THESIS FOR THE DEGREE OF LICENTIATE OF PHILOSOPHY

# On Artin Schemes of Tiled Orders

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**Abstract**

We associate a geometric object, the Artin scheme, to any "tiled" order in a matrix algebra. We assume for simplicity that the base ring is a discrete valuation ring containing a field and we calculate the dimensions of the cotangent spaces at closed points of the Artin scheme. As a consequence, we conclude that the order is hereditary if and only if the dimensions of the cotangent spaces are minimal.

**Keywords:** representable functor, Brauer-Severi scheme, Artin scheme, order, cotangent space.

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## 1. Introduction

Let $K$ be a field and $V$ be an $n$-dimensional vector space over $K$. To $V$ we can associate a projective space $P(V)$. In the classical definition $P(V)$ parametrizes the 1-dimensional subspaces of $V$. In this paper we will, however, use the definition in [7] and let $P(V)$ parametrize the subspaces of $V$ of codimension 1.

The $K$-algebra $A := \mathrm{End}_K(V)$ of $K$-endomorphisms of $V$ is a central simple algebra of dimension $n^2$ as vector space over $K$. It is easy to describe the left ideals of $A$. Each left ideal has the form $\mathrm{Hom}_K(V, W)$ for a unique subvectorspace $W$ of $V$. In particular, this gives a bijection between the $K$-points on $P(V)$ and the left ideals $I \subseteq A$ such that $A/I$ has dimension $n$ over $K$.

More generally, one can consider a covariant functor $\mathcal{F} : K\text{-}\mathbf{Alg} \to \mathbf{Sets}$ from the category of commutative $K$-algebras to the category of sets. To each $K$-algebra $K'$ we associate the set of all left ideals $I' \subseteq A' := K' \otimes_K A$ such that $A'/I'$ is a projective $K'$-module of constant rank $n$. This functor $\mathcal{F}$ may also be regarded as a contravariant functor from the category of affine $K$-schemes and then extended to a functor defined on the category of all $K$-schemes. This extended functor is represented by the $K$-scheme $P(V)$.

One may replace $K$ by an arbitrary commutative ring $R$ with 1 and $V$ by a projective $R$-module $M$ of rank $n$. Then $\Lambda := \mathrm{End}_R(M)$ is an Azumaya algebra which is projective of rank $n^2$ as $R$-module. In the same way as above we can consider the functor of left ideals of corank $n$, $\mathcal{F} : R\text{-}\mathbf{Alg} \to \mathbf{Sets}$. This functor is represented by a (generalized) projective space $P(M)$ as showed by Grothendieck. In particular if $M = R^n$ and $\Lambda = M_n(R)$, then $\mathcal{F}$ is represented by $\mathbb{P}_R^{n-1}$.

More generally Grothendieck showed (see [6]) that $\mathcal{F}$ is representable for all Azumaya algebras $\Lambda$ and he called the corresponding scheme $X_\Lambda$ the Severi-Brauer scheme of $\Lambda$. In the case of a central simple algebra over a field $K$ one gets Severi-Brauer varieties over $K$, which were studied by Châtelet already in the 1940's. We shall follow the terminology in [1] and call $X_\Lambda$ the Brauer-Severi scheme of $\Lambda$ and $\mathcal{F}$ the Brauer-Severi functor of $\Lambda$.

Let $R$ be a Dedekind domain with perfect residue fields and with quotient field $K$ and let $A$ be a central simple $K$-algebra . In [1] Artin studies the Brauer-Severi functor of maximal $R$-orders $\Lambda$. He notes that this functor is represented by a projective $R$-scheme $X$ and that it may have several connected components if $\Lambda$ is ramified. One of these components $X^0$ contains the generic fiber of $X$ over $R$, which is nothing but the Brauer-Severi $K$-variety of $A$. Artin then goes on and studies $X^0$ and shows that it is regular. This result was generalized to hereditary orders by Frossard [5].

To show that $X^0$ is regular, Artin first reduces to the case where $R$ is a complete discrete valuation ring. It is known that $\Lambda$ remains hereditary after unramified extensions (see [10]) and that $A$ has an unramified splitting field. It is therefore sufficient to study the split case where $A = M_n(K)$, which we assume from now on.

The hereditary $R$-orders in $A$ are well understood (see [15]). They form a subclass of the tiled orders. An $R$-order $\Lambda$ in $A = M_n(K)$ is said to be tiled (see [11]) if there is a set of $n$ primitive idempotents $e_1, \ldots, e_n \in \Lambda$ with $e_1 + \ldots + e_n = 1$. Artin used these idempotents to embed $X^0$ as a closed subscheme of a multiprojective space over $R$. He used thereby the fact that $X^0$ represents the subfunctor $\mathcal{F}^0 \subseteq \mathcal{F}$ of ideals $I \subseteq \Lambda$ such that $e_i\Lambda/e_iI$ is of rank 1 for each $i = 1, \ldots, n$. His equations for $X^0$ are multilinear.

Salberger showed (see section 4) how to represent $\mathcal{F}^0$ by a multiprojective $R$-scheme $X^0$ for arbitrary tiled $R$-orders. He interpreted such orders as groupoid rings twisted by 2-cocycles and obtained multilinear equations similar to those of Artin. We shall therefore call $X^0$ the Artin subscheme of the Brauer-Severi scheme $X$. The coefficients in Salberger's equations are given by the 2-cocycle of the groupoid defining the tiled order.

We shall in this paper use these equations to study the geometry of the Artin subscheme $X^0$ of the Brauer-Severi scheme $X$ of an arbitrary tiled order. The original aim was to show that the only tiled orders for which $X^0$ is regular are the hereditary orders. This would have been a converse to Frossard's result.

We did not succeed in doing this. Instead, we prove a somewhat weaker result (Theorem 1), in the case where $R$ contains a field $k$. It says that a tiled order $\Lambda \subseteq M_n(K)$ is hereditary if the tangent space dimensions of the closed points of $X$ are less or equal to $n$. Furthermore we give in Proposition 13 a condition for $\Lambda$ which implies that $X$ is singular.

The paper is organized in the following way:

In Section 2 we recall the definitions of Zariski sheaves and representable functors.

In Section 3 we introduce the Grassmann and the Brauer-Severi functors. We include a proof of the representability of the Brauer-Severi functor for $R$-algebras $\Lambda$, which are finitely generated and projective as $R$-modules.

In Section 4 we construct tiled orders with multiplication rules determined by certain groupoid 2-cocycles. We present equations for the Brauer-Severi scheme $X$ of such orders.

In Section 5, we study these orders over discrete valuation rings containing an algebraically closed field. We describe the subclasses of groupoid 2-cocycles giving rise to hereditary orders and "triangular" orders. We then study the geometry of the closed fiber of the Artin subscheme $X^0 \subseteq X$ for such orders and give a condition on the 2-cocycle for $X^0$ to be regular. Next, we investigate the cotangent spaces at certain closed points of $X^0$. We show how the dimension of the cotangent space can be determined from the 2-cocycle. We also give a sufficient condition for $X^0$ to be singular.

Finally, we give the main result, which gives a relation between hereditary orders $\Lambda$ and the dimensions of the cotangent spaces at closed points of $X^0$.

## 2. Representable functors

Let $\mathbf{C}$ be a category and let $\widehat{\mathbf{C}}$ denote the category $\mathbf{Func}(\mathbf{C}^{\mathrm{op}}, \mathbf{Sets})$ of contravariant functors from $\mathbf{C}$ to the category $\mathbf{Sets}$ of sets. For any $X \in \mathrm{Obj}(\mathbf{C})$ let $h_X \in \mathrm{Obj}(\widehat{\mathbf{C}})$ be the contravariant functor sending $Z$ to the set $\mathrm{Mor}_{\mathbf{C}}(Z, X)$ of morphisms from $Z$ to $X$ in $\mathbf{C}$. There is then a canonical covariant functor $h : \mathbf{C} \to \widehat{\mathbf{C}}$ which sends $X \in \mathrm{Obj}(\mathbf{C})$ to $h_X \in \mathrm{Obj}(\widehat{\mathbf{C}})$ and $f \in \mathrm{Mor}(X, Y)$ to the natural transformation $h(f) : h_X \to h_Y$ defined elementwise by composition, that is $h(f)(g : Z \to X) = f \circ g : Z \to Y$.

**Lemma 1** (Yoneda)**.**

(i) *For any functor $\mathcal{F} \in \mathrm{Obj}(\widehat{\mathbf{C}})$ and any $X \in \mathrm{Obj}(\mathbf{C})$ there is a natural bijection between the set $\mathcal{F}(X)$ and the set of natural transformations from $h_X$ to $\mathcal{F}$.*

(ii) *The functor $h$ is fully faithful.*

*Proof.* See [4] pp.252-253.■

Thus the category $\mathbf{C}$ is equivalent to a full subcategory of $\widehat{\mathbf{C}}$, where full means that $\mathrm{Mor}_{\mathbf{C}}(X, Y) \simeq \mathrm{Mor}_{\widehat{\mathbf{C}}}(h_X, h_Y)$ for all $X, Y \in \mathrm{Obj}(\mathbf{C})$.

**Definition 1.** *A functor $\mathcal{F} \in \widehat{\mathbf{C}}$ is said to be representable if there is an $X \in \mathrm{Obj}(\mathbf{C})$ such that $h_X \simeq \mathcal{F}$ in $\widehat{\mathbf{C}}$. In this case we also say that $X$ represents $\mathcal{F}$.*

A natural transformation $\tau : \mathcal{E} \to \mathcal{F}$ in $\widehat{\mathbf{C}}$ is called a monomorphism, and $\mathcal{E}$ a subfunctor of $\mathcal{F}$, if $\tau_X : \mathcal{E}(X) \to \mathcal{F}(X)$ is injective for all $X \in \mathrm{Obj}(\mathbf{C})$.

Let $R$ be a commutative ring with 1. An $R$-scheme is a morphism of schemes $\varphi : X \to \mathrm{Spec}\ R$ and an $R$-morphism $\phi$ from $\varphi : X \to \mathrm{Spec}\ R$ to $\psi : Y \to \mathrm{Spec}\ R$ is a commutative diagram of schemes

$$\begin{array}{ccc} X & \xrightarrow{\ \ \phi\ \ } & Y \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathrm{Spec}\ R & \end{array}$$

By abuse of notation we usually write $X$ for an $R$-scheme and $\phi : X \to Y$ for an $R$-morphism. We denote by $\mathbf{Sch}/R$ the category of $R$-schemes.

