stochastic programming, we can analyze the question of existence of the limiting optimal designs if the sequence of the empirical probability distributions converges, i.e., as the number of the load cases converges to infinity!

When the probability measure is not discrete, the problem is posed in an infinite-dimensional space, whence it is important to construct finite-dimensional approximations to the problem. The most popular approach to solving stochastic optimization problems is based on the idea of approximating the probability measure with a sequence of discrete probability measures with finite support, resulting in a desired sequence of finite-dimensional optimization problems.

Unfortunately, in our case the straightforward implementation of such strategy is impossible, owing to the lack of standard constraint qualifications by the feasible set of the problem. Therefore, we approximate the given topology optimization problem by a sequence of simpler sizing optimization problems (ε-perturbations), and then discretize the latter problems. The "naive" ε-perturbation involving only a change in the lower bounds on design variables studied in [Ach98, PaP00, EPP02] is sufficient for solving stochastic compliance minimization problems. For stress constrained weight minimization we propose a new penalty function based alternative to the perturbation method by Cheng and Guo [ChG97] (see also [Pet01, PaP00, EPP02]).

The outline of this paper is as follows. After introducing the necessary mechanical principles and formulating the problems, in Section 3.2 we formulate the penalty function approach to the stress constrained weight minimization. We illustrate this new approximation scheme with small numerical examples. Section 3.3 is dedicated to the construction of the finite-dimensional discretizations of the sizing approximations of the topology compliance and weight minimization problems. We end the paper with a numerical example (Section 3.4), illustrating all the steps of the proposed solution approach.

3.1.1 Mechanical equilibrium

In this subsection we introduce the notation and mechanical principles necessary to state the problems we are going to analyze.

Given positions of the nodes the *design* (and topology in particular) of a truss can be described by the following sets of *design* variables:

- $x_i \geq 0$, $i = 1, \dots, m$, representing the volume of material, allocated to the bar i in the structure;
- $X_j \geq 0$, $j = 1, \dots, r_2$, representing the volume of material, allocated to the cable j.

We introduce two index sets of the present (or active) members in the structure: $\mathcal{I}(\boldsymbol{x}) = \{\, i = 1, \dots, m \mid x_i > 0 \,\}$ and $\mathcal{J}(\boldsymbol{X}) = \{\, j = 1, \dots, r_2 \mid X_j > 0 \,\}$.

Let $(\Omega, \mathfrak{S}, \mathrm{P})$ be a complete probability space. Given a particular design the status of the linear elastic mechanical system is governed by the principle of minimum complementary energy $(\mathcal{C})_{(\boldsymbol{x},\boldsymbol{X})}(\omega)$ (in our case it is the $(\boldsymbol{x}, \boldsymbol{X}, \omega)$-parametric minimization problem):

$$\begin{cases} \min\limits_{(\boldsymbol{s},\boldsymbol{S},\boldsymbol{\lambda})} \mathcal{E}(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{s}, \boldsymbol{S}, \boldsymbol{\lambda}, \omega) := \dfrac{1}{2} \sum\limits_{i=1}^{m} \dfrac{s_i^2}{E(\omega) x_i} + \boldsymbol{g}_1^T(\omega)\boldsymbol{\lambda} \\ \qquad\qquad + \sum\limits_{j=1}^{r_2} \Big(\dfrac{(L_j(\omega) S_j)^2}{2E_c(\omega) X_j} + (\boldsymbol{g}_2(\omega))_j S_j \Big), \\ \text{s.t.} \begin{cases} \boldsymbol{C}_1^T(\omega)\lambda + \sum\limits_{i \in \mathcal{I}(\boldsymbol{x})} \boldsymbol{B}_i^T(\omega) s_i + \sum\limits_{j \in \mathcal{J}(\boldsymbol{X})} S_j \gamma_j(\omega) = \boldsymbol{f}(\omega), \\ \boldsymbol{\lambda} \geq 0, \\ \boldsymbol{S}_{\mathcal{J}(\boldsymbol{X})} \geq 0, \end{cases} \end{cases}$$

where the functions in the problem have the following meaning from a mechanical point of view:

- $E(\omega)$ and $E_c(\omega)$ are Young's moduli for the structure and cable materials respectively;
- $\boldsymbol{B}_i(\omega)$ is the kinematic transformation matrix for the bar i;
- $\gamma_j(\omega)$ is the unit direction vector of the cable j;
- $(\boldsymbol{g}_2(\omega))_j$ is the initial slack of the cable j;
- $L_j(\omega)$ is the length of the cable j;
- $\boldsymbol{C}_1(\omega)$ is the quasi-orthogonal kinematic transformation matrix for rigid obstacles;
- $\boldsymbol{g}_1(\omega) \geq 0$ is the vector of the initial gaps;
- $\boldsymbol{f}(\omega)$ is the vector of external forces.

For the problem to be tractable we assume that all functions listed above are $\mathfrak{S}$-measurable. We further assume that the matrix $\boldsymbol{C}_1$ is quasi-orthogonal, that is, that $\boldsymbol{C}_1\boldsymbol{C}_1^T = \boldsymbol{I}$, the unit matrix of the corresponding size. This condition is fulfilled if at each node either there is at most one rigid support or multiple supports "act" in orthogonal directions to each other.

The variables in the problem $(\mathcal{C})_{(\boldsymbol{x},\boldsymbol{X})}(\omega)$ have the following interpretation:
- s_i is the tensile force in the bar i times its length;
- S_j is the tensile force in the cable j;
- $\boldsymbol{\lambda}$ is the vector of contact forces.

Note, that from the quasi-orthogonality of $\boldsymbol{C}_1$ it follows that $\boldsymbol{\lambda}$ is uniquely determined by $(\boldsymbol{s}, \boldsymbol{S})$ and depends continuously on them:

$$\boldsymbol{\lambda} = \boldsymbol{C}_1(\omega)\Big(\boldsymbol{f}(\omega) - \sum_{i\in\mathcal{I}(\boldsymbol{x})} \boldsymbol{B}_i^T(\omega)s_i - \sum_{j\in\mathcal{J}(\boldsymbol{X})} S_j\boldsymbol{\gamma}_j(\omega)\Big). \tag{3.1}$$

These facts will be used often without backward reference.

3.1.2 Stochastic minimum compliance problem

We are now ready to state the first problem considered in this paper — the general stochastic minimum compliance problem $(\mathcal{P}_1)$:

$$\begin{cases} \min\limits_{(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}(\cdot),\boldsymbol{S}(\cdot))} c^f(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}(\cdot),\boldsymbol{S}(\cdot),\boldsymbol{\lambda}(\cdot)) := \\ \qquad\qquad \displaystyle\int_\Omega \mathcal{E}(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}(\boldsymbol{\omega}),\boldsymbol{S}(\omega),\boldsymbol{\lambda}(\omega),\omega)\,\mathrm{P}(d\omega) \\ \text{s.t.} \begin{cases} \underline{\boldsymbol{x}} \leq \boldsymbol{x} \leq \overline{\boldsymbol{x}}, & \mathbf{1}_m^T\boldsymbol{x} \leq v, \\ \underline{\boldsymbol{X}} \leq \boldsymbol{X} \leq \overline{\boldsymbol{X}}, & \mathbf{1}_{r_2}^T\boldsymbol{X} \leq V, \\ (\boldsymbol{s}(\boldsymbol{\omega}),\boldsymbol{S}(\omega),\boldsymbol{\lambda}(\omega)) \text{ solves } (\mathcal{C})_{(\boldsymbol{x},\boldsymbol{X})}(\omega), \text{ P-a.s.,} \end{cases} \end{cases}$$

where v and V are the limits on the amount of cable and structure material correspondingly.

In topology optimization we set lower bounds $\underline{\boldsymbol{x}} = \mathbf{0}$ and $\underline{\boldsymbol{X}} = \mathbf{0}$.

3.1.3 Stochastic stress constrained weight minimization problem

We formulate problem $(\mathcal{P}_2)$ as follows:

$$\begin{cases} \min_{(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}(\cdot),\boldsymbol{S}(\cdot))} w(\boldsymbol{x},\boldsymbol{X}) := \rho_1 \mathbf{1}_m^T \boldsymbol{x} + \rho_2 \mathbf{1}_{r_2}^T \boldsymbol{X} \\ \text{s.t.} \begin{cases} \underline{\boldsymbol{x}} \le \boldsymbol{x} \le \overline{\boldsymbol{x}}, \\ \underline{\boldsymbol{X}} \le \boldsymbol{X} \le \overline{\boldsymbol{X}}, \\ |s_i(\omega)| \le \overline{\sigma}_1 x_i, & i = 1,\dots,m, & \text{P-a.s.}, \\ L_j S_j(\omega) \le \overline{\sigma}_2 X_j, & j = 1,\dots,r_2, & \text{P-a.s.}, \\ (\boldsymbol{s}(\boldsymbol{\omega}), \boldsymbol{S}(\omega), \boldsymbol{\lambda}(\omega)) \text{ solves } (\mathcal{C})_{(\boldsymbol{x},\boldsymbol{X})}(\omega), & & \text{P-a.s.}, \end{cases} \end{cases}$$

where $\overline{\sigma}_1$ and $\overline{\sigma}_2$ are the maximal allowable effective stresses in, and ρ_1 and ρ_2 the densities of, the structure and the cable materials, respectively.

In topology optimization we set lower bounds $\underline{\boldsymbol{x}} = \mathbf{0}$ and $\underline{\boldsymbol{X}} = \mathbf{0}$.

3.2 Penalization

A continuation algorithm based on so-called ε-perturbation is a popular numerical approach for solving stress constrained weight minimization problems. The method was first proposed by Cheng and Guo [ChG97] for trusses, and it corresponds to solving a sequence of sizing structural optimization problems obtained from the original one by substituting lower bound on the design variables with ε^2 and relaxing the stress constraints by ε. The perturbation parameter ε (whence the name — ε-perturbation) is eventually reduced to zero, and the optimal solution from the previous iteration is used as a starting point for the new iteration. Petersson [Pet01] established the convergence of the optimal solutions to perturbed problems towards an optimal solution of the original problem, Patriksson and Petersson [PaP00] generalized the procedure for stochastic weight minimization of trusses including unilateral constraints and discrete probability measures, and Evgrafov et al. [EPP02] proved the convergence of the method for general probability measures in the special case of trusses without unilateral constraints under stochastic loading conditions.

Being an interesting theoretical result, it is difficult to transfer the ε-perturbation result to efficient algorithms. Stolpe and Svanberg [StS01] discussed the possible discontinuity of the optimal solutions to ε-perturbed problems with respect to ε, which might cause the continuation approach to fail in practice. Furthermore, the presence of stress constraints in the problem does not allow us to use the many numerical algorithms that were designed for MPECs without state constraints (e.g., the BT algorithm [OKZ98, Chapter 7] or the implicit programming algorithm [LPR96, Section 6.1]). Specifically in the stochastic setting, the presence of stress constraints does not allow us to construct discretizations for the problem. Even though the stress constraints must hold with probability one they can be violated on some set of measure zero, which may happen to contain our discretization points.

The goal of this section is to introduce an alternative convergent scheme, which in addition to adding the small lower bounds on the design variables moves the stress constraints into the objective function using a convex penalty function. Let

$$G(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{s}, \boldsymbol{S}) := \sum_{i=1}^{m} \frac{[|s_i| - \overline{\sigma}_1 x_i]_+^2}{x_i} + \sum_{j=1}^{r_2} \frac{[L_j S_j - \overline{\sigma}_2 X_j]_+^2}{X_j}.$$

Using the usual convention $0/0 = 0$ and $a/0 = \infty$ for any $a > 0$, the function G can be evaluated at any nonnegative design $(\boldsymbol{x}, \boldsymbol{X})$. Furthermore, G is l.s.c. on $\mathbb{R}_+^m \times \mathbb{R}_+^{r_2} \times \mathbb{R}^m \times \mathbb{R}_+^{r_2}$.

Let $\mu : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be an arbitrary function such that $\lim_{\varepsilon \to +0} \mu(\varepsilon) = +\infty$ and $\lim_{\varepsilon \to +0} \varepsilon\mu(\varepsilon) = 0$. Consider the penalized problem $(\bar{\mathcal{P}}_2^\varepsilon)$:

$$\begin{cases} \min\limits_{(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}(\cdot),\boldsymbol{S}(\cdot))} w^\varepsilon(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{s}(\cdot), \boldsymbol{S}(\cdot)) := \\ \qquad\qquad w(\boldsymbol{x}, \boldsymbol{X}) + \mu(\varepsilon) \displaystyle\int_\Omega G(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{s}(\omega), \boldsymbol{S}(\omega))\, \mathrm{P}(d\omega) \\ \text{s.t.} \begin{cases} \varepsilon \mathbf{1}_m \le \boldsymbol{x} \le \overline{\boldsymbol{x}} + \varepsilon \mathbf{1}_m, \\ \varepsilon \mathbf{1}_{r_2} \le \boldsymbol{X} \le \overline{\boldsymbol{X}} + \varepsilon \mathbf{1}_{r_2}, \\ (\boldsymbol{s}(\boldsymbol{\omega}), \boldsymbol{S}(\omega), \boldsymbol{\lambda}(\omega)) \text{ solves } (\mathcal{C})_{(\boldsymbol{x},\boldsymbol{X})}(\omega), \quad \text{P-a.s.} \end{cases} \end{cases}$$

The problem $(\bar{\mathcal{P}}_2^\varepsilon)$ is an instance of a stochastic MPEC (or stochastic bilevel programming problem). Even the existence of optimal solutions for this class of problems is a non-trivial fact, which is caused by the implicit characterization of the feasible set. Examples of MPEC having non-closed feasible sets are easy to construct [LPR96, Example 1.1.2].

The closedness properties of design-to-force mappings were investigated by Petersson [Pet01] for trusses, then generalized for trusses with unilateral constraints by Patriksson and Petersson [PaP00], and for stochastic settings by Evgrafov et al. [EPP02]. We cite some auxiliary results from the latter paper, which will be used in the development that follows.

Proposition 3.2.1 ([EPP02, Proposition 2.3]). *Let a sequence of nonnegative designs* $\{(\boldsymbol{x}_k, \boldsymbol{X}_k)\}$ *converge to* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}})$. *Suppose that* $(\boldsymbol{s}_k(\cdot), \boldsymbol{S}_k(\cdot), \boldsymbol{\lambda}_k(\cdot))$ *solves* $(\mathcal{C})_{(\boldsymbol{x}_k,\boldsymbol{X}_k)}(\cdot)$, *and that the sequence of energy expectations is bounded:*

$$\int_\Omega \mathcal{E}(\boldsymbol{x}_k, \boldsymbol{X}_k, \boldsymbol{s}_k(\omega), \boldsymbol{S}_k(\omega), \boldsymbol{\lambda}_k(\omega), \omega)\, \mathrm{P}(d\omega) \le C < \infty.$$

Then there exists a solution $(\bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot), \bar{\boldsymbol{\lambda}}(\cdot))$ *to the problem* $(\mathcal{C})_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}})}(\cdot)$, *and* $\{(\boldsymbol{s}_k(\cdot), \boldsymbol{S}_k(\cdot), \boldsymbol{\lambda}_k(\cdot))\}$ *almost surely converges to* $(\bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot), \bar{\boldsymbol{\lambda}}(\cdot))$.

The following energy estimation will be useful to obtain closedness properties of the feasible set of the penalized problem $(\bar{\mathcal{P}}_2^\varepsilon)$ and to verify the existence of solutions.

Lemma 3.2.2. *Suppose that the functions* $\boldsymbol{B}_i(\cdot)$, $\gamma_j(\cdot)$, $\boldsymbol{C}_1(\cdot)$, $\boldsymbol{f}(\cdot)$, $\boldsymbol{g}_1(\cdot)$, $\boldsymbol{g}_2(\cdot)$, $1/E(\cdot)$, $\boldsymbol{L}^2(\cdot)/E_c(\cdot)$ *are essentially bounded by some constant* $K > 0$. *Let* $(\boldsymbol{x}, \boldsymbol{X})$ *be a positive design such that* $(\boldsymbol{x}, \boldsymbol{X}) \leq (\overline{\boldsymbol{x}}, \overline{\boldsymbol{X}}) < \infty$ *and the problem* $(\mathcal{C})_{(\boldsymbol{x}, \boldsymbol{X})}(\cdot)$ *has a solution* $(\boldsymbol{s}(\cdot), \boldsymbol{S}(\cdot), \boldsymbol{\lambda}(\cdot))$. *Then there are constants* $K_1 > 0$ *and* $K_2 > 0$, *depending on* K, $\overline{\boldsymbol{x}}$, $\overline{\boldsymbol{X}}$, $\overline{\sigma}_1$, $\overline{\sigma}_2$, m *and* r_2 *only, such that*

$$\mathcal{E}(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{s}(\cdot), \boldsymbol{S}(\cdot), \boldsymbol{\lambda}(\cdot), \cdot) \leq K_1 + K_2 G(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{s}(\cdot), \boldsymbol{S}(\cdot)).$$

Proof. The inequality follows from the assumed essential boundedness of the functions defining the problem $(\mathcal{C})_{(\boldsymbol{x}, \boldsymbol{X})}(\cdot)$ and inequalities of the following type:

$$\sum_{i=1}^{m} \frac{s_i^2(\cdot)}{2E(\cdot) x_i} \leq \frac{1}{E(\cdot)} \sum_{i=1}^{m} \left[\overline{\sigma}_1^2 x_i + \frac{[|s_i(\cdot)| - \overline{\sigma}_1 x_i]_+^2}{x_i} \right],$$

following from the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, which holds for arbitrary $a, b \in \mathbb{R}$. In this way we can bound all the terms appearing in the definition of $\mathcal{E}$. #

Proposition 3.2.3. *Suppose that the problem* $(\bar{\mathcal{P}}_2^\varepsilon)$ *is feasible and proper. Then it possesses at least one optimal solution.*

Proof. Lemma 3.2.2 gives us the energy estimation, necessary for the application of Proposition 3.2.1. Then we can follow the proof of Theorem 2.4 in [EPP02] to reach the conclusion. #

Now we come to the main result of this section, asserting that all limit points of the family of optimal solutions to $(\bar{\mathcal{P}}_2^\varepsilon)$ as ε decreases to zero are in fact optimal solutions to the limiting problem $(\mathcal{P}_2)$. The proof of this result depends on the following assumption of locally directionally Lipschitz behaviour of the design-to-force mapping.

