Some Applications of Weighted Integral Formulas

Elin Götmark

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Abstract

We estimate the degree of polynomial solutions of the Bézout equation by means of the Koszul complex, and find explicit solutions by means of integral formulas. We also give an explicit proof of Briançon-Skoda's theorem by means of integral formulas. Next we construct integral formulas for sections of the line bundles of $\mathbb{P}^n$, which also gives rise to a Koppelman formula in $\mathbb{P}^n$. As an application we obtain some (known) vanishing theorems.

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Contents

1 Introduction

This work contains several seemingly unrelated parts, which are nevertheless connected by the use of integral representation formulas. Section 2 reviews integral formulas in $\mathbb{C}^n$, beginning with a short history and ending with some material from [1], which will be used in the following sections.

In Section 3 we look at the problem of estimating the degrees of polynomials which are solutions of a Bézout equation. We do this first by means of the Koszul complex, and then proceed to find explicit solutions by the use of integral formulas.

The next section looks at how to prove Briançon-Skoda's theorem by means of integral formulas. Like the previous section it also involves solving a division problem, but this time locally instead of globally.

In Section 5 we find integral formulas on the complex projective space $\mathbb{P}^n$ by taking inspiration from the procedure in [1]. In Section 6 we construct Koppelman formulas for $\mathbb{C}^n$, then in Section 7 we do the same for $\mathbb{P}^n$, and then as an application prove some (known) vanishing theorems.

The formalism we use is built on a foundation of differential forms, and we assume a knowledge of these. We also, from Section 5 onwards, assume a basic knowledge of $\mathbb{P}^n$ (the complex projective space) and vector bundles. Some remarks on notation: By $f(z) \lesssim g(z)$ we mean that $f(z) \leq Cg(z)$ for some constant C. If α is a differential form, then we define $\alpha_n = \alpha^n/n!$. By $\widehat{dz_i}$ we mean $dz_1 \wedge \ldots dz_{i-1} \wedge dz_{i+1} \wedge \ldots \wedge dz_n$.

2 Integral formulas on $\mathbb{C}^n$

integralform

The simplest example of an integral representation formula is the well-known Cauchy integral formula in one complex variable, which says that

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{\phi(\zeta)d\zeta}{\zeta - z}$$

if ϕ is holomorphic and $z \in D$. Note that the kernel is holomorphic outside z, and that it works for any domain D. In several complex variables things are more complicated, and one has much more freedom to construct different kinds of kernels. We will begin with a short history of the development of integral formulas, taken from [13]. Many of the kernels and formulas we mention will occur later in the text.

We have the Bochner-Martinelli kernel (discovered in 1938 by Martinelli and independently by Bochner in 1943), which works for any domain but is not holomorphic. A more general kernel, the Cauchy-Fantappiè-Leray kernel, includes the Bochner-Martinelli kernel as a special case, but can also be used e. g. to obtain a holomorphic kernel for convex domains. The kernel was discovered by Leray in 1959, but in the name he honored Cauchy and Fantappiè as influential mathematicians in the field. Koppelman rediscovered the same kernel in 1967, and shortly afterwards he introduced formulas to represent forms of degree $(0, q)$; the so-called Koppelman formulas. In 1969 another kernel was found by Henkin and independently by Ramirez, called the Henkin-Ramirez kernel, which is important because it is a holomorphic kernel on a strictly pseudoconvex domain. Since Henkin and Ramirez there have been many further developments. Integral formulas have many applications, they have been used e. g. to find explicit solutions to the $\bar{\partial}$-equation or to solve the Levi problem.

I will now present the ideas for constructing integral formulas which are contained in [1]. In the original article there are more illustrating examples, but on the other hand I have more detailed proofs. These formulas will be used in various contexts in the rest of the paper. First, let $\mathcal{E}_{p,q}(U)$ denote the space of smooth (p,q)-forms on the open set $U \subset \mathbb{C}^n$. Then we define $\mathcal{L}^m(U)$ to be $\oplus_{k=0}^n \mathcal{E}_{k,k+m}(U)$ for any m, and $\mathcal{L}^m_{curr}(U)$ to be the corresponding space of currents. If $f \in \mathcal{L}^m$ and $g \in \mathcal{L}^k$, then $f \wedge g \in \mathcal{L}^{m+k}$.

Now, for a fixed $z \in \mathbb{C}^n$, let $\delta_{\zeta - z}$ be contraction with the vector field

$$2\pi i \sum_1^n (\zeta_i - z_i)\frac{\partial}{\partial \zeta_i}.$$

This contraction anticommutes with the $\bar{\partial}$ operator (it is easy to prove this by checking on forms of the type $f(\zeta)d\zeta_I \wedge d\bar{\zeta}_J$). We now define $\nabla = \nabla_{\zeta - z} =$

$\delta_{\zeta-z} - \bar\partial$, which is an operator from $\mathcal{L}^m$ to $\mathcal{L}^{m+1}$ for all m. In fact, $\nabla \circ \nabla = 0$ so that

$$\ldots \to \mathcal{L}^m(U) \to \mathcal{L}^{m+1}(U) \to \ldots \tag{1}$$

complex

is a complex. Moreover, ∇ obeys Leibniz's rule, that is

$$\nabla(f \wedge g) = \nabla f \wedge g + (-1)^m f \wedge \nabla g$$

for $f \in \mathcal{L}^m$. We also have Stokes' theorem,

$$\int_{\partial D} f = -\int_D \nabla f,$$

if $f \in \mathcal{L}^{-1}$ and D has smooth boundary, which is seen by noting that $\int_D \delta_\eta f = 0$ and using the ordinary Stokes' theorem.

For a current $T \in \mathcal{L}^m_{curr}$ and a test form $\phi \in \mathcal{L}^{-m-1}$, we define

$$\nabla T.\phi = (-1)^m T.\nabla\phi. \tag{2}$$

tovsippa

This is a good definition since it holds if T is given by a smooth form, which is seen by applying Leibniz's rule and Stokes' formula.

We want to solve the equation

$$\nabla_{\zeta-z} u(\zeta) = 1 - [z] \tag{3}$$

vild

in $\mathcal{L}^{-1}_{curr}(\Omega)$, where Ω is some open set containing z. Here $[z]$ denotes the Dirac measure at z, viewed as a current of degree (n,n). This is really a set of equations, namely

$$\delta_{\zeta-z} u_1 = 1, \quad \ldots, \quad \delta_{\zeta-z} u_{k+1} - \bar\partial u_k = 0, \quad \ldots, \quad \bar\partial u_n = [z],$$

where u_k denotes the component of bidegree $(k, k-1)$. The equations are to be understood in the current sense. If u is such a solution, we will call it a Cauchy form. Now let $D \subset \Omega$ be a set with smooth boundary such that $z \in D$. If u_n is smooth, we can take it to be the kernel in our integral formula, i. e.

$$\phi(z) = \int_{\partial D} \phi(\zeta) u_n \tag{4}$$

adoxa

if ϕ is holomorphic in D. Later, in Proposition 2.3 (lunglav), we will see that it is enough to find a smooth solution u to $\nabla_{\zeta-z} u(\zeta) = 1$ in $\mathcal{L}^{-1}(U)$, where U is a neighborhood of ∂D not containing z; then u will in fact satisfy (4) (adoxa).

The following proposition gives us an example of a Cauchy form.

f\"altsippa **Proposition 2.1.** *Let $z \in \mathbb{C}^n$ be fixed and let*

$$b(\zeta) = \frac{1}{2\pi i}\frac{\partial|\zeta - z|^2}{|\zeta - z|^2}.$$

Then the form

$$u = \frac{b}{\nabla b} = \frac{b}{1 - \bar\partial b} = b \wedge \sum_1^n (\bar\partial b)^{k-1} \tag{5}$$ angst

will satisfy vild *(3) in $\mathbb{C}^n$.*

Proof. First we want to show that $\nabla u = 1$ when $\zeta \neq z$. It is easy to see that $\delta_{\zeta - z} b = 1$; furthermore $\delta_{\zeta-z}\bar\partial b = -\bar\partial \delta_{\zeta - z} b = \bar\partial 1 = 0$. Using this, we get $\delta_{\zeta-z} u_{k+1} - \bar\partial u_k = \delta_{\zeta-z}(b \wedge (\bar\partial b)^k) - \bar\partial(b \wedge (\bar\partial b)^{k-1}) = (\bar\partial b)^k - (\bar\partial b)^k = 0$. As a special case, we see that $\bar\partial u_n = 0$ since $u_{n+1} = 0$.

To see what happens over the singularity, we prove first that $\bar\partial u_n = [z]$. It is well-known that this is so, since our u_n is the well-known Bochner-Martinelli kernel. To prove it, assume for simplicity that $z = 0$, and take a test function $\phi(\zeta)$. Then $|u_n| \lesssim |\zeta|^{-2n+1}$ close to the origin, and thus u_n is integrable, so we have

$$\int \bar\partial\phi(\zeta) \wedge u_n = \lim_{\epsilon\to 0}\int_{|\zeta|>\epsilon} \bar\partial\phi(\zeta)\wedge u_n. \tag{6}$$ bort

Since $\bar\partial u_n = 0$ outside the origin, we can use Stokes' theorem on the integral on the right hand side of bort (6), and get

$$\left(\frac{1}{2\pi i}\right)^n \int_{|\zeta|=\epsilon} \phi(\zeta) \wedge \frac{\partial|\zeta|^2}{|\zeta|^2} \wedge \left(\bar\partial\frac{\partial|\zeta|^2}{|\zeta|^2}\right)^{n-1}. \tag{7}$$ backsippa

For the next step, we note that in the integrand we can replace $\bar\partial(\partial|\zeta|^2/|\zeta|^2)$ with $\partial\bar\partial|\zeta|^2/|\zeta|^2$, since the term containing $\partial|\zeta|^2$ will vanish because there already is one such factor in the integrand. We continue from backsippa (7) and get

$$\begin{aligned} &\left(\frac{1}{2\pi i}\right)^n \frac{1}{\epsilon^{2n}} \int_{|\zeta|=\epsilon} \phi(\zeta)\wedge\partial|\zeta|^2 \wedge \left(\bar\partial\partial|\zeta|^2\right)^{n-1} = \\ = \; &\left(\frac{1}{2\pi i}\right)^n \frac{1}{\epsilon^{2n}} \left(\int_{|\zeta|<\epsilon} \bar\partial\phi(\zeta)\wedge\partial|\zeta|^2\wedge\left(\bar\partial\partial|\zeta|^2\right)^{n-1} + \right. \\ + \; &\left.\int_{|\zeta|<\epsilon} \phi \wedge \left(\bar\partial\partial|\zeta|^2\right)^n\right), \end{aligned} \tag{8}$$ mosippa

where we have used Stokes' theorem. Now we have two integrals, and we will show that the first one converges to zero and the other one to $\phi(z)$ when $\epsilon \to 0$. The integrand of the first one is $\mathcal{O}(|\zeta|)$, so we can estimate the absolute value of that integral with $\mathrm{Vol}(B(0,\epsilon)) \cdot \mathcal{O}(\epsilon)/\epsilon^{2n}$, which goes to zero when $\epsilon \to 0$.

We note that $n! \cdot dV = (i\partial\bar{\partial}|\zeta|^2/2)^n$, so the second integral in (mosippa 8) is equal to

$$\begin{aligned}
&\left(\frac{1}{2\pi i}\right)^n \frac{1}{\epsilon^{2n}} \int_{|\zeta|<\epsilon} \phi(\zeta) \wedge \left(\bar{\partial}\partial|\zeta|^2\right)^n = \\
= \;& \left(\frac{-i}{2\pi}\right)^n \frac{1}{\epsilon^{2n}} \int_{|\zeta|<\epsilon} (\phi(0) + \mathcal{O}(|\zeta|)) \wedge (-1)^n \left(\partial\bar{\partial}|\zeta|^2\right)^n \\
= \;& \frac{n!}{\pi^n \epsilon^{2n}} \int_{|\zeta|<\epsilon} (\phi(0) + \mathcal{O}(|\zeta|)) dV. \qquad (9)
\end{aligned}$$

baka

But the volume of $B(0,\epsilon)$ in $\mathbb{C}^n$ is equal to $\pi^n\epsilon^{2n}/n!$, and then we see that (baka 9) converges to $\phi(0)$, since the term containing $\mathcal{O}(|\zeta|)$ will go to zero as $\epsilon \to 0$.

To conclude the proof, we also need to prove that $\delta_\zeta u_{k+1} = \bar{\partial} u_k$ in the current sense, for $k < n$. To this end, take a test form ϕ. Then we have

$$\begin{aligned}
&\int \bar{\partial} u_k \wedge \phi = \int u_k \wedge \bar{\partial}\phi = \lim_{\epsilon\to 0} \int_{|\zeta|>\epsilon} u_k \wedge \bar{\partial}\phi = \\
= \;& \lim_{\epsilon\to 0} \left(\int_{|\zeta|>\epsilon} \bar{\partial} u_k \wedge \phi + \int_{|\zeta|=\epsilon} u_k \wedge \phi \right). \qquad (10)
\end{aligned}$$

nipsippa

The second integral in (nipsippa 10) will go to zero as $\epsilon \to 0$, since $\mathrm{Vol}(\{|\zeta| = \epsilon\}) = \mathcal{O}(\epsilon^{2n-1})$ and $u_k \wedge \phi = \mathcal{O}(|\zeta|^{-2k+1})$, and the first one is equal to

$$\lim_{\epsilon\to 0} \int_{|\zeta|>\epsilon} \bar{\partial} u_k \wedge \phi = \lim_{\epsilon\to 0} \int_{|\zeta|>\epsilon} \delta_\zeta u_{k+1} \wedge \phi = \int \delta_\zeta u_{k+1} \wedge \phi,$$

where the last equality is true since $u_{k+1} = \mathcal{O}(|\zeta|^{-(2k+1)})$ and $\delta_\zeta u_{k+1} = \mathcal{O}(|\zeta|^{-2k})$. □

We observe that since u is a Cauchy form, the complex (complex 1) is exact if $z \notin U$. This is because if we take $f \in \mathcal{L}^m$ such that $\nabla f = 0$, then $u \wedge f \in \mathcal{L}^{m-1}$ and $\nabla(u \wedge f) = f$.