We now want to characterize the representable functors in $\widehat{\mathbf{C}}$ for the category $\mathbf{C} = \mathbf{Sch}/R$. One property they have is the following. Let $X, Y \in \mathrm{Obj}(\mathbf{C})$ and let $\bigcup_\alpha V_\alpha$, $V_\alpha \in \mathrm{Obj}(\mathbf{C})$, be a Zariski open covering of $Y$. Then $h_X(Y)$ is an equalizer in the diagram

$$h_X(Y) \to \prod_{\alpha} h_X(V_\alpha) \rightrightarrows \prod_{\alpha,\beta} h_X(V_\alpha \cap V_\beta)$$

where the two arrows to the right maps $(\phi_\alpha)$ to $(\phi_\alpha|_{V_\alpha \cap V_\beta})$ and $(\phi_\beta)$ to $(\phi_\beta|_{V_\alpha \cap V_\beta})$ respectively. Another way to express this is that $h_X$ induces a sheaf of sets on each scheme $Y \in \mathrm{Obj}(\mathbf{C})$.

**Definition 2.** *A contravariant functor* $\mathcal{F} : \mathbf{Sch}/R \to \mathbf{Sets}$ *is called a Zariski sheaf if it induces a sheaf of sets on each* $R$*-scheme* $Y$.

Let $\mathbf{AffSch}/R$ denote the full subcategory of $\mathbf{Sch}/R$, whose objects are the affine $R$-schemes. Consider the category $\mathbf{Func}((\mathbf{AffSch}/R)^{\mathbf{op}}, \mathbf{Sets})$. In this category we define Zariski sheaves, but with respect to the principal open subsets $D(f) := \{\mathfrak{p} \in \text{Spec } S; f \notin \mathfrak{p}\}$ where $f \in S$ and where $S$ is an $R$-algebra. These subsets form a basis for the Zariski topology on Spec $S$ with $D(f) \cap D(g) = D(fg)$ for all $f, g \in S$. Note that the ring of regular functions on $D(f)$ is the localisation $S_f$ (see [8], section II.2).

**Definition 3.** *A functor* $\mathcal{G} \in \mathbf{Func}((\mathbf{AffSch}/R)^{\mathbf{op}}, \mathbf{Sets})$ *is called a Zariski sheaf if* $\mathcal{G}(\text{Spec } S)$ *is an equalizer in the diagram*

$$\mathcal{G}(\text{Spec } S) \to \prod_i \mathcal{G}(\text{Spec } S_{f_i}) \rightrightarrows \prod_{i,j} \mathcal{G}(\text{Spec } S_{f_i f_j})$$

*for any set of elements* $f_i \in S$ *with* Spec $S = \bigcup_i D(f_i)$. *The morphisms are induced by the ring homomorphisms* $S \to S_{f_i}$, $S_{f_i} \to S_{f_i f_j}$ *and* $S_{f_j} \to S_{f_i f_j}$ *respectively.*

Let $\mathcal{F}_0 \in \mathbf{Func}((\mathbf{AffSch}/R)^{\mathbf{op}}, \mathbf{Sets})$ denote the functor obtained by restricting $\mathcal{F}$ to affine $R$-schemes.

**Proposition 1.** *The map* $\mathcal{F} \to \mathcal{F}_0$ *is an equivalence between the subcategory of Zariski sheaves in* $\mathbf{Func}((\mathbf{Sch}/R)^{\mathbf{op}}, \mathbf{Sets})$ *and the subcategory of Zariski sheaves in* $\mathbf{Func}((\mathbf{AffSch}/R)^{\mathbf{op}}, \mathbf{Sets})$.

*Proof.* See [4], Proposition I-12.■

The category $\mathbf{AffSch}/R$ is contravariantly equivalent to the category $R$-**Alg** of commutative $R$-algebras with 1. We may thus by Proposition 1 identify contravariant functors from $\mathbf{Sch}/R$ to **Sets** with covariant functors from $R$-**Alg** to **Sets**. To simplify we use the same notation $\mathcal{F}$ for both of them. Furthermore, we write $\mathcal{F}(S)$ instead of $\mathcal{F}(\text{Spec } S)$.

It is not the case that every Zariski sheaf is representable. However we will see in Lemma 2 that for a Zariski sheaf representability is a "local" property. To understand this, we must extend some notions from the category $\mathbf{C}$ to the category $\widehat{\mathbf{C}}$.

**Definition 4.** *A subfunctor* $\mathcal{E}$ *of a contravariant functor* $\mathcal{F} : \mathbf{Sch}/R \to \mathbf{Sets}$ *is called open if for any* $h_X \to \mathcal{F}$, $X \in \text{Obj}(\mathbf{C})$, *the pullback of the diagram* $\mathcal{E} \hookrightarrow \mathcal{F} \leftarrow h_X$ *is isomorphic to* $h_U$ *where* $U \hookrightarrow X$ *is an open immersion. In the same way we say that* $\mathcal{E}$ *is closed if* $U$ *is a closed subscheme of* $X$.

This definition coincides with the definition of open(closed) subscheme in the case $\mathcal{F}$ is representable (see [4], p.255).

**Definition 5.** *A collection* $\{\mathcal{F}_i\}$ *of open subfunctors of* $\mathcal{F}$ *is called an open covering of* $\mathcal{F}$ *if for each scheme* $X$ *the set* $\{U_i\}$, *where* $h_{U_i}$ *is the pullback of* $\mathcal{F}_i \hookrightarrow \mathcal{F} \leftarrow h_X$, *is a covering of* $X$.

We have already seen that representable functors must be Zariski sheaves. Furthermore, a functor of the form $h_X$ has an open covering of representable functors, namely itself. More interesting is the following converse statement.

**Lemma 2.** *Let $\mathcal{F} : \mathbf{Sch}/R \to \mathbf{Sets}$. If $\mathcal{F}$ is a Zariski sheaf and has an open covering of representable subfunctors then $\mathcal{F}$ is representable.*

*Proof.* See [12], Lemma 1.3.

## 3. The Brauer-Severi functor

We are now in a position to discuss the representability of two specific functors, the Grassmann functor and the Brauer-Severi functor. We shall use the following notation. $R$ is a commutative ring with 1, $L$ an $R$-module and $G_n(L,R)$ is the set of all $R$-submodules $M \subseteq L$ such that $L/M$ is a projective $R$-module of constant rank $n$. Furthermore let $\mathcal{G}_n(L,R)$ denote the covariant functor which to each $R$-algebra $S$ associates the set $G_n(L \otimes S, S)$. To see that $\mathcal{G}_n(L,R)$ is a Zariski sheaf, let Spec $R = \bigcup_i D(f_i)$, $f_i \in R$, be a covering of principal open subsets and consider the diagram

$$\begin{array}{ccccc} M & \dashrightarrow & \bigoplus M_{f_i} & \rightrightarrows & \bigoplus M_{f_i f_j} \\ \vdots & & \downarrow & & \downarrow \\ L & \longrightarrow & \bigoplus L_{f_i} & \rightrightarrows & \bigoplus L_{f_i f_j} \end{array}$$

where $M$ is the equalizer of the first row. Since the second row is an equalizer there is a unique $R$-module homomorphism $M \to L$ such that the diagram commutes. The cokernels of the vertical maps yield a new equalizer

$$L/M \longrightarrow \bigoplus L_{f_i}/M_{f_i} \rightrightarrows \bigoplus L_{f_i f_j}/M_{f_i f_j},$$

with $L_{f_i}/M_{f_i} \simeq (L/M)_{f_i}$. Hence $M \in G_n(L,R)$ if $M_{f_i} \in G_n(L_{f_i}, R_{f_i})$ for all $i$.

The functor $\mathcal{G}_n(L,R)$ gives rise to a contravariant functor from $\mathbf{C} = \mathbf{Sch}/R$ to $\mathbf{Sets}$ (also denoted $\mathcal{G}_n(L,R)$). We call it the Grassmann functor. It was first studied systematically by Grothendieck in [7].

**Example 1.** Let $L := R^m$. Then $\mathcal{G}_1(L,R)$ is represented by the scheme $\mathbb{P}_R^{m-1}$ (see [8], section II.7.1). In this case the quotient $S$-modules $(L \otimes S)/M$, $M \in \mathcal{G}_1(L,R)(S)$ are projective of constant rank 1. Such modules will be called invertible in the sequel.

If $M \in G_n(L,R)$ and $P = L/M$, then the surjection $q : L \to P$ induces a surjective $R$-homomorphism $q_n : \bigwedge^n L \to \bigwedge^n P$ (see [14], Appendix C) and hence an element $M_n = \text{Ker } q_n \in G_1(\bigwedge^n L, R)$. This map $G_n(L,R) \to G_1(\bigwedge^n L, R)$ is functorial and gives a monomorphism of functors $\mathcal{G}_n(L,R) \to \mathcal{G}_1(\bigwedge^n L, R)$.

**Proposition 2.** *The functor $\mathcal{G}_n(L,R)$ is a closed subfunctor of $\mathcal{G}_1(\bigwedge^n L, R)$ with respect to the embedding above.*

*Proof.* See [12] and [7], §9. ■

Thus, if $L$ is free of rank $m$ and $n \leq m$, then $\mathcal{G}_1(\bigwedge^n L, R)$ is represented by $\mathbb{P}^N_R$, where $N = \binom{m}{n} - 1$ and $\mathcal{G}_n(L, R)$ by a closed subscheme $X_{\mathcal{G}_n(L,R)}$ of $\mathbb{P}^N_R$ defined by the quadratic Plücker equations (see [9] pp.119-122 and [4] pp.107-110).

We now consider a particular subfunctor of the Grassmann functor.

**Definition 6.** *Let $R$ be a commutative ring with 1, $\Lambda$ an $R$-algebra (not necessarily commutative) and $P$ a left $\Lambda$-module which is projective of rank $n$ as $R$-module. The Brauer-Severi functor is the subfunctor $\mathcal{B}_n(\Lambda, R)$ of $\mathcal{G}_n(\Lambda, R)$ of left ideals $I \subseteq \Lambda$.*

The following proofs are due to Salberger.