Assumption 3.2.4. Let $(\boldsymbol{x}, \boldsymbol{X}) \geq \boldsymbol{0}$ be a design for which the problem $(\mathcal{C})_{(\boldsymbol{x}, \boldsymbol{X})}(\omega)$ has a solution $(\boldsymbol{s}(\omega), \boldsymbol{S}(\omega), \boldsymbol{\lambda}(\omega))$ for almost any ω. Let $(\Phi, \Gamma) > \boldsymbol{0}$ be arbitrary in $\mathbb{R}^m \times \mathbb{R}^{r_2}$ and for $\varepsilon > 0$ set $(\boldsymbol{x}_\varepsilon, \boldsymbol{X}_\varepsilon) := (\boldsymbol{x}, \boldsymbol{X}) + \varepsilon(\Phi, \Gamma)$. Denote by $(\boldsymbol{s}_\varepsilon(\omega), \boldsymbol{S}_\varepsilon(\omega), \boldsymbol{\lambda}_\varepsilon(\omega))$ the corresponding optimal solution to $(\mathcal{C})_{(\boldsymbol{x}_\varepsilon, \boldsymbol{X}_\varepsilon)}(\omega)$. Then, there exist $\varepsilon_0 > 0$ and a nonnegative function $\tau(\omega) \in L_2(\Omega, \mathrm{P})$ such that the inequality

$$\|(\boldsymbol{s}(\omega), \boldsymbol{S}(\omega), \boldsymbol{\lambda}(\omega)) - (\boldsymbol{s}_\varepsilon(\omega), \boldsymbol{S}_\varepsilon(\omega), \boldsymbol{\lambda}_\varepsilon(\omega))\| \leq \varepsilon \tau(\omega)$$

holds for almost all $\omega \in \Omega$ and all $0 < \varepsilon < \varepsilon_0$.

Assumption 3.2.4 is known to hold for trusses without unilateral constraints and general probability measures [EPP02, Lemma 3.3], and for trusses with unilateral contraints and discrete probability measures [PaP00, Theorem 3.3].

We denote the optimal value of a problem $(\mathcal{P})$ by $\mathrm{val}(\mathcal{P})$.

Theorem 3.2.5. *Suppose that the functions $\boldsymbol{B}_i(\cdot)$, $\gamma_j(\cdot)$, $\boldsymbol{C}_1(\cdot)$, $\boldsymbol{f}(\cdot)$, $\boldsymbol{g}_1(\cdot)$, $\boldsymbol{g}_2(\cdot)$, $1/E(\cdot)$, $\boldsymbol{L}^2(\cdot)/E_c(\cdot)$ are essentially bounded, and let Assumption 3.2.4 hold at each feasible design in $(\mathcal{P}_2)$. Suppose further that for some $\varepsilon_0 > 0$ there is a solution $(\boldsymbol{x}_0, \boldsymbol{X}_0, \boldsymbol{s}_0(\cdot), \boldsymbol{S}_0(\cdot))$, which is feasible in $(\mathcal{P}_2)$ with $(\boldsymbol{x}_0, \boldsymbol{X}_0) \geq \varepsilon_0 \mathbf{1}_{m+r_2}$. Then for any $0 < \varepsilon \leq \varepsilon_0$ the problem $(\bar{\mathcal{P}}_2^{\varepsilon})$ has an optimal solution $(\boldsymbol{x}_\varepsilon, \boldsymbol{X}_\varepsilon, \boldsymbol{s}_\varepsilon(\cdot), \boldsymbol{S}_\varepsilon(\cdot))$. Any limit point of the sequence $\{(\boldsymbol{x}_\varepsilon, \boldsymbol{X}_\varepsilon, \boldsymbol{s}_\varepsilon(\cdot), \boldsymbol{S}_\varepsilon(\cdot))\}$ as ε goes to zero is an optimal solution to $(\mathcal{P}_2)$.*

Proof. The point $(\boldsymbol{x}_0, \boldsymbol{X}_0, \boldsymbol{s}_0(\cdot), \boldsymbol{S}_0(\cdot))$ is feasible in $(\bar{\mathcal{P}}_2^{\varepsilon})$ for any $0 < \varepsilon \leq \varepsilon_0$, with an objective value $w^{\varepsilon}(\boldsymbol{x}_0, \boldsymbol{X}_0, \boldsymbol{s}_0(\cdot), \boldsymbol{S}_0(\cdot)) = w(\boldsymbol{x}_0, \boldsymbol{X}_0)$. Proposition 3.2.3 implies the existence of a sequence $\{(\boldsymbol{x}_\varepsilon, \boldsymbol{X}_\varepsilon, \boldsymbol{s}_\varepsilon(\cdot), \boldsymbol{S}_\varepsilon(\cdot))\}$ of optimal solutions to $(\bar{\mathcal{P}}_2^{\varepsilon})$. The compactness of the design space implies the existence of a subsequence $\{(\boldsymbol{x}_{\varepsilon_k}, \boldsymbol{X}_{\varepsilon_k})\}$ converging to a limit $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}) \geq \mathbf{0}_{m+r_2}$. Since $w^{\varepsilon_k}(\boldsymbol{x}_{\varepsilon_k}, \boldsymbol{X}_{\varepsilon_k}, \boldsymbol{s}_{\varepsilon_k}(\cdot), \boldsymbol{S}_{\varepsilon_k}(\cdot)) \leq \mathrm{val}(\mathcal{P}_2)$, we can apply Lemma 3.2.2 and Proposition 3.2.1 to conclude that the sequence $\{(\boldsymbol{s}_{\varepsilon_k}(\cdot), \boldsymbol{S}_{\varepsilon_k}(\cdot))\}$ converges to a limit $(\bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot))$, solving the problem $(\mathcal{C})_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}})}(\cdot)$. The boundedness of $\{\mu(\varepsilon_k) \int_\Omega G(\boldsymbol{x}_{\varepsilon_k}, \boldsymbol{X}_{\varepsilon_k}, \boldsymbol{s}_{\varepsilon_k}(\cdot), \boldsymbol{S}_{\varepsilon_k}(\cdot))\, \mathrm{P}(d\omega)\}$ as ε_k goes to zero, the nonnegativity and l.s.c. property of G, and Fatou's Lemma, imply that the point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}, \bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot))$ satisfies the stress constraints, whence is feasible in $(\mathcal{P}_2)$. Then, the following inequalities hold:

$$\begin{aligned} \mathrm{val}(\mathcal{P}_2) \leq w(\bar{x}, \bar{X}) &\leq \liminf_{k\to\infty} w^{\varepsilon_k}(\boldsymbol{x}_{\varepsilon_k}, \boldsymbol{X}_{\varepsilon_k}, \boldsymbol{s}_{\varepsilon_k}(\cdot), \boldsymbol{S}_{\varepsilon_k}(\cdot)) \\ &= \liminf_{k\to\infty} \mathrm{val}(\bar{\mathcal{P}}_2^{\varepsilon_k}). \end{aligned} \tag{3.2}$$

On the other hand, for any point $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{s}}(\cdot), \widetilde{\boldsymbol{S}}(\cdot))$ that is feasible in $(\mathcal{P}_2)$ and for large enough k, Assumption 3.2.4 implies that the point $(\widetilde{\boldsymbol{x}} + \varepsilon_k \mathbf{1}_m, \widetilde{\boldsymbol{X}} + \varepsilon_k \mathbf{1}_{r_2}, \widetilde{\boldsymbol{s}}_{\varepsilon_k}(\cdot), \widetilde{\boldsymbol{S}}_{\varepsilon_k}(\cdot))$, where $(\widetilde{\boldsymbol{s}}_{\varepsilon_k}(\cdot), \widetilde{\boldsymbol{S}}_{\varepsilon_k}(\cdot), \widetilde{\boldsymbol{\lambda}}_{\varepsilon_k}(\cdot))$ solves $(\mathcal{C})_{(\widetilde{\boldsymbol{x}}+\varepsilon_k\mathbf{1}_m, \widetilde{\boldsymbol{X}}+\varepsilon_k\mathbf{1}_{r_2})}(\cdot)$, is feasible in the problem $(\bar{\mathcal{P}}_2^{\varepsilon_k})$ with the objective value $w^{\varepsilon_k}(\widetilde{\boldsymbol{x}} + \varepsilon_k \mathbf{1}_m, \widetilde{\boldsymbol{X}} + \varepsilon_k \mathbf{1}_{r_2}, \widetilde{\boldsymbol{s}}_{\varepsilon_k}(\cdot), \widetilde{\boldsymbol{S}}_{\varepsilon_k}(\cdot)) \leq w(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{X}}) + \varepsilon_k(m\rho_1 + r_2\rho_2) + K\varepsilon_k\mu(\varepsilon_k)$ for a suitable positive constant K, thus showing the reverse inequality:

$$\limsup_{k\to\infty} \mathrm{val}(\bar{\mathcal{P}}_2^{\varepsilon_k}) \leq \mathrm{val}(\mathcal{P}_2);$$

together with (3.2) this establishes the optimality of $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}, \bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot))$ in $(\mathcal{P}_2)$. #

3.2.1 Numerical examples

We illustrate Theorem 3.2.5 with small numerical examples.

Example 3.2.6 (One-bar truss with a cable). Figure 3.1 shows a simple one-dimensional structure introduced and analyzed in [PaP00] that consists of a bar suspended with one cable. Suppose that $\Omega = [-1, 2]$, P is the

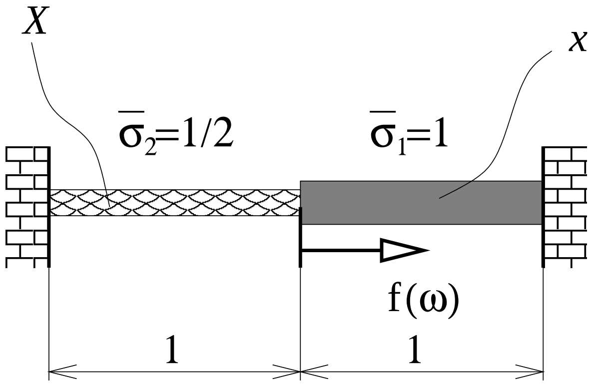

Fig. 3.1: The cable suspended one-bar truss.

uniform distribution on $[-1/2, 1]$, $f(\omega) = \omega$, $E = E_c = 1$, $\rho_1 = \rho_2 = 1$, $\overline{x} = 1$ and $\overline{X} = 2$. Every feasible design (x, X) must satisfy $x > 0$ (because $\mathrm{P}(f < 0) > 0$). Solving the equilibrium problem, we can see that the design-to-force mapping

$$s(\omega) = \begin{cases} \frac{x\omega}{x+X}, & \text{for } \omega \geq 0, \\ \omega, & \text{otherwise}, \end{cases} \qquad S(\omega) = \begin{cases} \frac{X\omega}{x+X}, & \text{for } \omega \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

satisfies the Assumption 3.2.4 for every feasible design, thus we can expect the convergence of solutions to penalized problems towards the solution of the original problem.

After eliminating the state variables one obtains the following optimization problem:

$$\begin{cases} \min\limits_{(x,X)} x + X \\ \text{s.t.} \begin{cases} 0 \leq x \leq 1, & \\ 0 \leq X \leq 2, & \\ \dfrac{x\omega}{x+X} \leq x, & \text{P-a.s.,} \\ -\omega \leq x, & \text{P-a.s.,} \\ 0 \leq \dfrac{X\omega}{x+X} \leq X/2, & \text{P-a.s.} \end{cases} \end{cases}$$

Figure 3.2 shows the feasible domain for the design variables (x, X).

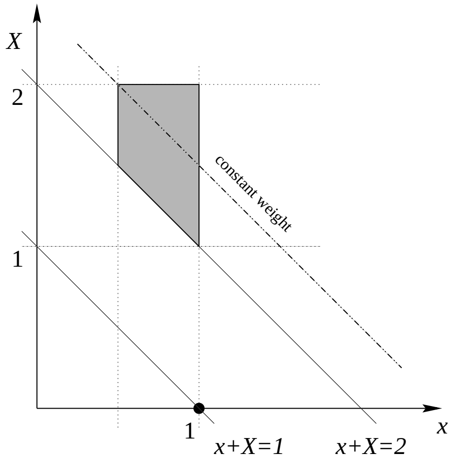

Fig. 3.2: The admissible design domain. The optimal solution is at the black circle.

Note that this domain consists of the union of a two-dimensional, convex domain and the isolated optimal point $(x^*, X^*) = (1, 0)$ with corresponding optimal weight $w^* = 1$.

Now we set $\mu(\varepsilon) = \varepsilon^{-0.8}$ and consider perturbation parameters $\varepsilon_k = 2^{-k}$. The corresponding penalized problem is:

$$\begin{cases} \min\limits_{(x,X)} x + X + \varepsilon_k^{-0.8}\Bigg(x \displaystyle\int\limits_{\{\,x+X\le\omega\le 1\,\}} \left[\frac{\omega}{x+X} - 1\right]^2 d\omega \\ \qquad\qquad\qquad + \displaystyle\int\limits_{\{-1/2\le\omega\le -x\,\}} \left[-\,\omega - x\right]^2 d\omega \\ \qquad\qquad\qquad + X \displaystyle\int\limits_{\{\,(x+X)/2\le\omega\le 1\,\}} \left[\frac{\omega}{x+X} - 1/2\right]^2 d\omega \Bigg) \\ \text{s.t.} \begin{cases} \varepsilon_k \le x \le 1 + \varepsilon_k, \\ \varepsilon_k \le X \le 2 + \varepsilon_k. \end{cases} \end{cases}$$

The behaviour of the optimal solutions to the penalized problems is shown in Figure 3.3. The sequence converges to the optimal solution of $(\mathcal{P}_2)$, as was predicted by Theorem 3.2.5. We note that this problem possesses uncountably many local minima, corresponding to the weight $x + X = 2$, but our approximation scheme recovers the globally optimal solution.

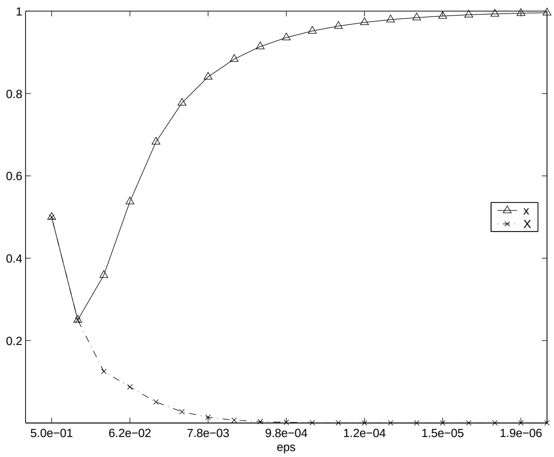

Fig. 3.3: The convergence of the optimal solutions to the penalized problems.

Example 3.2.7 (4-bar truss). Consider the problem of minimizing the weight of the 4-bar structure shown in Figure 3.4 (this problem also appear in [EPP02]). The stress limit for each bar is $\overline{\sigma} = 1$, and the Young's modulus is $E = 1$. Assume that the upper design bounds are inactive, and that the force vector $\boldsymbol{f}(\omega)$ equals $(2\cos(\omega), 1.5\sin(\omega))^T$. The probability measure is the uniform one on $[0, \pi]$. Since the initial structural topology as well as the loading conditions are symmetric, we can expect symmetric optimal solutions (i.e., $x_1^* = x_4^*$, $x_2^* = x_3^*$). Figure 3.5 shows the projection of the set of feasible designs onto the linear subset $\{\, \boldsymbol{x} \in \mathbb{R}^4 \mid x_1 = x_4, x_2 = x_3 \,\}$. Note that the feasible set is not a finite union of polyhedra, because we work with an infinite number of load cases (compare with the similar Problem 1 in [StS01]). There are three local minima, two of which (including the globally optimal solution, $\boldsymbol{x}^* = (0, 2.5, 2.5, 0)^T$) are singular. The nonsingular non-global, local minimum of the original problem is the global minimum for the "naively" perturbed problem for all small values of ε. Therefore, we cannot approximate the globally optimal solution by the "naive" ε-perturbation.

Our penalization scheme recovers the global optimal solution. Figure 3.6 shows the convergence of the optimal solutions to the ε-perturbed problems towards the solution of the original problem, as ε decreases to zero (variables x_3 and x_4 are not shown, owing to the symmetry of the calculated optimal solutions). We have used the nested formulation (with eliminated state variables), adaptive numerical quadratures to calculate the penalty function and a finite difference approximation of the derivatives to solve the problem using a sequential quadratic programming (SQP) algorithm. The penalty parameter was taken to be $\mu(\varepsilon) = \varepsilon^{-0.8}$.

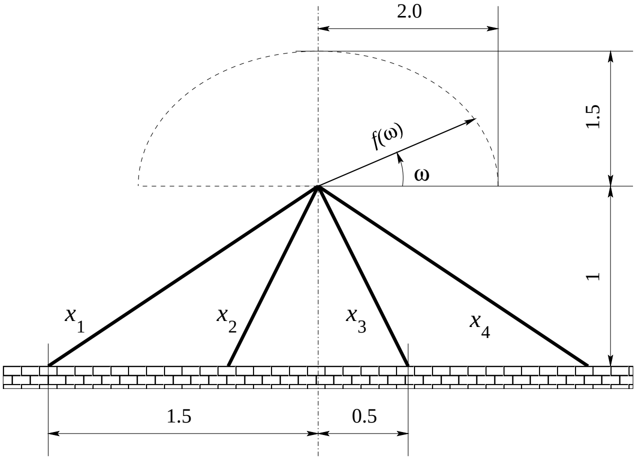

Fig. 3.4: The 4-bar truss problem.

Of course, applying a nonlinear programming algorithm to a penalized problem we can only expect a local solution, owing to the nonconvexity of the problem. For example, it is possible to obtain all three local minima for the 4-bar truss problem by providing different starting points to an SQP algorithm.

3.3 Discretization

The most popular method to solve a stochastic programming problem involving a non-discrete probability measure is to approximate it with a sequence of finite-dimensional problems with discrete measures. To implement such a procedure one needs to construct two objects:

- a discrete measure $\widetilde{\mathrm{P}}$, in some sense close to the original one. The popular choice is to start from a finite partition $\mathcal{A} = \{\Omega_i \mid \cup_i \Omega_i = \Omega\}$ and define $\widetilde{\mathrm{P}}_i = \mathrm{P}(\Omega_i)$. This involves the calculation of probabilities $\mathrm{P}(\Omega_i)$, which, ideally, we would like to avoid and replace by some estimations of $\mathrm{P}(\Omega_i)$;
- an approximation $\widetilde{f}$ to a random element f of the original problem. One possible choice is to define $\widetilde{f} = E(f \mid \mathcal{A})$, which involves the computation of conditional means. In the case when Ω itself is a subset of a metric space we can define a vector $\widetilde{\omega} = E(\omega \mid \mathcal{A})$, and consider $\widetilde{f}_i = f(\widetilde{\omega}_i)$. This approximation still involves the computation of conditional means, although simpler than in the previous case. In the computational scheme, we would instead like to choose sampling points $\widetilde{\omega}_i \in \Omega_i$ and set, as before, $\widetilde{f}_i = f(\widetilde{\omega}_i)$.