The following proposition tells us how to find other Cauchy forms (it is Proposition 2.2 on page 6 of [MA 1]).

carex **Proposition 2.2.** *Suppose $u \in \mathcal{L}^{-1}(\Omega \setminus \{z\})$ solves $\nabla u = 1$ in $\Omega \setminus \{z\}$, and that $|u_k| \lesssim |\zeta - z|^{-(2k-1)}$. Then u satisfies $\nabla_{\zeta - z} u = 1 - [z]$ in Ω.*

Proof. Let u^1 be the form in Proposition 2.1, and let u^2 be a form satisfying the conditions in the proposition. For simplicity, assume again that $z = 0$. Then $u^1 \wedge u^2 = \mathcal{O}(|\zeta|^{-(2n-2)})$ near the origin, and $\nabla(u^1 \wedge u^2) = u^2 - u^1$ pointwise outside the origin. We want to show that $\nabla(u^1 \wedge u^2) = u^2 - u^1$ holds in the current sense. Take a test form $\phi \in \mathcal{L}^1(\Omega)$. Then in light of (2) we want to show that

$$-\int (u^1 \wedge u^2) \wedge \nabla \phi = \int (u^1 - u^2) \wedge \phi. \tag{11}$$ maskros

Using firstly that $u^1 \wedge u^2$ is integrable, and secondly Stokes' theorem, the right hand side of (11) is equal to

$$\begin{aligned} &- \lim_{\epsilon \to 0} \int_{|\zeta| > \epsilon} (u^1 \wedge u^2) \wedge \nabla \phi = \\ &= \lim_{\epsilon \to 0} \left(\int_{|\zeta| = \epsilon} u^1 \wedge u^2 \wedge \phi + \int_{|\zeta| > \epsilon} \nabla(u^1 \wedge u^2) \wedge \phi \right). \end{aligned} \tag{12}$$ mikroskop

The first of the integrals in (12) will go to zero when $\epsilon \to 0$, since $u^1 \wedge u^2 = \mathcal{O}(|\zeta|^{-2n+2})$ and $Vol(\{|\zeta| = \epsilon\}) = \mathcal{O}(\epsilon^{2n-1})$. As for the first integral, we get

$$\lim_{\epsilon \to 0} \int_{|\zeta| > \epsilon} \nabla(u^1 \wedge u^2) \wedge \phi = \lim_{\epsilon \to 0} \int_{|\zeta| > \epsilon} (u^1 - u^2) \wedge \phi = \int (u^1 - u^2) \wedge \phi$$

using the fact that $u^1 - u^2$ is integrable. Thus, we know that $\nabla(u^1 \wedge u^2) = u^2 - u^1$ as currents. It follows that $\nabla u^1 = \nabla u^2$, and since u^1 is a Cauchy form, u^2 must be one too. □

We are now ready for the following proposition (Proposition 2.1 on page 5 of [1]):

lunglav **Proposition 2.3.** *Suppose that $z \in D$ and $z \notin U \supset \partial D$. If $u \in \mathcal{L}^m_{curr}(U)$ and $\nabla_{\zeta - z} u = 1$, then $\bar{\partial} u_n = 0$, and all such u_n define the same Dolbeault cohomology class $\omega_{\zeta - z}$ in U and any representative for $\omega_{\zeta - z}$ occurs in this way. If u_n is smooth and $\phi \in \mathcal{O}(\overline{D})$, then*

$$\phi(z) = \int_{\partial D} \phi(\zeta) u_n. \tag{13}$$ propekv

Proof. Obviously if $\nabla_{\zeta - z} u = 1$, then $\bar\partial u_n = 0$. Now we take u'_n to be the top-degree term of u', where $\nabla u' = 1$. We must prove that u'_n is in the same cohomology class as u_n. But $\nabla(u - u') = 0$, and since $z \notin U$ we can find a solution w to $\nabla w = u - u'$. Then $u'_n = u_n + \bar\partial w_n$, which is what we wanted to prove.

To prove the other direction, let $u'_n = u_n + \bar\partial \psi$, where ψ is an $(n, n-2)$ form, be another representative of the cohomology class. Then $u' = u - \nabla\psi$ solves $\nabla u' = 1$, and u'_n is in fact the top-degree term of such a solution. Let u be the Bochner-Martinelli form; then u_n satisfies (13). If we use Stokes' theorem on the boundary of ∂D, which is empty, it follows that $\int_{\partial D} \bar\partial\psi = 0$, and thus u'_n satisfies (13). □

This proposition shows that one can see the kernel itself as just the top-degree term in a larger form, and that all kernels occur in such a way. This approach has advantages for example when we construct weighted formulas.

Definition 1. A smooth form $g \in \mathcal{L}^0$ such that $\nabla g = 0$ and $g_0(z) = 1$ is a weight.

The main example of a weight is $1 + \nabla q$, where $q \in \mathcal{L}^{-1}$. Now we have the following proposition (Proposition 3.1 on page 7 in [1]):

viktint **Proposition 2.4.** *If g is a weight in Ω, $D \subset\subset \Omega$, and u solves $\nabla u = 1$ in a neighborhood of ∂D, then*

$$\phi(z) = \int_{\partial D} \phi(\zeta)(u \wedge g)_n + \int_D \phi g_n$$

if $\phi \in \mathcal{O}(\overline{D})$.

Proof. We have $\nabla(u \wedge g) = \nabla u \wedge g = (1 - [z]) \wedge g = g - [z]$. Then $\bar\partial(u \wedge g)_n = g_n - [z]$, and the proposition follows. □

We will make use of this proposition in Section 3.2.

3 Estimating the degree of polynomial solutions of the Bézout equation

We propose to use methods from complex analysis to make estimates of the degree of polynomials which are related to the division problem in Hilbert's Nullstellensatz. The setup is as follows: We suppose that $f(z) = (f_1(z), ..., f_m(z))$ is a tuple of polynomials which have no common zeroes in $\mathbb{C}^n$, and such that $\deg(f_i) \le d$. Now we want to find polynomials $p(z) = (p_1(z), \dots, p_m(z))$ such that $f \cdot p = 1$. We know that this is possible by Hilbert's Nullstellensatz, but we also want to get an upper bound on their degree.

The first break-through in this problem was by Brownawell in the 1987 paper [8]. The main part of his paper was to obtain the inequality

$$1/|f(z)| \lesssim |z|^M \tag{14}$$

by algebraic means with $M = (n-1)d^\mu - 1$, where $\mu = \min(m, n)$. Using this inequality it was proven that $\deg(p_i) \le \mu(d^\mu + d)$, by means of a result of Skoda [15] based on Hörmander's work on L^2-estimates and the $\bar\partial$-equation. An improvement on this was given in 1988 by Kollár [11], where the estimates were improved to $\deg(f_i p_i) \le d^\mu$ in the case of $d > 2$. In 1997, Sombra [17] proved that $\deg(f_i p_i) \le 2d^\mu$ with no restrictions on d. Their proofs were algebraic.

The result of Skoda that Brownawell used was

Theorem 3.1. *Let $f(z) = (f_1(z), ..., f_m(z))$ be a tuple of polynomials such that $\deg(f_i) \le d$, $|f(z)| > 0$ in $\mathbb{C}^n$, and $1/|f(z)| \lesssim |z|^M$. Then there exist $p(z) = (p_1(z), \dots, p_m(z))$ such that $f \cdot p = 1$ and $\deg(p_i) \le \mu(M + d) - d$, where $\mu = \min(m, n+1)$. The result is true for $M \ge -d$, and if $m > n$ we have the additional requirement that $(n+1)M \ge -n$.*

This result can be sharpened by the use of residue currents, as in [2], where the same estimate is obtained with $\mu = \min(m, n)$. In Section 3.1 we will show that it is possible to obtain Theorem 3.1 by means of the Koszul complex. In Section 3.2 we will obtain an explicit solution $p(z)$ by means of integral formulas, with an estimate of $\deg p$ which is only slightly worse than the one in Theorem 3.1.

3.1 Using the Koszul complex

The Koszul complex method has been widely used in complex analysis since Hörmander first used it this context in [10]). We let E be a trivial vector bundle over $\mathbb{C}^n$ with the global frame $\{e_i\}$, and let E^* with frame e_i^* be the dual bundle. We then regard $f(z) = \sum_1^m f_i(z)e_i^*$ as a section of E^*, and define the operation δ_f on sections of E as contraction with f. In other words, if $s = \sum_1^m s_i e_i$ is a section of E, then $\delta_f(s) = \sum_1^m s_i f_i$. The operation is extended to sections of $\bigwedge E$ by Leibniz' rule. We introduce the operator $\nabla = \delta_f - \bar\partial$ and aim to solve the equation $\nabla u = 1$. To this end, we define

$$\sigma = \sum_1^m \sigma_i = \sum_1^m \frac{\bar f_i}{|f|^2} e_i$$

and set $u = \sigma/\nabla\sigma = \sigma \wedge \sum_0^\infty (\bar\partial\sigma)^k$, in a similar way as in (5). Then $\nabla u = 1$, using a "telescoping sum" argument. But in fact, the sum is not endless – we have $\bar\partial u_\mu = 0$ where $\mu = \min(m, n+1)$. It is obvious that $\bar\partial u_{n+1} = 0$ since u_{n+1} is a $(0,n)$-form. On the other hand, $\bar\partial u_m = \delta_f u_{m+1} = 0$ since the degree of u_{m+1} in $\bigwedge E$ is too high. We can rewrite $\nabla u = 1$ as the system of equations

$$\begin{aligned} \bar\partial u_\mu &= 0 \\ \delta_f u_\mu &= \bar\partial u_{\mu-1} \\ &\vdots \\ \delta_f u_2 &= \bar\partial u_1 \\ \delta_f u_1 &= 1. \end{aligned}$$

From these equations we see that it is possible to solve the system of equations

$$\begin{aligned} \bar\partial w_\mu &= u_\mu \\ \bar\partial w_{\mu-1} &= u_{\mu-1} + \delta_f w_\mu \\ &\vdots \\ \bar\partial w_1 &= u_1 + \delta_f w_2. \end{aligned} \tag{15}$$

k-eqns

We want to obtain a holomorphic solution ψ to $f\psi = 1$; actually we can use $\psi = u_1 + \delta_f w_2$. The problem then is to estimate the degree of ψ. This can be done by first obtaining an L^2-estimate of u_μ, then w_μ, then $w_{\mu-1}$, etc, working through the equations all the way down to ψ by using the following theorem.

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Theorem 3.2. *Let g be a smooth $(0, q+1)$-form on $\mathbb{C}^n$, with $\bar\partial g = 0$ and*

$$\int \frac{|g|^2}{(1+|\zeta|^2)^{k+n+\epsilon}} < \infty, \tag{16}$$

where $k \leq -2n$ if $q = n-1$, and $k \in \mathbb{Z}$ for other q. Then there is a smooth $(0,q)$-form u which solves $\bar{\partial} u = g$, such that

$$\int \frac{|u|^2}{(1+|\zeta|^2)^{k+1+n+\epsilon}} < \infty.$$

A proof of this theorem is given in AB [3]. The first problem, then, is to estimate the absolute value of

$$u_\mu = \sigma \wedge (\bar{\partial}\sigma)^{\mu-1} = \sigma \wedge \sum_{|I|=\mu-1} \bar{\partial}\sigma_I \wedge e_I,$$

where $\bar{\partial}\sigma_I = \bar{\partial}\sigma_{i_1} \wedge \ldots \wedge \bar{\partial}\sigma_{i_{\mu-1}}$. There is a crude way of doing this, where we simply observe that $|\bar{\partial}\sigma| \leq |\bar{\partial} f|/|f|^2 \lesssim |z|^{2M+d-1}$, which would imply that $|u_\mu| \lesssim |z|^{2M(\mu-1)+M+(\mu-1)(d-1)}$. We can do better than this, but then we need some background on positive differential forms. The concept of an (n,n)-form being positive is well-defined, since complex manifolds have a well-defined orientation. For a (p,p)-form, positivity can be defined in two equivalent ways:

Definition 2. Let α be a (p,p)-form. Then $\alpha \geq 0$ if $\alpha \wedge i\gamma_1 \wedge \bar{\gamma}_1 \wedge \ldots \wedge i\gamma_{n-p} \wedge \bar{\gamma}_{n-p} \geq 0$ for all $(1,0)$-forms $\gamma_1, \ldots, \gamma_{n-p}$. Equivalently, $\alpha \geq 0$ if the restriction to every subspace of dimension p is positive.

herm **Proposition 3.3.** *Let $\alpha = \sum_{j,k} \alpha_{j,k} i dz_j \wedge d\bar{z}_k$ be a $(1,1)$-form. Then $\alpha \geq 0$ if and only if $\xi \mapsto \sum_{j,k} \alpha_{j,k}\xi_j\bar{\xi}_k$ is a positive semi-definite Hermitian form.*

From this it follows that $i\partial\bar{\partial} v \geq 0$ if v is a plurisubharmonic function.

