On a Class of Renewal Processes in a Random Environment

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Abstract

This paper deals with a generalization of the class of renewal processes with absolutely continuous lifelength distribution. A random environment is modelled by a positive recurrent birth and death process on a finite state space. The generalization is made by using a stochastic failure rate function, governed by the environment process and based on an underlying set of deterministic failure rate functions. Renewal processes (RPRE processes) are defined in this environment by using a certain Poisson embedding of the stochastic failure rate function. The existence of a stationary RPRE process is investigated by considering an embedded regenerative process, and asymptotics is proved by establishing an exact coupling. Particular attention is paid to the case when the underlying deterministic failure rates are increasing or decreasing. In that case couplings are established by using Poisson embedding, giving domination results, stochastical monotonicity properties, and rate results. A version of Blackwell's theorem is proved.

Keywords and phrases. Failure rates, Poisson embedding, random environment, coupling, asymptotics, rate of convergence, stochastical domination, stochastical monotonicity.

Contents

1 Introduction

Considerable attention has been paid to stochastic processes in a random environment. The processes studied include random walks, Markov processes, branching processes, diffusion processes, queues and lifelengths. Such models are used to describe systems which evolve in a fluctuating environment. The latter is usually only observable through its impact on the system which it affects. One example is when the capacity of a queueing system is affected by an underlying parallel process; such systems have been studied, in which the arrival rate or the service rates have been stochastic processes. The definition of the random environment differs from one model to another, but the prototypical assumption is that one or more of the parameters of the model constitute a stochastic process. As a concrete example from the vast literature, Nyberg [7] studied birth and death processes with intensities governed by a certain time homogeneous Markov process.

Although there is an extensive literature on random environment models, there seems to be very little work on renewal processes in a random environment. There has been much successful research on generalizations of renewal processes; however, only a few (ONE?) papers have introduced a parallel random environment process. Baxter and Li [4] have worked explicitly with a random environment modelled by a stochastic process. A martingale approach enables them to prove asymptotics (weak convergence towards stationarity), and some standard limit theorems of renewal theory such as the key renewal theorem and Blackwell's theorem. Their assumptions about the random environment and how it affects the renewal process differs significantly from the assumptions in this paper, however.

In this paper we study a generalization of a class of renewal processes to one consisting of intensity-governed point processes; they can be thought of as renewal processes in a random environment (RPRE processes). The class of renewal processes considered consists of those with absolutely continuous lifelength distribution, a property making it possible to define the lifelength failure rate function. The random environment is modelled by a birth and death process $Z = (Z_t)_0^\infty$, with a finite state space $\mathsf{S} = \{0, \dots M\}$, for some $M \in \mathbb{Z}_+$. Renewal processes are then defined in this environment by allowing Z_t to affect the intensity for a point at t. A standard renewal process has the intensity $r(A_t)$ for a point at time t, (where A_t is the time elapsed since the last renewal), meanwhile renewal processes in a random environment are defined in such way that the intensity for a point at time t is equal to $r^{(Z_t)}(A_t)$, i.e., we use a stochastic failure rate function; here $\{r^{(i)}(\cdot); i \in \mathsf{S}\}$ is a collection of failure rate functions. Rigorous definitions are made in Section 2, using a particular Poisson thinning technique (Poisson embedding of the failure rate functions.)

The aim of this paper is to use the coupling method to investigate different aspects of this class of RPRE processes, such as asymptotics, rate of convergence towards stationarity, stochastic monotonicity and stochastic domination. The study is broad rather than deep, and has more the nature of a survey over the possibilities of studying the RPRE class through the Poisson embedding technique. We are particularly interested in the IFR and DFR cases, i.e., when the failure rates $r^{(i)}(\cdot)$ are increasing and decreasing, respectively, for all $0 \leq i \leq M$. In these cases different kinds of couplings are carried out, by using Poisson embedding technique. We will be very brief in the case with general failure rates. There are reasons for this; one is that the Poisson embedding technique doesn't seem to be the right tool without some monotonicity conditions on the failure rates. Furthermore, the IFR and DFR classes are wide, and are probably natural to consider in many applications.