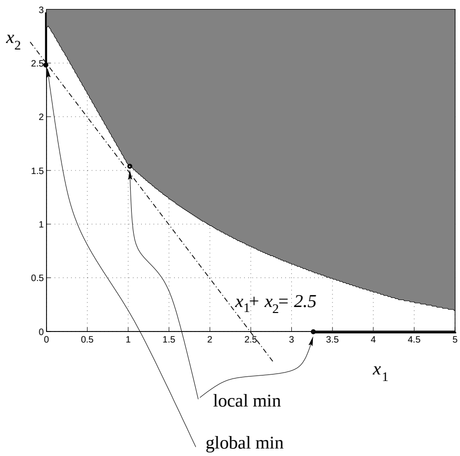

Fig. 3.5: The feasible design domain of the 4-bar truss problem.

Examples of discretizations of stochastic programming problems with recourse (the most studied class of stochastic programming problems) based on the ideas of conditional means can be found in [Ols76b, Ols76a] and in [BiW86]. One generic scheme which allows us to avoid the computations of conditional means is the "method of mechanical quadratures" [Vai71], which we will use to approximate integrals in our problems. It was successfully used to discretize stochastic programming problems with complete recourse in [Lep90].

To characterize the convergence of the solutions to the discretized problems towards a solution to the original one, we use the general notion of discrete convergence, introduced by Stummel [Stu73] and Vainikko [Vai78]. See also [Vas82, Lep90, Lep94] for the applications to stochastic programming problems, [Pan79, Lis87, Lis90] for the application to variational inequalities in Banach spaces, and [Lep93, Lep96] for applications to control problems. With some identifications we can embed the method of mechanical quadratures into the framework of discrete limit spaces. In spirit, the following analysis is close to that of Lepp [Lep90] for the stochastic programming problems with recourse.

It seems impossible to apply discretization schemes directly to the structural topology optimization problems, owing to the discontinuity of the lower-level objective function $\mathcal{E}$ at the points, where the topology changes. Therefore, we discretize sizing approximations of the topology optimization problems instead.

In this section we assume that all state variables are elements of $L_\infty(\Omega, \mathrm{P})$. Measurability of the state variables is necessary for the model to be tractable in the probabilistic setting; it is a very mild condition and

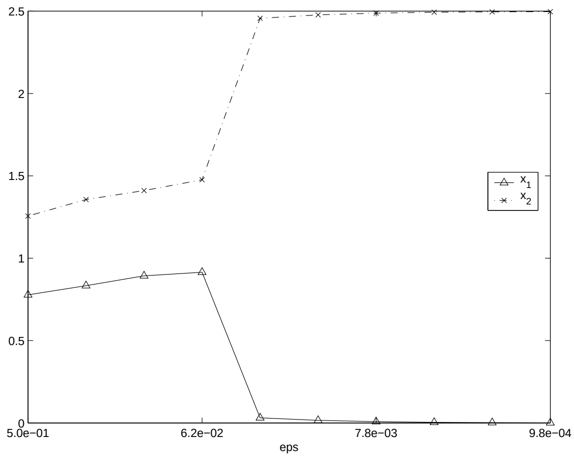

Fig. 3.6: Convergence of the ε-perturbations for the 4-bar truss problem.

holds under measurability assumptions on the problem's data [EPP02, Corollary 2.2]. A more questionable assumption is the essential boundedness of the state variables. But since we formulate our problems in the framework of linear elasticity, all loads and stresses considered must be bounded.

In the remainder of the section we list the main assumptions and results, while the interested reader can find the complete development with proofs in the appendix.

Suppose that Ω is a compact metric space with a metric denoted by $\rho(\cdot,\cdot)$. Let $\mathfrak{S} \supset \mathcal{B}(\Omega)$, $\mathrm{P}(\{\,\omega \mid \rho(\omega,\omega_0) < r\,\}) = \mathrm{P}(\{\,\omega \mid \rho(\omega,\omega_0) \le r\,\}) > 0$ for any $\omega_0 \in \Omega$, $r > 0$, and P is a regular measure.

Consider a sequence of *partitions* of Ω, $\mathcal{A}^k = \{\,A_1^k,\dots,A_k^k\,\}$, satisfying the following properties:

(M1) $\mathrm{P}(A_l^k) > 0$,
(M2) $\cup_{l=1}^k A_l^k = \Omega$,
(M3) $A_i^k \cap A_j^k = \emptyset$, $i \ne j$,
(M4) $\lim_{k\to\infty} \mathrm{diam}(A_l^k) = 0$,
(M5) $\mathrm{P}(\partial A_l^k) = 0$.

Note, that the collection of the sets $\{\mathcal{A}^k\}$, satisfying the properties (M1)–(M5) generates an algebra $\mathfrak{S}_0 \subset \mathfrak{S}$.

Define a sequence of discrete measures P_k with support $\operatorname{supp}\mathrm{P}_k = \{\,\omega_1^k,\dots,\omega_k^k\,\}$, satisfying the following properties:

(M6) $\omega_l^k \in A_l^k$,
(M7) $\lim_{k\to\infty} \max_{1\le l\le k} \mathrm{P}_k(\omega_l^k)/\mathrm{P}(A_l^k) = 1$.

We further assume the following:

(D1) the functions $E(\cdot)$, $E_c(\cdot)$, $\boldsymbol{g}_1(\cdot)$, $\boldsymbol{g}_2(\cdot)$, $\boldsymbol{L}_j(\cdot)$ and $\boldsymbol{f}(\cdot)$ are $\mathfrak{S}_0$-measurable;
(D2) the functions $\boldsymbol{f}(\cdot), \boldsymbol{g}_1(\cdot), \boldsymbol{g}_2(\cdot), 1/E(\cdot), \boldsymbol{L}_j^2(\cdot)/E_c(\cdot)$ are bounded;
(D3) the functions $\boldsymbol{C}_1(\cdot), \boldsymbol{B}_i(\cdot), \boldsymbol{\gamma}_j(\cdot)$ are constants;
(D4) the functions $E(\cdot)$ and $E_c(\cdot)/\boldsymbol{L}_j^2(\cdot)$ are bounded.

We denote by $(\mathcal{C})^k_{(\boldsymbol{x},\boldsymbol{X})}(\omega_l^k)$ the following equilibrium principle:

$$\begin{cases} \min\limits_{(\boldsymbol{s},\boldsymbol{S},\boldsymbol{\lambda})} \mathcal{E}(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s},\boldsymbol{S},\boldsymbol{\lambda},\omega_l^k) := \dfrac{1}{2}\sum\limits_{i=1}^m \dfrac{s_i^2}{E(\omega_l^k)x_i} + \boldsymbol{g}_1^T(\omega_l^k)\boldsymbol{\lambda} \\ \qquad\qquad + \sum\limits_{j=1}^{r_2}\Big(\dfrac{(L_j(\omega_l^k)S_j)^2}{2E_c(\omega_l^k)X_j} + (\boldsymbol{g}_2(\omega_l^k))_j S_j\Big), \\ \text{s.t.} \begin{cases} \boldsymbol{C}_1^T\lambda + \sum\limits_{i\in\mathcal{I}(\boldsymbol{x})} \boldsymbol{B}_i^T s_i + \sum\limits_{j\in\mathcal{J}(\boldsymbol{X})} S_j\gamma_j = \boldsymbol{f}(\omega_l^k), \\ \boldsymbol{\lambda} \ge 0, \\ \boldsymbol{S}_{\mathcal{J}(\boldsymbol{X})} \ge 0. \end{cases} \end{cases}$$

Let $(\mathcal{P}_1^\varepsilon)$ be an ε-perturbation of the problem $(\mathcal{P}_1)$:

$$\begin{cases} \min\limits_{(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}(\cdot),\boldsymbol{S}(\cdot))} c^f(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}(\cdot),\boldsymbol{S}(\cdot),\boldsymbol{\lambda}(\cdot)) \\ \text{s.t.} \begin{cases} \varepsilon\mathbf{1}_m \le \boldsymbol{x} \le \overline{\boldsymbol{x}}, \quad \mathbf{1}_m^T\boldsymbol{x} \le v, \\ \varepsilon\mathbf{1}_{r_2} \le \boldsymbol{X} \le \overline{\boldsymbol{X}}, \quad \mathbf{1}_{r_2}^T\boldsymbol{X} \le V, \\ (\boldsymbol{s}(\boldsymbol{\omega}),\boldsymbol{S}(\omega),\boldsymbol{\lambda}(\omega)) \text{ solves } (\mathcal{C})_{(\boldsymbol{x},\boldsymbol{X})}(\omega), \text{ P-a.s.} \end{cases} \end{cases}$$

Any limit point of the family of optimal solutions to $(\mathcal{P}_1^\varepsilon)$ as ε converges to zero is an optimal solution to the limiting problem $(\mathcal{P}_1)$, owing to [EPP02, Theorem 3.1].

In the following two theorems we construct discretizations for the problems $(\mathcal{P}_1^\varepsilon)$ and $(\bar{\mathcal{P}}_2^\varepsilon)$. The formal definition of weak* discrete convergence is given in the appendix (Definition 3.A.2). We note that from the weak* discrete convergence of the sequence $\{(\boldsymbol{x}_k^*, \boldsymbol{X}_k^*, \boldsymbol{s}_k^*(\cdot), \boldsymbol{S}_k^*(\cdot))\}$ follows the (usual) convergence of the optimal designs.

Theorem 3.3.1. *Consider the following sequence* $\{(\mathcal{P}_1^\varepsilon)^k\}$ *of discretiza-*

tions of the problem $(\mathcal{P}_1^\varepsilon)$:

$$\begin{cases} \min\limits_{(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}_k(\cdot),\boldsymbol{S}_k(\cdot))} c^f(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}_k(\cdot),\boldsymbol{S}_k(\cdot),\boldsymbol{\lambda}_k(\cdot)) := \\ \qquad \sum_{l=1}^{k} \mathcal{E}(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}_k(\omega_l^k),\boldsymbol{S}_k(\omega_l^k),\boldsymbol{\lambda}(\omega_l^k),\omega_l^k)\,\mathrm{P}_k(\omega_l^k) \\ \text{s.t.} \begin{cases} \varepsilon\mathbf{1}_m \le \boldsymbol{x} \le \overline{\boldsymbol{x}}, \quad \mathbf{1}_m^T\boldsymbol{x} \le v, \\ \varepsilon\mathbf{1}_{r_2} \le \boldsymbol{X} \le \overline{\boldsymbol{X}}, \quad \mathbf{1}_{r_2}^T\boldsymbol{X} \le V, \\ (\boldsymbol{s}_k(\omega_l^k),\boldsymbol{S}_k(\omega_l^k),\boldsymbol{\lambda}_k(\omega_l^k)) \text{ solves } (\mathcal{C})^k_{(\boldsymbol{x},\boldsymbol{X})}(\omega_l^k), \\ \qquad l = 1,\dots,k. \end{cases} \end{cases}$$

Suppose that the assumptions (M1)–(M7) *and* (D1)–(D4) *hold. Suppose further that there exists an optimal solution* $(\boldsymbol{x}^*,\boldsymbol{X}^*,\boldsymbol{s}^*(\cdot),\boldsymbol{S}^*(\cdot))$ *to the problem* $(\mathcal{P}_1^\varepsilon)$ *such that the energy functional* $\mathcal{E}(\boldsymbol{x}^*,\boldsymbol{X}^*,\boldsymbol{s}^*(\cdot),\boldsymbol{S}^*(\cdot),\boldsymbol{\lambda}^*(\cdot),\cdot)$ *is essentially bounded. Finally, assume that the problems* $(\mathcal{C})^k_{(\boldsymbol{x}^*,\boldsymbol{X}^*)}(\omega_l^k)$ *are feasible for any* k, $1 \le l \le k$.

Let $\{(\boldsymbol{x}_k^*,\boldsymbol{X}_k^*,\boldsymbol{s}_k^*(\cdot),\boldsymbol{S}_k^*(\cdot))\}$ *be a sequence of optimal solutions to* $\{(\mathcal{P}_1^\varepsilon)^k\}$. *Then any weak** *discrete limit point of this sequence solves the limiting problem* $(\mathcal{P}_1^\varepsilon)$.

Theorem 3.3.2. *Consider the following sequence* $\{(\bar{\mathcal{P}}_2^\varepsilon)^k\}$ *of discretizations of the problem* $(\bar{\mathcal{P}}_2^\varepsilon)$:

$$\begin{cases} \min\limits_{(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}_k(\cdot),\boldsymbol{S}_k(\cdot))} w^\varepsilon(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}_k(\cdot),\boldsymbol{S}_k(\cdot)) := w(\boldsymbol{x},\boldsymbol{X}) + \\ \qquad \mu(\varepsilon)\sum_{l=1}^{k} G(\boldsymbol{x},\boldsymbol{X},\boldsymbol{s}_k(\omega_l^k),\boldsymbol{S}_k(\omega_l^k))\,\mathrm{P}_k(\omega_l^k) \\ \text{s.t.} \begin{cases} \varepsilon\mathbf{1}_m \le \boldsymbol{x} \le \overline{\boldsymbol{x}} + \varepsilon\mathbf{1}_m, \\ \varepsilon\mathbf{1}_{r_2} \le \boldsymbol{X} \le \overline{\boldsymbol{X}} + \varepsilon\mathbf{1}_{r_2}, \\ (\boldsymbol{s}_k(\omega_l^k),\boldsymbol{S}_k(\omega_l^k),\boldsymbol{\lambda}_k(\omega_l^k)) \text{ solves } (\mathcal{C})^k_{(\boldsymbol{x},\boldsymbol{X})}(\omega_l^k), \\ \qquad l = 1,\dots,k. \end{cases} \end{cases}$$

Suppose that the assumptions (M1)–(M7) *and* (D1)–(D4) *hold. Suppose further that there exists an optimal solution* $(\boldsymbol{x}^*,\boldsymbol{X}^*,\boldsymbol{s}^*(\cdot),\boldsymbol{S}^*(\cdot))$ *to the problem* $(\bar{\mathcal{P}}_2^\varepsilon)$ *such that the energy functional* $\mathcal{E}(\boldsymbol{x}^*,\boldsymbol{X}^*,\boldsymbol{s}^*(\cdot),\boldsymbol{S}^*(\cdot),\boldsymbol{\lambda}^*(\cdot),\cdot)$ *is essentially bounded. Finally, assume that the problems* $(\mathcal{C})^k_{(\boldsymbol{x}^*,\boldsymbol{X}^*)}(\omega_l^k)$ *are feasible for any* k, $1 \le l \le k$.

Let $\{(\boldsymbol{x}_k^*,\boldsymbol{X}_k^*,\boldsymbol{s}_k^*(\cdot),\boldsymbol{S}_k^*(\cdot))\}$ *be a sequence of optimal solutions to* $\{(\bar{\mathcal{P}}_2^\varepsilon)^k\}$. *Then any weak** *discrete limit point of this sequence solves the limiting problem* $(\bar{\mathcal{P}}_2^\varepsilon)$.

In Theorems 3.3.1 and 3.3.2 the requirement of feasibility of $(\mathcal{C})^k_{(\boldsymbol{x},\boldsymbol{X})}(\cdot)$ for positive designs may be related to a "richness" of the ground structure (a truss with the topology where all members are present). For example, we can start from a ground structure which is able to sustain any load and thus satisfy the assumption.

3.4 Numerical example

We consider the problem of finding a minimum weight of the cable suspended crane shown in Figure 3.7. In this example, the force is a unit vector with the direction uniformly distributed on $[-3\pi/4, -\pi/4]$. The

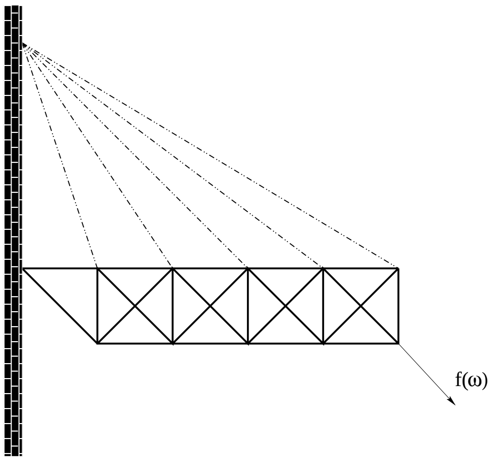

Fig. 3.7: The ground structure for the weight minimization problem.

number of bars is $m = 23$, and the number of cables is $r_2 = 5$. We set $\rho_1 = \rho_2 = 1.0$, the maximal cross-sectional area for both cables and bars equals 1.0, the maximal stresses are $\overline{\sigma}_1 = 1.4$ and $\overline{\sigma}_2 = 0.8$, Young's moduli are $E = E_c = 1.0$, and the initial slacks $\boldsymbol{g}_1 = \mathbf{0}$. For the penalty function, we use $\mu(\varepsilon) = \varepsilon^{-0.8}$, and then start with $\varepsilon = 0.05$ and successively multiply it by a factor of 0.6, until it gets as small as $5.0 \cdot 10^{-4}$.

We have solved the nested formulation of the problem $(\bar{\mathcal{P}}_2^{\varepsilon})$, i.e., we have eliminated the state variables and treated them as functions of design. To solve for optimal solutions, we have used a sequential quadratic programming algorithm, and to obtain first-order information we have used finite-difference approximations. The starting point was the ground structure. This numerical approach can only be applied to trusses of small size, and the construction of efficient algorithms to solve problems of this type is one of the topics of the ongoing research.

In Table 3.1 we report the optimal weights for discretizations with k varying from 1 to 625. The norm of the differences between optimal de-

k	w^*	$\|\boldsymbol{x}_k^* - \boldsymbol{x}_{625}^*\|$	$\|\boldsymbol{X}_k^* - \boldsymbol{X}_{625}^*\|$
1	10.3753	1.44	1.30
5	13.2174	$1.16 \cdot 10^{-1}$	$1.23 \cdot 10^{-1}$
25	13.3063	$5.29 \cdot 10^{-3}$	$2.26 \cdot 10^{-3}$
125	13.3118	$4.63 \cdot 10^{-4}$	$2.47 \cdot 10^{-4}$
625	13.312	-	-

Tab. 3.1: Convergence of the optimal desings and the objective values.

k	$\mathrm{P}_{625}(\sigma_{1\%})$	$\mathrm{P}_{625}(\sigma_{5\%})$	$\mathrm{P}_{625}(\sigma_{10\%})$
1	0.9984	0.976	0.952
5	0.8512	0.168	0.12
25	0.432	0.0	0.0
125	0.430	0.0	0.0
625	0.4304	0.0	0.0

Tab. 3.2: Stress violations.

signs for various values of k and the optimal design for $k = 625$ is also included. Further increases of k do not lead to any significant changes in the optimal design, therefore we conclude that the optimal design corresponding to $k = 625$ is a good approximation of the limiting optimal design. Table 3.2 shows the stress constraint violations, where $\sigma_{\alpha\%}$ denotes the event "at least one structure member (bar or cable) violates the stress constraint by at least $\alpha\%$". From the tables one can observe convergence of the optimal designs, even though we are not sure whether the Assumption 3.2.4 is satisfied in this problem. High values of $\mathrm{P}_{625}(\sigma_{1\%})$ even for $k = 625$ are due to our treatment of stress constraints via a penalty function, which allows small violations. In accordance with Theorem 3.2.5, this value reduces with a further reduction of the penalty parameter ε, as shown in Figure 3.8.