Proposition 3.4. *If $\alpha_1, \ldots, \alpha_k$ are positive $(1,1)$-forms, then $\alpha_1 \wedge \ldots \wedge \alpha_k$ is also positive.*

The form $\beta = \frac{i}{2}\partial\bar{\partial}|\zeta|^2$ is a positive $(1,1)$-form, and $\beta_n = dV$, where dV is the volume form of $\mathbb{C}^n$ with the ordinary Lebesgue measure. We also have:

scal **Proposition 3.5.** *If α_1 and α_2 are $(p,0)$-forms, then $\langle \alpha_1, \alpha_2 \rangle dV = c\alpha_1 \wedge \bar{\alpha}_2 \wedge \beta_{n-p}$, where $c = (-1)^{p(p-1)/2}(i/2)^p$.*

We return to our goal – to estimate u_μ. To do this, we try to find the infimum of all r such that

$$\int \frac{|u_\mu|^2}{(1+|\zeta|^2)^{r+n}} dV < \infty. \tag{17}$$ choklad

One can think of r as approximating the polynomial degree of u_μ. Now, we have

$$|u_\mu|^2 dV \leq |\sigma|^2 \sum_{|I|=\mu-1} |\bar{\partial}\sigma_I|^2 dV. \tag{18}$$ $\boxed{\texttt{namarie}}$

We use Proposition 3.5 to rewrite our integrand:

$$\begin{aligned}
\sum_{|I|=\mu-1} |\bar{\partial}\sigma_I|^2 dV &= c \sum_{|I|=\mu-1} \partial\bar{\sigma}_I \wedge \bar{\partial}\sigma_I \wedge \beta_{n-\mu+1} = \\
= \quad c' &\sum_{|I|=\mu-1} i\partial\bar{\sigma}_{i_1} \wedge \bar{\partial}\sigma_{i_1} \wedge \ldots \wedge i\partial\bar{\sigma}_{i_\mu} \wedge \bar{\partial}\sigma_{i_\mu} \wedge \beta_{n-\mu+1} \lesssim \\
\lesssim \quad &\left(\sum_j i\partial\bar{\sigma}_j \wedge \bar{\partial}\sigma_j\right)^{\mu-1} \wedge \beta_{n-\mu+1}. \qquad (19)
\end{aligned}$$ $\boxed{\texttt{solsken}}$

Observe that c' must be a positive number, since the sums on line two and on the left hand side of line one are both positive forms (as is the form on line three). By (18) and (19) we get

$$\begin{aligned}
&\int \frac{|u_\mu|^2}{(1+|\zeta|^2)^{r+n}} dV \leq \\
\lesssim \quad &\int \left(\frac{1}{1+|\zeta|^2}\right)^{r+n} |\sigma|^2 \left(\sum_j i\partial\bar{\sigma}_j \wedge \bar{\partial}\sigma_j\right)^{\mu-1} \wedge \beta_{n-\mu+1}. \qquad (20)
\end{aligned}$$ $\boxed{\texttt{ineq}}$

To proceed, we need a lemma. We will only indicate the proof, as it is mostly raw calculations. Also see page 94 of [4].

Lemma 3.6. *Let f be as before. Then*

$$\sum_j i\partial\bar{\sigma}_j \wedge \bar{\partial}\sigma_j \leq \frac{4}{\epsilon^2} \frac{i\partial\bar{\partial}|f|^\epsilon}{|f|^{2+\epsilon}}. \tag{21}$$ $\boxed{\texttt{morot}}$

Proof. We first prove that

$$\frac{i\partial|f|^2 \wedge \bar{\partial}|f|^2}{|f|^2} \leq i\partial\bar{\partial}|f|^2. \tag{22}$$ $\boxed{\texttt{punchkotte}}$

by expanding the expression $i\partial\bar{\partial}\log|f|^2 \geq 0$, which holds because $\log|f|^2$ is a plurisubharmonic function. To prove the statement in the lemma, first expand the left hand side of (21) and get

$$\sum_j i\partial\bar{\sigma}_j \wedge \bar{\partial}\sigma_j \leq \frac{i\partial\bar{\partial}|f|^2}{|f|^4}.$$

Then we show that

$$\frac{i\partial\bar{\partial}|f|^2}{|f|^4} \leq \frac{4}{\epsilon^2}\frac{i\partial\bar{\partial}|f|^\epsilon}{|f|^{2+\epsilon}}$$

by expanding $i\partial\bar{\partial}|f|^\epsilon = i\partial(\bar{\partial}(|f|^2)^{\epsilon/2})$, which is a positive form since $|f|^\epsilon$ is plurisubharmonic. Then use (22) [punchkotte]. □

Now, if we start with (20) [ineq] and use the lemma we just proved, we get

$$\int |\sigma|^2 \left(\frac{1}{1+|\zeta|^2}\right)^{r+n} \left(\sum_j i\partial\bar{\sigma}_j \wedge \bar{\partial}\sigma_j\right)^{\mu-1} \wedge \beta_{n-\mu+1} \leq$$

$$\lesssim \int |\sigma|^2 \left(\frac{1}{1+|\zeta|^2}\right)^{r+n} \left(\frac{i\partial\bar{\partial}|f|^\epsilon}{|f|^{2+\epsilon}}\right)^{\mu-1} \wedge \beta_{n-\mu+1}.$$

Note that the last estimate is gained at the price of getting a large constant depending on ϵ in front of the integral. But since all we care about is the polynomial growth of the integrand, this is not important. Using our hypothesis (14) [s\"otnos] and the estimate $|\sigma| \leq 1/|f| \lesssim |\zeta|^M$, we get

$$\int |\sigma|^2 \left(\frac{1}{1+|\zeta|^2}\right)^{r+n} \left(\frac{i\partial\bar{\partial}|f|^\epsilon}{|f|^{2+\epsilon}}\right)^{\mu-1} \wedge \beta_{n-\mu+1} \lesssim$$

$$\lesssim \int \left(\frac{1}{1+|\zeta|^2}\right)^{r-(\mu-1)M-M+n-(\mu-1)M\epsilon/2} \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-1} \wedge \beta_{n-\mu+1}.$$

Now we want to use integration by parts. To this end, we take a smooth cutoff function $\chi = \chi(|\zeta|^2)$ such that

$$\chi = \begin{cases} 1 & |\zeta| < 1 \\ 0 & |\zeta| > 2. \end{cases}$$

Then let $\chi_R(\zeta) = \chi(\zeta/R)$. We will integrate

$$\int \chi_R \left(\frac{1}{1+|\zeta|^2}\right)^{r-\mu M+n-(\mu-1)M\epsilon/2} \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-1} \wedge \beta_{n-\mu+1}$$

by parts, and then let $R \to \infty$. To simplify the notation, first set

$$\alpha = \left(\frac{1}{1+|\zeta|^2}\right)^{r-\mu M+n-(\mu-1)M\epsilon/2}. \tag{23}$$

[m\r ankaka]

We want to move $\partial\bar{\partial}$ on the first $|f|^\epsilon$ to the other factors. This will not affect $(\partial\bar{\partial}|f|^\epsilon)^{\mu-2}$'s or $\beta_{n-\mu+1}$, since they are closed, but only α and χ. We have:

$$
\begin{aligned}
&\int \chi_R \alpha \wedge i\partial\bar{\partial}|f|^\epsilon \wedge \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-2} \wedge \beta_{n-\mu+1} = \\
= &\int i\partial\bar{\partial}\chi_R \wedge \alpha |f|^\epsilon \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-2} \wedge \beta_{n-\mu+1} + \\
+ &\int i\partial\chi_R \wedge \bar{\partial}\alpha \wedge |f|^\epsilon \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-2} \wedge \beta_{n-\mu+1} + \\
+ &\int i\partial\alpha \wedge \bar{\partial}\chi_R \wedge |f|^\epsilon \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-2} \wedge \beta_{n-\mu+1} + \\
+ &\int \chi_R i\partial\bar{\partial}\alpha \wedge |f|^\epsilon \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-2} \wedge \beta_{n-\mu+1} \\
= &\ I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{24}
$$

partint

Now, we want to examine each of these four integrals. Note that the integrands of the first three ones have support on $\{R < |\zeta| < 2R\}$; we will choose an r such that these integrals vanish when $R \to \infty$. We first consider I_4, which has support on $\{|\zeta| < R\}$, and to do that we have to look at $i\partial\bar{\partial}\alpha$ (cf. (23)). m\r ankaka

alfa **Lemma 3.7.** *We have the estimate*

$$
i\partial\bar{\partial}\left(\frac{1}{1+|\zeta|^2}\right)^k \leq 2k(k+1)\left(\frac{1}{1+|\zeta|^2}\right)^{k+1}\beta.
$$

Proof. We have:

$$
\begin{aligned}
&i\partial\bar{\partial}\left(\frac{1}{1+|\zeta|^2}\right)^k = \\
= \ &ik(1+|\zeta|^2)^{-k-2}\left((k+1)\partial|\zeta|^2 \wedge \bar{\partial}|\zeta|^2 - (1+|\zeta|^2)\partial\bar{\partial}|\zeta|^2\right).
\end{aligned}
\tag{25}
$$

vatten

Since $i\partial\bar{\partial}|\zeta|^2$ is a positive form, we can just omit the last term when we make an estimate. Also, $\frac{i}{2}\partial|\zeta|^2 \wedge \bar{\partial}|\zeta|^2 \leq |\zeta|^2\beta$, which can be seen by choosing $\zeta = (\zeta_1, 0, \ldots, 0)$, since the forms in question are unchanged by unitary mappings. Thus the lemma follows from (25). vatten $\square$

We can see now that

$$
I_4 \lesssim \int \chi \left(\frac{1}{1+|\zeta|^2}\right)^{r-\mu M+n+1-(\mu-1)M\epsilon/2} \wedge |f|^\epsilon \wedge \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-2} \wedge \beta_{n-\mu+2},
$$

i. e. we have one more factor β and one less factor $i\partial\bar{\partial}|f|^\epsilon$ compared with what we had before the integration by parts. We make the estimate $|f|^\epsilon \lesssim (1+|\zeta|^2)^{d\epsilon/2}$, which gives us

$$I_4 \lesssim \int \chi \left(\frac{1}{1+|\zeta|^2}\right)^{r-\mu M+n+1-(\mu-1)M\epsilon/2-d\epsilon/2} \wedge \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-2} \wedge \beta_{n-\mu+2}.$$

Now we integrate this by parts again. This will give rise to three new "boundary integrals" (we will look closer at these later) and another integral which, like I_4, has support on $\{|\zeta| < R\}$. We then repeat the procedure until we are down to

$$\int \chi\beta_n \left(\frac{1}{1+|\zeta|^2}\right)^{r-\mu M+n+\mu-1-(\mu-1)M\epsilon/2-d(\mu-1)\epsilon/2}.$$

This integral is convergent if

$$r > \mu M - (\mu-1) + \epsilon(\mu-1)(M+d)/2, \tag{26}$$

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so the integral I_4 also converges for this r.

We also have to make sure that the integrals I_1, I_2 and I_3 go to zero as $R \to \infty$. Take I_1, for example, which contains the $(1,1)$-form $i\partial\bar{\partial}\chi_R$. Looking closer at this form, we see that

$$\partial\bar{\partial}\chi_R = \partial\bar{\partial}\chi\left(\frac{|\zeta|^2}{R^2}\right) = \chi''\left(\frac{|\zeta|^2}{R^2}\right)\frac{\partial|\zeta|^2 \wedge \bar{\partial}|\zeta|^2}{R^4} + \chi'\left(\frac{|\zeta|^2}{R^2}\right)\frac{\partial\bar{\partial}|\zeta|^2}{R^2}.$$

Using the same idea as in the proof of Lemma alfa 3.7, we can make the estimate

$$i\partial\bar{\partial}\chi_R \leq \frac{1}{R^2}\phi_1(|\zeta|^2/R^2)\beta,$$

where ϕ_1 is some positive rotation-invariant function that has support on the annulus $1 \leq |\zeta| \leq 2$ and $\phi_1(\zeta) \geq \chi''(\zeta) + \chi'(\zeta)$.

Now we can say that

$$I_1 = \int i\partial\bar{\partial}\chi_R \wedge \left(\frac{1}{1+|\zeta|^2}\right)^k |f|^\epsilon \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-2} \wedge \beta_{n-\mu+1} \leq$$
$$\int \frac{1}{R^2}\phi_1(|\zeta|^2/R^2) \cdot \left(\frac{1}{1+|\zeta|^2}\right)^{k-d\epsilon/2} \left(i\partial\bar{\partial}|f|^\epsilon\right)^{\mu-2} \wedge \beta_{n-\mu+2}. \tag{27}$$

This looks like what we had before the integration by parts, except that the cutoff function has support on $\{R \leq |\zeta| \leq 2R\}$ instead, and that we one more β and one less $i\partial\bar{\partial}|f|^{\epsilon}$. And, of course, that we have an extra $1/R^2$ to help with the convergence. Now, we can just repeat the integration by parts, if we check that I_2 and I_3 will behave nicely as well.

I_2 contains the form

$$\begin{aligned} i\partial\chi\left(|\zeta|^2/R^2\right) \wedge \bar{\partial}\left(\tfrac{1}{1+|\zeta|^2}\right)^k &= \\ = i\chi'\left(|\zeta|^2/R^2\right)\tfrac{1}{R^2}\partial|\zeta|^2 \wedge \left(\tfrac{-k}{1+|\zeta|^2}\right)^{k+1}\bar{\partial}|\zeta|^2 &\lesssim \\ \lesssim \phi_2(|\zeta|^2/R^2)\left(\tfrac{1}{1+|\zeta|^2}\right)^k \tfrac{1}{R^2}\beta, & \end{aligned} \tag{28}$$

where ϕ_2 is a positive function that has support on the annulus $1 \leq |\zeta| \leq 2$ and $\phi_2(|\zeta|^2/R^2) \geq \chi'\left(|\zeta|^2/R^2\right)$. Just as with I_1, we have got one more β, one less $i\partial\bar{\partial}|f|^{\epsilon}$, and an extra $1/R^2$. Now we repeat the integration by parts. The integral I_3 is in fact identical to I_2.