**Lemma 3.** *Let $M \subseteq L$ be an inclusion of $R$-modules such that $P = L/M$ is invertible and let $\varphi : L \to L$ be an $R$-homomorphism. Then $\varphi(M) \subseteq M$ if and only if $l \otimes \varphi(l')$ and $\varphi(l) \otimes l'$ have the same images in $P \otimes P$ for all $l, l' \in L$.*

*Proof.* $\Rightarrow$; If $\varphi(M) \subseteq M$ then $\varphi$ induces $\bar{\varphi} : P \to P$. Put $p = l + M$ and $p' = l' + M$. Since $\mathrm{End}_R(P) = R$, we can find $r \in R$ such that $\bar{\varphi}(p) = rp$ and $\bar{\varphi}(p') = rp'$. Thus $p \otimes \bar{\varphi}(p') = p \otimes rp' = rp \otimes p' = \bar{\varphi}(p) \otimes p'$.

$\Leftarrow$; Let $m \in M$ and $q = \varphi(m) + M$ in $P$. We want to show that $q = 0$. Since $P$ is invertible this follows if $p \otimes q = 0$ for all $p \in P$. Let $p = l + M$. By assumption $l \otimes \varphi(m)$ and $\varphi(l) \otimes m$ have the same images in $P \otimes P$ so that $p \otimes q = 0$. ■

**Corollary 1.** *Let $L, M$ and $\varphi$ be as in Lemma 3, with the extra assumption that $L$ is a free $R$-module with basis $e_1, \ldots, e_n$. Then $\varphi(M) \subseteq M$ if and only if $e_j \otimes \varphi(e_k)$ and $\varphi(e_j) \otimes e_k$ have the same images in $P \otimes P$ for all $j, k \in \{1, \ldots, n\}$.*

Since, by assumption, $\Lambda$ is locally free, the representability of $\mathcal{B}_n(\Lambda, R)$ will follow from Lemma 2 if we can represent $\mathcal{B}_n(\Lambda, R)$ in the case when $\Lambda$ is free.

**Proposition 3.** *Let $\Lambda$ be an $R$-algebra which is free as $R$-module. Then $\mathcal{B}_n(\Lambda, R)$ is represented by a closed subscheme $X_{\mathcal{B}_n(\Lambda,R)}$ of $X_{\mathcal{G}_n(\Lambda,R)}$.*

*Proof.* Using the embedding of Proposition 2, we may reduce to the case when $n = 1$. It is thus enough to show that $\mathcal{B}_1(\Lambda, R)$ is representable. Let $S$ be an $R$-algebra. An $S$-module inclusion $M \subseteq \Lambda \otimes_R S$, where $M \in \mathcal{G}_1(\Lambda, R)(S)$, is an element of $\mathcal{B}_1(\Lambda, R)(S)$ precisely when $M$ is a left ideal of $\Lambda \otimes_R S$. This is the case precisely when $e_l M \subseteq M$ for all $e_l$ in an $S$-basis of $\Lambda \otimes_R S$. Let

$$\varphi = \begin{bmatrix} a^l_{11} & \cdots & a^l_{1m} \\ \vdots & \ddots & \vdots \\ a^l_{m1} & \cdots & a^l_{mm} \end{bmatrix}$$

be the matrix of the $S$-module homomorphism $\varphi$ induced by $e_l$ in the basis $e_1, \ldots, e_m$. By applying Corollary 1 to the equalities

$$e_j \otimes \varphi(e_k) = \sum_{i=1}^{m} a_{ik}^l e_j \otimes e_i$$
$$\varphi(e_j) \otimes e_k = \sum_{i=1}^{m} a_{ij}^l e_i \otimes e_k$$

we obtain that the images $p_i = e_i + M$ must satisfy the tensor relations

$$\sum_{i=1}^{m} a_{ik}^l p_j \otimes p_i = \sum_{i=1}^{m} a_{ij}^l p_i \otimes p_k. \tag{1}$$

for all $j, k \in \{1, \ldots, m\}$. Let $Y \subseteq \mathbb{P}_R^{m-1}$ be the closed subscheme corresponding to the homogeneous ideal generated by the elements $\sum_{i=1}^m a_{ik}^l x_i x_j = \sum_{i=1}^m a_{ij}^l x_i x_k$, $l \in \{1, \ldots, m\}$. Then an $R$-morphism of schemes Spec $S \to \mathbb{P}_R^{m-1}$, corresponding to a quotient $S$-module $P = (\Lambda \otimes S)/M$, factors through $Y$ if and only if the global sections $p_1, \ldots, p_n \in P$ satisfy the tensor relations (1) for $l \in \{1, \ldots, m\}$. Hence the $R$-scheme $X_{\mathcal{B}_1(\Lambda,R)} := Y$ represents $\mathcal{B}_1(\Lambda, R)$. ■

## 4. Artin schemes of tiled orders

In this section we consider a certain open and closed subscheme, the Artin subscheme, of the Brauer-Severi scheme in the case $\Lambda$ is a certain groupoid algebra. We give Salberger's equations for the Artin scheme and show that the groupoid algebras give rise to tiled orders.

Let $\mathbb{Z}_n := \{1, 2, \ldots, n\}$ and $G$ be the groupoid with elements in $\mathbb{Z}_n \times \mathbb{Z}_n$ and the following (partial) law of composition. The product $(i,j)(k,l)$ is defined if and only if $j = k$, and $(i,j)(j,l) := (i,l)$. Let $G$ act trivially on the commutative ring $R$ and let $\tau : G \times G \to R$ be a multiplicative 2-cocycle. This means that

$$\tau_{\alpha,\beta\gamma}\tau_{\beta,\gamma} = \tau_{\alpha\beta,\gamma}\tau_{\alpha,\beta}$$

for all $\alpha, \beta, \gamma \in G$ whenever the products $\alpha\beta$ and $\beta\gamma$ are defined. We may also regard $\tau$ as a function $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R$, $(i,j,k) \mapsto \tau_{(i,j),(j,k)}$, and we shall in the sequel write $\tau_{ijk}$ for $\tau(i,j,k)$. The cocycle condition may be rewritten as

$$\tau_{ijl}\tau_{jkl} = \tau_{ikl}\tau_{ijk}.$$

The 2-cocycle is said to be normalized if $\tau_{iij} = \tau_{ijj} = 1$ for all $i, j \in \mathbb{Z}_n$.

Let $\Lambda_\tau$ (or simply $\Lambda$) denote the $R$-algebra with $\Lambda = \bigoplus_{1 \le i,j \le n} R\epsilon_{ij}$ as $R$-module and with multiplication rules $\epsilon_{ij}\epsilon_{jk} = \tau_{ijk}\epsilon_{ik}$ and $\epsilon_{ij}\epsilon_{kl} = 0$ if $j \neq k$. The associativity of this multiplication follows from the cocycle condition, and makes $\Lambda$ into an $R$-algebra. If $\tau_{ijk} = 1$ for all $i, j, k \in \mathbb{Z}_n$, then $\Lambda = M_n(R)$ and $\epsilon_{ij}$, $1 \le i, j \le n$, is the standard $R$-basis of $M_n(R)$.

**Lemma 4.** *Let $\tau$ and $\sigma$ be 2-cocycles as above. Suppose that there exists a function $u : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R^*$, $(i,j,k) \mapsto u_{ijk}$, to the multiplicative group $R^*$ of invertible elements of $R$ such that*

$$\tau_{ijk} = u_{ijk}\sigma_{ijk}$$

*for all $i,j,k \in \mathbb{Z}_n$. Then $\tau$ and $\sigma$ induce isomorphic $R$-algebras, $\Lambda_\tau$ and $\Lambda_\sigma$. In particular, if $\tau_{ijk} \in R^*$ for all $i,j,k \in \mathbb{Z}_n$, then $\Lambda_\tau \simeq M_n(R)$ as $R$-algebras.*

*Proof.* The elements $u_{ijk}$ form a cocycle for $G$ with values in the group $R^*$. Fix $l$ and let $v_{ij} = u_{ijl}$. Then

$$u_{ijk} = \frac{v_{ij}v_{jk}}{v_{ik}}$$

for all $i,j,k \in \mathbb{Z}_n$. The map $\epsilon_{ij} \mapsto v_{ij}\epsilon'_{ij}$ induces an $R$-algebra isomorphism from $\Lambda_\tau = \bigoplus_{1\le i,j\le n} R\epsilon_{ij}$ to $\Lambda_\sigma = \bigoplus_{1\le i,j\le n} R\epsilon'_{ij}$. ■

We want to determine equations for the Brauer-Severi scheme $X$ of $\Lambda$. This scheme may consist of several connected components (see [1]). Artin studied the following open and closed subscheme $X^0$ of $X$. Consider the universal $\mathcal{O}_X$ quotient module $\mathcal{P}$, representing the functor $\mathcal{B}_n(\Lambda, R)$. As an $\mathcal{O}_X$-module $\mathcal{P}$ has a decomposition $\bigoplus_{i=1}^n \mathcal{P}_i$, and each $\mathcal{P}_i$ has constant rank on the connected components of $X$ (see [8] p.109 and [2] pp.109-110). Let $X^0$ denote the subscheme of $X$ where $\operatorname{rank}(\mathcal{P}_i) = 1$ for all $i$. We shall in the sequel call this subscheme $X^0$ the *Artin subscheme of* $X$ or simply the *Artin scheme of* $\Lambda$.

The following key lemma is due to Salberger.