Two optimal designs corresponding to $k = 1$ and $k = 625$ are shown in the Figure 3.9, and their behaviour under various loading conditions is shown in the Figure 3.10.

3.5 Concluding remarks and further research

In Section 3.2 we have introduced a principally new approach to solve stress constrained topology weight minimization problems. Even though we do not get an a priori estimate of the violation of the stress constraints in this way (as we do while using an ordinary ε-perturbation approach), this approach has several practical advantages. The penalized formula-

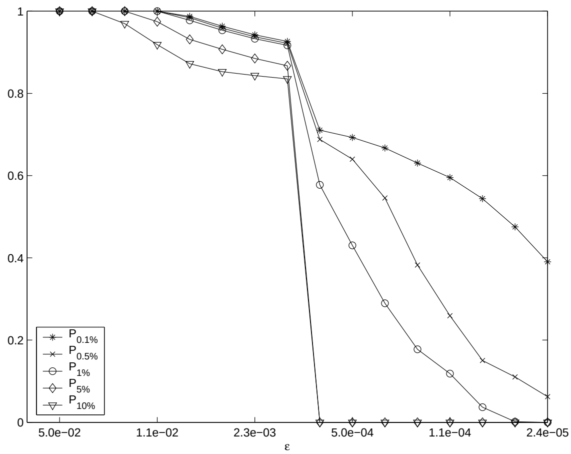

Fig. 3.8: The decrease of $\mathrm{P}_{625}(\sigma_{\alpha\%})$ as ε decreases for various values of α ($k = 625$).

tion does not have any state constraints, which makes it suitable for the discretization of the "continuous" stochastic programming problem. Furthermore, many numerical algorithms for bilevel programming problems, which work only for problems without state constraints, can be applied.

It is still an open question whether the *simultaneous* decrease of the perturbation parameter ε and increase of the number of discretization points leads to the convergence of the optimal solutions to discretized sizing optimization problems towards optimal solutions to stochastic topology compliance minimization problems. In the case of stress constrained weight minimization such convergence does not necessarily hold because of the known discontinuity of solutions with respect to changes in probability measure [EPP02]. Thus, for stochastic weight minimization, an actual question is the finding of an alternative formulation, which has stable solutions with respect to modelling errors.

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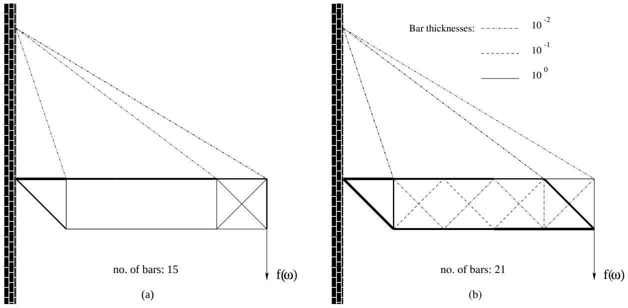

Fig. 3.9: The optimal designs for the weight minimization problem corresponding to (a) $k = 1$ and (b) $k = 625$. Line thicknesses are proportional to cross-sectional areas.

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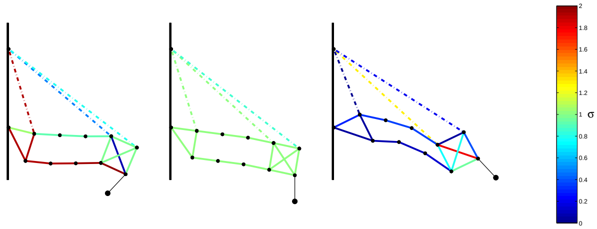
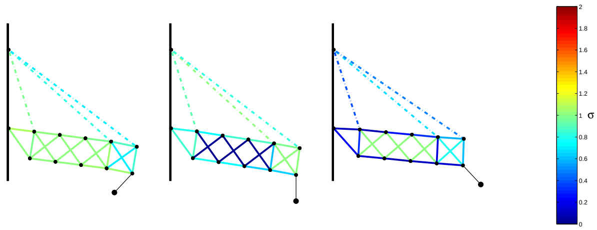


(a)



(b)

Fig. 3.10: Stresses and displacements for various random forces for optimal designs, corresponding to (a) $k = 1$ and (b) $k = 625$. Note: for the sake of better vizualization of stresses, line thicknesses are *not* proportional to cross-sectional areas.

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3.A The proof of convergence of discretizations

3.A.1 Discrete limit spaces

Consider a sequence of finite-dimensional spaces $\{\ell^k_\infty(\mathrm{P}_k)\}$ and a Banach space $L_\infty(\mathrm{P})$. Further consider a sequence of linear bounded *connection* operators $\mathfrak{p}_k : L_\infty(\mathrm{P}) \to \ell^k_\infty(\mathrm{P}_k)$ satisfying the following property:

$$\forall\, s(\cdot) \in L_\infty(\mathrm{P}) : \lim_{k\to\infty} \|\mathfrak{p}_k(s(\cdot))\| = \|s(\cdot)\|.$$

In the same way, we can consider a sequence of finite-dimensional spaces $\{\ell^k_1(\mathrm{P}_k)\}$ and a Banach space $L_1(\mathrm{P})$, together with a sequence of linear connection operators $\mathfrak{q}_k : L_1(\mathrm{P}) \to \ell^k_1(\mathrm{P}_k)$, satisfying the corresponding property.

Definition 3.A.1. The sequence $\{s_k(\cdot) \in \ell^k_\infty(\mathrm{P}_k)\}$ (strongly) discretely converges to a limit $s(\cdot) \in L_\infty(\mathrm{P})$, if $\lim_{k\to\infty} \|s_k(\cdot) - \mathfrak{p}_k(s(\cdot))\| = 0$.

A completely analogous definition could be given in the case of L_1-spaces.

Definition 3.A.2. The sequence $\{s_k(\cdot) \in \ell^k_\infty(\mathrm{P}_k)\}$ weakly* discretely converges to a limit $s(\cdot) \in L_\infty(\mathrm{P})$, if for any sequence $\{u_k(\cdot) \in \ell^k_1(\mathrm{P}_k)\}$ (strongly) discretely converging to a limit $u(\cdot) \in L_1(\mathrm{P})$, it holds that $\lim_{k\to\infty}\langle u_k(\cdot), s_k(\cdot)\rangle = \langle u(\cdot), s(\cdot)\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the standard pairing between the corresponding dual spaces.

For further details about discrete limit spaces, the reader is referred to the original works of Stummel [Stu73] and Vainikko [Vai78].

3.A.2 Method of mechanical quadratures

The main reason for introducing the partitions $\mathcal{A}^k$ and the measures P_k with the properties (M1)–(M7) (cf. Section 3.3) is the following result:

Proposition 3.A.3 ([Vai71, Corollary 1]). *The equality*

$$\lim_{k\to\infty}\sum_{l=1}^{k} f(\omega^k_l)\mathrm{P}_k(\omega^k_l) = \int_\Omega f(\omega)\,\mathrm{P}(d\omega)$$

holds for an arbitrary bounded $\mathrm{P}|_{\mathfrak{S}_0}$*-integrable function function* $f : \Omega \to \mathbb{R}$ *iff the properties* (M1)–(M7) *hold.*

Thus we can approximate any integral involving $\mathrm{P}|_{\mathfrak{S}_0}$-integrable functions.

To put the method into a framework of discrete limit spaces, we need to define corresponding connection operators. We define $\mathfrak{p}_k : L_\infty(\Omega,\mathrm{P}) \to \ell^k_\infty(\mathrm{P}_k)$ in the following way:

$$[\mathfrak{p}_k(s(\cdot))](\omega^k_l) := \mathrm{P}(A^k_l)^{-1}\int_{A^k_l} s(\omega)\,\mathrm{P}(d\omega).$$

Then we define operators $\mathfrak{q}_k : L_1(\Omega,\mathrm{P}) \to \ell^k_1(\mathrm{P}_k)$ on the everywhere dense subset $S_0 \subset L_1(\Omega,\mathrm{P})$ of $\mathfrak{S}_0$-measurable functions by:

$$[\mathfrak{q}_k(u(\cdot))](\omega^k_l) := u(\omega^k_l),$$

and extend them onto the whole space $L_1(\Omega,\mathrm{P})$ by continuity. For the proof of the norm-consistency and the properties of the connection operators, the reader is referred to [Lep88, Lep90].

3.A.3 Auxiliary results

In this section we adapt two auxiliary results due to Lepp [Lep90] to our notation. First we restate some assumptions for further reference.

Consider a function $f : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}$ and suppose that the following assumptions hold:

(A3) The function $f(\cdot,\cdot,\omega)$ is convex and differentiable for each $\omega \in \Omega$. Suppose further that $f(\boldsymbol{x},\boldsymbol{s},\cdot)$ is $\mathfrak{S}_0$-measurable and bounded for each $(\boldsymbol{x},\boldsymbol{s}) \in \mathbb{R}^n \times \mathbb{R}^m$. Moreover, for each bounded set $B \subset \mathbb{R}^n \times \mathbb{R}^m$ there corresponds a constant $\beta \in \mathbb{R}$ such, that $|f(\boldsymbol{x},\boldsymbol{s},\omega)| \leq \beta$ for all $(\boldsymbol{x},\boldsymbol{s}) \in B$.

(A4) The functions $f'_{\boldsymbol{x}}(\cdot,\cdot,\omega)$, $f'_{\boldsymbol{s}}(\cdot,\cdot,\omega)$ are continuous, and the functions $f'_{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{s},\cdot)$, $f'_{\boldsymbol{s}}(\boldsymbol{x},\boldsymbol{s},\cdot)$ are bounded and $\mathfrak{S}_0$-measurable. Moreover, to each bounded set $B \subset \mathbb{R}^n \times \mathbb{R}^m$ there correspond bounded and $\mathfrak{S}_0$-measurable functions $\gamma : \Omega \to \mathbb{R}$, $\delta : \Omega \to \mathbb{R}$, such that $|f'_{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{s},\cdot)| \leq \gamma(\cdot)$, $|f'_{\boldsymbol{s}}(\boldsymbol{x},\boldsymbol{s},\cdot)| \leq \delta(\cdot)$ for all $(\boldsymbol{x},\boldsymbol{s}) \in B$.

Proposition 3.A.4 ([Lep90, Proposition 3.1]). *Suppose that a function f satisfies assumptions* (A3) *and* (A4). *Suppose further that $(\boldsymbol{x}_k, \boldsymbol{s}_k(\cdot)) \in \mathbb{R}^n \times \ell^k_\infty(\mathrm{P}_k)$ is a bounded sequence such that $f(\boldsymbol{x}_k, \boldsymbol{s}_k(\cdot), \cdot) \leq 0$, P_k-a.s. Then any weak* discrete limit point $(\boldsymbol{x}, \boldsymbol{s}(\cdot)) \in \mathbb{R}^n \times L_\infty(\mathrm{P})$ satisfies $f(\boldsymbol{x}, \boldsymbol{s}(\cdot), \cdot) \leq 0$, P-a.s.*

Proposition 3.A.5 ([Lep90, Proposition 3.2]). *Suppose that a function f satisfies assumptions* (A3) *and* (A4).

(i) *Suppose that $(\boldsymbol{x}_k, \boldsymbol{s}_k(\cdot)) \in \mathbb{R}^n \times \ell^k_\infty(\mathrm{P}_k)$ weakly* discretely converges to a limit $(\boldsymbol{x}, \boldsymbol{s}(\cdot)) \in \mathbb{R}^n \times L_\infty(\mathrm{P})$. Then the following inequality holds:*

$$\liminf_{k\to\infty} \sum_{l=1}^{k} f(\boldsymbol{x}_k, \boldsymbol{s}_k(\omega^k_l), \omega^k_l)\, \mathrm{P}_k(\omega^k_l) \geq \int_\Omega f(\boldsymbol{x}, \boldsymbol{s}(\omega), \omega)\, \mathrm{P}(d\omega).$$

(ii) *Suppose that $(\boldsymbol{x}_k, \boldsymbol{s}_k(\cdot)) \in \mathbb{R}^n \times \ell^k_\infty(\mathrm{P}_k)$ (strongly) discretely converges to a limit $(\boldsymbol{x}, \boldsymbol{s}(\cdot)) \in \mathbb{R}^n \times L_\infty(\mathrm{P})$. Then the following inequality holds:*

$$\limsup_{k\to\infty} \sum_{l=1}^{k} f(\boldsymbol{x}_k, \boldsymbol{s}_k(\omega^k_l), \omega^k_l)\, \mathrm{P}_k(\omega^k_l) \leq \int_\Omega f(\boldsymbol{x}, \boldsymbol{s}(\omega), \omega)\, \mathrm{P}(d\omega). \tag{3.3}$$

Remark 3.A.5.1. Almost the same arguments as those in the proof of Proposition 3.A.5 (ii) can be used to prove the following fact: if a function f satisfies assumptions (A3) and (A4) and $(\boldsymbol{x}_k, \boldsymbol{s}_k(\cdot)) \in \mathbb{R}^n \times \ell^k_1(\mathrm{P}_k)$ (strongly) discretely converges to a limit $(\boldsymbol{x}, \boldsymbol{s}(\cdot)) \in \mathbb{R}^n \times L_1(\mathrm{P})$ such that $\boldsymbol{s}(\cdot)$ is essentially bounded, then inequality (3.3) holds.

3.A.4 Discretization of $(\mathcal{C})_{(x,X)}$

In this section we study the convergence of the solutions to discretizations of the subproblem $(\mathcal{C})_{(\boldsymbol{x},\boldsymbol{X})}(\cdot)$ appearing as a constraint in our problems. We note that we cannot apply Theorem 3.1 in [Lep90] because the Slater condition is not necessarily fulfilled in our problem. We use Hoffman's error bound [Hof52] instead.

In the same way as the notion of discrete convergence generalizes the notion of convergence onto sequences of discrete spaces, we may generalize the notion of closedness of point-to-set mappings. The following two propositions assert such "discrete closedness" and "discrete continuity" of the mapping $(\boldsymbol{x}, \boldsymbol{X}) \to \{\text{ solution of } (\mathcal{C})_{(\boldsymbol{x},\boldsymbol{X})}(\cdot)\,\}$ on a set $(\boldsymbol{x}, \boldsymbol{X}) \geq \varepsilon \mathbf{1}_{m+r_2}$, for fixed $\varepsilon > 0$, with respect to weak* discrete convergence.

Proposition 3.A.6. *Suppose that the assumptions* (M1)–(M7) *and* (D1)–(D3) *(cf. Section 3.3) hold. Consider an arbitrary sequence of positive designs* $(\boldsymbol{x}_k, \boldsymbol{X}_k) \geq \varepsilon \mathbf{1}_{m+r_2}$ *converging to* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}})$. *Suppose that the sequence* $\{(\boldsymbol{s}_k^*(\cdot), \boldsymbol{S}_k^*(\cdot), \boldsymbol{\lambda}_k^*(\cdot))\}$ *of solutions to* $(\mathcal{C})^k_{(\boldsymbol{x}_k,\boldsymbol{X}_k)}(\cdot)$ *weakly* discretely converges to* $(\bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot), \bar{\boldsymbol{\lambda}}(\cdot))$. *Suppose further that the optimal solution to the problem* $(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}})}(\cdot)$ *exists. Then* $(\bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot), \bar{\boldsymbol{\lambda}}(\cdot))$ *is the solution of* $(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}})}(\cdot)$.

Proof. The proof is in three steps. First, we need to establish the feasibility of the limit $(\bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot), \bar{\boldsymbol{\lambda}}(\cdot))$ in the problem $(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}})}(\cdot)$. Then we show that the objective value at the limit point is no greater than the lower limit of the sequence of optimal values of discrete problems. Finally, we prove that any feasible solution is a strong discrete limit of the feasible solutions to discrete problems, thus showing that the limit point is indeed optimal.

Assumptions (D1)–(D3) imply assumptions (A3) and (A4) for the constraints. Thus we can apply Proposition 3.A.4 to verify the feasibility of $(\bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot), \bar{\boldsymbol{\lambda}}(\cdot))$ in the problem $(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}})}(\cdot)$.

Denote the optimal solution to the limit problem $(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}})}(\cdot)$ by $(\boldsymbol{s}^*(\cdot), \boldsymbol{S}^*(\cdot), \boldsymbol{\lambda}^*(\cdot))$. Since the inequality $0 \leq \mathcal{E}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}, \boldsymbol{s}^*(\cdot), \boldsymbol{S}^*(\cdot), \boldsymbol{\lambda}^*(\cdot), \cdot) \leq \mathcal{E}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}, \bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot), \bar{\boldsymbol{\lambda}}(\cdot), \cdot)$ holds and the latter function is essentially bounded, so is the optimal value of the problem $(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}})}(\cdot)$. For each $\mathfrak{S}_0$-measurable set D consider the problem $(\mathcal{C})_{(x,X;D)}$:

$$
\begin{cases}
\displaystyle\min_{(\boldsymbol{s}(\cdot),\boldsymbol{S}(\cdot),\boldsymbol{\lambda}(\cdot))} c^f_D(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{s}(\cdot), \boldsymbol{S}(\cdot), \boldsymbol{\lambda}(\cdot)) := \\
\qquad\qquad \displaystyle\int_\Omega \chi_D(\omega)\mathcal{E}(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{s}(\boldsymbol{\omega}), \boldsymbol{S}(\omega), \boldsymbol{\lambda}(\omega), \omega)\, \mathrm{P}(d\omega) \\
\text{s.t.} \begin{cases}
\boldsymbol{C}_1^T \boldsymbol{\lambda}(\omega) + \displaystyle\sum_{i=1}^m \boldsymbol{B}_i^T \boldsymbol{s}_i(\omega) + \sum_{j=1}^{r_2} \boldsymbol{S}_j(\omega)\boldsymbol{\gamma}_j = \boldsymbol{f}(\omega), & \text{P-a.s.,} \\
\boldsymbol{\lambda}(\omega) \geq 0, & \text{P-a.s.,} \\
\boldsymbol{S}(\omega) \geq 0, & \text{P-a.s.}
\end{cases}
\end{cases}
$$

Owing to the integrability of the objective value $\mathcal{E}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}, \boldsymbol{s}^*(\cdot), \boldsymbol{S}^*(\cdot), \boldsymbol{\lambda}^*(\cdot), \cdot)$, $(\boldsymbol{s}^*(\cdot), \boldsymbol{S}^*(\cdot), \boldsymbol{\lambda}^*(\cdot))$ solves the problem $(\mathcal{C})_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}; D)}$ for any $D \in \mathfrak{S}_0$. This property also holds for the discretizations $(\mathcal{C})^k_{(\boldsymbol{x}, \boldsymbol{X})}(\cdot)$ and $(\mathcal{C})^k_{(\boldsymbol{x}, \boldsymbol{X}; D)}$ (the latter is defined in a similar way). On the other hand, the optimal value of the problem $(\mathcal{C})_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}})}(\cdot)$ is measurable with respect to the completion of the σ-algebra, generated by $\mathfrak{S}_0$ [CaV77, Lemma III.39]. Thus if $(\boldsymbol{s}^*(\cdot), \boldsymbol{S}^*(\cdot), \boldsymbol{\lambda}^*(\cdot))$ solves $(\mathcal{C})_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}; D)}$ for each $\mathfrak{S}_0$-measurable D then it solves $(\mathcal{C})_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}})}(\cdot)$ as well.