We have shown that for each extra β, we get an extra factor $1/R^2$. When we have done partial integrations until no $i\partial\bar{\partial}|f|^{\epsilon}$ remains, the integrand will be (except for a cutoff-function)

$$\frac{1}{R^{2(\mu-1)}}\left(\frac{1}{1+|\zeta|^2}\right)^{r-\mu M+n-\epsilon(\mu-1)(M+d)/2},$$

which we can just as well write as

$$\left(\frac{1}{1+|\zeta|^2}\right)^{r-\mu M+n+\mu-1-\epsilon(\mu-1)(M+d)/2}.$$

If we choose $r > \mu M - (\mu-1) + \epsilon(\mu-1)(M+d)/2$, as above, we see that all the integrals with cutoff functions that have support on $R \leq |\zeta| \leq 2R$ will go to zero, which is what we wanted to prove. To sum up, we have determined that

$$\int \frac{|u_\mu|^2}{(1+|\zeta|^2)^{r+n+\epsilon'}} dV < \infty \tag{29}$$

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for $r \geq \mu M - (\mu-1)$, where $\epsilon' = \epsilon(\mu-1)(M+d)/2$ is chosen to be smaller than 1. We will continue by working back through the equations (k-eqns 15) to get an estimate for ψ. We apply Theorem godis 3.2 to $\bar{\partial}w_\mu = u_\mu$ and get

$$\int \frac{|w_\mu|^2}{(1+|\zeta|^2)^{r+n+1+\epsilon'}} < \infty.$$

Next we want to estimate $w_{\mu-1}$. We know that $\bar{\partial} w_{\mu-1} = u_{\mu-1} + \delta_f w_\mu$, and since $u_{\mu-1}$ grows much less than u_μ and the application of δ_f amounts to multiplying with a polynomial of degree d, we see that

$$\int \frac{|\bar{\partial} w_{\mu-1}|^2}{(1+|\zeta|^2)^{r+n+1+d+\epsilon'}} < \infty.$$

By Theorem 3.2 it follows that

$$\int \frac{|w_{\mu-1}|^2}{(1+|\zeta|^2)^{r+n+2+d+\epsilon'}} < \infty.$$

Then we continue in the same way until we get

$$\int \frac{|\psi|^2}{(1+|\zeta|^2)^{r+n+(\mu-1)+d(\mu-1)+\epsilon'}} < \infty$$

for $\psi = u_1 + \delta_f w_2$, where $r \geq \mu M - (\mu - 1)$. But since ψ is a holomorphic function, it follows from Liouville's theorem that it must be a polynomial of degree at most

$$\begin{aligned} & r + (\mu-1) + d(\mu-1) + \epsilon' = \mu M - (\mu-1) + (\mu-1) + d(\mu-1) + \epsilon' = \\ = \; & \mu(M+d) - d + \epsilon'. \end{aligned} \tag{30}$$

Since ϵ' can be chosen arbitrarily small, we conclude that $\deg \psi$ is at most $\mu(M+d) - d$.

Let us discuss how small M can be for this to be true. First, we rewrite (14) as

$$\frac{|f|}{(1+|z|)^d} \gtrsim \frac{1}{(1+|z|)^{M+d}}.$$

Since f is a tuple of polynomials with no common zeroes in $\mathbb{C}^n$, we have $M + d \geq 0$, so $M \geq -d$. The other condition in Theorem 3.1 arises because when we use Theorem 3.2 on (29), we need to have $r \geq -2n$, which means that $\mu M - (\mu - 1) \geq -2n$. We need only make sure that this condition holds if $m > n$, though, since otherwise the form u_μ is not of degree $(0, n)$. In this case the condition is $(n+1)M - n \geq -2n$ or $(n+1)M \geq -n$.

3.2 Using integral formulas

It is also possible to get an explicit solution to the division problem by using the integral formulas of Berndtsson [6]. We can then reuse some of the estimates in the section on the Koszul complex section to obtain an estimate of the degree of the solutions which is only slightly worse. First some preliminaries. We regard f again as a tuple, not as a section of a vector bundle. Let $\sigma = (\bar{f}_1/|f|^2, \dots, \bar{f}_m/|f|^2)$, $\mu = \min(n+1, m)$, and let $h(\zeta, z) = (h_1, \dots, h_m)$ be a vector of holomorphic $(1,0)$-forms such that $\delta_{\zeta - z} h_j = f_j(\zeta) - f_j(z)$. The h_i are called Hefer forms, and can be explicitly constructed since the f_i's are polynomials. Note that the coefficients of the Hefer forms will then be polynomials of degree $d-1$ in both z and ζ. We have the theorem

Theorem 3.8. *Let $f(z) = (f_1(z), ..., f_m(z))$ be a tuple of polynomials such that $\deg(f_i) \le d$, $|f(\zeta)| > 0$ in $\mathbb{C}^n$, and $1/|f(z)| \lesssim |z|^M$. Then*

$$p_i(z) = \int \sigma_i \sum_{k=0}^{\mu} c_n \left(i\bar{\partial} \frac{\bar{\zeta} \cdot d\zeta}{1 + |\zeta|^2} \right)^{n-k} \wedge (h \cdot \bar{\partial}\sigma)^k \left(\frac{1 + z \cdot \bar{\zeta}}{1 + |\zeta|^2} \right)^{r-n+k} (f(z) \cdot \sigma)^{\mu-k-1}$$

where $c_n = \binom{r}{n-k}\binom{\mu}{k}$. We also have the estimate $\deg p_i \le \mu(M + d) - d + (d-1)(\mu - 1)$. The result is true for $M \ge -d$, and if $m > n$ we have the additional requirement that $(n+1)M \ge -n$.

Proof. From Proposition viktint 2.4 we know that if $g(\zeta, z)$ is a weight, then

$$1 = \int_{\partial D} (u \wedge g)_{n,n-1} + \int_D g_{n,n}. \tag{31}$$ intform

The indices indicate the bidegree, u is the Bochner-Martinelli kernel, and in our application, $D = B(0, R)$. We want to use a weight g such that the first integral disappears when $R \to \infty$, and such that $f(z)$ is a factor in g. We will then estimate the degree of the rest of the second term, which will be our $p(z)$. Consider

$$\begin{aligned} g_1 &= \left(\frac{1 + z \cdot \bar{\zeta}}{1 + |\zeta|^2} + i\bar{\partial} \frac{\bar{\zeta} \cdot d\zeta}{1 + |\zeta|^2} \right)^r = \left(1 - \nabla \frac{\bar{\zeta} \cdot d\zeta}{1 + |\zeta|} \right)^r \\ g_2 &= (f(z) \cdot \sigma + h \cdot \bar{\partial}\sigma)^\mu = (1 - \nabla(h \cdot \sigma))^\mu. \end{aligned} \tag{32}$$ vikt

which are weights. Now let $g = g_1 \wedge g_2$. Then g will be our desired weight for some value of r, as g_1 will make the boundary integral go to zero, and $f(z)$ is a factor in g_2 since $(\bar{\partial}\sigma)^\mu = 0$, just as in the section on the Koszul complex. The idea now is to find an r such that the first integral in (intform 31) goes to zero, and the second converges, as $R \to \infty$. Then the degree of $p(z)$ must be $r - n + d(\mu - 1)$.

We start by looking at the second integral. Writing it explicitly, we see that it is equal to

$$\int (f(z)\cdot\sigma - h\cdot\bar{\partial}\sigma)^{\mu}\wedge\left(\frac{1+z\cdot\bar{\zeta}}{1+|\zeta|^2}+\omega\right)^{r}. \tag{33}$$

Note that $\omega = i\bar{\partial}\bar{\zeta}\cdot d\zeta/(1+|\zeta|^2) = i\partial\bar{\partial}\log(1+|\zeta|^2)$, which is the second term in g_1, is a strictly positive $(1,1)$-form. It gives rise in a natural way to a Hermitian metric on $T_{1,0}$, which induces a Riemannian metric, dependent on ω, for which the volume form is $dV = \left(\frac{1}{2}\right)^n\omega_n$. The term in the integrand of (33) which has the largest polynomial growth is

$$\sigma(h\cdot\bar{\partial}\sigma)^{\mu-1}\wedge\left(\frac{1+z\cdot\bar{\zeta}}{1+|\zeta|^2}\right)^{r-n+\mu-1}\omega_{n-\mu+1}.$$

If we use the scalar product induced by ω instead of β, then by Proposition 3.5 we have

$$\begin{aligned}(h\cdot\bar{\partial}\sigma)^{\mu-1}\wedge\omega_{n-\mu+1} &= \sum_{|I|=\mu} h_{i_1}\wedge\bar{\partial}\sigma_{i_1}\wedge\ldots\wedge h_{i_{\mu-1}}\wedge\bar{\partial}\sigma_{i_{\mu-1}}\wedge\omega_{n-\mu+1} = \\ &= \pm\sum_{|I|=\mu-1} h_{i_1}\wedge\ldots\wedge h_{i_{\mu-1}}\ldots\bar{\partial}\sigma_{i_1}\wedge\ldots\wedge\bar{\partial}\sigma_{i_{\mu-1}}\wedge\omega_{n-\mu+1} = \\ &= \pm\sum_{|I|=\mu-1}\langle h_I,\bar{\partial}\sigma_I\rangle_\omega dV. \end{aligned}\tag{34}$$

Thus, our integral converges if

$$\int\left|\sigma\left(\frac{1+z\cdot\bar{\zeta}}{1+|\zeta|^2}\right)^{r-n+\mu-1}\sum_{|I|=\mu}\langle h_I,\bar{\partial}\sigma_I\rangle_\omega\right| dV<\infty. \tag{35}$$

To simplify our formulas, we observe that

$$\left|\frac{1+z\cdot\bar{\zeta}}{1+|\zeta|^2}\right|\lesssim\frac{1}{1+|\zeta|}.$$

We use first the triangle inequality on (35), and then the Cauchy-Schwarz inequality three times in a row – first on the scalar product inside the sum, then on the sum itself, and finally on the whole integral:

$$\left(\int\left|\sigma\left(\frac{1}{1+|\zeta|}\right)^{r-n+\mu-1}\sum_{|I|}\langle h_I,\bar{\partial}\sigma_I\rangle_\omega\right| dV\right)^2 \le$$

$$
\begin{aligned}
&\leq \left(\int |\sigma| \left(\frac{1}{1+|\zeta|} \right)^{r-n+\mu-1} \sum_{|I|} |\langle h_I, \bar{\partial}\sigma_I \rangle_\omega| dV \right)^2 \leq \\
&\leq \left(\int |\sigma| \left(\frac{1}{1+|\zeta|} \right)^{r-n+\mu-1} \sum_{|I|} |h_I|_\omega |\bar{\partial}\sigma_I|_\omega dV \right)^2 \leq \\
&\leq \left(\int |\sigma| \left(\frac{1}{1+|\zeta|} \right)^{r-n+\mu-1} \sqrt{\sum_{|I|} |h_I|^2_\omega} \sqrt{\sum_{|I|} |\bar{\partial}\sigma_I|^2_\omega dV} \right)^2 \lesssim \\
&\lesssim \int \left(\frac{1}{1+|\zeta|^2} \right)^k \sum_{|I|} |h_I|^2_\omega dV \times \\
\times \quad &\int |\sigma|^2 \sum_{|I|} |\bar{\partial}\sigma_I|^2_\omega \left(\frac{1}{1+|\zeta|^2} \right)^{r-n+\mu-1-k} dV
\end{aligned}
\tag{36}
$$

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We need to determine what k must be for the first integral in the product to converge. We rewrite the integrand as

$$
\left(\frac{1}{1+|\zeta|^2} \right)^k \sum_{|I|} c_I h_I \wedge \bar{h}_I \wedge \omega_{n-\mu+1},
$$

where the c_I are constants. Each h_i is a $(1,0)$-form with polynomial degree $d-1$. Also

$$
\omega \leq \frac{\beta}{1+|\zeta|^2}, \tag{37}
$$

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so to compensate, $k = (d-1)(\mu-1) - (n-\mu+1) + n = d(\mu-1)$. The second integral in the product (36) is almost the same integral as (17) in the previous section. The main difference is that we have ω instead of β. Using (37), we see that

$$
\omega_{n-\mu+1} \leq \frac{\beta_{n-\mu+1}}{(1+|\zeta|^2)^{n-\mu+1}},
$$

and so $r-n+\mu-1-k+n-\mu+1 = \mu M - (\mu-1) + \epsilon + n$, where we get the right hand side from (26). That is, $r = \mu M + n + (d-1)(\mu-1) + \epsilon$, so $\deg p \leq r - n + d(\mu-1) = \mu(M+d) - d + (d-1)(\mu-1)$. □

4 Proving Briançon-Skoda's theorem by means of integral formulas

We can use the same integral formulas as in the previous section to prove Briançon-Skoda's theorem (see BS [16]):

chirka **Theorem 4.1.** *Let* $f_1, \dots, f_m, \phi$ *be germs of holomorphic functions at* $0 \in \mathbb{C}^n$*, and assume that*

$$|\phi| \lesssim |f|^{\mu+r-1}, \tag{38}$$ elbereth

where $\mu = \min(m, n)$*. Then* ϕ *belongs to the ideal* $(f)^r$*.*

By $\phi \in (f)$ we mean that there exists a tuple $p(z)$ of holomorphic functions such that $\phi = f \cdot p$, and $(f)^r$ is then the ideal generated by all products of r elements of (f).

The original statement of Briançon-Skoda is that $\overline{(f)}^{\mu+r-1} \subseteq (f)^r$. Here $\overline{(f)}$ is the integral closure of (f), and $\phi \in \overline{(f)}$ is equivalent with $|\phi| \lesssim |f|$. The original theorem follows from our theorem in the following way: take $\phi \in \overline{(f)}^{\mu+r-1}$. Then ϕ is generated by elements of the type $\phi_1 \cdot \ldots \cdot \phi_{\mu+r-1}$, where $\phi_i \in \overline{(f)}$, meaning that $|\phi| \lesssim |f|$. So we get $|\phi| \lesssim |f|^{\mu+r-1}$, which by Theorem chirka 4.1 implies that $\phi \in (f)^r$.

In the proof of the theorem we wish to use the same type of integral formulas as in the previous section. The difference is that there we wished to solve a global division problem without common zeroes, and the problem was the behavior at infinity. Now, we wish to solve a local division problem with zeroes, and we want to use the same ideas as in the previous section. We cannot use the same weight g_2 as before (see vikt (32)) since it has singularities where $f(\zeta) = 0$. So we start by finding a replacement to it. If we first define

$$\sigma^\epsilon(\zeta) = \frac{\bar{f}(\zeta)}{|f(\zeta)|^2 + \epsilon}$$

and then

$$g_2^\epsilon(\zeta, z) = f(z) \cdot \sigma^\epsilon(\zeta) + h(\zeta, z) \cdot \bar{\partial}\sigma^\epsilon(\zeta) + \frac{\epsilon}{|f(\zeta)|^2 + \epsilon},$$

then g_2^ϵ will be a weight. This is so because it is equal to $1 - \nabla(h \cdot \sigma^\epsilon)$, which is shown by an easy calculation. Note that when $\epsilon \to 0$, then $g_2^\epsilon \to f(z) \cdot \sigma + h \cdot \bar{\partial}\sigma = g_2$ where $f(\zeta) \neq 0$.