**Lemma 5.** *There is a natural bijection between the following two sets:*

(i) *Left ideals $I \subseteq \Lambda$ such that $P_i := \epsilon_{ii}\Lambda/\epsilon_{ii}I$ is an invertible $R$-module for each $i \in \mathbb{Z}_n$.*

(ii) *$n$-tuples of $M_1, \ldots, M_n \in G_1(R^n, R)$ such that*

$$\tau_{ijk}p_{ik} \otimes p_{jl} = \tau_{ijl}p_{il} \otimes p_{jk} \tag{2}$$

*in $P_i \otimes_R P_j$ for all $i,j,k,l \in \mathbb{Z}_n$, where $p_{ik} \in P_i := R^n/M_i$ is the image of $e_k = (0,\ldots,0,1,0,\ldots,0) \in R^n$ with 1 in the $k$'th position*

*Proof.* (i)⇒ (ii); Let $p_{ik}$ be the image of $\epsilon_{ik}$ in $P_i$. If we multiply with $\epsilon_{ij} = \epsilon_{ii}\epsilon_{ij}$ from the left, then we obtain an $R$-module homomorphism $\epsilon_{jj}\Lambda \to \epsilon_{ii}\Lambda$ which sends $\epsilon_{jj}I$ into $\epsilon_{ii}I$. Let $\gamma_{ij} : P_j \to P_i$ be the corresponding quotient homomorphism and $p_{jk} := \epsilon_{jk} + \epsilon_{jj}I$. Then,

$$\gamma_{ij}(p_{jk}) = \epsilon_{ij}\epsilon_{jk} + \epsilon_{ii}I = \tau_{ijk}\epsilon_{ik} + \epsilon_{ii}I = \tau_{ijk}p_{ik}.$$

As $P_j \otimes_R P_j$ is invertible, we have

$$p_{jk} \otimes p_{jl} = p_{jl} \otimes p_{jk}.$$

By applying $\gamma_{ij} \otimes id$ to this equality we obtain

$$\gamma_{ij}(p_{jk}) \otimes p_{jl} = \gamma_{ij}(p_{jl}) \otimes p_{jk}$$

and

$$\tau_{ijk} p_{ik} \otimes p_{jl} = \tau_{ijl} p_{il} \otimes p_{jk}$$

in $P_i \otimes_R P_j$.

(ii)$\Rightarrow$(i); Let $\theta_j : R^n \to \epsilon_{jj}\Lambda$ be the $R$-module isomorphism sending $(r_1, \ldots, r_n)$ to $r_1\epsilon_{j1} + \ldots + r_n\epsilon_{jn}$ and let $I_j = \theta_j(M_j)$. Then $P_j = R^n/M_j \simeq \epsilon_{jj}\Lambda/I_j$ is invertible as $R$-module. It is therefore sufficient to prove that $I = I_1 \oplus \ldots \oplus I_n$ is a left ideal in $\Lambda = \epsilon_{11}\Lambda \oplus \ldots \oplus \epsilon_{nn}\Lambda$. That is, we have to show that

$$\epsilon_{ij}(I) = \epsilon_{ij}(I_j) = \epsilon_{ij}(\theta_j(M_j)) \subseteq \theta_i(M_i) = I_i$$

for all $i, j \in \mathbb{Z}_n$. Suppose $\sum_{k=1}^n r_k e_k \in M_j$. Then $\sum_{k=1}^n r_k p_{jk} = 0$ so that $\sum_{k=1}^n r_k p_{jk} \otimes \tau_{ijl} p_{il} = 0$ for all $l \in \mathbb{Z}_n$. Applying (2) gives

$$\sum_{k=1}^n r_k \tau_{ijk} p_{ik} \otimes p_{jl} = \sum_{k=1}^n r_k p_{jk} \otimes \tau_{ijl} p_{il} = 0,$$

for all $l \in \mathbb{Z}_n$, which is possible only if $\sum_{k=1}^n r_k \tau_{ijk} p_{ik} = 0$. Hence

$$\sum_{k=1}^n r_k \tau_{ijk} e_k \in M_i$$

and

$$\epsilon_{ij}(\theta_j(\sum_{k=1}^n r_k e_k)) = \sum_{k=1}^n r_k \epsilon_{ij}\epsilon_{jk} = \sum_{k=1}^n r_k \tau_{ijk} \epsilon_{ik} \in \theta_i(M_i). \blacksquare$$

We may and shall apply Lemma 5 to $\Lambda \otimes_R S$ for commutative $R$-algebras $S$. We then obtain similar bijections between suitable left ideals in $\Lambda \otimes_R S$ and $n$-tuples of elements in $G_1(S^n, S)$ satisfying the same tensor relations. These bijections are functorial under homomorphisms of $R$-algebras.

**Corollary 2.** *The Artin scheme $X^0$ of the $R$-algebra $\Lambda_\tau$ is isomorphic to the $R$-subscheme $X'$ of $(\mathbb{P}_R^{n-1})^n$ defined by the multihomogeneous equations*

$$\tau_{ijk} x_{ik} x_{jl} = \tau_{ijl} x_{il} x_{jk}, \qquad i, j, k, l \in \mathbb{Z}_n.$$

*Proof.* The bijection in Lemma 5 extends to a isomorphism between two functors $R$-**Alg** $\to$ **Sets**. The first is represented by $X^0$ and the second by $X'$. $\blacksquare$

Obviously, if we fix the multiprojective coordinates $x_{ik}$, we can recover the elements $\tau_{ijk}$ from the equations of the scheme $X^0$, and hence it is possible to reconstruct the order $\Lambda$.

**Definition 7.** *Let $R$ be an integral domain with quotient field $K$ and $A$ be a split central simple $K$-algebra, that is $A \simeq M_n(K)$ for some $n$. An $R$-order (see [15]) in $A$ is a subring $\Lambda$ of $A$ containing the unit element $1_A$ of $A$ such that $\Lambda$ is a full $R$-lattice in $A$. An $R$-order $\Lambda$ in $A \simeq M_n(K)$ is called a tiled*

*R-order if there exist primitive orthogonal idempotents* $\epsilon_{11}, \ldots, \epsilon_{nn} \in \Lambda$ *such that* $\sum_{i=1}^{n} \epsilon_{ii} = 1_A$.

**Lemma 6.** *Let R be an integral domain with quotient field K and* $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R \setminus \{0\}$ *be a normalized cocycle. Then the following holds.*

(i) *There exists a function* $u : \mathbb{Z}_n \times \mathbb{Z}_n \to R \setminus \{0\}$, $(i, j) \mapsto u_{ij}$ *with* $u_{ii} = 1$ *for all* $i \in \mathbb{Z}_n$ *and*

$$\tau_{ijk} = \frac{u_{ij} u_{jk}}{u_{ik}}.$$

(ii) *Let* $A = A_\tau$ *be the K-algebra with* $A = \bigoplus_{1 \le i,j \le n} K e_{ij}$ *as vector space over K and with multiplication rules* $e_{ij} e_{jk} = \tau_{ijk} e_{ik}$ *and* $e_{ij} e_{kl} = 0$ *if* $j \ne k$. *Then* $A \simeq M_n(K)$ *as K-algebras.*

(iii) $\Lambda = \Lambda_\tau = \bigoplus_{1 \le i,j \le n} R e_{ij}$ *is a tiled R-order in A containing the primitive orthogonal idempotents* $e_{11}, \ldots, e_{nn}$.

*Proof.*

(i) Choose $u_{ij} = \tau_{ijl}$ for some fixed $l \in \mathbb{Z}_n$.

(ii) This is a special case of Lemma 4.

(iii) It is clear that $\Lambda$ is a full $R$-lattice in $A$ and that $\Lambda$ is closed under multiplication. ■

The orders in the last lemma were studied in the thesis of P. Lundström [13] under the name of Brauer orders. The interpretation in terms of 2-cocycles of the groupoid is due to Salberger.

## 5. Local studies of certain schemes

To simplify the further investigation of the Artin subscheme $X^0$ of the Brauer-Severi scheme $X$, we shall in this section make the following assumptions on the base ring $R$. We suppose that $R$ is a discrete valuation ring containing an algebraically closed field $k$, which is isomorphic to the residue field of $R$. We denote by $t$ an arbitrary but fixed generator of the maximal ideal $\mathfrak{m}$ of $R$. We are interested in the regularity of $X^0$. Let us therefore recall some definitions and results concerning regularity and Kähler differentials.

**Definition 8.** *A local ring* $(B, \mathfrak{m})$ *with residue field F is called a regular local ring if*

$$\dim_F(\mathfrak{m}/\mathfrak{m}^2) = \dim B$$

*where the first dimension is the dimension as vector space over F and the second dimension is the Krull dimension of the ring.*

Note that $\dim B \le \dim_F(\mathfrak{m}/\mathfrak{m}^2)$ holds for all local rings.

**Proposition 4.** *Let* $(B, \mathfrak{m})$ *be a local ring, which contains a field k isomorphic to its residue field* $B/\mathfrak{m}$. *Then there is an isomorphism of vector spaces over k,*

$$\mathfrak{m}/\mathfrak{m}^2 \simeq \Omega_{B/k} \otimes_B k.$$

*In particular, if $B$ is a discrete valuation ring, then $\Omega_{B/k} \otimes_B k$ is a one-dimensional vector space over $k$ generated by $dt$ for any $t \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

*Proof.* See [8] p.174.■

**Proposition 5.** *Let $A$ be a commutative $R$-algebra, let $I$ be an ideal of $A$, and let $\bar{A} = A/I$. Then there is a natural exact sequence of $\bar{A}$-modules:*

$$I/I^2 \xrightarrow{d} \Omega_{A/R} \otimes_A \bar{A} \longrightarrow \Omega_{\bar{A}/R} \longrightarrow 0$$

*Proof.* See [8] p.173 or [3] p.389.■

**Proposition 6.** *Let $A$ be a commutative $R$-algebra and $S$ be a multiplicative system of $A$. Then,*

$$\Omega_{S^{-1}A/R} \simeq S^{-1}\Omega_{A/R}$$

*Proof.* See [8] p.173 or [3] p.397. ■

**Proposition 7.** *If $A := R[x_1, \dots, x_n]$ is a polynomial ring over a commutative $k$-algebra $R$, then*

$$\Omega_{A/k} \simeq (A \otimes_R \Omega_{R/k}) \oplus (\oplus_{i=1}^n A dx_i)$$

*Proof.* See [3] p.394.■

**Corollary 3.** *Let $k$ be a field and $R$ be a discrete valuation ring containing $k$ with residue field isomorphic to $k$. Let $A = R[x_1, \dots, x_n]$ be a polynomial ring over $R$ and $I \subseteq A$ be an ideal generated by some polynomials $q_1, \dots, q_m \in A$. Let $\bar{A} = A/I$ and $B = S^{-1}\bar{A}$ for some multiplicative system $S$ of $\bar{A}$. Finally, let $t$ be a generator for the maximal ideal of $R$. Then,*

$$\Omega_{B/k} \simeq [Bdt \oplus (\oplus_{i=1}^n B dx_i)]/\langle dq_1, \dots, dq_n \rangle.$$

*Proof.* Combine the previous 4 propositions.■

**Definition 9.** *Let $(Z, \mathcal{O}_Z)$ be a scheme.*

(i) *A point $p$ of $Z$ is a regular point if the local ring $(\mathcal{O}_{Z,p}, \mathfrak{m})$ at $p$ is a regular local ring.*

(iii) *The scheme $Z$ is regular if all of its points are regular.*

In our case it is enough to check the regularity at the closed points. The Artin scheme $X^0$ is Noetherian, and such a scheme is regular if and only if it is regular at all its closed points (see [14], Theorem 19.3).