Thus we assume that $(\boldsymbol{s}^*_k(\cdot), \boldsymbol{S}^*_k(\cdot), \boldsymbol{\lambda}^*_k(\cdot))$ solves the problem $(\mathcal{C})^k_{(\boldsymbol{x}_k, \boldsymbol{X}_k; D)}$, and we will prove that $(\bar{\boldsymbol{s}}(\cdot), \bar{\boldsymbol{S}}(\cdot), \bar{\boldsymbol{\lambda}}(\cdot))$ solves $(\mathcal{C})^k_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}; D)}$ for a fixed, but arbitrary $\mathfrak{S}_0$-measurable D.

Assumptions (D1), (D2), $\mathfrak{S}_0$-measurability of D and inequality $(\boldsymbol{x}_k, \boldsymbol{X}_k) \geq \varepsilon \mathbf{1}_{m+r_2}$ (and thus $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}) \geq \varepsilon \mathbf{1}_{m+r_2}$) allow us to invoke Proposition 3.A.5(i) to conclude:

$$
\begin{aligned}
&\operatorname{val}(\mathcal{C})_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}; D)} \\
&\leq \int_\Omega \chi_D(\omega) \mathcal{E}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}, \bar{\boldsymbol{s}}(\omega), \bar{\boldsymbol{S}}(\omega), \bar{\boldsymbol{\lambda}}(\omega), \omega) \, \mathrm{P}(d\omega) \\
&\leq \liminf_{k\to\infty} \sum_{l=1}^{k} \chi_D(\omega_l^k) \mathcal{E}(\boldsymbol{x}_k, \boldsymbol{X}_k, \\
&\qquad\qquad \boldsymbol{s}_k(\omega_l^k), \boldsymbol{S}_k(\omega_l^k), \boldsymbol{\lambda}_k(\omega_l^k), \omega_l^k) \, \mathrm{P}_k(\omega_l^k) \\
&= \liminf_{k\to\infty} \operatorname{val}(\mathcal{C})^k_{(\boldsymbol{x}_k, \boldsymbol{X}_k; D)}.
\end{aligned} \tag{3.4}
$$

Now, let $(\widetilde{\boldsymbol{s}}_k(\omega_l^k), \widetilde{\boldsymbol{S}}_k(\omega_l^k), \widetilde{\boldsymbol{\lambda}}_k(\omega_l^k))$ be a projection of $\mathfrak{p}_k\{(\widetilde{\boldsymbol{s}}(\cdot), \widetilde{\boldsymbol{S}}(\cdot), \widetilde{\boldsymbol{\lambda}}(\cdot))\}(\omega_l^k)$ onto the set:

$$
\begin{aligned}
\{ (\boldsymbol{s}, \boldsymbol{S}, \boldsymbol{\lambda}) \in \mathbb{R}^{m+r_2+r_1} \mid \boldsymbol{C}_1^T \boldsymbol{\lambda} + \sum_{i=1}^{m} \boldsymbol{B}_i^T \boldsymbol{s}_i + \sum_{j=1}^{r_2} \boldsymbol{S}_j \boldsymbol{\gamma}_j = \boldsymbol{f}(\omega_l^k), \\
\boldsymbol{S} \geq 0, \boldsymbol{\lambda} \geq 0 \},
\end{aligned}
$$

for an arbitrary point $(\widetilde{\boldsymbol{s}}(\cdot), \widetilde{\boldsymbol{S}}(\cdot), \widetilde{\boldsymbol{\lambda}}(\cdot))$ that is feasible in $(\mathcal{C})_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}}; D)}$. Then $(\widetilde{\boldsymbol{s}}_k(\omega_l^k), \widetilde{\boldsymbol{S}}_k(\omega_l^k), \widetilde{\boldsymbol{\lambda}}_k(\omega_l^k))$ is feasible in $(\mathcal{C})^k_{(\boldsymbol{x}_k, \boldsymbol{X}_k; D)}$. Since assumption (D3) holds we can use Hoffman's error bound for linear systems [Hof52] to obtain the following estimation:

$$
\begin{aligned}
\|(\widetilde{\boldsymbol{s}}_k(\omega_l^k), \widetilde{\boldsymbol{S}}_k(\omega_l^k), \widetilde{\boldsymbol{\lambda}}_k(\omega_l^k)) - \mathfrak{p}_k\{(\widetilde{\boldsymbol{s}}_k(\cdot), \widetilde{\boldsymbol{S}}_k(\cdot), \widetilde{\boldsymbol{\lambda}}_k(\cdot))\}(\omega_l^k)\| \\
\leq K \|\boldsymbol{f}(\omega_l^k) - \mathfrak{p}_k\{\boldsymbol{f}(\cdot)\}(\omega_l^k)\|,
\end{aligned}
$$

for some constant $K > 0$ independent from k, l and $\boldsymbol{f}$. Both sequences $\{\mathfrak{p}_k(\boldsymbol{f})\}$ and $\{\mathfrak{q}_k(\boldsymbol{f})\}$ strongly discretely converge to $\boldsymbol{f}$ in discrete L_1-

sense [Vai78, p. 649], and thus we can establish the strong discrete convergence of $\{(\widetilde{\boldsymbol{s}}_k(\cdot), \widetilde{\boldsymbol{S}}_k(\cdot), \widetilde{\boldsymbol{\lambda}}_k(\cdot))\}$ to $(\widetilde{\boldsymbol{s}}_k(\cdot), \widetilde{\boldsymbol{S}}_k(\cdot), \widetilde{\lambda}_k(\cdot))$:

$$\lim_{k\to\infty} \|(\widetilde{\boldsymbol{s}}_k(\cdot), \widetilde{\boldsymbol{S}}_k(\cdot), \widetilde{\boldsymbol{\lambda}}_k(\cdot)) - \mathfrak{p}_k(\widetilde{\boldsymbol{s}}_k(\cdot), \widetilde{\boldsymbol{S}}_k(\cdot), \widetilde{\boldsymbol{\lambda}}_k(\cdot))\|_{\ell_1^k} \leq \lim_{k\to\infty} K\|\mathfrak{q}_k(\boldsymbol{f}(\cdot)) - \mathfrak{p}_k(\boldsymbol{f}(\cdot))\|_{\ell_1^k} = 0.$$

Then, we use Remark 3.A.5.1 to get:

$$\begin{aligned}
&\mathrm{val}(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}};D)} \\
&\geq \limsup_{k\to\infty} \sum_{l=1}^{k} \chi_D(\omega_l^k)\mathcal{E}(\boldsymbol{x}_k, \boldsymbol{X}_k, \\
&\qquad \widetilde{\boldsymbol{s}}_k(\omega_l^k), \widetilde{\boldsymbol{S}}_k(\omega_l^k), \widetilde{\boldsymbol{\lambda}}_k(\omega_l^k), \omega_l^k)\,\mathrm{P}_k(\omega_l^k) \\
&\geq \limsup_{k\to\infty} \mathrm{val}(\mathcal{C})^k_{(\boldsymbol{x}_k,\boldsymbol{X}_k;D)},
\end{aligned}$$

which together with (3.4) completes the proof. #

Proposition 3.A.7. *Suppose that the assumptions* (M1)–(M7) *and* (D1)–(D4) *hold. Consider an arbitrary sequence of positive designs* $(\boldsymbol{x}_k, \boldsymbol{X}_k) \geq \varepsilon\mathbf{1}_{m+r_2}$ *converging to* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{X}})$*, and suppose that the problems* $(\mathcal{C})^k_{(\boldsymbol{x}_k,\boldsymbol{X}_k)}(\cdot)$ *and* $(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}})}(\cdot)$ *are feasible. Suppose further that the optimal value of the problem* $(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}})}(\cdot)$ *is essentially bounded.*

Then the sequence of solutions $(\boldsymbol{s}_k^*(\cdot), \boldsymbol{S}_k^*(\cdot), \boldsymbol{\lambda}_k^*(\cdot))$ *to* $(\mathcal{C})^k_{(\boldsymbol{x}_k,\boldsymbol{X}_k)}$ *strongly discretely converges to the solution* $(\boldsymbol{s}^*(\cdot), \boldsymbol{S}^*(\cdot), \boldsymbol{\lambda}^*(\cdot))$ *of the problem* $(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}})}(\cdot)$.

Proof. The additional assumption (D4) implies the strong convexity of the objective function with respect to $(\boldsymbol{s}, \boldsymbol{S})$ locally uniformly with respect to $(\boldsymbol{x}, \boldsymbol{X})$. Since $\boldsymbol{\lambda}$ depends linearly on $(\boldsymbol{s}, \boldsymbol{S})$ (cf. (3.1)) we may assume the strong convexity of $\mathcal{E}$ with respect to state variables on a feasible set locally uniformly with respect to $(\boldsymbol{x}, \boldsymbol{X})$. According to [Lep94, Remark 2] under the strong convexity of the objective function weak* discrete convergence of the solutions to the problems $(\mathcal{C})^k_{(\boldsymbol{x}_k,\boldsymbol{X}_k)}(\cdot)$ implies strong discrete convergence. Since the solution to the problem $(\mathcal{C})_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{X}})}(\cdot)$ is unique, the weak* discrete compactness of the sequence $\{(\boldsymbol{s}_k^*(\cdot), \boldsymbol{S}_k^*(\cdot), \boldsymbol{\lambda}_k^*(\cdot))\}$ and Proposition 3.A.6 imply that the sequence $\{(\boldsymbol{s}_k^*(\cdot), \boldsymbol{S}_k^*(\cdot), \boldsymbol{\lambda}_k^*(\cdot))\}$ weakly* discretely converges to $(\boldsymbol{s}^*(\cdot), \boldsymbol{S}^*(\cdot), \boldsymbol{\lambda}^*(\cdot))$. Since weak* discrete compactness follows from boundedness [Vai78, Proposition 10], it is sufficient to show the boundedness of $\{(\boldsymbol{s}_k^*(\cdot), \boldsymbol{S}_k^*(\cdot), \boldsymbol{\lambda}_k^*(\cdot))\}$.

From the proof of Proposition 3.A.6 we know that there exists a sequence $\{(\boldsymbol{s}_k(\cdot), \boldsymbol{S}_k(\cdot), \boldsymbol{\lambda}_k(\cdot))\}$ feasible in $\{(\mathcal{C})^k_{(\boldsymbol{x}_k,\boldsymbol{X}_k)}(\cdot)\}$, and strongly discretely converging to $(\boldsymbol{s}^*(\cdot), \boldsymbol{S}^*(\cdot), \boldsymbol{\lambda}^*(\cdot))$ in L_1-sense. Owing to the optimality of $(\boldsymbol{s}_k^*(\cdot), \boldsymbol{S}_k^*(\cdot), \boldsymbol{\lambda}_k^*(\cdot))$ in $(\mathcal{C})^k_{(\boldsymbol{x}_k, \boldsymbol{X}_k)}(\cdot)$ the inequality $\mathcal{E}(\boldsymbol{x}_k, \boldsymbol{X}_k, \boldsymbol{s}_k^*(\cdot), \boldsymbol{S}_k^*(\cdot), \boldsymbol{\lambda}_k^*(\cdot), \cdot) \leq \mathcal{E}(\boldsymbol{x}_k, \boldsymbol{X}_k, \boldsymbol{s}_k(\cdot), \boldsymbol{S}_k(\cdot), \boldsymbol{\lambda}_k(\cdot), \cdot)$ holds. For any $\mathfrak{S}_0$-measurable set D according to Remark 3.A.5.1 we then have:

$$\begin{aligned}\limsup_{k\to\infty} \sum_{i=1}^{k} \chi_D(\omega_l^k)\, \mathcal{E}(\boldsymbol{x}_k, \boldsymbol{X}_k,&\\ \boldsymbol{s}_k(\omega_l^k), \boldsymbol{S}_k(\omega_l^k), \boldsymbol{\lambda}_k(\omega_l^k), \omega_l^k)\mathrm{P}_k(\omega_l^k)&\\ \leq \int_\Omega \chi_D(\omega)\mathcal{E}(\boldsymbol{x}^*, \boldsymbol{X}^*, \boldsymbol{s}^*(\omega), \boldsymbol{S}^*(\omega), \boldsymbol{\lambda}^*(\omega), \omega)\, \mathrm{P}(d\omega),&\end{aligned}$$

from which we deduce that the sequence $\{\mathcal{E}(\boldsymbol{x}_k, \boldsymbol{X}_k, \boldsymbol{s}_k(\cdot), \boldsymbol{S}_k(\cdot), \boldsymbol{\lambda}_k(\cdot), \cdot)\}$ is bounded. Since $\mathcal{E}$ is weakly coercive in $(\boldsymbol{s}, \boldsymbol{S}, \boldsymbol{\lambda})$ locally uniformly w.r.t. $(\boldsymbol{x}, \boldsymbol{X})$ on a set $\{\, (x, X) \geq \varepsilon \mathbf{1}_{m+r_2} \,\}$, and always nonnegative, the boundedness of $(\boldsymbol{s}_k^*(\cdot), \boldsymbol{S}_k^*(\cdot), \boldsymbol{\lambda}_k^*(\cdot))$ follows. #

3.A.5 Discretization of $(\mathcal{P}_1)$

Proof of Theorem 3.3.1. Applying Proposition 3.A.7 to the sequence $\{(\boldsymbol{x}^*, \boldsymbol{X}^*, \boldsymbol{s}_k(\cdot), \boldsymbol{S}_k(\cdot), \boldsymbol{\lambda}_k(\cdot))\}$, where $(\boldsymbol{s}_k(\cdot), \boldsymbol{S}_k(\cdot), \boldsymbol{\lambda}_k(\cdot))$ solves $(\mathcal{C})^k_{(\boldsymbol{x}^*, \boldsymbol{X}^*)}(\cdot)$, and then using Proposition 3.A.5(ii), we obtain the following inequality:

$$\begin{aligned}&\limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_1^\varepsilon)^k\\ &\leq \limsup_{k\to\infty} \sum_{l=1}^{k} \mathcal{E}(\boldsymbol{x}^*, \boldsymbol{X}^*, \boldsymbol{s}_k(\omega_l^k), \boldsymbol{S}_k(\omega_l^k), \boldsymbol{\lambda}_k(\omega_l^k), \omega_l^k)\, \mathrm{P}(\omega_l^k)\\ &\leq \mathrm{val}(\mathcal{P}_1^\varepsilon).\end{aligned} \tag{3.5}$$

We may assume that the original sequence weakly* discretely converges to a limit $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{s}}(\cdot), \widetilde{\boldsymbol{S}}(\cdot))$. Its feasibility is implied by Proposition 3.A.6 and the continuity of the design constraints. From Proposition 3.A.5(i) we get the reverse inequality:

$$\begin{aligned}&\mathrm{val}(\mathcal{P}_1^\varepsilon) \leq c^f(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{s}}(\cdot), \widetilde{\boldsymbol{S}}(\cdot), \widetilde{\boldsymbol{\lambda}}(\cdot))\\ &\leq \liminf_{k\to\infty} \sum_{l=1}^{k} \mathcal{E}(\boldsymbol{x}_k^*, \boldsymbol{X}_k^*, \boldsymbol{s}_k^*(\omega_l^k), \boldsymbol{S}_k^*(\omega_l^k), \boldsymbol{\lambda}_k^*(\omega_l^k), \omega_l^k)\, \mathrm{P}(\omega_l^k)\\ &= \liminf_{k\to\infty} \mathrm{val}(\mathcal{P}_1^\varepsilon)^k,\end{aligned}$$

which, together with (3.5), completes the proof. #

3.A.6 Discretization of $(\mathcal{P}_2)$

Proof of Theorem 3.3.2. The proof is similar to the one of Theorem 3.3.1 and is therefore omitted. #

3.A.7 Existence of weak* limit points

To establish the convergence of the discretization scheme it remains to establish the existence of weak* discrete limit points for the sequence of solutions to discretized problems. A sufficient condition for such an existence is the boundedness of the sequence.

Proposition 3.A.8. *Suppose that the assumptions* (M1)–(M7) *and* (D1)–(D4) *hold. Suppose further that the functions $E(\cdot)$, $E_c(\cdot)$, and $\boldsymbol{L}(\cdot)$ are independent from ω. Let $\{(\boldsymbol{x}_k, \boldsymbol{X}_k)\}$ be a bounded sequence of designs such that $(\boldsymbol{x}_k, \boldsymbol{X}_k) \geq \varepsilon \mathbf{1}_{m+r_2}$ for some $\varepsilon > 0$ and all k. Suppose that the problem $(\mathcal{C})_{(\boldsymbol{x},\boldsymbol{X})}(\omega)$ is feasible for any $\omega \in \Omega$ for $(\boldsymbol{x}, \boldsymbol{X}) > 0$. Then the sequence of solutions $\{(\boldsymbol{s}_k(\cdot), \boldsymbol{S}_k(\cdot), \boldsymbol{\lambda}_k(\cdot))\}$ to the problems $\{(\mathcal{C})^k_{(\boldsymbol{x}_k,\boldsymbol{X}_k)}(\cdot)\}$ is bounded.*

Proof. Owing to [BGK+83, Theorem 5.5.2] the solution $(\widetilde{\boldsymbol{s}}_1(\cdot), \widetilde{\boldsymbol{S}}_1(\cdot), \widetilde{\boldsymbol{\lambda}}_1(\cdot))$ of $(\mathcal{C})_{(\boldsymbol{x}_1,\boldsymbol{X}_1)}(\cdot)$ is a piecewise affine function of $(\boldsymbol{f}(\cdot), \boldsymbol{g}_1(\cdot), \boldsymbol{g}_2(\cdot))$, whence it is bounded.