We also need to construct a weight with compact support. To do this, let u be a Cauchy form, that is $\nabla u = 1 - [0]$, and take a cut-off function with compact support χ that is 1 in a neighborhood of the origin. Then $g_1 = 1 - \nabla(u - \chi u)$ is a weight whose compact support includes 0, since

$$g_1 = 1 - (1 - [0]) - \bar{\partial}\chi \wedge u + \chi(1 - [0]) = \chi - \bar{\partial}\chi \wedge u.$$

According to Proposition 2.4, we have

$$\begin{aligned} \phi(z) &= \int (g_2^\epsilon)^{l+r} \wedge g_1 \wedge \phi = \\ &= \binom{l+r}{r} \int (f(z)\cdot\sigma^\epsilon)^r \wedge (g_2^\epsilon)^l \wedge g_1 \wedge \phi + \\ &+ \int \sum_{k=0}^{r-1} \binom{l+r}{k} (f(z)\cdot\sigma^\epsilon)^k \left(h\cdot\bar{\partial}\sigma^\epsilon + \frac{\epsilon}{|f|^2+\epsilon}\right)^{l+r-k} \wedge g_1 \wedge \phi. \end{aligned} \tag{39}$$

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Note that the boundary integral disappears since g_1 has compact support and that the first integral in the right hand side is an element in $(f)^r$ (if it converges). To prove the theorem, we look at the cases $m > n$ and $m \le n$ separately. The first case will be proved by taking $l = n$ in (39) and proving that the second integral goes to 0 when $\epsilon \to 0$ (Proposition 4.2), and that the first integral is convergent (Proposition 4.6) after we factor out the $f(z)$. The second case is proved in a similar way in Proposition 4.4 and Proposition 4.5 by taking $l = m - 1$ in (39). We begin with the case $m > n$.

ida **Proposition 4.2.** *If (38) holds and $m > n$, then*

$$\int \sum_{k=0}^{r-1} \binom{n+r}{k} (f(z)\cdot\sigma^\epsilon)^k \left(h\cdot\bar{\partial}\sigma^\epsilon + \frac{\epsilon}{|f|^2+\epsilon}\right)^{n+r-k} \wedge g_1 \wedge \phi \tag{40}$$

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converges to zero when $\epsilon \to 0$.

Proof. Since the zero set of f might be complicated, we use Hironaka's theorem on resolution of singularities to obtain a zero set with normal crossings. More precisely, according to Hironaka's theorem, if we take a small enough neighborhood U of the origin, there exists an n-dimensional manifold U_1 and a proper analytic map $\Pi_1 : U_1 \to U$, with the following properties: if we let $Y = \{z : f_1(z)\cdot\ldots\cdot f_m(z) = 0\}$ and $Y_1 = \Pi_1^{-1}(Y)$, then $\Pi_1 : U \setminus Y \to U_1 \setminus Y_1$ is biholomorphic and Y_1 has (finitely many) normal crossings in U_1.

If g_1 has support in U and K is the integrand of (40), then (40) is equal to the integral $\int_{U_1} \pi_1^* K$. Since there might be many crossings, we take a partition of unity $\{\rho_j\}$ such that the support of each ρ_j contains only one crossing. The partition of unity is finite, so we choose one of the integrals $\int_{U_1} \rho_j \pi_1^* K$ and check if it converges to zero. In the support of ρ_j we can find local coordinates τ_k such that we can write $\Pi_1^* f_j = a_j \mu_j$, where the a_j are non-vanishing and the μ_j are monomials in τ_k.

We can simplify the problem even more, since given a finite number of monomials $\mu_1, \ldots, \mu_m$ defined in $\operatorname{supp}(\rho_j)$, there exists a toric manifold U_2 and a proper holomorphic map $\Pi_2 : U_2 \to \operatorname{supp}(\rho_j)$, such that Π_2 is biholomorphic outside the coordinate axes, and locally it is true that for some i, $\Pi_2^* \mu_i$ will divide all the other $\Pi_2^* \mu_j$. Also the $\Pi_2^* \mu_j$'s will still be monomials. (For more on these techniques, see [5].) So we get

$$\int \rho_j \pi_1^* K = \int_{U_2} \pi_2^* (\rho_j \pi_1^* K).$$

Then we find another partition of unity $\{\rho'_j\}$, where the support of each ρ'_j is such that some $\Pi_2^* \mu_i$ divides all the others. The partition of unity is finite since $\bar{U}_2$ is compact. We choose some ρ'_k and look at

$$\int_{U_2} \rho'_k \pi_2^* (\rho_j \pi_1^* K). \tag{41}$$

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If this integral goes to zero, then we will be done. What it boils down to is that we can assume that f, after the pullbacks and partitions of unity, can be replaced with $f_0 f'$, where $f' = (f'_1, \ldots, f'_m)$ has no common zeroes in the neighborhood where we integrate. We can also assume that f_0 is a monomial, i. e. $f_0 = z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}$. Now look at the integrand of (41): both ρ'_k and $\pi_2^*(\rho_j)$ are just smooth functions. But each term in $\pi_2^*(\pi_1^* K)$ we can write as

$$\left(f(z) \cdot \frac{\overline{f_0 f'}}{|f_0 f'|^2 + \epsilon} \right)^k \left(\pi_2^*(\pi_1^* h) \cdot \bar{\partial} \frac{\overline{f_0 f'}}{|f_0 f'|^2 + \epsilon} + \frac{\epsilon}{|f_0 f'|^2 + \epsilon} \right)^{n+r-k} \wedge \pi_2^* \pi_1^* (\phi g_1).$$

The form $\pi_2^*(\pi_1^* h)$ is a holomorphic $(1,0)$-form if h is, and $\phi \circ \pi_1 \circ \pi_2$ satisfies (38) if ϕ does. The form $\pi_2^* \pi_1^* g_1$ is also smooth, so the only important thing that has changed is that we can replace f with $f_0 f'$ - it does not really matter which smooth forms are involved, since we are anyway integrating over a compact set. To avoid a complicated notation, we will proceed with the original integral, with f replaced with $f_0 f'$.

First observe that

$$\left(h \cdot \bar{\partial} \sigma^\epsilon \right)^{n+1} = 0 \tag{42}$$

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for degree reasons. For $k \leq n$ we have

$$\left(h \cdot \bar\partial \sigma^\epsilon\right)^k = \left(\bar\partial\left[\frac{\bar f_0}{|f_0|^2|f'|^2+\epsilon}\cdot(h\cdot \bar f')\right]\right)^k = \left(\frac{\bar f_0}{|f_0|^2|f'|^2+\epsilon}\right)^k (h\cdot\bar\partial \bar f')^k +$$

$$+k\bar\partial\left(\frac{\bar f_0}{|f_0|^2|f'|^2+\epsilon}\right)\left(\frac{\bar f_0}{|f_0|^2|f'|^2+\epsilon}\right)^{k-1} \wedge (h\cdot \bar f')\wedge(h\cdot\bar\partial\bar f')^{k-1}. \tag{43}$$

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Since f_0 is a monomial, we have $|f_0| \lesssim |df_0|$ close to the origin. Then we can show (we omit some calculations) that

$$\left|\left(h\cdot\bar\partial\sigma^\epsilon\right)^k\right| \lesssim \frac{|\bar\partial \bar f_0||f_0|^{k-1}}{(|f_0|^2+\epsilon)^k}. \tag{44}$$

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Recall that we want to look at the integral (40). With (42) in mind, we can show that the term in the integrand with the worst singularity is

$$\phi g_1 \wedge (f(z)\cdot\sigma^\epsilon)^{r-1}(h\cdot\bar\partial\sigma^\epsilon)^n \cdot \frac{\epsilon}{|f|^2+\epsilon}. \tag{45}$$

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By (44) we can estimate the integral over the absolute value of (45) with a constant times

$$\epsilon\int_{|\zeta|<1}\frac{|\bar f_0|^{2n+2r-3}|\bar\partial\bar f_0|}{(|f_0|^2+\epsilon)^{n+r}}, \tag{46}$$

faktura

if we assume that the support of the weight g_1 is the unit ball and use the estimate (38).

Recall that we can assume that $f_0 = z_1^{k_1}\cdot\ldots\cdot z_n^{k_n}$. We will look at the term of $|\bar\partial\bar f_0|$ containing the first of the partial derivatives, which is equal to

$$\epsilon\int_{|z|<1}\frac{|z_1|^{k_1(2n+2r-2)-1}\cdot\ldots\cdot|z_n|^{k_n(2n+2r-2)}}{(|z_1|^{2k_1}\cdot\ldots\cdot|z_n|^{2k_n}+\epsilon)^{n+r}}. \tag{47}$$

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We will use the dominated convergence theorem to show that (47) converges to zero when $\epsilon\to 0$. First,

$$\frac{|z_1|^{2k_1}\cdot\ldots\cdot|z_n|^{2k_n}}{(|z_1|^{2k_1}\cdot\ldots\cdot|z_n|^{2k_n}+\epsilon)}\le 1 \quad\text{and}\quad \frac{\epsilon}{(|z_1|^{2k_1}\cdot\ldots\cdot|z_n|^{2k_n}+\epsilon)}\le 1,$$

so the integrand of I_k can be estimated with $|z_1|^{-1}$ which is integrable. Moreover, when $\epsilon\to 0$, the integrand goes to zero pointwis, so the integral (40) goes to zero. □

We have now shown that the second term of (39) vanishes if $m > n$, so we look at the same problem in the case $m\le n$, and postpone the discussion of whether the first integral in (39) is convergent. First we have a lemma:

lingon **Lemma 4.3.** *We have*

$$(h \cdot \bar\partial \sigma^\epsilon)^m = \epsilon \frac{(h \cdot \bar\partial \bar f)^m}{(|f|^2+\epsilon)^{m+1}}. \tag{48}$$

Proof. We first make the calculation

$$(h \cdot \bar\partial \sigma^\epsilon)^m = \left(\frac{h \cdot \bar\partial \bar f}{|f|^2+\epsilon} - \frac{(h\cdot \bar f)\bar\partial |f|^2}{(|f|^2+\epsilon)^2} \right)^m$$

$$= \left(\frac{h \cdot \bar\partial \bar f}{|f|^2+\epsilon} \right)^m - m \frac{((h\cdot\bar\partial \bar f)^{m-1} h \cdot \bar f)\bar\partial |f|^2}{(|f|^2+\epsilon)^{m+1}}$$

$$= \frac{((h\cdot\bar\partial \bar f)^{m-1}}{(|f|^2+\epsilon)^{m+1}} \big(h\cdot \bar\partial \bar f(|f|^2+\epsilon) - m(h\cdot \bar f)\bar\partial|f|^2 \big)$$

$$= \epsilon \frac{(h \cdot \bar\partial \bar f)^m}{(|f|^2+\epsilon)^{m+1}} + \frac{((h\cdot\bar\partial \bar f)^{m-1}}{(|f|^2+\epsilon)^{m+1}} \big(h\cdot \bar\partial \bar f(|f|^2+\epsilon) - m(h\cdot \bar f)\bar\partial|f|^2 \big). \tag{49}$$ niniel

We must show that the second term of (49) is zero. But we have

$$(h\cdot \bar\partial \bar f)^{m-1} = (m-1)! \sum_1^m \widehat{h_j \wedge \bar\partial \bar f_j}$$

and then the second term will be equal to

$$(m-1)! \left(\sum_1^m (h\cdot\bar\partial \bar f)^m |f|^2 - m \sum_1^m f_j \bar f_j (h \cdot \bar\partial \bar f)^m \right) = 0.$$

□

Going back to our integrals, we look at the second integral in (39) in the case $l = m-1$.

sus **Proposition 4.4.** *If (38) holds and $m \le n$, then*

$$\int \sum_{k=0}^{r-1} \binom{m+r-1}{k} (f(z)\cdot \sigma^\epsilon)^k \left(h \cdot \bar\partial \sigma^\epsilon + \frac{\epsilon}{|f|^2+\epsilon} \right)^{m+r-1-k} \wedge g_1 \wedge \phi$$

converges to zero when $\epsilon \to 0$.

Proof. Like in the previous proof, we can replace f with $f_0 f'$. The worst term in the integrand is

$$\begin{aligned}(f(z)\cdot\sigma^\epsilon)^{r-1}(h\cdot\bar\partial\sigma^\epsilon)^m\wedge g_1\wedge\phi &=\\ = \quad \epsilon\phi g_1\wedge\frac{(f(z)\cdot\bar f')^{r-1}\bar f_0^{r-1}(h\cdot\bar\partial\bar f)^m}{(|f|^2+\epsilon)^{r+m}}&, \end{aligned} \tag{50}$$ label

where we have used Lemma 4.3. As in the proof of Proposition 4.2, we can estimate the integral of the absolute value of (50) with

$$\epsilon\int_{|\zeta|<1}\frac{|f_0|^{2m+2r-3}|\bar\partial\bar f_0|}{(|f_0|^2+\epsilon)^{r+m}}.$$

The rest follows as in the proof of Proposition 4.2. □

We turn to the next proposition, which will show that the first integral in (39) is convergent when $l=m-1$, if we first factor out the $f(z)$'s.

polyfoni **Proposition 4.5.** *Assume that (38) holds and that $m\le n$. Then*

$$\sum_{|I|=r}\sigma_I^\epsilon\wedge(g_2^\epsilon)^{m-1}\wedge g_1\wedge\phi, \tag{51}$$ russin

where $\sigma_I^\epsilon=\sigma_{i_1}^\epsilon\dots\sigma_{i_r}^\epsilon$, converges to an integrable function when $\epsilon\to 0$.