As we have assumed that $R$ is a discrete valuation ring containing an algebraically closed field $k$ isomorphic to the residue field of $R$, all closed points on the Artin scheme $X^0$ are $k$-rational.

**Proposition 8.** *Let $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R \setminus \{0\}$ be a multiplicative 2-cocycle as in section 4 and let $\Lambda$ be the tiled $R$-order $\Lambda_\tau$ described in Lemma 6. Then $X^0$ is isomorphic to the subscheme of $(\mathbb{P}_R^{n-1})^n$ defined by the multihomogeneous equations*

$$t^{[i,j,k]} x_{ik} x_{jl} = t^{[i,j,l]} x_{il} x_{jk} \ , \qquad i, j, k, l \in \mathbb{Z}_n$$

*for non-negative integers* $[i,j,k]$ *satisfying*

$$\begin{aligned}[i,i,j]=[i,j,j] &= 0\\ [i,j,k]+[i,k,l] &= [i,j,l]+[j,k,l]\end{aligned}$$

*for all* $i,j,k\in\mathbb{Z}_n$.

*Proof.* Since $R$ is a discrete valuation ring each $\tau_{ijk}$ in the equations of $X^0$ may be written $\tau_{ijk}=u_{ijk}t^{[i,j,k]}$, where $u_{ijk}$ is a unit in $R$. Then $\Lambda_\tau\simeq\Lambda_\sigma$ for $\sigma_{ijk}=t^{[i,j,k]}$. Hence $X^0$ is isomorphic to the Artin scheme of $\Lambda_\sigma$ with the equations asserted above. The assertions about the non-negative integers $[i,j,k]$ follow directly from the fact that $\sigma$ is a normalized multiplicative 2-cocycle. ■

By letting $i=k$ and/or $j=l$ we get the identities

$$\begin{aligned}[i,j,i] &= [i,j,l]+[j,i,l]\\ [i,j,k]+[i,k,j] &= [j,k,j]\\ [i,j,i] &= [j,i,j]\end{aligned}$$

which we shall use frequently.

Since the value $\tau_{ijk}=t^{[i,j,k]}$ is determined by the integers $[i,j,k]$ we consider the additive 2-cocycle function $f:\mathbb{Z}_n\times\mathbb{Z}_n\times\mathbb{Z}_n\to\mathbb{Z}_{\geq 0}$, $(i,j,k)\mapsto[i,j,k]$, rather than the function $\tau$. We shall call $f$ a cocycle function.

We first consider cocycle functions $f$ satisfying the following additional assumptions.

**Hypothesis 1** (H1). $f(i,j,i)=[i,j,i]\geq 1$ *for any two different* $i,j\in\mathbb{Z}_n$.

Assume H1 and consider, for any $i\in\mathbb{Z}_n$, the relation

$$j\leq_i k \text{ if } [i,j,k]=0.$$

This is a partial order on the set $\mathbb{Z}_n$ since

(i) $j\leq_i j$ for all $j$ since $[i,j,j]=0$.
(ii) If $j\leq_i k$ and $k\leq_i l$ then $j\leq_i l$ since $[i,j,k]+[i,k,l]\geq[i,j,l]$.
(iii) If $j\leq_i k$ and $k\leq_i j$ then $j=k$ since $[i,j,k]+[i,k,j]=[j,k,j]\neq 0$ if $j\neq k$.

In the same way one can verify that the relation $i\leq^k j$ if $[i,j,k]=0$ is a partial order on the set $\mathbb{Z}_n$. These observations are due to Salberger.

**Hypothesis 2** (H2). *If* $n\geq 2$ *there is for each* $i\in\mathbb{Z}_n$ *another element* $i'\in\mathbb{Z}_n$ *such that* $i'\leq_i j$ *for all* $j\in\mathbb{Z}_n\setminus\{i\}$.

Note that $i'$ is uniquely determined by $i$ if H1 holds (use (iii)). We shall call $i'$ the successor of $i$ and use the notation $(i')'=i''$, $(i'')'=i^{(3)}$ and so on.

**Lemma 7.** *Assume* H1 *and* H2. *Then the successor map* $\mathbb{Z}_n\to\mathbb{Z}_n$, $i\mapsto i'$, *consists of exactly one cycle.*

*Proof.* If $n = 1$, there is nothing to prove. If $n \geq 2$, let $i, i', i'', \ldots, i^{(s)} = i$ be a cycle. If $s < n$, let $l \notin \{i, i', i'', \ldots, i^{(s)}\}$ and consider the order $\leq^l$. By the definition of successor we have

$$i \leq^l i' \leq^l i'' \leq^l \ldots \leq^l i^{(s)} = i$$

so that $i = i'$. This is impossible, whence $s = n$.■

To study the relation between the orders $\leq_i$ and the cycle constructed by means of the successor map, we introduce the following non-symmetric "distance" function $d$, which was suggested by Salberger.

**Definition 10.** *Let $i, j \in \mathbb{Z}_n$. Define $d(i,j) \in \{0, \ldots, n-1\}$ so that $j$ is the $d(i,j)'$th successor of $i$ and put $d(i,i) = 0$.*

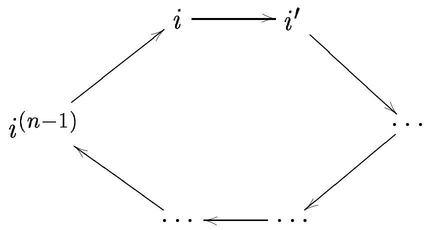
**Lemma 8.** *Assume* H1 *and* H2 *and let $i, j, k \in \mathbb{Z}_n$. Then $j \leq_i k \Leftrightarrow d(i,j) \leq d(i,k)$.*

*Proof.* By Lemma 7 it is enough to prove that $j \leq_i j'$ for $i, j \in \mathbb{Z}_n$ with $i \neq j'$. We use induction with respect to $d(i,j) = d$. If $d = 0$, then $i = j$ and $[i, j, j'] = 0$. If $d > 0$ then $d(i', j) = d - 1$ and $j \leq_{i'} j'$, that is $[i', j, j'] = 0$, by the induction assumption. Also $[i, i', j'] = 0$ by the definition of $i'$. Hence

$$[i, i', j] + [i, j, j'] = [i, i', j'] + [i', j, j'] = 0$$

and $[i, j, j'] = 0$ as was to be proved.■

Lemma 8 may be visualised in the following way. Let




be the cycle of the set $\mathbb{Z}_n$ corresponding to the successor operation. If we remove $i$ from this, we obtain the total order induced by $\leq_i$. Conversely, the cycle can be constructed from the order $\leq_i$, for any $i \in \mathbb{Z}_n$, by connecting the maximal and the minimal elements.

**Corollary 4.** *Assume* H1 *and* H2, *and let $i, j, k \in \mathbb{Z}_n$ be such that $i \neq j, k$. Then,*

(a) $\min\{[i,j,k], [i,k,j]\} = 0$,
(b) $\max\{[i,j,k], [i,k,j]\} = [j,k,j]$,
(c) $[i,j,i] = [i,k,i]$.

*Proof.*

(a) $(\mathbb{Z}_n \setminus \{i\}, \leq_i)$ is a totally ordered set by Lemma 8.
(b) $\max\{[i,j,k], [i,k,j]\} = [i,j,k] + [i,k,j] = [j,k,j]$.
(c) We may by (a) assume that $[i,j,k] = 0$. Now use that

$$[i,j,k] + [i,k,i] = [i,j,i] + [j,k,i].$$■

**Proposition 9.** *Assume* H1 *and* H2*. Then $f$ assumes exactly two values, if $n \geq 2$.*

*Proof.* By part (a) and (b) of Corollary 4, it suffices to show that $[i, j, i] = [k, l, k]$ for all $i, j, k, l \in \mathbb{Z}_n$ with $i \neq j$ and $k \neq l$. This follows from part (c) of Corollary 4 and the identity $[i, j, i] = [j, i, j]$. ■

Let us consider the case when H1 does not hold. It is then possible that $[i, j, i] = 0$ for $i \neq j$. We use again an idea of Salberger and consider the relation

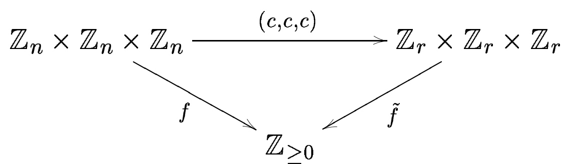
$$i \simeq j \text{ if } [i, j, i] = 0$$

on $\mathbb{Z}_n$. This is an equivalence relation since $[j, k, j] = [i, k, j] + [i, j, k]$ and both numbers to the right are zero if $[i, j, i] = [i, k, i] = 0$. Hence the relation induces a partition

$$\mathbb{Z}_n = \bigcup_{i=1}^{r} B_i$$

of $\mathbb{Z}_n$ into $r$ equivalence classes $B_1, \ldots, B_r$. Let $c : \mathbb{Z}_n \to \mathbb{Z}_r$ be the map defined by $c(k) = i$ if $k \in B_i$. We shall call this map a *class map* for $f$.