Then we can use the locally Lipschitz continuity result [PaP00, Theorem 3.4] to bound the sequence $\{(\widetilde{\boldsymbol{s}}_k(\cdot), \widetilde{\boldsymbol{S}}_k(\cdot), \widetilde{\boldsymbol{\lambda}}_k(\cdot))\}$ of solutions to $(\mathcal{C})_{(\boldsymbol{x}_k,\boldsymbol{X}_k)}(\cdot)$:

$$\begin{aligned}&\|(\widetilde{\boldsymbol{s}}_k(\omega), \widetilde{\boldsymbol{S}}_k(\omega), \widetilde{\boldsymbol{\lambda}}_k(\omega)) - (\widetilde{\boldsymbol{s}}_1(\omega), \widetilde{\boldsymbol{S}}_1(\omega), \widetilde{\boldsymbol{\lambda}}_1(\omega))\| \\ &\leq \tau(\omega)\|(\boldsymbol{x}_k, \boldsymbol{X}_k) - (\boldsymbol{x}_1, \boldsymbol{X}_1)\| \cdot \|(\boldsymbol{u}_k(\omega), \boldsymbol{e}_k(\omega)\|,\end{aligned}$$

and thus the boundedness of $\{(\boldsymbol{s}_k(\cdot), \boldsymbol{S}_k(\cdot), \boldsymbol{\lambda}_k(\cdot))\}$ will follow if we can show the boundedness of $\tau(\omega)$ and of the sequence of Lagrange multipliers $\{(\boldsymbol{u}_k(\cdot), \boldsymbol{e}_k(\cdot))\}$ for the constraints of $(\mathcal{C})_{(\boldsymbol{x}_k,\boldsymbol{X}_k)}(\cdot)$.

For positive designs we can assume that both the optimal solutions and Lagrange multipliers for the problem $(\mathcal{C})_{(\boldsymbol{x},\boldsymbol{X})}(\cdot)$ are unique. Therefore, owing to [BGK+83, Theorem 5.5.1] they continuously depend on $(\boldsymbol{f}(\cdot), \boldsymbol{g}_1(\cdot), \boldsymbol{g}_2(\cdot))$. Proposition 3 in [LuT97] then implies that τ can be choosen independent from ω.

The boundedness of $\{(\boldsymbol{u}_k(\cdot), \boldsymbol{e}_k(\cdot))\}$ follows from [Hag79, Theorem 3.1], which asserts the Lipschitz dependence of solutions to quadratic problems on a bounded set of parameters, provided a uniform strong convexity of the objective function and a uniform linear independence constraint qualification conditions hold. We note that the assumptions of the theorem follow from the boudedness of $\{(\boldsymbol{x}_k, \boldsymbol{X}_k)\}$ and $(\boldsymbol{f}(\cdot), \boldsymbol{g}_1(\cdot), \boldsymbol{g}_2(\cdot))$, the quasi-orthogonality of $\boldsymbol{C}_1$, and the boundedness away from zero of $\{(\boldsymbol{x}_k, \boldsymbol{X}_k)\}$. #

Paper 4

STABLE RELAXATIONS OF STOCHASTIC STRESS CONSTRAINED WEIGHT MINIMIZATION PROBLEMS

Anton Evgrafov* and Michael Patriksson*

Abstract

The problem of finding a truss of minimal weight subject to stress constraints and stochastic loading conditions is considered. We demonstrate that this problem is ill-posed by showing that the optimal solutions change discontinuously as small changes in the modelling of uncertainty are introduced. We propose a relaxation of this problem, which is stable with respect to such errors. We establish a classic ε-perturbation result for the relaxed problem, and propose a solution scheme based on discretizations of the probability measure. Using Chebyshev's inequality we give an a priori estimation of the probability of stress constraint violations in terms of the relaxation parameter. The convergence of the relaxed optimal designs towards the original (non-relaxed) optimal designs as the relaxation parameter decreases to zero is established.

Key words: Stochastic programming, robust optimization, ε-perturbation, stress constraints, discretization

4.1 Introduction

We consider the problem of finding a truss of minimal weight subject to stress constraints and stochastic loading conditions. The reason for introducing the stochasticity into the problem is that uncertainty due to loading conditions unknown in advance has to be taken into account to obtain robust optimal solutions. On the other hand, Evgrafov et al. [EPP02] showed that optimal solutions to stochastic stress constrained

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weight minimization problem change discontinuously as small changes in the modelling of uncertainty are introduced. Therefore, the optimal solutions are not robust with respect to modelling errors, and their quality is very hard to estimate.

In this paper we impose the stress constraints in a relaxed manner, which makes the weight minimization problem stable with respect to changes in probability measure. By adjusting the relaxation parameter one can ensure that stress constrains are noticeably violated with an arbitrarily small probability, and that the relaxed optimal designs are close to the original (non-relaxed) optimal designs.

Given positions of the nodes the *design* (and topology in particular) of a truss can be described by *design* variables $x_i \geq 0$, $i = 1, \dots, m$, representing the volume of material, allocated to the bar i in the structure. We introduce an index set $\mathcal{I}(\boldsymbol{x}) = \{\, i = 1, \dots, m \mid x_i > 0 \,\}$ of the present (or active) members in the structure.

Let $(\Omega, \mathfrak{S}, \mathrm{P})$ be a complete probability space. The stochastic stress constrained minimization problem can be formulated as follows:

$$(\mathcal{P}_2) \begin{cases} \min\limits_{(\boldsymbol{x}, \boldsymbol{s}(\cdot))} w(\boldsymbol{x}) := \mathbf{1}_m^T \boldsymbol{x} \\ \text{s.t.} \begin{cases} \underline{\boldsymbol{x}} \leq \boldsymbol{x}, \\ |s_i(\omega)| \leq \overline{\sigma}_1 x_i, \qquad i = 1, \dots, m, \qquad \text{P-a.s.}, \\ \boldsymbol{s}(\omega) \text{ solves } (\mathcal{C})_{\boldsymbol{x}}(\omega), \qquad \text{P-a.s.}, \end{cases} \end{cases}$$

where the minimization problem $(\mathcal{C})_{\boldsymbol{x}}(\omega)$ is the principle of minimum complementary energy:

$$(\mathcal{C})_{\boldsymbol{x}}(\omega) \begin{cases} \min\limits_{\boldsymbol{s}} \mathcal{E}(\boldsymbol{x}, \boldsymbol{s}) := \frac{1}{2} \sum\limits_{i=1}^{m} \frac{s_i^2}{E x_i}, \\ \text{s.t.} \left\{ \sum\limits_{i \in \mathcal{I}(\boldsymbol{x})} \boldsymbol{B}_i^T s_i = \boldsymbol{f}(\omega). \right. \end{cases}$$

The data in the problem has the following meaning from a mechanical point of view:

- E is the Young's modulus for the structure material;
- $\boldsymbol{B}_i$ is the kinematic transformation matrix for the bar i;
- $\boldsymbol{f}(\omega)$ is the vector of external forces.

For the problem to be tractable we assume that the function $\boldsymbol{f}(\cdot)$ is $\mathfrak{S}$-measurable. The variable s_i in the problem $(\mathcal{C})_{\boldsymbol{x}}(\omega)$ is the tensile force in the bar i times its length.

4.2 A (short) quest for a correct relaxation

One does not need a specially constructed example to demonstrate the discontinuity of optimal solutions to the stress constrained weight minimization problem. The problem instance below is probably the simplest example one could imagine.

Example 4.2.1 (One-bar truss). Figure 4.1 shows a simple one-dimensional structure that consists of a single bar.

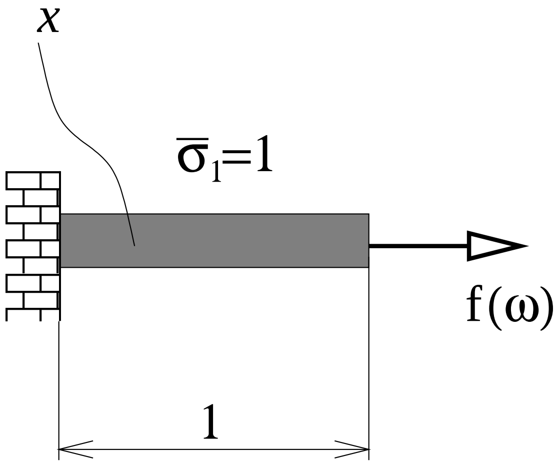

Fig. 4.1: The one-bar truss.

Suppose that $\Omega = [0, 2]$, $\boldsymbol{f}(\omega) = \omega$, $\overline{\sigma}_1 = 1$, and $\underline{\boldsymbol{x}} = 0$. Let $\mathrm{P}^{(1)}$ be a uniform distribution on $[0, 1]$, $\mathrm{P}^{(2)}$ be a uniform distribution on $[1, 2]$, and $\mathrm{P}_k = (k-1)/k\mathrm{P}^{(1)} + 1/k\mathrm{P}^{(2)}$. The sequence $\{\mathrm{P}_k\}$ weakly converges to $\mathrm{P} = \mathrm{P}^{(1)}$, and each measure possesses a density.

The structure is statically determinable, thus the force $\boldsymbol{s}(\omega)$ is independent from $\boldsymbol{x}$ and equals to $\boldsymbol{f}(\omega)$. The optimal solution $\boldsymbol{x}^*$ of $(\mathcal{P}_2)$ equals 1, while each optimal solution $\boldsymbol{x}^*_k$ of $(\mathcal{P}_2)^k$ equals 2. Therefore, the sequence $\{\boldsymbol{x}^*_k\}$ does not converge to $\boldsymbol{x}^*$ as k goes to infinity as one wants.

It is difficult to imagine the existence of any mild conditions under which the stochastic stress constrained weight minimization problem is stable, when it is unstable even for the extremely simple structure of Example 4.2.1. Thus, it is reasonable to construct a relaxation of the problem, having the following properties:

(i) it is possible to recover a solution to the original problem as a limit point of the solutions to the relaxed problems as a relaxation parameter goes to 0;

(ii) the relaxed problem is stable with respect to changes in the probability measure;
(iii) it is possible to estimate the violation of the relaxed constraints; and
(iv) it is possible to numerically solve the relaxed problem.

One straightforward approach, which obviously satisfies the requirement (iii), is to choose a relaxation parameter $\delta > 0$ and to require that $\mathrm{P}(|s_i(\omega)| \leq \overline{\sigma}_1 x_i + \delta) = 1$, $i = 1, \ldots, m$. This approach is used when sizing approximations to the deterministic case of the problem $(\mathcal{P}_2)$ are considered [ChG97, Pet01]. To show why such a relaxation of the problem is not enough, we consider the following example.

Example 4.2.2 (Two-bar truss). Figure 4.2 shows a simple structure that consists of two bars.

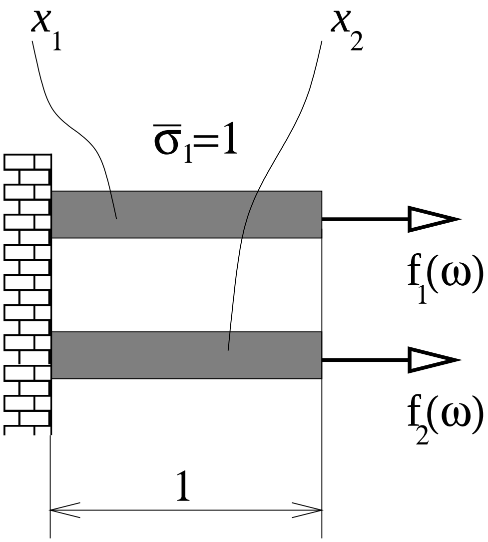

Fig. 4.2: The two-bar truss.

Suppose that $m = 2$, $\Omega = [0, 2]$, $f_2(\omega) = \omega - 1$, $\overline{\sigma}_1 = 1$, $\underline{\boldsymbol{x}} = \mathbf{0}$, and

$$f_1(\omega) = \begin{cases} \omega, & \text{if } 0 \leq \omega \leq 0.5, \\ 1 - \omega, & \text{if } 0.5 < \omega \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathrm{P}_k^{(1)}$ be a uniform distribution on $[0, 1/k]$, $\mathrm{P}^{(2)}$ be a uniform distribution on $[1, 2]$, and $\mathrm{P}_k = 1/k^2 \mathrm{P}^{(1)} + (k^2 - 1)/k^2 \mathrm{P}^{(2)}$. The sequence $\{\mathrm{P}_k\}$ weakly converges to $\mathrm{P} = \mathrm{P}^{(2)}$ and each measure possesses a density.

As before, the force vector $\boldsymbol{s}(\omega)$ is independent from the design and equals $\boldsymbol{f}(\omega)$. The optimal solution to the non-relaxed problem $(\mathcal{P}_2)^k$ is $\boldsymbol{x}_k^* = (1/k, 1)^T$; thus the sequence of solutions $\{\boldsymbol{x}_k^*\}$ converges to the optimal solution $\boldsymbol{x}^* = (0, 1)^T$ of the non-relaxed problem $(\mathcal{P}_2)$ as k goes to infinity for this example.

For any "small" $\delta > 0$ the optimal solution of the relaxed problem $(\mathcal{P}_2^\delta)$ exists and equals $\boldsymbol{x}^\delta = (0, 1-\delta)^T$. On the other hand, for $k > 1/\delta$ the feasible design space of the problem $(\mathcal{P}_2^\delta)^k$ is $(0,\infty) \times [1-\delta, \infty)$, and the objective function $w(\boldsymbol{x})$ does not attain its infimum on this set. Therefore, there is no optimal solution to the relaxed problem $(\mathcal{P}_2^\delta)^k$!

Example 4.2.2 clearly shows that the requirement (ii) is violated by the "straightforward" relaxation of stress constraints.

To introduce the "correct" relaxation scheme, for positive designs $\boldsymbol{x}$, we consider a convex, non-negative and differentiable function, which was used Evgrafov and Patriksson [EvP02] to construct a penalty function for the stress constrained weight minimization problem:

$$G(\boldsymbol{x}, \boldsymbol{s}) := \sum_{i=1}^{m} \frac{[|s_i| - \overline{\sigma}_1 x_i]_+^2}{x_i}.$$

Using the usual convention $0/0 = 0$ and $a/0 = \infty$ for any $a > 0$, the function G can be evaluated at any non-negative design $\boldsymbol{x}$, and, furthermore, it is l.s.c. on $\mathbb{R}_+^m \times \mathbb{R}^m$.

Now, for a positive relaxation parameter $\delta > 0$ consider the following minimization problem:

$$(\mathcal{P}_2^\delta)\begin{cases} \min\limits_{(\boldsymbol{x},\boldsymbol{s}(\cdot))} \; w(\boldsymbol{x}) \\ \text{s.t.} \begin{cases} \underline{\boldsymbol{x}} \le \boldsymbol{x}, \\ \displaystyle\int_\Omega G(\boldsymbol{x}, \boldsymbol{s}(\omega))\,\mathrm{P}(d\omega) \le \delta, \\ \boldsymbol{s}(\omega) \text{ solves } (\mathcal{C})_{\boldsymbol{x}}(\omega), \quad \text{P-a.s.} \end{cases} \end{cases}$$

Owing to the measurability of the solutions to $(\mathcal{C})_{\boldsymbol{x}}(\cdot)$ (cf. [EPP02, Corollary 2.2]), the problem $(\mathcal{P}_2^\delta)$ is indeed a relaxation of $(\mathcal{P}_2)$ (in the sense that the feasible set of the former problem contains that of the latter), and $(\mathcal{P}_2^0) = (\mathcal{P}_2)$.

Furthermore, owing to Chebyshev's inequality, for any $c > 0$ the following inequality holds:

$$\mathrm{P}\big(|s_i(\omega)| \ge \overline{\sigma}_1 x_i + c\big) \le \frac{\delta x_i}{c^2}, \tag{4.1}$$

i.e., by choosing a small δ the probability of violating any stress constraint can be made arbitrarily small. Therefore, the proposed relaxation satisfies the requirement (iii).

The rest of the paper is organised as follows. In Section 4.3 we investigate the properties of the feasible set of the problem $(\mathcal{P}_2^\delta)$, and show that it satisfies a Slater-type constraint qualification. Section 4.4 addresses the existence of solutions for the problem. In Sections 4.5, 4.6 and 4.7 we show that the problem $(\mathcal{P}_2^\delta)$ possesses the properties we listed; in particular, Theorem 4.5.1 verifies the property (i), and Theorem 4.6.3 addresses the stability requirement (ii). Using Theorems 4.5.2 and 4.7.1 we can approximate the problem with a sequence of simple differentiable and finite-dimensional subproblems; this gives us the property (iv). At last, we illustrate the theory with a numerical example in Section 4.8.

4.3 Auxiliary results

In this section we collect auxiliary results necessary for the following development.

The lemma below asserts the continuity of the mapping $\boldsymbol{x} \to \boldsymbol{s}(\cdot)$, where $\boldsymbol{s}(\cdot)$ solves $(\mathcal{C})_{\boldsymbol{x}}(\cdot)$, restricted to the feasible set of the problem $(\mathcal{P}_2^\delta)$. It is an important part of the proof of existence of solutions to $(\mathcal{P}_2^\delta)$, as it enables us to choose a feasible state corresponding to the limit of the design variables.

Lemma 4.3.1. *Suppose that the sequence $\{(\boldsymbol{x}_k, \boldsymbol{s}_k(\cdot))\}$ to $(\mathcal{P}_2^\delta)$ has design components converging to a limit $\boldsymbol{x}_0$. Then the sequence of state variables* P*-a.s. converges to a limit $\boldsymbol{s}_0(\cdot)$ solving $(\mathcal{C})_{\boldsymbol{x}_0}(\cdot)$ as k goes to infinity.*

Proof. The sequence of designs is bounded, so we can use [EvP02, Lemma 2.2] to conclude that the sequence of energy estimations $\{\int_\Omega \mathcal{E}(\boldsymbol{x}_k, \boldsymbol{s}_k(\omega))\, \mathrm{P}(d\omega)\}$ is bounded. Now the claim follows from [EPP02, Proposition 2.3]. #

The following lemma is the crucial technical tool. It shows that a Slater-type constraint qualification holds for the relaxed stress constraints.