Proof. As in the proof of Proposition 4.2, we choose one of the integrals resulting from the pullbacks and partitions of unity. The calculations are again very similar to that proof and we will only state that we can dominate the absolute value of (51) with

$$\frac{|f_0|^{2m+2r-3}|\bar\partial\bar f_0|}{(|f_0|^2+\epsilon)^{r+m-1}}. \tag{52}$$ kram

This function is dominated by $|z|^{-1}$ in the same way as (46) is. Thus (52) converges to an integrable function. Does it follow that the original function converges to something integrable as well? To get the original integral back, we must start with terms like $\rho_k'\pi_2^*(\rho_j\pi_1^*K)$, where K is the original integral. Each of these terms is integrable. Then we must apply $(\pi_1)_*$ and $(\pi_2)_*$ and sum over j and k. But outside the sets where we have singularites, our projections Π_1 and Π_2 are in fact biholomorphisms, which means that the original integral will be integrable there. Note that the sets where we have singularites are zero sets. Let s be the integrand that we want to show is integrable, and let $\chi_{1/n}$ be cutoff functions with support in $U\backslash Y$ that increase pointwise to χ with support in U. Then

$$\sup_n \int |s\chi_{1/n}| < \infty$$

since the projections are biholomorphic in the support of every $\chi_{1/n}$. Then $\lim_{n\to\infty} \int |s\chi_{1/n}|$ must exist and be equal to $\int |s\chi|$ by the monotone convergence theorem, which means that the original integral is integrable. □

We turn to the case $m > n$, where we have to show that the first term of (39), with $l = n$, converges when $\epsilon \to 0$. In this case the integrand will not converge to an integrable function, but rather to a current operating on ϕ. First we note that $g_1 = \chi - \bar{\partial}\chi \wedge u$, and $\bar{\partial}\chi = 0$ close to 0, so the terms containing $\bar{\partial}\chi$ will be convergent. This means we only have to look at the term containing χ. Then we can say the following:

petasites **Proposition 4.6.** *If we assume that we can replace f with $f_0 f'$, where $|f'| > 0$ close to the origin and f_0 is a monomial, we have*

$$\lim_{\epsilon\to 0}\int \sum_{|I|=r} \phi\chi\sigma_I^{\epsilon} \wedge (h\cdot\bar{\partial}\sigma^{\epsilon})^n = \left[\frac{1}{f_0^{r+n}}\right]\cdot\left(\sum_{|I|=r}\phi\chi\frac{\bar{f}'_I(\bar{\partial}\bar{f}'\cdot h)^n}{|f'|^{2(r+n)}}\right) + \\ + \;\bar{\partial}\left[\frac{1}{f_0^{r+n}}\right]\cdot\left(\sum_{|I|=r}\phi\chi\frac{\bar{f}'_I(\bar{\partial}\bar{f}'\cdot h)^{n-1}(\bar{f}'\cdot h)}{|f'|^2(r+n)}\right) \qquad (53)$$ simpsons

where $\left[1/f_0^{r+n}\right]$ denotes the principal value current of $1/f_0^{r+n}$.

This proposition does not mean that the original integral is equal to these currents, only that after we do pullbacks and partitions of unity as in the proof of Proposition 4.2, then one of the resulting integrals can be written using the currents above. From this we can draw the conclusion that the original integral converges to *something*, though the limit of the integrand is not integrable. But actually, this is all we need, since the integral will depend holomorphically on z, and thus it will finish the proof of Briançon-Skoda's theorem. To prove Proposition 4.6, we need the following lemma [14]:

wombat **Lemma 4.7.** *If ϕ is a test form, f_0 is a holomorphic monomial and α is a smooth non-zero function, then*

$$\lim_{\epsilon\to 0}\int\left(\frac{\bar{f}_0}{|f_0|^2\alpha+\epsilon}\right)^k \wedge \phi = \left[\frac{1}{f_0^k}\right].\frac{\phi}{\alpha}.$$

Proof. (of Proposition 4.6) By (43) we have

$$\int \sum_{|I|=r} \phi\chi\sigma_I^\epsilon \wedge (h\cdot\bar\partial\sigma^\epsilon)^n = \int \sum_{|I|=r} \phi\chi \bar f_I' (\bar\partial \bar f' \wedge h)^n \left(\frac{\bar f_0}{|f|^2+\epsilon}\right)^{r+n} +$$

$$+ \quad \int \sum_{|I|=r} \phi\chi \bar f_I' (\bar\partial \bar f' \wedge h)^{n-1} \wedge (\bar f' \wedge h) \left(\frac{\bar f_0}{|f|^2+\epsilon}\right)^{r+n-1} \wedge \bar\partial \frac{\bar f_0}{|f|^2+\epsilon}. \quad (54)$$

Using Lemma 4.7 we can see that the first integral on the right hand side of (54) converges to the first integral on the right hand side of (53). As for the other integral, first we note that

$$\left(\frac{\bar f_0}{|f|^2+\epsilon}\right)^{r+n-1} \wedge \bar\partial \frac{\bar f_0}{|f|^2+\epsilon} = \frac{1}{r+n} \bar\partial \left(\frac{\bar f_0}{|f|^2+\epsilon}\right)^{r+n},$$

and then another application of the lemma shows that the second integral on the right hand side of (54) converges to the second integral on the right hand side of (53). □

5 Integral formulas on $\mathbb{P}^n$

intformpn

We would like to find integral formulas on for sections of line bundles on $\mathbb{P}^n$. Such formulas on $\mathbb{P}^n$ have been considered before in [12], where they were constructed by using known formulas in $\mathbb{C}^{n+1}$, and in [7], where they were constructed directly on $\mathbb{P}^n$. I will also construct formulas directly on $\mathbb{P}^n$, but by using an analogue to the method used in Section 2, which allows for greater flexibility.

First we define $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ by $\pi(\zeta) = [\zeta]$, where $[\zeta]$ denotes the set of all non-zero multiples of ζ. $\mathbb{P}^n$ can be covered with a set of coordinate neighborhoods $\{U_j\}$, where for example $U_0 = \{[\zeta] : \zeta_0 \neq 0\}$, and the local coordinates in U_0 are given by $\pi_0(\zeta_0, \dots, \zeta_n) = (\zeta_1/\zeta_0, \dots, \zeta_n/\zeta_0) = (\zeta_1', \dots, \zeta_n')$.

We want to characterize differential forms in $\mathbb{P}^n$. Let us take a differential $(1,0)$-form $\alpha(\zeta') = f(\zeta_1', \dots, \zeta_n')d\zeta_i'$ in U_0 and look at $\pi_0^*\alpha(\zeta')$. This will be a differential form in $\mathbb{C}^{n+1}$:

$$\pi^*\alpha = f\left(\frac{\zeta_1}{\zeta_0}, \dots, \frac{\zeta_n}{\zeta_0}\right) d\left(\frac{\zeta_{i_1}}{\zeta_0}\right) = f\left(\frac{\zeta_1}{\zeta_0}, \dots, \frac{\zeta_n}{\zeta_0}\right)\frac{1}{\zeta_0^2}(\zeta_0 d\zeta_{i_1} - \zeta_{i_1} d\zeta_0) \quad (55)$$

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and similarly with $(0,1)$-forms. The pullback of a form of degree (p,q) in $\mathbb{P}^n$ is given by the wedge product of factors like the one in (55).

Definition 3. A projective form is a differential form in $\mathbb{C}^{n+1}$ that arises from the pullback of a differential form in $\mathbb{P}^n$.

By δ_ζ and $\delta_{\bar\zeta}$ we mean contraction with the vector fields

$$\sum_0^n \zeta_i \frac{\partial}{\partial \zeta_i} \quad \text{and} \quad \sum_0^n \bar\zeta_i \frac{\partial}{\partial \bar\zeta_i}.$$

Note that $\delta_\zeta \bar\partial = -\bar\partial \delta_\zeta$. If $\delta_\zeta \alpha = 0$, we have $\alpha = \delta_\zeta \beta(\zeta)$, since we can take $\beta(\zeta) = (\sum \bar\zeta_j d\zeta_j / |\zeta|^2) \wedge \alpha$.

yuki **Proposition 5.1.** *Take a differential form $\alpha(\zeta)$ in $\mathbb{C}^{n+1}$. Then α is a projective form if and only if $\delta_\zeta\alpha(\zeta) = 0$, $\delta_{\bar\zeta}\alpha(\zeta) = 0$ and α is zero-homogeneous (i e $\alpha(c\zeta) = \alpha(\zeta)$ for $c \in \mathbb{C}$).*

Proof. We begin by proving the proposition for 1-forms. A projective form of degree $(1,0)$ is a sum of terms of the type (55), which are homogeneous. We also have $\delta_\zeta(\zeta_0 d\zeta_{i_1} - \zeta_{i_1} d\zeta_0) = 0$. The case with $(0,1)$-forms is similar.

Conversely, take a $(1,0)$-form $\alpha(\zeta)$ such that $\delta_\zeta\alpha(\zeta) = 0$ and $\alpha(\zeta)$ is zero-homogeneous. We want to show that $\alpha(\zeta)$ is projective. First we find a

(2,0)-form $\beta(\zeta)$ such that $\alpha(\zeta) = \delta_\zeta \beta(\zeta)$. The form $\beta(\zeta)$ consists of terms of the type $f(\zeta)d\zeta_i \wedge d\zeta_j$; let us look at one of these terms. Clearly, if α is zero-homogeneous, β will be as well, which means that f is homogeneous of degree -2. Then we have

$$\begin{aligned} &\delta_\zeta f(\zeta)d\zeta_i \wedge d\zeta_j = f(\zeta)(\zeta_i d\zeta_j - \zeta_j d\zeta_i) = \\ &= \frac{1}{\zeta_i^2} f\left(\frac{\zeta_1}{\zeta_i}, \ldots, \overset{\text{place } i_k}{1}, \ldots, \frac{\zeta_n}{\zeta_i}\right)(\zeta_i d\zeta_j - \zeta_j d\zeta_i), \end{aligned} \tag{56}$$

which is the pullback under π_i of $f(\zeta'_1, \ldots, 1, \ldots, \zeta'_n)d\zeta'_j$. The case for (0,1)-forms is similar. We have proved that the projective 1-forms are exactly the ones that satisfy the conditions in the proposition. It follows that the exterior algebra generated by the projective 1-forms must be exactly the forms that satisfy the conditions. □

Differential forms can also take values in some line bundle over $\mathbb{P}^n$. Let α take values in $L^m = \mathcal{O}(m)$ (locally, sections in this line bundle correspond to m-homogeneous functions). Then the pullback of α in $\mathbb{C}^{n+1}$ will be an m-homogeneous form, and we will still have $\delta_\zeta \alpha(\zeta) = 0$ and $\delta_{\bar\zeta}\alpha(\zeta) = 0$.

We want to find integral formulas for sections of the line bundles on $\mathbb{P}^n$, in a similar way as before. In $\mathbb{C}^n$ we used the operator $\nabla = \delta_{\zeta - z} - \bar\partial$, and in $\mathbb{P}^n$ we will replace $\delta_{\zeta - z}$ with δ_η, which is contraction with the section

$$\eta = 2\pi i z \cdot \frac{\partial}{\partial \zeta} = 2\pi i \sum_0^n z_i \frac{\partial}{\partial \zeta_i}$$

where z is an fixed point of $\mathbb{C}^{n+1} \setminus \{0\}$. Note that if $[\zeta] = [z]$, then δ_η is zero on all projective forms, according to Proposition yuki 5.1.

calo **Proposition 5.2.** *The section η takes values in $T(\mathbb{P}^n_{[\zeta]}) \otimes L^{-1}_{[\zeta]} \otimes L^1_{[z]}$. Expressed in the local coordinates in U_0, we have*

$$\eta = 2\pi i \frac{z_0}{\zeta_0} \sum_1^n (z'_i - \zeta'_i) \frac{\partial}{\partial \zeta'_i}.$$

By saying that η is a section of $L^1_{[z]}$, we simply mean that η is 1-homogeneous in z.

Proof. Without loss of generality, suppose that $z_0 \neq 0$. We want to know the image (or push-forward) of $\partial/\partial\zeta_i$ under π_0. Take a function F on $\mathbb{P}^n$. Then

$$\pi_0(\frac{\partial}{\partial\zeta_i})F(\zeta') = \frac{\partial}{\partial\zeta_i}F\circ\pi_0(\zeta) = \frac{\partial}{\partial\zeta_i}F(\zeta_1/\zeta_0,\dots\zeta_n/\zeta_0) =$$

$$= \begin{cases} \frac{1}{\zeta_0}\frac{\partial}{\partial\zeta_i'}F(\zeta') & \text{if } i\neq 0 \\ -\sum_{j=1}^{n}\frac{\zeta_i}{\zeta_0^2}\frac{\partial}{\partial\zeta_j'}F(\zeta') & \text{if } i=0 \end{cases} \tag{57}$$

using the complex chain rule and the fact that $\frac{\partial}{\partial\zeta_i}(\bar\zeta_j/\bar\zeta_0)=0$ for all i,j. That is, we have in local coordinates

$$\frac{\partial}{\partial\zeta_i} = \frac{1}{\zeta_0}\frac{\partial}{\partial\zeta_i'} \text{ if } i\neq 0, \text{ and } \frac{\partial}{\partial\zeta_0} = -\sum_{j=1}^{n}\frac{\zeta_i}{\zeta_0^2}\frac{\partial}{\partial\zeta_j'}.$$

Substituting into η, we get

$$\eta = 2\pi i z_0\left(\sum_1^n z_i'\frac{\partial}{\partial\zeta_i} + \frac{\partial}{\partial\zeta_0}\right) = 2\pi i\frac{z_0}{\zeta_0}\left(\sum_1^n z_i'\frac{\partial}{\partial\zeta_i'} - \sum_1^n \zeta_i'\frac{\partial}{\partial\zeta_i'}\right) =$$

$$= 2\pi i\frac{z_0}{\zeta_0}\sum_1^n(z_i'-\zeta_i')\frac{\partial}{\partial\zeta_i'}. \tag{58}$$

capella

□

Set $\nabla = \delta_\eta - \bar\partial$, where δ_η will act in a natural way as a contraction on differential forms on $\mathbb{P}^n$. As in the previous section, we want to solve $\nabla u = 1 - [[z]]$, where $[[z]]$ is the Dirac measure at the point $[z]$. To find such a u, we start with the form $v = \bar z\cdot d\zeta = \sum_0^n \bar z_i d\zeta_i$, which has the property that $\delta_\eta v = 2\pi i|z|^2 \neq 0$ when $[\zeta]=[z]$. The problem is that v is just a form on $\mathbb{C}^{n+1}$ and may not be a projective form. To remedy that, we will project it onto the subspace of projective forms. According to Proposition 5.1 (yuki), a projective $(1,0)$ form α is characterized by $\delta_\zeta\alpha = 0$. Thus, we can describe the $(1,0)$-forms on $\mathbb{C}^{n+1}$ at some point $\zeta\neq 0$ as the sum of the space of projective forms and the span of $\bar\zeta\cdot d\zeta$ (which is a form such that $\delta_\zeta\bar\zeta\cdot d\zeta = |\zeta|^2\neq 0$). The projection $\hat v$ of v onto the projective forms can then be written as

$$\hat v = v - \frac{\langle v, \bar\zeta\cdot d\zeta\rangle}{|\bar\zeta\cdot d\zeta|^2}\bar\zeta\cdot d\zeta = \bar z\cdot d\zeta - \frac{\bar z\cdot\zeta}{|\zeta|^2}\bar\zeta\cdot d\zeta. \tag{59}$$

shuichi

If $u = \hat v/\nabla\hat v$, we have

nablau **Proposition 5.3.** *We have* $\nabla_f u = 1 - [[z]]$.