**Lemma 9.** *There exists a cocycle function $\tilde{f} : \mathbb{Z}_r \times \mathbb{Z}_r \times \mathbb{Z}_r \to \mathbb{Z}_{\geq 0}$ such that the diagram below commutes.*




*Proof.* Let $i_1, i_2, j_1, j_2, k_1, k_2$ be element in $\mathbb{Z}_n$ such that $i_1 \simeq i_2$, $j_1 \simeq j_2$ and $k_1 \simeq k_2$. We must show that

$$[i_1, j_1, k_1] = [i_2, j_2, k_2].$$

This follows from the equalities

$$\begin{aligned} [i_1, i_2, j_1] + [i_1, j_1, k_1] &= [i_1, i_2, k_1] + [i_2, j_1, k_1] \\ [i_1, j_1, j_2] + [i_1, j_2, k_1] &= [i_1, j_1, k_1] + [j_1, j_2, k_1] \\ [i_1, j_1, k_1] + [i_1, k_1, k_2] &= [i_1, j_1, k_2] + [j_1, k_1, k_2]. \blacksquare \end{aligned}$$

Note that $\tilde{f}$, by construction, satisfies hypothesis H1. Hence the results in Lemma 7, Lemma 8, Corollary 4 and Proposition 9 holds for $\tilde{f}$ if we assume that it satisfies hypothesis H2.

We next give a matrix representation of triangular orders. For this we need to order the index set $\mathbb{Z}_n$ as follows.

**Definition 11.** *Let $f : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_{\geq 0}$ be a cocycle function and $c : \mathbb{Z}_n \to \mathbb{Z}_r$ be a class map for $f$. Suppose that the quotient cocycle $\tilde{f}$ satisfies* H2*. Then $(f, c)$ is on standard form if*

(i) $c(i) < c(j) \Rightarrow i < j$

(ii) *The ordering on $\mathbb{Z}_r$ is given by $\leq_1$ (see p.11).*

**Definition 12.** *Let $R$ be a discrete valuation ring with quotient field $K$ and $t$ a generator of the maximal ideal $\mathfrak{m}$ of $R$. Then an $R$-order $\Lambda' \subseteq M_n(K)$ is called triangular if $\Lambda' = \bigoplus_{1\leq i,j\leq n} Rt^{[i,j]}e_{ij}$ for a function*

$$[\ ,\ ] : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_{\geq 0}, (i,j) \mapsto [i,j]$$

*with at most one value $m \neq 0$ and with $[i,j] = 0$ for $i \leq j$. Here $e_{ij}$, $1 \leq i,j \leq n$ is the standard basis for $M_n(K)$.*

Note that $t^{[i,j]}e_{ij}t^{[j,k]}e_{jk} \in Rt^{[i,k]}e_{ij}$ and hence that $[i,j]+[j,k]-[i,k] \geq 0$ for all $i,j,k \in \mathbb{Z}_n$.

**Proposition 10.** *Let $f : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_{\geq 0}$, $(i,j,k) \mapsto [i,j,k]$, be a cocycle function such that $\tilde{f}$ satisfies* H2. *Let $\tau_{ijk} = t^{[i,j,k]}$. Then $\Lambda := \Lambda_\tau = \bigoplus_{1\leq i,j\leq n} R\epsilon_{ij}$ is isomorphic to a triangular $R$-order $\Lambda' \subseteq M_n(K)$.*

*Proof.* We may, after a permutation of $\mathbb{Z}_n$ and renumeration of the classes, assume that $(f,c)$ is on standard form. Put $[i,j] = [1,i,j]$ for all $i,j \in \mathbb{Z}_n$. Then $[i,j] \in \mathbb{Z}_{\geq 0}$ with $[i,j]+[j,k]-[i,k] \geq 0$ and $[i,j] = 0$ for $i \leq j$. Moreover, by Proposition 9 we obtain that $[\ ,\ ] : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_{\geq 0}$ assumes at most two values. Hence $\Lambda' = \bigoplus_{1\leq i,j\leq n} Rt^{[i,j]}e_{ij}$ is a triangular $R$-order. The $R$-module homomorphism $\Lambda \to \Lambda'$ with $\epsilon_{ij} \mapsto t^{[i,j]}e_{ij}$ gives an $R$-algebra isomorphism from $\Lambda$ to $\Lambda'$.■

**Remark 1.** Let $\Lambda'$ be a triangular order with corresponding equivalence classes $B_1, \ldots, B_r$, written in the order $\leq_1$, and let $|B_i| = n_i$. Then $\Lambda'$ has a matrix representation

$$\Lambda' = \begin{bmatrix} [R]_{11} & [R]_{12} & [R]_{13} & \cdots & [R]_{1(r-1)} & [R]_{1r} \\ [(t)^m]_{21} & [R]_{22} & [R]_{23} & \cdots & [R]_{1(r-1)} & [R]_{2r} \\ [(t)^m]_{31} & [(t)^m]_{32} & [R]_{33} & \cdots & [R]_{1(r-1)} & [R]_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [(t)^m]_{r1} & [(t)^m]_{r2} & [(t)^m]_{r3} & \cdots & [(t)^m]_{1(r-1)} & [R]_{rr} \end{bmatrix}$$

where $[I]_{ij}$ is an $n_i \times n_j$-matrix with elements from the ideal $I$ on each place. If $m = 1$, and $n_1 = \ldots = n_r$, then these are the orders studied in [1].

We now shall follow the calculations of Artin [1] (see also [5]) and describe the irreducible components of the closed fiber $Y$ of the Artin scheme $X^0$ for a triangular order $\Lambda$.

Note that if $i \simeq j$, $i,j \in \mathbb{Z}_n$, then we have the equation

$$x_{ik}x_{jl} = x_{il}x_{jk}$$

for all $k,l \in \mathbb{Z}_n$. Hence the projective coordinates of $x_{jk}$ are uniquely determined by the projective coordinates of $x_{ik}$. Thus it is sufficient to consider one element in each equivalence class of $\simeq$. We can identify the scheme $X^0$ with a closed subscheme of $(\mathbb{P}_R^{n-1})^r$ in the following way. Fix a transversal

$$T := \{b_1, \ldots, b_r\}$$

of representatives for the classes $B_1, \ldots, B_r$. We shall in the sequel often use $h, i$ or $j$ to denote an element of $T$.

**Proposition 11.** *Let $\Lambda = \Lambda_\tau$ for a cocycle $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R \setminus \{0\}$ of the form $\tau_{ijk} = t^{[i,j,k]}$. Let $X^0$ be the Artin scheme of $\Lambda$. Then the closed fiber $Y$ of $X^0$ is defined by the multihomogeneous equations*

$$\begin{array}{rcll} x_{il}x_{jk} & = & 0 & \textit{if} \quad [i,j,k] > 0 \textit{ and } [i,j,l] = 0 \\ x_{ik}x_{jl} & = & x_{il}x_{jk} & \textit{if} \quad [i,j,k] = [i,j,l] = 0 \end{array}$$

*where $i, j \in T$ and $k, l \in \mathbb{Z}_n$.*

*Proof.* This is a consequence of Corollary 2. ■

We divide the closed fiber $Y$ into its irreducible components.

**Proposition 12.** *Let $\Lambda$ and $X^0$ be as in Proposition 11. Then the closed fiber $Y$ of $X^0$ can be written $Y = \bigcup_{h \in T} Y_h$ where $Y_h$ is the scheme defined by the multihomogeneous equations*

$$\begin{array}{rcll} x_{jk} & = & 0 & \textit{if} \quad [h,j,k] > 0 \\ x_{ik}x_{jl} & = & x_{il}x_{jk} & \textit{if} \quad [i,j,k] = [i,j,l] = 0 \end{array}$$

*where $i, j \in T$ and $k, l \in \mathbb{Z}_n$.*

*Proof.* To verify that $Y_h \subseteq Y$, let $p \in Y_h$. We must show that $x_{il}(p)x_{jk}(p) = 0$ if $[i,j,k] > 0$ and $[i,j,l] = 0$. One of the integers $[i,j,h]$ and $[j,i,h]$ must be non-zero. By symmetry we may assume that $[j,i,h] > 0$ and $[j,h,i] = 0$. Also, $[j,i,k] = 0$ since $[i,j,k] > 0$. Hence,

$$[j,h,k] + [h,i,k] = [j,h,i] + [j,i,k] = 0.$$

This implies that $[j,h,k] = 0$ and $[h,j,k] > 0$ since $h \neq j$. Thus $x_{jk}(p) = 0$ and $x_{jk}(p)x_{il}(p) = 0$.

To show that $Y \subseteq \bigcup_{h \in T} Y_h$, let $p$ be a point on $Y$. We must find a number $h \in T$ such that $x_{jk}(p) = 0$ for all $j \in T$, $k \in \mathbb{Z}_n$, with $[h,j,k] > 0$. We introduce the following relation, suggested by Salberger, on elements in $T$. Put $i \leq j$ if $i = j$ or if there is an $l$ such that $[i,j,l] = 0$ and $x_{il}(p) \neq 0$. Then:

(i) $\leq$ is antisymmetric. For suppose $i \neq j$ and that there exist $k, l$ such that $x_{il}(p) \neq 0$, $[i,j,l] = 0$ and $x_{jk}(p) \neq 0$, $[j,i,k] = 0$. Then $[i,j,k] > 0$ which contradicts the equations of Proposition 11. Hence $i = j$.

(ii) $\leq$ is transitive. For suppose $i_1 \leq i_2$ and $i_2 \leq i_3$. Then there exist $k, l$ with $x_{i_1 l}(p) \neq 0$, $[i_1, i_2, l] = 0$ and $x_{i_2 k}(p) \neq 0$, $[i_2, i_3, k] = 0$. Then also $[i_1, i_2, k] = 0$ since otherwise $x_{i_1 l}(p)x_{i_2 k}(p) = 0$ by Proposition 11. Hence

$$[i_1, i_2, i_3] + [i_1, i_3, k] = [i_1, i_2, k] + [i_2, i_3, k] = 0$$

so $[i_1, i_3, k] = 0$ and $i_1 \leq i_3$.

Let $h$ be a minimal element of $\leq$ and $j \in T$, $k \in \mathbb{Z}_n$, such that $[h,j,k] > 0$. Then $x_{jk}(p) = 0$ since $h$ is minimal. ■

For a triangular order $\Lambda$ the cocycle function $\tilde{f}$ satisfies hypothesis H1 and H2. We can thus define the successor operation on the set $T$ of representatives. Consider the sequence of $r-1$ rational maps

$$\mathbb{P}_k^{n-1} \xrightarrow{pr_1} \mathbb{P}_k^{n-1} \xrightarrow{pr_2} \cdots \xrightarrow{pr_{r-1}} \mathbb{P}_k^{n-1}$$

where $pr_s$ kills the coordinates $x_{i^{(s)}k}$ where $[i, i^{(s)}, k] > 0$. The closure of the graph in $(\mathbb{P}_k^{n-1})^r$ of these maps is isomorphic to $Y_i$ (compare with the construction in [1]).