Lemma 4.3.2. *Suppose that $(\boldsymbol{x}, \boldsymbol{s}(\cdot))$ is a solution that is feasible in $(\mathcal{P}_2^\delta)$ and is such that $\int_\Omega G(\boldsymbol{x}, \boldsymbol{s}(\cdot))\, \mathrm{P}(d\omega) > 0$. Fix an arbitrary $\varepsilon > 0$. Then it is possible to find a feasible point $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{s}}(\cdot))$ such that $\widetilde{\boldsymbol{x}} > \boldsymbol{x}$, $\|\widetilde{\boldsymbol{x}} - \boldsymbol{x}\| < \varepsilon$, and $\int_\Omega G(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{s}}(\omega))\, \mathrm{P}(d\omega) < \int_\Omega G(\boldsymbol{x}, \boldsymbol{s}(\omega))\, \mathrm{P}(d\omega)$.*

Proof. Let $\bar{\boldsymbol{x}} = (1 + \varepsilon/3)\boldsymbol{x}$, and let $\bar{\boldsymbol{s}}(\cdot)$ be the solution to $(\mathcal{C})_{\bar{\boldsymbol{x}}}(\cdot)$. Then $\bar{\boldsymbol{s}}(\cdot) = \boldsymbol{s}(\cdot)$ and, since $\mathrm{P}\{G(\boldsymbol{x}, \boldsymbol{s}(\omega)) > 0\} > 0$, there is an index i such that $x_i > 0$ and $\mathrm{P}\{[|s_i(\cdot)| - \overline{\sigma}_1 x_i]_+^2 / x_i > 0\} > 0$. The continuity of $\int_\Omega [|s_i(\omega)| - \overline{\sigma}_1 x_i]_+^2 / x_i\, \mathrm{P}(d\omega)$ w.r.t. x_i implies that $\int_\Omega G(\bar{\boldsymbol{x}}, \bar{\boldsymbol{s}}(\omega))\, \mathrm{P}(d\omega) < \int_\Omega G(\boldsymbol{x}, \boldsymbol{s}(\omega))\, \mathrm{P}(d\omega)$.

For some positive $p \geq 3$ to be determined later set $\widetilde{\boldsymbol{x}} = \bar{\boldsymbol{x}} + \varepsilon/p \cdot \mathbf{1}_m$ and let $\widetilde{\boldsymbol{s}}(\cdot)$ be the solution of $(\mathcal{C})_{\widetilde{\boldsymbol{x}}}(\cdot)$. Using the directionally Lipschitz continuous dependence of solutions to $(\mathcal{C})_{\boldsymbol{x}}(\cdot)$ on $\boldsymbol{x}$ (cf. [EPP02, Lemma 3.3]), the continuity of $[|\widetilde{s}_i| - \overline{\sigma}_1\widetilde{x}_i]_+^2/\widetilde{x}_i$ for i such that $\bar{x}_i > 0$, and the inequality

$$\frac{[|\widetilde{s}_i(\cdot)| - \overline{\sigma}_1\widetilde{x}_i]_+^2}{\widetilde{x}_i} \leq \frac{p(\tau + \overline{\sigma}_1)^2\|\widetilde{\boldsymbol{x}} - \bar{\boldsymbol{x}}\|^2}{\varepsilon} = \frac{\varepsilon(\tau + \overline{\sigma}_1)^2\|\mathbf{1}_m\|^2}{p},$$

for i such that $\bar{x}_i = 0$, we conclude that it is possible to choose a large enough p such that the inequality $\int_\Omega G(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{s}}(\omega))\,\mathrm{P}(d\omega) < \int_\Omega G(\boldsymbol{x}, \boldsymbol{s}(\omega))\,\mathrm{P}(d\omega)$ holds. #

4.4 Existence of optimal solutions

From now on we make the following blanket assumptions:
(B1) for every positive design $\boldsymbol{x}$ the problem $(\mathcal{C})_{\boldsymbol{x}}(\omega)$ is feasible for almost any ω;
(B2) the problem $(\mathcal{C})_{\mathbf{0}_m}(\omega)$ is infeasible with a positive probability.

The first assumption is related to the "richness" of the ground structure and is easy to satisfy in practice. For example, one can start from a ground structure that is able to sustain *any* load. The second assumption eliminates the possibility of the empty structure being the optimal solution.

In view of Example 4.2.2 it is of prime importance to establish the existence of optimal solutions to the problem $(\mathcal{P}_2^\delta)$ for any $\delta > 0$.

Theorem 4.4.1. *For any $\delta > 0$ the problem $(\mathcal{P}_2^\delta)$ possesses at least one optimal solution $(\boldsymbol{x}^*, \boldsymbol{s}^*(\cdot))$.*

Proof. If there is at least one feasible solution $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{s}}(\cdot))$ then we can bound the design space by introducing additional constraint $w(\boldsymbol{x}) \leq w(\widetilde{\boldsymbol{x}})$. Then from any minimizing sequence one can choose a subsequence with converging design components. Lemma 4.3.1 ensures that the corresponding subsequence of forces converges, and the limit is then feasible in $(\mathcal{P}_2^\delta)$ owing to the lower semi-continuity and non-negativity of G and Fatou's Lemma. Since the objective function is continuous in both design and state variables (it is independent of the forces), the limiting point is also an optimal solution.

Thus it remains to find a feasible solution. Following the proof of Lemma 4.3.2, we see that if $\widetilde{\boldsymbol{s}}(\cdot)$ solves $(\mathcal{C})_{\mathbf{1}_m}(\cdot)$, then it solves $(\mathcal{C})_{2^q\cdot\mathbf{1}_m}(\cdot)$ for any $q \geq 0$ as well. Thus we can make the value of $\int_\Omega G(2^q \cdot \mathbf{1}_m, \boldsymbol{s}(\omega))\,\mathrm{P}(d\omega)$ arbitrarily small (but nonnegative), if we choose a "large enough" q. Hence the point $(2^q \cdot \mathbf{1}_m, \widetilde{\boldsymbol{s}}(\omega))$ is feasible in $(\mathcal{P}_2^\delta)$ for some q. #

Remark 4.4.1.1. For any optimal solution $(\boldsymbol{x}^*, \boldsymbol{s}^*(\cdot))$ to $(\mathcal{P}_2^\delta)$ the equality $\int_\Omega G(\boldsymbol{x}^*, \boldsymbol{s}^*(\omega))\,\mathrm{P}(d\omega) = \delta$ holds.

Proof. If the strict inequality $\int_\Omega G(\boldsymbol{x}^*, \boldsymbol{s}^*(\omega))\,\mathrm{P}(d\omega) < \delta$ held, then for some $0 < \mu < 1$ we would have $\int_\Omega G(\mu\boldsymbol{x}^*, \boldsymbol{s}^*(\omega))\,\mathrm{P}(d\omega) < \delta$ as well. Furthermore, $\boldsymbol{s}^*(\cdot)$ solves $(\mathcal{C})_{\mu\boldsymbol{x}^*}(\cdot)$, and $0 < w(\mu\boldsymbol{x}^*) < w(\boldsymbol{x}^*)$ (cf. assumption (B2)). The latter inequality contradicts the optimality of $(\boldsymbol{x}^*, \boldsymbol{s}^*(\cdot))$ in $(\mathcal{P}_2^\delta)$. #

4.5 Continuity with respect to lower bounds and relaxation parameter

An additional motivation for considering the relaxed problems $(\mathcal{P}_2^\delta)$ is given by the following result, which ensures that by reducing the relaxation parameter to zero one recovers optimal solutions to the original problem $(\mathcal{P}_2)$.

We denote by $\mathrm{val}(\mathcal{P})$ the optimal value of any problem $(\mathcal{P})$.

Theorem 4.5.1. *Suppose that the problem $(\mathcal{P}_2)$ possesses an optimal solution, and let the sequence $\{\delta_k\}$ monotonically decrease to zero. Then any limit point of the sequence of optimal solutions $\{\boldsymbol{x}^*_{\delta_k}, \boldsymbol{s}^*_{\delta_k}(\cdot)\}$ (and there is at least one) is an optimal solution to $(\mathcal{P}_2)$.*

Proof. The inequality

$$\mathrm{val}(\mathcal{P}_2) \geq \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_2^{\delta_k}) \tag{4.2}$$

obviously holds.

On the other hand, the optimal solution to $(\mathcal{P}_2)$ is feasible in each problem $(\mathcal{P}_2^{\delta_k})$. In particular it means that the sequence of optimal designs $\{\boldsymbol{x}^*_{\delta_k}\}$ is bounded and has a limit point $\widetilde{\boldsymbol{x}}$. Lemma 4.3.1 implies that the corresponding sequence of forces $\{\boldsymbol{s}^*_{\delta_k}(\cdot)\}$ converges to a limit $\widetilde{\boldsymbol{s}}(\cdot)$ solving the problem $(\mathcal{C})_{\widetilde{\boldsymbol{x}}}(\cdot)$. The non-negativity and lower semicontinuity of G, and Fatou's Lemma, imply that $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{s}}(\cdot))$ is feasible in $(\mathcal{P}_2)$, and thus we get:

$$\mathrm{val}(\mathcal{P}_2) \leq w(\widetilde{\boldsymbol{x}}) = \liminf_{k\to\infty} w(\boldsymbol{x}^*_{\delta_k}) = \liminf_{k\to\infty} \mathrm{val}(\mathcal{P}_2^{\delta_k}).$$

Together with (4.2), this proves the claim. #

The function G, defining the constraints of our problem, is not upper-semicontinuous at the designs which are not strictly positive. Therefore, to apply numerical algorithms we would like to introduce a positive lower bound $\varepsilon\mathbf{1}_m$ on the design variables and eventually reduce ε to zero. This method, called ε-perturbation, is classic in topology optimization and is known to converge for compliance minimization problems [Ach98, PaP00, EPP02]. On the other hand, for stress constrained weight minimization this simple procedure cannot approximate some optimal solutions, owing to the phenomena known as "stress singularities" and "singular topologies" [SvG68, Kir90, ChJ92, RoB94]. More sophisticated numerical approaches are known to overcome this difficulty, for example the ε-perturbation by Cheng and Guo [ChG97] (see also [Pet01, PaP00, EPP02]) and a penalty function approach by Evgrafov and Patriksson [EvP02]. It turns out that for our relaxation the simple approach outlined above is sufficient. To be more precise, for $\varepsilon > 0$ consider the following ε-perturbation of the problem $(\mathcal{P}_2^\delta)$:

$$(\mathcal{P}_2^{\delta,\varepsilon})\begin{cases} \min\limits_{(\boldsymbol{x},\boldsymbol{s}(\cdot))} w(\boldsymbol{x}) \\ \text{s.t.} \begin{cases} \underline{\boldsymbol{x}} + \varepsilon \mathbf{1}_m \le \boldsymbol{x}, \\ \displaystyle\int_\Omega G(\boldsymbol{x}, \boldsymbol{s}(\omega))\,\mathrm{P}(d\omega) \le \delta, \\ \boldsymbol{s}(\omega) \text{ solves } (\mathcal{C})_{\boldsymbol{x}}(\omega), \quad \text{P-a.s.} \end{cases} \end{cases}$$

Theorem 4.5.2. *Let $\{(\boldsymbol{x}_\varepsilon^*, \boldsymbol{s}_\varepsilon^*(\cdot))\}$ be a sequence of optimal solutions to the problems $\{(\mathcal{P}_2^{\delta,\varepsilon})\}$. Then any limit point of the sequence $\{(\boldsymbol{x}_\varepsilon^*, \boldsymbol{s}_\varepsilon^*(\cdot))\}$ as ε goes to zero (and there is at least one) is an optimal solution to the problem $(\mathcal{P}_2^\delta)$. Furthermore:*

$$\mathrm{val}(\mathcal{P}_2^\delta) = \inf_{\varepsilon>0} \mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon}) = \lim_{\varepsilon\to 0} \mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon}).$$

Proof. For any $\varepsilon_0 > 0$ there is a point $(\boldsymbol{x}_0, \boldsymbol{s}_0(\cdot))$, which for every $\varepsilon \in (0, \varepsilon_0)$ is feasible in each problem $(\mathcal{P}_2^{\delta,\varepsilon})$. In particular, it means that the sequence of optimal designs $\{\boldsymbol{x}_\varepsilon^*\}$ is bounded and has a limit point. We make another observation, namely that for $\varepsilon_1 < \varepsilon_2$ it holds that $\mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon_1}) \le \mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon_2})$.

Suppose that $\lim_{k\to\infty} \boldsymbol{x}_{\varepsilon_k}^* = \widetilde{\boldsymbol{x}}$ for some sequence ε_k converging to zero. Lemma 4.3.1 implies that the corresponding sequence of forces $\{\boldsymbol{s}_{\varepsilon_k}^*(\cdot)\}$ converges to a limit $\widetilde{\boldsymbol{s}}(\cdot)$ solving the problem $(\mathcal{C})_{\widetilde{\boldsymbol{x}}}(\cdot)$. The nonnegativity and lower semi-continuity of G, and Fatou's Lemma, imply that $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{s}}(\cdot))$ is feasible in $(\mathcal{P}_2^\delta)$, and thus we get:

$$\mathrm{val}(\mathcal{P}_2^\delta) \le w(\widetilde{\boldsymbol{x}}) = \lim_{k\to\infty} w(\boldsymbol{x}_{\varepsilon_k}^*) = \inf_{\varepsilon>0} \mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon}). \tag{4.3}$$

On the other hand, Lemma 4.3.2 implies that any feasible solution $(\boldsymbol{x}, \boldsymbol{s}(\cdot))$ to $(\mathcal{P}_2^\delta)$ such that $\int_\Omega G(\boldsymbol{x}, \boldsymbol{s}(\omega))\,\mathrm{P}(d\omega) > 0$ holds can be arbitrarily closely approximated by feasible points of $(\mathcal{P}_2^{\delta,\varepsilon})$. In particular, any optimal solution to $(\mathcal{P}_2^\delta)$ can be approximated in such a way, to give us the reverse inequality:

$$\mathrm{val}(\mathcal{P}_2^\delta) = \lim_{k\to\infty} w(\boldsymbol{x}_{\varepsilon_k}) \ge \inf_{\varepsilon>0} \mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon}).$$

Together with (4.3), this proves the claim. #

The following proposition enables us to approximate the optimal value of $(\mathcal{P}_2^\delta)$ from below in a different way.

Proposition 4.5.3. *Let the sequence* $\{\delta_k\}$ *monotonically increase to* $\delta_\infty > 0$. *Then* $\mathrm{val}(\mathcal{P}_2^{\delta_\infty}) = \lim_{k\to\infty} \mathrm{val}(\mathcal{P}_2^{\delta_k})$.

Proof. Obviously, the inequality

$$\mathrm{val}(\mathcal{P}_2^{\delta_\infty}) \le \liminf_{k\to\infty} \mathrm{val}(\mathcal{P}_2^{\delta_k}) \tag{4.4}$$

holds.

On the other hand, Lemma 4.3.2 implies that any solution $(\boldsymbol{x}, \boldsymbol{s}(\cdot))$ that is feasible in $(\mathcal{P}_2^{\delta_\infty})$ and is such that $\int_\Omega G(\boldsymbol{x}, \boldsymbol{s}(\omega))\,\mathrm{P}(d\omega) > 0$ can be arbitrarily closely approximated by feasible points of $(\mathcal{P}_2^{\delta_k})$ for "large enough" k. In particular, any optimal solution to $(\mathcal{P}_2^{\delta_\infty})$ can be approximated in such a way, which gives us the reverse inequality:

$$\mathrm{val}(\mathcal{P}_2^{\delta_\infty}) \ge \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_2^{\delta_k}).$$

Together with (4.4), this proves the claim. #

4.6 Continuity with respect to changes in probability measure

In this section we prove the main result of the paper, showing that for fixed $\delta > 0$ the optimal solutions to the problem $(\mathcal{P}_2^\delta)$ change continuously as the probability measure changes. Throughout the section we assume that Ω is a compact metric space, $\mathfrak{S} = \mathcal{B}(\Omega)$ and the source of uncertainty $\boldsymbol{f}(\cdot)$ is assumed to be a continuous function.

Continuity allows us to omit the adverb "almost" when we talk about solutions of $(\mathcal{C})_{\boldsymbol{x}}(\cdot)$ for positive designs $\boldsymbol{x}$.

Proposition 4.6.1. *For positive design* $\boldsymbol{x}$ *and* each $\omega \in \Omega$ *the problem* $(\mathcal{C})_{\boldsymbol{x}}(\omega)$ *has a unique solution* $\boldsymbol{s}(\omega)$, *which is a continuous function of* ω.

Proof. We made an assumption that the problem $(\mathcal{C})_{\boldsymbol{x}}(\omega)$ is feasible for any ω for a positive design $\boldsymbol{x}$. The claim then follows from [EPP02, Corollary 4.1]. #

Consider a sequence of probability measures $\{\mathrm{P}_k\}$ defined on $\mathcal{B}(\Omega)$, together with a sequence of optimization problems:

$$(\mathcal{P}_2^\delta)^k \begin{cases} \displaystyle\min_{(\boldsymbol{x},\boldsymbol{s}(\cdot))} \; w(\boldsymbol{x}) \\ \text{s.t.} \begin{cases} \underline{\boldsymbol{x}} \le \boldsymbol{x}, \\ \displaystyle\int_\Omega G(\boldsymbol{x}, \boldsymbol{s}(\omega))\,\mathrm{P}_k(d\omega) \le \delta, \\ \boldsymbol{s}(\omega) \text{ solves } (\mathcal{C})_{\boldsymbol{x}}(\omega), \quad \mathrm{P}_k\text{-a.s.} \end{cases} \end{cases}$$

Without any further regularity assumptions on the probability measure we can prove the following inequality.

Lemma 4.6.2. *Suppose that the sequence of probability measures* $\{\mathrm{P}_k\}$ *weakly converges to* P. *Then* $\mathrm{val}(\mathcal{P}_2^\delta) \geq \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_2^\delta)^k$.

Proof. Fix arbitrary positive numbers $\varsigma < \delta$ and $\varepsilon > 0$. Consider an optimal solution $(\boldsymbol{x}^*, \boldsymbol{s}^*(\cdot))$ to $(\mathcal{P}_2^{\varsigma,\varepsilon})$. Owing to Proposition 4.6.1, $\boldsymbol{s}^*(\cdot)$ is a continuous function. Furthermore, since the energy $\mathcal{E}(\boldsymbol{x}^*, \boldsymbol{s}^*(\cdot))$ is continuous, we can deduce that $G(\boldsymbol{x}^*, \boldsymbol{s}^*(\cdot))$ is continuous as well. Since $\{\mathrm{P}_k\}$ weakly converges to P we conclude that $(\boldsymbol{x}^*, \boldsymbol{s}^*(\cdot))$ is feasible in $(\mathcal{P}_2^\delta)^k$ for large enough k, and

$$\mathrm{val}(\mathcal{P}_2^{\varsigma,\varepsilon}) \geq \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_2^\delta)^k$$

holds.