Proof. We can assume that $z = (1, 0, \ldots, 0)$. The proposition will follow from Proposition carex 2.2, if we show that $|u_k| \lesssim |\zeta'|^{-2k+1}$ close to $\zeta' = 0$. We have

$$u_k = \frac{\hat{v} \wedge (\bar{\partial}\hat{v})^{k-1}}{(\delta_\eta \hat{v})^k}.$$

Since $\hat{v}(z) = 0$ and $\hat{v}$ is smooth, we have $|\hat{v}| \lesssim |\zeta'|$. Furthermore, $\delta_\eta \hat{v} = (|z|^2|\zeta|^2 - |\bar{z} \cdot \zeta|^2)/|\zeta|^2 = (|\zeta|^2 - |\zeta_0|^2)/|\zeta|^2 = |\zeta'|^2/(1 + |\zeta'|^2) \geq |\zeta'|^2/2$ close to z, so we can make the estimate

$$\left| \frac{\hat{v} \wedge (\bar{\partial}\hat{v})^{k-1}}{(\delta_\eta \hat{v})^k} \right| \lesssim |\zeta'|^{-2k+1}$$

close to $\zeta' = 0$, which concludes the proof. □

It is interesting to compare our kernel with the Bochner-Martinelli kernel. Obviously, they are not the same locally. On the other hand, our kernel u has the property of being invariant under linear transformations that preserve the metric, which is not the case for the Bochner-Martinelli kernel. More precisely, we have:

Proposition 5.4. *Let A be a unitary linear transformation on $\mathbb{C}^{n+1}$. Then u, expressed as a differential form on $\mathbb{C}^{n+1}$, is invariant under pullback of A.*

Proof. We look first at the pullback of $\hat{v}$ (see shuichi 59); it is equal to

$$A^*\hat{v} = A\bar{z} \cdot dA\zeta - \frac{A\bar{z} \cdot A\zeta}{|A\zeta|^2} A\bar{\zeta} \cdot dA\zeta.$$

Since $dA\zeta = Ad\zeta$ (remember that $d\zeta = (d\zeta_0, \ldots, d\zeta_n)$) and A is orthogonal, we have $A^*\hat{v} = \hat{v}$. Then we recall that $u = \hat{v}/\nabla\hat{v}$, so one has to check that $A^*\delta_\eta\hat{v} = \delta_\eta\hat{v}$ and $A^*\bar{\partial}\hat{v} = \bar{\partial}\hat{v}$, for example. This is easily done. □

We can use u to construct integral formulas for line bundles on $\mathbb{P}^n$. Note that u_n is of bidegree $(n, n-1)$, and takes values in the line bundle $L^n_{[\zeta]} \otimes L^{-n}_{[z]}$. To integrate with respect to ζ, we need to pair u_n with a section ϕ in $L^{-n}_{[\zeta]}$, so that their product will be a differential form on $\mathbb{P}^n$ that takes values in the trivial line bundle. Thus, if ϕ is holomorphic and $[z] \in D \subset \mathbb{P}^n$ we have

$$\int_{\partial D} \phi u_n = \int_D \phi \bar{\partial} u_n = \phi([z])$$

by Stokes' formula and Proposition nablau 5.3.

However, this only gives us an integral formula for sections of $L_{[\zeta]}^{-n}$. To get one for sections of other line bundles, we need to use weighted formulas (like we did in Section intsect 3.2 in order to handle functions that grew too quickly at infinity).

Definition 4. We say that $g \in \mathcal{L}^0$ is a weight if $\nabla g = 0$, $g_0([z]) = 1$ and g_k takes values in $L_{[\zeta]}^{k-i} \otimes L_{[z]}^{i-k}$ for some i. Here g_k has bidegree (k,k) as a differential form.

Just as before, the wedge product of two weights will again be a weight, and we have $\nabla(g \wedge u) = g - [[z]]$. We will show that

$$\alpha = \frac{z\bar{\zeta}}{|\zeta|^2} - 2\pi i \bar{\partial} \frac{\bar{\zeta} \cdot d\zeta}{|\zeta|^2} = \frac{z \cdot \bar{\zeta}}{|\zeta|^2} + 2\pi i \partial\bar{\partial} \log|\zeta|^2,$$

is a weight, and then we can use α^{n+r} to integrate sections of L^r. Note first that $z \cdot \bar{\zeta}/|\zeta|^2$ takes values in $L_{[\zeta]}^{-1} \otimes L_{[z]}^{1}$, and that $\bar{\partial}(\bar{\zeta} \cdot d\zeta/|\zeta|^2)$ is a (1,1)-form that takes values in the trivial bundle. Clearly, it is 0-homogeneous, and further we have $\delta_\zeta \bar{\partial}\bar{\zeta} \cdot d\zeta/|\zeta|^2 = -\bar{\partial}\delta_\zeta \bar{\zeta} \cdot d\zeta/|\zeta|^2 = -\bar{\partial}1 = 0$. Further, $\nabla\alpha = 0$ and $\alpha_0([z]) = 1$, thus α is indeed a weight.

For general weights, and for α in particular, we have the following proposition:

Proposition 5.5. *Take a weight g such that $g_{n,n}$ takes values in $L_{[\zeta]}^{-r} \otimes L_{[z]}^{r}$ (and $g_{k,k}$ takes values in $L_{[\zeta]}^{k-r-n} \otimes L_{[z]}^{-k+r+n}$). If a section ϕ of L^r is holomorphic in Ω, $[z] \in D \subset\subset \Omega$, and K is such that $\nabla_\eta K = g$ in a neighborhood of ∂D then*

$$\phi([z]) = \int_{\partial D} \phi \wedge K_n + \int_D \phi g_n.$$

So by using $g = \alpha^{n+r}$, we now have integral formulas for sections of L^r where $r \geq -n$.

Example 1. If ϕ is a global section of L^r, then

$$\phi([z]) = \int_{\mathbb{P}^n} \phi g_n = \int_{\mathbb{P}^n} \phi \alpha_{n,n}^{r+n}$$

What is this, explicitly? First, we know that

$$\alpha_{n,n}^{n+r} = (2\pi i)^n \binom{r+n}{n} \left(\frac{z \cdot \bar{\zeta}}{|\zeta|^2}\right)^r \left(\partial\bar{\partial}\log(|\zeta|^2)\right)^n,$$

so we get

$$\phi([z]) = (2\pi i)^n \binom{r+n}{n} \int_{\mathbb{P}^n} \left(\frac{z \cdot \bar{\zeta}}{|\zeta|^2} \right)^r \left(\partial \bar{\partial} \log(|\zeta|^2) \right)^n \phi([\zeta]).$$

This formula is also obtained in [7].

6 The Koppelman formula in $\mathbb{C}^n$

We will now look at Koppelman formulas in $\mathbb{C}^n$; the account is inspired by Section 9, page 16 in [1]. One can regard this section as a continuation of Section 2. The idea here is that while z has been a constant before, we now want to regard it as a variable.

Let Ω be a domain in $\mathbb{C}^n$ and let $\eta = z - \zeta$, where $(\zeta, z) \in \Omega \times \Omega$. Consider the subbundle $E^* = \{d\eta_1, \dots, d\eta_n\}$ of the cotangent bundle $T^*_{1,0}$ over $\Omega \times \Omega$. Let E be its dual bundle, with basis e_j, and let δ_η be contraction with the section

$$2\pi i \sum_1^n \eta_j e_j, \tag{60}$$

where $\{e_j\}$ is the dual basis to $\{d\eta_j\}$. Now we look at

$$\bigwedge (E^* \oplus T^*_{0,1}),$$

and let $\mathcal{L}_{p,q}$ denote the space of sections of this bundle with degree p in E^* and degree q in $T^*_{0,1}$. Set $\mathcal{L}^m = \bigoplus_p \mathcal{L}_{p,p+m}$ and $\nabla = \nabla_\eta = \delta_\eta - \bar\partial$, where $\bar\partial$ acts on $\mathbb{C}^n \times \mathbb{C}^n$. Then ∇ will map $\mathcal{L}^m$ to $\mathcal{L}^{m+1}$.

In Section 2 we wanted to solve $\nabla_{\zeta - z} u = 1 - [z]$. Note that z is the zero set of $\zeta - z$ in $\mathbb{C}^n$. Now instead we look at the zero set of η in $\mathbb{C}^n \times \mathbb{C}^n$, which is the diagonal $\Delta = \{\zeta = z\}$ of $\Omega \times \Omega$. We want to solve $\nabla_\eta u = 1 - [\Delta]$. In fact, we can use the Bochner-Martinelli kernel again: if

$$b(\zeta, z) = \frac{1}{2\pi i} \frac{\partial |\eta|^2}{|\eta|^2}$$

then we can set $u = b/\nabla_\eta b$.

Proposition 6.1. *If $u = u/\nabla_\eta b$, then $\nabla_\eta u = 1 - [\Delta]$.*

Proof. This proof is quite similar to the proof of Proposition 2.1. We will show one of the calculations required and leave the rest to the reader. Take a test form $\psi(\zeta, z)$ of bidegree (n, n) on $\Omega \times \Omega$. We want to show that

$$\int_{\zeta, z} \bar\partial \psi(\zeta, z) \wedge b \wedge (\bar\partial b)^{n-1} = \int_z \psi(z, z). \tag{61}$$

Since the integrand is integrable and $(\bar\partial b)^n = 0$ outside $[\Delta]$, the left hand side of (61) is equal to

$$\lim_{\epsilon \to 0} \int_{|\eta| > \epsilon} \bar\partial \psi \wedge b \wedge (\bar\partial b)^{n-1} = \lim_{\epsilon \to 0} \int_{|\eta| = \epsilon} \psi \wedge b \wedge (\bar\partial b)^{n-1}. \tag{62}$$

By using first the definition of b and then Stokes' theorem, the integral on the right hand side of (62) is equal to

$$\left(\frac{1}{2\pi i}\right)^n \frac{1}{\epsilon^{2n}} \int_{|\eta|=\epsilon} \psi \wedge \partial|\eta|^2 \wedge (\bar\partial\partial|\eta|^2)^{n-1} =$$
$$= \left(\frac{1}{2\pi i}\right)^n \frac{1}{\epsilon^{2n}} \left(\int_{|\eta|<\epsilon} \bar\partial\psi \wedge \partial|\eta|^2 \wedge (\bar\partial\partial|\eta|^2)^{n-1} + \int_{|\eta|<\epsilon} \psi \wedge (\bar\partial\partial|\eta|^2)^n \right). \quad (63)$$

The first of these integrals goes to zero in the same way as the first integral in (8), except that we also need to use the fact that ψ has compact support. As for the second integral, if we make the change of coordinates $(\eta, \rho) = (z-\zeta, z+\zeta)$, where we set $\tilde\psi(\eta,\rho) = \psi(\zeta, z)$, and then use Fubini's theorem, then we get

$$\left(\frac{1}{2\pi i}\right)^n \frac{1}{\epsilon^{2n}} \int_\rho \int_{|\eta|<\epsilon} \tilde\psi(\eta,\rho)(\bar\partial\partial|\eta|^2)^n = \int_\rho \tilde\psi(0,\rho) = \int_z \psi(z,z),$$

by using Proposition 2.1. □

By a proof very similar to the proof of Proposition 2.2, we get the following proposition:

isis **Proposition 6.2.** *Suppose* $u \in \mathcal{L}^{-1}(\Omega \times \Omega \setminus \Delta)$ *solves* $\nabla_\eta u = 1$, *and that* $|u_k| \lesssim |\eta|^{-(2k-1)}$. *Then* $\nabla_\eta u = 1 - [\Delta]$.

Weights are defined as before:

Definition 5. A form $g \in \mathcal{L}^0(\Omega\times\Omega)$ is a weight if $g_0 \equiv 1$ on Δ and $\nabla_\eta g = 0$.

If g is a weight, then we can solve $\nabla_\eta v = g - [\Delta]$ by setting $v = g \wedge u$. If $K = (u \wedge g)_n$ and $P = g_n$, then $\bar\partial K = [\Delta] - P$. Then we can prove

koala **Proposition 6.3 (Koppelman's formula).** *If* $D \subset\subset \Omega$ *and* $\phi \in \mathcal{E}_{p,q}(\bar D)$ *we have*

$$\begin{aligned}\phi(z) &= (-1)^{p+q} \int_{\partial D} \phi \wedge K + (-1)^{p+q+1} \int_D \bar\partial\phi \wedge K + \\ &+ (-1)^{p+q} \bar\partial_z \int_D \phi \wedge K + \int_D \phi \wedge P, \end{aligned} \quad (64)$$ iris

where the integrals are taken over the ζ *variable.*

Proof. First assume that ϕ has compact support in D, so that the the integral over the boundary is zero. Take a test form $\psi(z)$ in Ω. Then we have

$$\begin{aligned}
& (-1)^{p+q}\int_z \psi\wedge\Big(-\int_\zeta \bar\partial\phi\wedge K+\bar\partial_z\int_\zeta \phi\wedge K\Big)+\int_{z,\zeta}\psi\wedge\phi\wedge P= \\
= \;& (-1)^{p+q}\Big(-\int_{z,\zeta}\psi\wedge d\phi\wedge K+(-1)^{2n-p-q+1}\int_{z,\zeta} d\psi\wedge\phi\wedge K\Big)+ \\
+ \;& \int_{z,\zeta}\psi\wedge\phi\wedge P=-\int_{z,\zeta} d(\psi\wedge\phi)\wedge K+\int_{z,\zeta}\psi\wedge\phi\wedge P= \\
= \;& \int_{z,\zeta}\psi\wedge\phi\wedge dK+\int_{z,\zeta}\psi\wedge\phi\wedge P=\int_z\psi\wedge\phi, \qquad (65)
\end{aligned}$$

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where we use Stokes' theorem repeatedly, and also that the degree of ψ must be $(n-p,n-q)$. If ϕ does not have compact support in D, we can make the decomposition $\phi=\phi_1+\phi_2$, where ϕ_1 has compact support, and $\phi_2(\zeta)=0$ in a neighborhood of z. Take a test form ψ with support in that same neighborhood. Then

$$\begin{aligned}
& \int_z\psi\wedge\int_{\partial D}\phi_2\wedge K=\int_{z,\zeta}\psi\wedge d_\zeta(\phi_2\wedge K)= \\
= \;& \int_{z,\zeta}\psi\wedge d(\phi_2\wedge K)-\int_{z,\zeta}\psi\wedge d_z(\phi_2\wedge K)= \\
= \;& \int_{z,\zeta}\psi\wedge d\phi_2\wedge K+(-1)^{p+q}\int_{z,\zeta}\psi\wedge\phi_2\wedge dK+(-1)^{p+q}\int_{z,\zeta}d\psi\wedge\phi_2\wedge K= \\
= \;& \int_z\psi\wedge\int_\zeta\bar\partial\phi_2\wedge K+(-1)^{p+q+1}\int_z\psi\wedge\int_\zeta\phi_2\wedge P- \\
- \;& \int_z\psi\wedge\bar\partial_z\int_\zeta\phi_2\wedge K.
\end{aligned}$$

This gives us a formula for ϕ_2. If we combine it with the formula (65) that we already have for ϕ_1, we will obtain (64). □

If we can get the first and fourth terms of the right hand side of Koppelman's formula to disappear, then we can take a closed form ϕ and get a solution of the $\bar\partial$-problem for ϕ. Of course, this cannot work for all domains D, since the $\bar\partial$-problem is not solvable for all domains.