Each $Y_i$ is then isomorphic to a sequence of $r-1$ blow-ups of the space $\mathbb{P}_k^{n-1}$ along regular subschemes. As noted by Artin [1] and Frossard [5], this scheme is regular of dimension $n-1$. Hence the singularities on $Y$ must belong to at least two irreducible components of $Y$.

**Proposition 13.** *Let $\Lambda = \Lambda_\tau$ for a cocycle $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R \setminus \{0\}$ of the form $\tau_{ijk} = t^{[i,j,k]}$. Suppose that the corresponding cocycle function $\tilde{f}$ satisfies* H2. *Let $X^0$ be the Artin scheme of $\Lambda$ and $Y$ the closed fiber of $X^0$. Let $p$ be a $k$-rational point of $Y$ and $C = \mathcal{O}_{Y,p}$. Then*

$$\dim_k(\Omega_{C/k} \otimes_C k(p)) \leq n$$

*with equality if and only if $p$ is in at least two components of $Y$.*

*Proof.* By Proposition 10, $\Lambda$ is isomorphic to a triangular order and we may thus assume that $\Lambda$ is of the form described in Remark 1. Also, since $Y$ only depends on $\Lambda/t\Lambda$, we may assume that $\Lambda$ is of the form

$$\Lambda = \begin{bmatrix} [R]_{11} & [R]_{12} & [R]_{13} & \dots & [R]_{1(r-1)} & [R]_{1r} \\ [(t)]_{21} & [R]_{22} & [R]_{23} & \dots & [R]_{1(r-1)} & [R]_{2r} \\ [(t)]_{31} & [(t)]_{32} & [R]_{33} & \dots & [R]_{1(r-1)} & [R]_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [(t)]_{r1} & [(t)]_{r2} & [(t)]_{r3} & \dots & [(t)]_{1(r-1)} & [R]_{rr} \end{bmatrix}$$

where $[I]_{ij}$ is an $n_i \times n_j$-matrix with elements from the ideal $I$ on each place. The result for such hereditary orders (see section 7) may be found in [5], section 2. The proof there is an obvious generalization of Artin's proof in [1], section 4.■

**Proposition 14.** *Let $\Lambda = \Lambda_\tau$ be as in the previous proposition with the additional assumption that $[i, j, i] \leq 1$ for all $i, j \in \mathbb{Z}_n$. Let $B = \mathcal{O}_{X,p}$ for a $k$-point $p$ of $X^0 \subseteq X$. Then*

$$\dim_k(\Omega_{B/k} \otimes_B k(p)) \leq n.$$

*Proof.* Note that by the exact sequence

$$tB/t^2B \xrightarrow{d} \Omega_{B/k} \otimes_B C \longrightarrow \Omega_{C/k} \longrightarrow 0$$

of Proposition 5 we have,

$$\dim_k(\Omega_{B/k} \otimes_B k(p)) \leq \dim_k(\Omega_{C/k} \otimes_C k(p)) + 1$$

where $C = \mathcal{O}_{Y,p}$. Hence, if $p$ only belongs to one irreducible component $Y_i$, then we are done. Suppose therefore that $p \in Y_i \cap Y_j$, $i \neq j$. We must verify that the differential $dt$ is a linear combination of the differentials which generate $\Omega_{C/k} \otimes_C k(p)$. Choose $k, l \in \mathbb{Z}_n$ such that $x_{ik}(p) \neq 0$, $[j,i,k] = 0$ and $x_{jl}(p) \neq 0$, $[i,j,l] = 0$. Hence $[i,j,k] = 1$ and

$$tx_{ik}x_{jl} = x_{il}x_{jk}.$$

Let $y_{il} = x_{il}/x_{ik}$ and $y_{jk} = x_{jk}/x_{jl}$. Then

$$dt = y_{il}dy_{jk} + y_{jk}dy_{il} = 0$$

in $\Omega_{B/k} \otimes_B k(p)$ since $y_{il}(p) = y_{jk}(p) = 0$. Thus

$$\dim_k(\Omega_{B/k} \otimes_B k(p)) = \dim_k(\Omega_{C/k} \otimes_C k(p)) \leq n. \blacksquare$$

We now study the cotangent spaces in the case where we may have $[i,j,k] \geq 2$. First suppose that $[i,j,i] \geq 1$ for all $i,j$. We shall work in the affine space where $x_{jj} \neq 0$, $j \in \mathbb{Z}_n$ with affine coordinates $y_{jk} = x_{jk}/x_{jj}$.

**Definition 13.** *Let $f$ be a cocycle function. A pair $(i,k) \in \mathbb{Z}_n \times \mathbb{Z}_n$ is called adjacent pair if $k$ is a minimal element of the partially ordered set $(\mathbb{Z}_n \setminus \{i\}, \leq_i)$, that is $[i,j,k] > 0$ for all $j \neq i,k$.*

There is a connection between the notion of adjacent pair and successor (as defined on p.12) of the orders $\leq_i$ as follows. Let $i'$ be a successor of $i$. Then $(i,i')$ is an adjacent pair since

$$[i,j,i'] = [i,j,i'] + [i,i',j] = [i',j,i'] > 0$$

for all $j \neq i,i'$. Conversely suppose $(i,k)$ is an adjacent pair. Then

$$[i,k,j] = [j,k,j] - [i,j,k] < [j,k,j]$$

for all $j \neq i,k$. Hence, if $[j,k,j] \leq 1$ for all $j \neq k$, then $k$ must be a successor of $i$.

**Proposition 15.** *Let $\Lambda = \Lambda_\tau$ for a cocycle $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R \setminus \{0\}$ of the form $\tau_{ijk} = t^{[i,j,k]}$ such that the corresponding $f$ satisfies* H1. *Let $p$ be the $k$-point of $X^0$ with $t(p) = x_{jk}(p) = 0$ for all $j,k \in \mathbb{Z}_n$, $j \neq k$. Then:*

- (i) *If $[i,j,i] = 1$ for some $i,j \in \mathbb{Z}_n$, then the vector space $\Omega_{B/k} \otimes_B k(p)$ has a $k$-basis consisting of the differentials $\{dy_{ik}\}$ where $(i,k)$ are the adjacent pairs.*
- (ii) *If $[i,j,i] \geq 2$ for all $i,j \in \mathbb{Z}_n$ then the vector space $\Omega_{B/k} \otimes_B k(p)$ has a $k$-basis consisting of the differentials $\{dy_{ik}\}$, where $(i,k)$ are the adjacent pairs, together with the differential $dt$.*

*Proof.* $\Omega_{B/k} \otimes_B k(p)$ is generated by the differentials $dy_{ik}$, $i,k \in \mathbb{Z}_n$, and $dt$ with relations given by

$$d(t^{[i,j,k]}y_{ik}y_{jl}) = d(t^{[i,j,l]}y_{il}y_{jk})$$

where $i,j,k,l \in \mathbb{Z}_n$. These relations can be rewritten as

$$y_{ik}y_{jl}dt^{[i,j,k]} + t^{[i,j,k]}(y_{ik}dy_{jl} + y_{jl}dy_{ik}) = y_{il}y_{jk}dt^{[i,j,l]} + t^{[i,j,l]}(y_{il}dy_{jk} + y_{jk}dy_{il}).$$

The relations with $i = j$ or $k = l$ may be omitted. If $\{i, j\} \cap \{k, l\} = \emptyset$, then the relations become $0 = 0$ in $\Omega_{B/k} \otimes_B k(p)$. If $j = l$ and $i \neq j, k$, $k \neq l$, then the relation is

$$t^{[i,j,k]} dy_{ik} = 0, \qquad i, j, k \in \mathbb{Z}_n. \tag{3}$$

If $j = l$ and $i = k$, then

$$dt^{[i,j,i]} = 0, \qquad i, j \in \mathbb{Z}_n. \tag{4}$$

All relations in $\Omega_{B/k} \otimes_B k(p)$ are then obtained from the relations (3) and (4) above. If $(i, k)$ is not an adjacent pair, there is $j \neq i, k$ such that $[i, j, k] = 0$ and hence $dy_{ik} = 0$. If $(i, k)$ is an adjacent pair, then there is no such $j$ and hence the differentials $dy_{ik}$ do not occur in any of the relations above. They are thus linearly independent in $\Omega_{B/k} \otimes_B k(p)$. If $[i, j, i] \geq 2$ for all $i \neq j$ then $dt$ does not occur in any relation so that (ii) holds. If $[i, j, i] = 1$ for some $i \neq j$, then $dt = 0$ which gives (i).■

**Proposition 16.** *Let $\Lambda = \Lambda_\tau$ for a cocycle $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R \backslash \{0\}$ of the form $\tau_{ijk} = t^{[i,j,k]}$ such that the corresponding $f$ satisfies* H1. *If $[i, j, i] \geq 2$ for all $i, j \in \mathbb{Z}_n$ then the Artin scheme $X^0$ of $\Lambda$ is singular.*

*Proof.* Let $p$ be the point defined by $t(p) = x_{jk}(p) = 0$ for all $j, k \in \mathbb{Z}_n$, $j \neq k$, and let $y_{jk} = x_{jk}/x_{jj}$. By Proposition 15, the maximal ideal $\mathfrak{m}$ of the local ring $B$ has the elements $y_{ik}$, where $(i, k)$ is an adjacent pair, and $t$ as a minimal set of generators. The ideal in $B$ generated by the $y_{ik}$'s, $(i, k)$ an adjacent pair, clearly contains $\mathfrak{m}_p^N$ where $N = \max\{[i, j, k]\}$. But then dim $B < \dim_k(\Omega_{B/k} \otimes_B k(p))$ (see [3], Corollary 10.7) so $X^0$ is singular at $p$ and therefore a singular scheme.■

We now remove the hypothesis that $[i, j, i] \geq 1$ for all $i, j \in \mathbb{Z}_n$. As noted on p.15 the scheme $X^0$ is isomorphic to the closed subscheme of $(\mathbb{P}_R^{n-1})^r$ given by the equations

$$t^{[i,j,k]} x_{ik} x_{jl} = t^{[i,j,l]} x_{il} x_{jk}$$

where $i, j \in T$ and $k, l \in \mathbb{Z}_n$. We consider the point $p \in X^0$ where $t(p) = x_{jk}(p) = 0$ for all pairs $j \in T$, $k \in \mathbb{Z}_n$, such that $j \neq k$.