Owing to Theorem 4.5.2 and Proposition 4.5.3, the following inequality holds:

$$\mathrm{val}(\mathcal{P}_2^\delta) = \inf_{\varepsilon>0} \inf_{\varsigma<\delta} \mathrm{val}(\mathcal{P}_2^{\varsigma,\varepsilon}) \geq \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_2^\delta)^k,$$

which is the desired result. #

To prove the reverse inequality we assume additional regularity properties on the sequence $\{\mathrm{P}_k\}$. Namely, we suppose that each measure P_k has a density $p_k(\cdot)$ with respect to a Lebesgue measure on Ω and that the sequence $\{p_k(\cdot)\}$ converges to a density $p(\cdot)$ of P Lebesgue-almost everywhere. This assumption is not very restrictive from the theoretical point of view, and it is usually assumed in engineering applications of probability theory.

Theorem 4.6.3. *Let* $\{(\boldsymbol{x}_k, \boldsymbol{s}_k(\cdot))\}$ *be a sequence of solutions to* $\{(\mathcal{P}_2^\delta)^k\}$. *Then any limit point (and there is at least one) of the sequence* $\{(\boldsymbol{x}_k, \boldsymbol{s}_k(\cdot))\}$ *is a solution to the limiting problem* $(\mathcal{P}_2^\delta)$.

Proof. As in the proof of Theorem 4.4.1 for large enough q we can find a point $(2^q \cdot \mathbf{1}_m, \widetilde{\boldsymbol{s}}(\cdot))$ that is feasible in $(\mathcal{P}_2^{\delta/2})$. Since $\{\mathrm{P}_k\}$ weakly converges to P and $\widetilde{\boldsymbol{s}}(\cdot)$ is continuous, for large enough k this point is feasible to $(\mathcal{P}_2^\delta)^k$. In particular, it means that the sequence $\{\boldsymbol{x}_k\}$ is bounded and has a limit point $\boldsymbol{x}_0$. Therefore, we may assume that the original sequence has converging design components.

The lower semi-continuity and non-negativity of G, and Fatou's

Lemma, imply:

$$\int_\Omega \liminf_{k\to\infty} G(\boldsymbol{x}_k,\boldsymbol{s}_k(\omega))p(\omega)\,d\omega \leq \int_\Omega \liminf_{k\to\infty}[G(\boldsymbol{x}_k,\boldsymbol{s}_k(\omega))p_k(\omega)]\,d\omega \leq \liminf_{k\to\infty}\int_\Omega G(\boldsymbol{x}_k,\boldsymbol{s}_k(\omega))p_k(\omega)\,d\omega \leq \delta.$$

Thus we see that the P-probability of the set $\Omega_f = \{\omega \in \Omega \mid \liminf_{k\to\infty} G(\boldsymbol{x}_k,\boldsymbol{s}_k(\omega)) < \infty\}$ is one. Using Lemma 4.3.1 we can verify the existence of a limiting state $\boldsymbol{s}_0(\cdot)$ corresponding to the design $\boldsymbol{x}_0$, and the P-a.s. convergence of $\boldsymbol{s}_k(\cdot)$ to this state. Using the lower semicontinuity of G, this implies:

$$\int_\Omega G(\boldsymbol{x}_0,\boldsymbol{s}_0(\omega))p(\omega)\,d\omega \leq \int_\Omega \liminf_{k\to\infty} G(\boldsymbol{x}_k,\boldsymbol{s}_k(\omega))p(\omega)\,d\omega \leq \liminf_{k\to\infty}\int_\Omega G(\boldsymbol{x}_k,\boldsymbol{s}_k(\omega))p_k(\omega)\,d\omega \leq \delta.$$

The latter inequality shows that $(\boldsymbol{x}_0,\boldsymbol{s}_0(\cdot))$ is feasible in $(\mathcal{P}_2^\delta)$, and thus:

$$\mathrm{val}(\mathcal{P}_2^\delta) \leq w(\boldsymbol{x}_0) \leq \liminf_{k\to\infty} \mathrm{val}(\mathcal{P}_2^\delta)^k.$$

Together with the estimation of $\limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_2^\delta)^k$ given by Lemma 4.6.2 this finishes the proof. #

To show the qualitative difference between the problems $(\mathcal{P}_2)$ and $(\mathcal{P}_2^\delta)$ we reconsider Example 4.2.1.

Example 4.6.4 (Example 4.2.1 revisited). Figure 4.3 shows the convergence of solutions to $(\mathcal{P}_2^\delta)^k$ to the solution of $(\mathcal{P}_2^\delta)$ as k increases to infinity for various values of δ, as predicted by Theorem 4.6.3.

On the other hand, for a fixed k, the solutions to $(\mathcal{P}_2^\delta)^k$ converge to the optimal solution $x_k^* = 2$ of $(\mathcal{P}_2)^k$ as δ decreases to zero, in accordance with Theorem 4.5.1. Similarly, optimal solutions to $(\mathcal{P}_2^\delta)$ converge to the optimal solution $x^* = 1$ of $(\mathcal{P}_2)$.

This example shows that one cannot in general expect convergence as δ goes to zero and k goes to infinity *simultaneously*.

4.7 Discretization

The most popular method to solve a stochastic programming problem involving a non-discrete probability measure is to approximate it with a

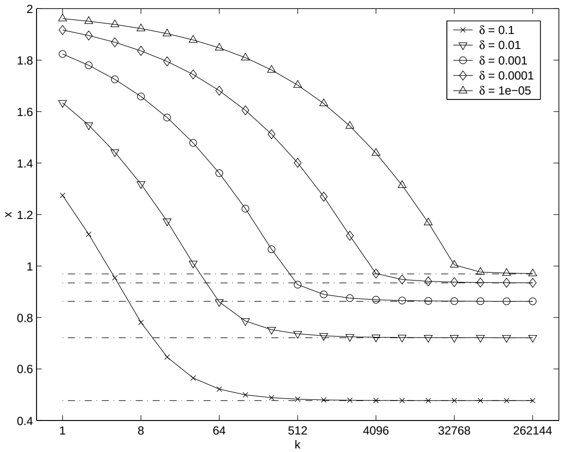

Fig. 4.3: Convergence of solutions to $(\mathcal{P}_2^\delta)^k$ to the solution of $(\mathcal{P}_2^\delta)$ for various values of δ.

sequence of finite-dimensional problems with discrete measures. Unfortunately, we cannot apply Theorem 4.6.3 to our situation, because the approximating discrete measures do not possess densities. Without this assumption, the implementation of such a strategy seems to be impossible, owing to the discontinuity of the function G defining the constraints of our problem. Therefore, we discretize the sizing approximations $(\mathcal{P}_2^{\delta,\varepsilon})$ of $(\mathcal{P}_2^\delta)$; Theorem 4.5.2 shows the viability of such an approach.

In this section we sketch one possible discretization approach, which does not require us to assume the continuity of the load vector $\boldsymbol{f}(\cdot)$ with respect to ω. Evgrafov and Patriksson [EvP02] used this approach to discretize sizing approximations to the stochastic compliance minimization problem and to the original (non-relaxed) stress constrained weight minimization problem. The interested reader is referred to the cited paper and references therein for the detailed development of the discretization theory.

Suppose that Ω is a compact metric space with a metric denoted by $\rho(\cdot,\cdot)$. Let $\mathfrak{S} \supset \mathcal{B}(\Omega)$, $\mathrm{P}(\{\,\omega \mid \rho(\omega,\omega_0) < r\,\}) = \mathrm{P}(\{\,\omega \mid \rho(\omega,\omega_0) \le r\,\}) > 0$ for any $\omega_0 \in \Omega$, $r > 0$, and P be a regular measure.

Consider a sequence of *partitions* of Ω, $\mathcal{A}^k = \{\,A_1^k,\dots,A_k^k\,\}$, satisfying the following properties for each k and $1 \le l \le k$:

(M1) $\mathrm{P}(A_l^k) > 0$,
(M2) $\cup_{l=1}^k A_l^k = \Omega$,
(M3) $A_i^k \cap A_j^k = \emptyset$, $i \neq j$,
(M4) $\lim_{k\to\infty} \operatorname{diam}(A_l^k) = 0$,
(M5) $\mathrm{P}(\partial A_l^k) = 0$.

Note that the collection of sets $\{\mathcal{A}^k\}$, satisfying the properties (M1)–(M5), generates an algebra $\mathfrak{S}_0 \subset \mathfrak{S}$.

Define a sequence of discrete measures P_k with support $\operatorname{supp} \mathrm{P}_k = \{\,\omega_1^k, \dots, \omega_k^k\,\}$, satisfying the following properties for each k and $1 \le l \le k$:

(M6) $\omega_l^k \in A_l^k$,

(M7) $\lim_{k\to\infty} \max_{1\le l\le k} \mathrm{P}_k(\omega_l^k)/\mathrm{P}(A_l^k) = 1$.

We further assume that

(D1) the function $\boldsymbol{f}(\cdot)$ is $\mathfrak{S}_0$-measurable and bounded.

We denote by $(\mathcal{C})_{\boldsymbol{x}}^k(\omega_l^k)$ the following equilibrium principle:

$$\begin{cases} \min\limits_{\boldsymbol{s}} \mathcal{E}(\boldsymbol{x}, \boldsymbol{s}) \\ \text{s.t.} \begin{cases} \sum\limits_{i \in \mathcal{I}(\boldsymbol{x})} \boldsymbol{B}_i^T s_i = \boldsymbol{f}(\omega_l^k). \end{cases} \end{cases}$$

In the following theorem we establish the convergence of discretizations for the problem $(\mathcal{P}_2^{\delta,\varepsilon})$. We note that from the weak* discrete convergence of the sequence $\{(\boldsymbol{x}_k^*, \boldsymbol{s}_k^*(\cdot))\}$ follows the (usual) convergence of the optimal designs.

Theorem 4.7.1. *Consider the following sequence $\{(\mathcal{P}_2^{\delta,\varepsilon})^k\}$ of discretizations of the problem $(\mathcal{P}_2^{\delta,\varepsilon})$:*

$$(\mathcal{P}_2^{\delta,\varepsilon})^k \begin{cases} \min\limits_{(\boldsymbol{x},\boldsymbol{s}(\cdot))} \; w(\boldsymbol{x}) \\ s.t. \begin{cases} \underline{\boldsymbol{x}} + \varepsilon \mathbf{1}_m \le \boldsymbol{x}, \\ \displaystyle\int_\Omega G(\boldsymbol{x}, \boldsymbol{s}(\omega))\, \mathrm{P}_k(d\omega) \le \delta, \\ \boldsymbol{s}(\omega_l^k) \text{ solves } (\mathcal{C})_{\boldsymbol{x}}^k(\omega_l^k), \quad l = 1, \dots, k. \end{cases} \end{cases}$$

Suppose that the assumptions (M1)–(M7) *and* (D1) *hold. Suppose further that there exists an optimal solution $(\boldsymbol{x}^*, \boldsymbol{s}^*(\cdot))$ to the problem $(\mathcal{P}_2^{\delta,\varepsilon})$ such that the energy functional $\mathcal{E}(\boldsymbol{x}^*, \boldsymbol{s}^*(\cdot))$ is essentially bounded.*

Owing to the positivity of $\boldsymbol{x}^*$ and assumption (B1) *the problems $(\mathcal{C})_{(\boldsymbol{x}^*)}^k(\omega_l^k)$ are feasible for any k, $1 \le l \le k$. Thus, there exists a sequence of optimal solutions to $\{(\mathcal{P}_2^{\delta,\varepsilon})^k\}$; we denote it by $\{(\boldsymbol{x}_k^*, \boldsymbol{s}_k^*(\cdot))\}$ Then any weak* discrete limit point of this sequence solves the limiting problem $(\mathcal{P}_2^{\delta,\varepsilon})$.*

Proof. We assume that the original sequence is weakly* convergent. The following two inequalities follow respectively from Propositions A.7

and A.8 in [EvP02]:

$$\mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon}) \le \liminf_{k\to\infty} \mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon})^k,$$
$$\mathrm{val}(\mathcal{P}_2^{\varsigma,\varepsilon}) \ge \limsup_{k\to\infty} \mathrm{val}(\mathcal{P}_2^{\delta,\varepsilon})^k,$$

for any $0 < \varsigma < \delta$. Then, the claim follows from Proposition 4.5.3. #

4.8 Numerical example

We consider the problem of finding a minimal weight of the beam-like structure shown in Figure 4.4. In this example, the forces of magnitude one act independently from each other, with the directions uniformly distributed in the intervals schematically shown in the figure. The number of bars in the ground structure is $m = 49$. We set $E = 1.0$, $\overline{\sigma}_1 = 1.0$, start with $\varepsilon = 0.05$ and successively multiply it with factor 0.6 until it gets as small as $5.0 \cdot 10^{-4}$.

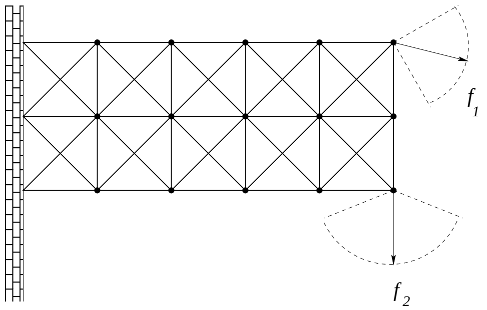

Fig. 4.4: The ground structure for the weight minimization problem.

We have solved the nested formulation of the problem (i.e., we have eliminated the state variables and treated them as functions of design) using an SQP-type algorithm. The starting point was the equally distributed material.

In Table 4.1 we report the optimal weights and statistics describing the violations of the stress constraints for various values of the number of discretization points k. The definitions of the statistics used are given below:

$$\max_\sigma := \max_{1\le\ell\le\widehat{k}} \max_{1\le i\le m} [|\sigma|_{i\ell} - \overline{\sigma}_1]_+,$$

$$\mathrm{avg}_\sigma := \sum_{\ell=1}^{\widehat{k}} \Big\{ \max_{1\le i\le m} [|\sigma|_{i\ell} - \overline{\sigma}_1]_+ \Big\} P_{\widehat{k}}(\omega_\ell^{\widehat{k}}),$$

k	w^*	$\max_\sigma$	avg_σ
1	33.599	745.5%	285.0%
25	45.447	18.95%	1.02%
625	45.967	13.27%	0.61%

Tab. 4.1: Results of numerical calculation.

where $\sigma_{i\ell}$ is a tensile stress in the bar i under the loading condition ℓ, and $\widehat{k} = 625$. The number $\max_\sigma$ characterises the maximal stress violation in the structure for all load cases, whereas avg_σ is an average (for all load cases) maximal (among the structure members) stress violation. The way we formulate stress constraints only guarantees that avg_σ is small when δ is small. Nevertheless, $\max_\sigma$ turns out to be not very big and seems to decrease with δ for this problem.

The reduction of the relaxation parameter δ to the value $1 \cdot 10^{-5}$ while keeping $k = 625$ gives us only a 3.6% increase in the optimal weight, whereas the corresponding numbers $\max_\sigma$ and avg_σ decrease drastically to 2.54% respectively $4 \cdot 10^{-2}\%$ (compare with the last row in Table 4.1).

Further increases of k do not lead to significant changes in the optimal design. Therefore, we assume that $k = 625$ is a reasonably good approximation of the problem's probability measure, and, in particular, use this approximation when calculating statistics $\max_\sigma$ and avg_σ.

Two optimal designs corresponding to $k = 1$ and $k = 625$ are shown in the Figure 4.5. It is interesting to note that the multiple-load optimal design has fewer bars than the corresponding average-load design. Their

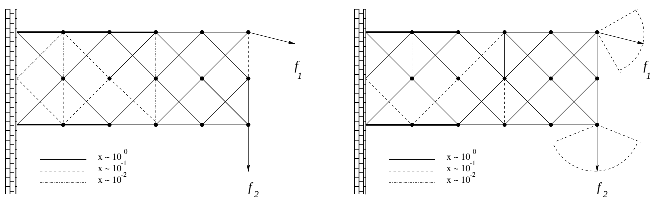

Fig. 4.5: The optimal designs for the weight minimization problem corresponding to (a) $k = 1$ and (b) $k = 625$. Line thicknesses are proportional to cross-sectional areas.

behaviour under various loading conditions is shown in the Figure 4.6.

4.9 Conclusions

The relaxation of the stress constrained weight minimization problem proposed in this paper offers a good trade-off between the strict satisfaction of the stress constraints and the robustness of the optimal solutions obtained with respect to changes in the modelling of uncertainty. The bound (4.1) on the constraint violations also allows one to choose a satisfactory value of δ before starting the optimization. For example, one can choose the boundary value c of the maximal acceptable violation of stress constraints, and then choose δ to be so small that the estimation $\delta\overline{x}/c^2$ of the probability of exceeding this boundary is negligible, where $\overline{x}$ is an upper bound for the design variables $\boldsymbol{x}$.

The ongoing research is concentrated on the development of efficient numerical methods for the problem $(\mathcal{P}_2^\delta)$ as well as on the possible extensions of the results for more general mechanical models (e.g., trusses with unilateral constraints, frames, possibly with flexible joints).

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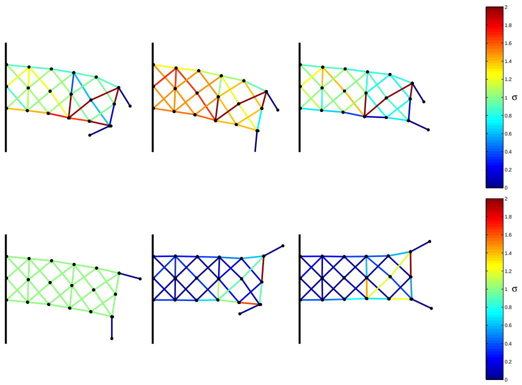
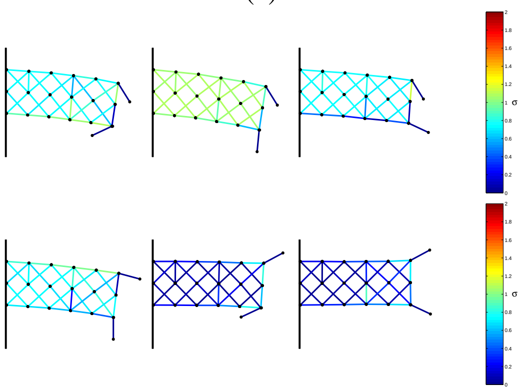


(a)



(b)

Fig. 4.6: Stresses and displacements for various random forces for optimal designs, corresponding to (a) $k = 1$ and (b) $k = 625$. Note: for the sake of a better visualization of stresses, line thicknesses are *not* proportional to cross-sectional areas.