7 Koppelman's formula on $\mathbb{P}^n$ and solutions to the $\bar\partial$-equation

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Now that we have obtained Koppelman's formula for $\mathbb{C}^n$, we would like to have a similar formula in $\mathbb{P}^n$. We start by taking

$$\eta = 2\pi i z \cdot \frac{\partial}{\partial \zeta} = 2\pi i \sum_0^n z_i \frac{\partial}{\partial \zeta_i} \tag{66}$$ tandborste

just as before, except that now it is a section over $\mathbb{P}^n \times \mathbb{P}^n$. We define $\nabla = \nabla_\eta = \delta_\eta - \bar\partial$, where $\bar\partial$ acts on both variables. Then we want $u = \hat v / \nabla \hat v$ to solve the equation $\nabla u = 1 - [\Delta]$. The current u_n will be of bidegree $(n, n-1)$ as a form on $\mathbb{P}^n \times \mathbb{P}^n$, but the differentials without bars on come only from $d\zeta_i$'s, since the vector field ([tandborste] 66) contains no dz_i. Compare this with the vector field ([kali] 60) that was used to construct Koppelman formulas on $\mathbb{C}^n$, which does contain dz_i's! On the other hand, u_n will be a sum of terms whose differentials with bars on are built out of every possible combination of $d\bar\zeta_i$'s and $d\bar z_i$'s (of degree $n-1$), because $\bar\partial$ acts on both variables. When we want to integrate a (n,n)-form ϕ against u_n, then it cannot contain any $d\zeta_i$'s. We must prove the following:

Proposition 7.1. *If the (n,n)-form $\phi(\zeta, z)$ takes values in $L^{-n}_{[\zeta]} \times L^n_{[z]}$ and contains no $d\zeta_i$'s, then we have*

$$\nabla_\eta u.\phi = (1 - [\Delta]).\phi.$$

In other words, in the right hand side we do not have the whole of $[\Delta]$, but only the part of $[\Delta]$ that contains no dz_i's.

Proof. The proposition will follow from Proposition [isis] 6.2, since the statement is local. It is enough to show that $|u_k| \lesssim |\eta|^{-(2k-1)}$ locally, meaning that the coefficients of u_k satisfy this estimate. The proof of this is essentially identical to the proof of Proposition [nablau] 5.3, except that one has a general z instead just $z = (1, 0, \ldots, 0)$. □

To get formulas for sections of other line bundles, we can use the same weight

$$\alpha = \frac{z \cdot \bar\zeta}{|\zeta|^2} - \frac{1}{2\pi i}\bar\partial \frac{\bar\zeta \cdot d\zeta}{|\zeta|^2}$$

as in Section [intformpn] 5, if we consider it as a form on $\mathbb{P}^n \times \mathbb{P}^n$. Then we get

$$\nabla(u \wedge \alpha^{n+r}) = \alpha^{n+r} \wedge (1 - [\Delta]) = \alpha^{n+r} - [\Delta] \qquad (67)$$ mango

since $(\alpha^{n+r})_{0,0}(\zeta, \zeta) = 1$. But note that $(\alpha^{n+r})_{0,0}$ is also a section of $L^{-(r+n)}_{[\zeta]} \times L^{r+n}_{[z]}$, which means that the current $[\Delta]$ on the right hand side of (67) must now be paired with a section of $L^{r}_{[\zeta]}$ when we wish to integrate. If we set $K = (u \wedge \alpha^{n+r})_{n,n-1}$ and $P = (\alpha^{n+r})_{n,n}$, then we have $\bar{\partial} K = [\Delta] - P$. With this, we can prove the following Koppelman's formula for sections of $L^{r}_{[\zeta]}$ in the same way as we proved Proposition 6.3.

ek **Proposition 7.2 (Koppelman's formula).** *If $D \subset\subset \Omega$ and $\phi \in \mathcal{E}_{p,q}(\bar{D})$ takes values in $L^{r}_{[\zeta]}$, we have*

$$\begin{aligned} \phi(z) &= (-1)^{p+q} \int_{\partial D} \phi \wedge K + (-1)^{p+q+1} \int_{D} \bar{\partial}\phi \wedge K + \\ &+ (-1)^{p+q} \bar{\partial}_z \int_{D} \phi \wedge K + \int_{D} \phi \wedge P, \end{aligned}$$

where the integrals are taken over the $[\zeta]$ variable.

Note that if we choose ϕ to be a global section and $D = \mathbb{P}^n$, then the boundary term will disappear. If we can also get $P = 0$, then we get a solution formula for the $\bar{\partial}$-equation. In the case when ϕ is a section of $L^{-n}_{[\zeta]}$, for example, P is automatically zero since we do not need any weight α. This shows that the cohomology group of the bundle of $(0, q)$-forms in $L^{-n}_{[\zeta]}$ is trivial. As an application of Koppelman's formula, we will now find which of the cohomology groups for the bundles of $(0, q)$- and (n, q)-forms in $L^{k}_{[\zeta]}$ are trivial. Of course, this is already well known, see for example Theorem 10.7 on p. 397 of [9], but this is a different way of proving it and also yields explicit formulas for the solutions of the $\bar{\partial}$-equation. We obtain the following:

chakobsa **Theorem 7.3.** *By using the Koppelman formula (7.2) one can show that the following cohomology groups are trivial:*
a) $H^{0,q}(\mathbb{P}^n, L^r)$ for $0 < q < n$ and all r;
b) $H^{n,q}(\mathbb{P}^n, L^r)$ for $0 < q < n$ and all r;
c) $H^{0,0}(\mathbb{P}^n, L^r)$ for $r < 0$;
d) $H^{0,n}(\mathbb{P}^n, L^r)$ for $r \geq -n$;
e) $H^{n,0}(\mathbb{P}^n, L^r)$ for $r \leq n$;
f) $H^{n,n}(\mathbb{P}^n, L^r)$ for $r > 0$.

This theorem was also proved by Berndtsson in [7] with essentially the same u (though derived in a different way) and the same Koppelman formula. Before the proof we need a lemma:

spiklav **Lemma 7.4.** *We have* $\bigwedge^n T^*_{0,1}(\mathbb{P}^n) \simeq L^{-n-1}$.

Proof. To prove the lemma, we observe that $\sum(-1)^j z_j \widehat{dz_j}$ is a global non-zero $(n,0)$-form that takes values in L^{n+1}. This means that the line bundle $\bigwedge^n T^*_{0,1}(\mathbb{P}^n) \otimes L^{n+1}$ is trivial, which means that $\bigwedge^n T^*_{0,1}(\mathbb{P}^n) \simeq L^{-n-1}$. □

Proof. (of Theorem chakobsa 7.3) As noted before, $H^{0,q}(\mathbb{P}^n, L^{-n})$ is trivial for $0 \leq q \leq n$ since no weight is needed, and we have $\phi = (-1)^{q+1}\bar{\partial} \int \phi \wedge u_n$. Further, if we let ϕ be a section of L^r, $r \geq -n$, be a $\bar{\partial}$-closed $(0,q)$-form where $q \neq 0$ then $\int \phi \wedge P = 0$ since if it were not, it would have bidegree $(0,0)$ as a section of $L^r_{[z]}$ (because the integrand does not contain any dz's or $d\bar{z}$'s). But remember that the left hand side of the Koppelman formula is just $\phi(z)$, which has bidegree $(0,q)$. Also, $\int \phi \wedge P$ cannot be cancelled out by any other term on the right hand side, since $\bar{\partial} \int \phi \wedge K$ cannot contain anything of bidegree $(0,0)$. Thus $\int \phi \wedge P = 0$, which means that $H^{0,q}(\mathbb{P}^n, L^r)$ is trivial for $0 < q \leq n$ and $r \geq -n$. We also get the formula $\phi(z) = (-1)^{q+1}\bar{\partial} \int \phi \wedge K$. The condition $r \geq -n$ comes about because we cannot raise α to a negative power. With $q = n$, this proves d).

How, then, do we investigate the line bundles L^r where $r < -n$? In fact, if we look at the proof of the Koppelman formula in Proposition koala 6.3, we see that the roles of $\phi(\zeta)$ and $\psi(z)$ are in fact symmetrical, and we can use this to get a Koppelman formula for $\psi(z)$ instead of $\phi(\zeta)$. Note that then ψ needs to have bidegree (n,q) where $0 \leq q \leq n$. If ϕ takes values in L^r, ψ has to take values in L^{-r}, so that we can obtain results for L^r with $r \leq n$. The case for L^n mirrors the one for L^{-n}: we see that $H^{n,q}(\mathbb{P}^n, L^n)$ is trivial for $0 \leq q \leq n$ since no weight is needed, and we have $\psi(\zeta) = (-1)^{n+q+1}\bar{\partial} \int_z \psi \wedge u_n$. If we take a section ψ of L^r, $r \leq n$, to be a $\bar{\partial}$-closed (n,q)-form where $q \neq n$ then $\int \psi \wedge P = 0$ since there are not enough $d\bar{z}$'s, which shows that $H^{n,q}(\mathbb{P}^n, L^r)$ is trivial for $0 \leq q < n$ and $r \leq n$. Also we get the formula $\psi(\zeta) = (-1)^{n+q+1}\bar{\partial} \int_z \psi \wedge K$. With $q = 0$, this proves e).

Furthermore, by Lemma spiklav 7.4 we have $\mathcal{E}_{0,q}(\mathbb{P}^n, L^r) \simeq \mathcal{E}_{n,q}(\mathbb{P}^n, L^{r+n+1})$. The isomorphism is given by taking $\phi(z) \in \mathcal{E}_{0,q}(\mathbb{P}^n, L^r)$ and $\omega(z) = \sum(-1)^j z_j \widehat{dz_j}$ and then simply taking the wedge product $\omega \wedge \phi$, which lies in $\mathcal{E}_{n,q}(\mathbb{P}^n, L^{r+n+1})$. Since we know that $H^{0,q}(\mathbb{P}^n, L^r)$ is trivial for $0 < q \leq n$ and $r \geq -n$, it follows that $H^{n,q}(\mathbb{P}^n, L^r)$ is trivial for $0 < q \leq n$ and $r > 0$. We can easily find an explicit formula for the solution, since

$$\omega \wedge \phi = (-1)^{q+1} \omega \wedge \bar{\partial} \int \phi \wedge K = \bar{\partial}(\omega \wedge \int \phi \wedge K).$$

If we combine this with the results of the previous paragraph, we see that $H^{n,q}(\mathbb{P}^n, L^r)$ is trivial for $0 < q < n$ and all r, which proves b). Also, if we let $q = n$, we see that $H^{n,n}(\mathbb{P}^n, L^r)$ is trivial for $r > 0$, which proves f).

Finally, we apply the isomorphism $\mathcal{E}_{0,q}(\mathbb{P}^n, L^r) \simeq \mathcal{E}_{n,q}(\mathbb{P}^n, L^{r+n+1})$ the other way around. Since we know that $H^{n,q}(\mathbb{P}^n, L^r)$ is trivial for $0 \leq q < n$ and $r \leq n$, it follows that $H^{0,q}(\mathbb{P}^n, L^r)$ is trivial for $0 \leq q < n$ and $r < 0$. With $q = 0$, we see that $H^{0,0}(\mathbb{P}^n, L^r)$ is trivial for $r < 0$, which proves c). Note that these are precisely the line bundles that lack holomorphic sections - quite naturally, since there is no way a section with bidegree $(0,0)$ can be $\bar{\partial}$-exact. If we combine the results of this paragraph with those of the first paragraph, we also see that $H^{0,q}(\mathbb{P}^n, L^r)$ is trivial for $0 < q < n$ and all r, which proves a).

Finding explicit solutions after using the isomorphism $\mathcal{E}_{0,q}(\mathbb{P}^n, L^r) \simeq \mathcal{E}_{n,q}(\mathbb{P}^n, L^{r+n+1})$ backwards is a little more difficult. We have on the one hand $\psi(\zeta) = \omega \wedge \psi'$, where ψ' is of bidegree $(0,q)$, and on the other hand $\psi = (-1)^{n+q+1}\bar{\partial} \int_z \psi \wedge K$. In other words, we want to factor out $\omega(\zeta)$ from $\bar{\partial} \int_z \psi \wedge K$. Since all the $d\zeta_i$'s are in $K(\zeta, z)$, this means that we want to write $K = \omega \wedge K'$. To do this in practice, we observe that $\sum \bar{\zeta}_i/|\zeta|^2 \wedge \omega = d\zeta_0 \wedge \ldots d\zeta_n$. Thus K' equals the $(0,q)$-part of $\sum \bar{\zeta}_i/|\zeta|^2 \wedge K$. □

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