By intersecting $X^0$ with hyperplanes passing through $p$, we shall reduce to the case treated in Proposition 16.

**Definition 14.** *Let $H_{jk}$ denote the hyperplane in $(\mathbb{P}_R^{n-1})^r$ where $x_{jk} = 0$.*

**Lemma 10.** *The hyperplane $H_{jk}$, $j \simeq k$, $j \neq k$, intersects $X^0$ transversally at $p$.*

*Proof.* Let $\mathfrak{m}$ be maximal ideal of the local ring $B$ at $p$ on $X^0$. We must verify that $x_{jk}$ is not in $\mathfrak{m}^2$. The crucial equations are

$$t^{[j,i,k]} x_{jk} x_{ii} = t^{[j,i,i]} x_{ik} x_{ji}, \tag{5}$$

for $i \in T$, $i \neq j$. Since

$$[j, i, k] + [i, j, k] = [i, j, i] \geq 1$$

and

$$[i,j,k] + [i,k,j] = [j,k,j] = 0$$

we have $[j,i,k] \geq 1$ so equation (5) is $0 = 0$ for all $i \in T$. ∎

Note that if $H_{jk}$ intersects $X^0$ transversally at $p$, then the cotangent space dimension at $p$ decreases by one. Also, if $p$ is a regular point on $X^0$ and $H_{jk}$ intersects transversally at $p$, then $p$ is a regular point on $X^0 \cap H_{jk}$ and the dimension of local rings at $p$ decreases by one (see [14], Theorem 14.2).

**Lemma 11.** *If $x_{jk}(p) = 0$, $j \simeq k$, for a $k$-point $p$ in $X^0$, then $x_{ik}(p) = 0$ for all $i \in T$.*

*Proof.* We have the equation

$$t^{[i,j,k]} x_{ik} x_{jj} = t^{[i,j,j]} x_{ij} x_{jk}$$

and since $j \simeq k$ it follows that $x_{ik}(p) = 0$. ∎

Let $H$ denote the multiprojective linear subspace of $(\mathbb{P}_R^{n-1})^r$ defined by $x_{jk} = 0$ for all $j \in T$, $k \in \mathbb{Z}_n$, such that $j \simeq k$, $j \neq k$. As a consequence of Lemma 10 and Lemma 11, the scheme $X^0 \cap H$ is the result of $n - r$ consecutive intersections of $X^0$ with hyperplanes intersecting transversally at $p$. Furthermore $X^0 \cap H$ is isomorphic to the closed subscheme of $(\mathbb{P}_R^{r-1})^r$ defined by the multihomogeneous equations

$$t^{[i,j,k]} x_{ik} x_{jl} = t^{[i,j,l]} x_{il} x_{jk}$$

where $i,j,k,l \in T$. Since $[i,j,i] \geq 1$ for all $i,j \in T$ the scheme $X^0 \cap H$ is of the type we investigated in Proposition 15.

**Proposition 17.** *Let $\Lambda = \Lambda_\tau$ for a cocycle $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R \setminus \{0\}$ of the form $\tau_{ijk} = t^{[i,j,k]}$. Suppose $[i,j,i] > 0$ for some $i,j \in \mathbb{Z}_n$ and $[k,l,k] \geq 2$ for all $k,l \in T$ such that $[k,l,k]$ is nonzero. Then the the Artin scheme $X^0$ of $\Lambda$ is singular.*

*Proof.* We apply Proposition 16 to the scheme $X^0 \cap H$ constructed above. Then $X^0 \cap H$ has a singular $k$-point. As $X^0$ intersects $H$ transversally at $p$, this point must be singular also on $X^0$. ∎

## 6. Hereditary orders

In this section we let $R$ be a discrete valuation ring with quotient field $K$ and $\Lambda$ an $R$-order in a split central simple $K$-algebra $A$.

**Definition 15.** *$\Lambda$ is called a left (right) hereditary order if every left (right) $\Lambda$-lattice is projective (see [15], p.130).*

It is known (see [15] p.307) that $\Lambda$ is left hereditary if and only if $\Lambda$ is right hereditary. We shall therefore use the term hereditary order. We recall the following structure theorem.

**Proposition 18.** *Let $\Lambda$ be a hereditary $R$-order and $R$ be a discrete valuation ring. Then there exists positive integers $\{n_1, \dots, n_r\}$ with sum $n$ and an isomorphism of $K$-algebras $A \simeq M_n(K)$ such that, under this isomorphism,*

$$\Lambda \simeq \begin{bmatrix} [R]_{11} & [R]_{12} & [R]_{13} & \dots & [R]_{1(r-1)} & [R]_{1r} \\ [(t)]_{21} & [R]_{22} & [R]_{23} & \dots & [R]_{1(r-1)} & [R]_{2r} \\ [(t)]_{31} & [(t)]_{32} & [R]_{33} & \dots & [R]_{1(r-1)} & [R]_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [(t)]_{r1} & [(t)]_{r2} & [(t)]_{r3} & \dots & [(t)]_{1(r-1)} & [R]_{rr} \end{bmatrix}$$

*and*

$$\mathrm{rad}(\Lambda) \simeq \begin{bmatrix} [(t)]_{11} & [R]_{12} & [R]_{13} & \dots & [R]_{1(r-1)} & [R]_{1r} \\ [(t)]_{21} & [(t)]_{22} & [R]_{23} & \dots & [R]_{1(r-1)} & [R]_{2r} \\ [(t)]_{31} & [(t)]_{32} & [(t)]_{33} & \dots & [R]_{1(r-1)} & [R]_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [(t)]_{r1} & [(t)]_{r2} & [(t)]_{r3} & \dots & [(t)]_{1(r-1)} & [(t)]_{rr} \end{bmatrix}$$

*where $[I]_{ij}$ are $n_i \times n_j$-matrixes with elements from the ideal $I$ on each place. Conversely every such order $\Lambda$ is hereditary.*

*Proof.* This a special case of Theorem (39.14) in [15] in the case where $R$ is a complete. But the completeness is not needed for split $K$-algebras.■

As a consequence of the proposition above we note that for a hereditary $R$-order $\Lambda$ the dual lattice $\tilde{\Lambda} := \{x \in A; \mathrm{tr}(x\Lambda) \subseteq R\}$ is equal to $t^{-1}\mathrm{rad}(\Lambda)$.

The following result is essentially due to P. Lundström [13], p.72, but we give a proof of Salberger.

**Proposition 19.** *Let $\Lambda = \Lambda_\tau$ for a cocycle $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R \setminus \{0\}$ of the form $\tau_{ijk} = t^{[i,j,k]}$. Then $\Lambda$ is hereditary if and only if $[i,j,i] \le 1$ for all $i,j \in \mathbb{Z}_n$.*

*Proof.* If $[i,j,i] \le 1$ for all $i,j \in \mathbb{Z}_n$ we have seen in Remark 1 that $\Lambda$ has a representation as in Proposition 18. Hence $\Lambda$ is hereditary.

For the converse, consider the non-degenerate symmetric bilinear form $b : A \times A \to K$, $(x,y) \to \mathrm{tr}(xy)$ (see [15], section 9). Since

$$b(\epsilon_{ij}, \epsilon_{kl}) = \mathrm{tr}(\epsilon_{ij}\epsilon_{kl}) = \begin{cases} 0 & \text{if } (i,j) \neq (l,k) \\ t^{[i,j,i]} & \text{if } (i,j) = (l,k) \end{cases}$$

the dual basis $\{\widetilde{\epsilon_{ij}}\}_{1 \le i,j \le n}$ of the basis $\{\epsilon_{ij}\}_{1 \le i,j \le n}$ has the form $\widetilde{\epsilon_{ij}} = t^{-[i,j,i]}\epsilon_{ji}$. If $\Lambda$ is hereditary, then $\widetilde{\epsilon_{ij}} = t^{-[i,j,i]}\epsilon_{ji} \in t^{-1}\mathrm{rad}(\Lambda)$ so that $\epsilon_{ji} \in t^{[i,j,i]-1}\mathrm{rad}(\Lambda)$. Since the set $\{\epsilon_{ij}\}$ is an $R$-basis for $\Lambda$ this is possible only if $[i,j,i] \le 1$. ■

**Theorem 1.** *Let $R$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$. Suppose that $R$ contains an algebraically closed field $k$ such that $R = k + \mathfrak{m}$. Let $\Lambda = \Lambda_\tau$ for a cocycle $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to R \setminus \{0\}$ and let $X^0$ be the Artin scheme of $\Lambda$. Then $\Lambda$ is hereditary if and only if the dimension of the cotangent space $\Omega_{B/k} \otimes_B k(p)$ at any closed point $p$ of $X^0$ is at most $n$.*

*Proof.* By Lemma 4 we may assume that $\tau_{ijk} = t^{[i,j,k]}$ for an additive cocycle function $f : \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_{\geq 0}$, $(i,j,k) \mapsto [i,j,k]$.

If $\Lambda$ is hereditary, then $\dim_k(\Omega_{B/k} \otimes_B k(p)) \leq n$ at any closed point $p \in X^0$, by Proposition 14.

If $\Lambda$ is not hereditary there exists $i, j \in \mathbb{Z}_n$, such that $[i,j,i] \geq 2$. Let $p$ be the $k$-point where $t(p) = x_{jk}(p) = 0$ for all $j \in T$, $k \in \mathbb{Z}_n$ such that $j \neq k$ and let $B = \mathcal{O}_{X,p}$. By Proposition 15 the cotangent space at $p$ considered as a point of $X^0 \cap H$ has dimension greater or equal to $r + 1$. Since $X^0 \cap H$ is obtained by $n - r$ consecutive intersections of $X^0$ by hyperplanes $H_{jk}$ intersecting transversally at $p$, the cotangent space dimension $\dim_k(\Omega_{B/k} \otimes_B k(p))$ at $p \in X^0$ is greater or equal to $(r + 1) + (n - r) = n + 1$. ■

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