The rest of the paper is organized as follows. The RPRE processes are defined in Section 2 and notation is settled. In Section 3 we study the impact of good and bad environment processes on the point process, and we prove stochastical domination results. Section 4 deals with conditions for monotonicity properties; only the DFR case is studied. Section 5 deals stationarity and with certain embedded regenerative processes. They are studied in order to establish sufficient conditions for the existence of stationary RPRE processes. In Section 6 we prove asymptotical results in the total variation norm, by establishing exact couplings. In Section 7 we study moments of the coupling epoch, rendering rates results, and in Section 8 comparisons of mean measures are made, resulting in a random environment version of Blackwell's theorem. Some simulations are made throughout the work; they are described in Section 9.

2 Preliminaries

Before a rigorous definition of the class of point processes to be studied, we present some background theory and examples, and establish some notation.

2.1 Failure rates and Poisson embedding

Suppose that the non-negative random variable Y has absolutely continuous distribution F, i.e, F has a density f w.r.t. the Lebesgue measure. Then we define *the failure rate function* r by

$$r(x) = \frac{f(x)}{1 - F(x)}.$$

If Y is the lifelength of an item, for example an electric component or a light bulb, we can interpret $r(t)$ as being the failure intensity for a component with age t, since

$$\mathbb{P}(Y_i \in (t, t + dt) \,\|\, Y_i > t) = r(t)\,dt.$$

One way of generating such a random variable given the failure rate function is with use of *Poisson embedding.* Recall the well-known relation

$$1 - F(t) = e^{-R(t)}$$

where

$$R(t) = \int_0^t r(u)du.$$

Now let ξ be a two-dimensional Poisson process in $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$ with expectation measure ℓ_+, the Lebesgue measure restricted to $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$. We define for any set $B \in \mathcal{B}(\mathbb{R}_+^2)$

$$B_t = \{(x, y) \in B : x \leq t\}$$

and

$$\tau_B(\xi) = \inf\{t \geq 0;\ \xi(B_t) > 0\},$$

abbreviated by τ_B when it is understood which Poisson process we use. Then with

$$B = \{(x, y) \in \mathbb{R}_+^2 :\ y \leq r(x)\}$$

and

$$Y = \tau_B$$

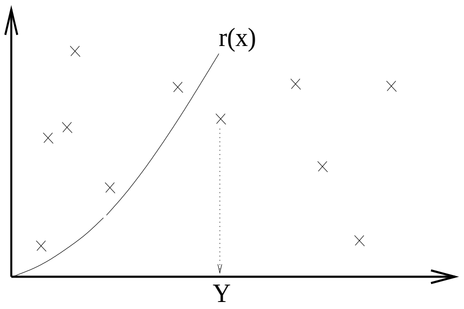

Figure 1: *The first point under the failure rate function r has the x-coordinate distributed according to F.*

we get $Y \stackrel{\mathcal{D}}{=} F$ because

$$\mathbb{P}(Y > t) = \mathbb{P}(\xi(B_t) = 0) = \exp(-\ell_+(B_t)) = \exp(-R(t)) =$$

$$1 - F(t),$$

where we used that the Lebesgue measure of B_t equals $\int_0^t r(u)du = R(t)$.

By using the technique described above, we can construct a renewal process with initial age $a \geq 0$, and with lifelength distribution F. We need the residual lifelength distribution F_a, which is defined through

$$F_a(t) = \begin{cases} 1 & \text{if } F(a) = 1 \\ (F(a+t) - F(a))/(1 - F(a)) & \text{if } F(a) < 1. \end{cases}$$

Notice that F_a has failure rate function $r_a = r(a + \cdot)$. Let

$$B_0 = \{(x, y) \in \mathbb{R}_+^2 \, ; \, y \leq r_a(x)\},$$

and let

$$S_0 = \begin{cases} 0 & \text{if } a = 0 \\ \tau_{B_0} & \text{if } a > 0. \end{cases}$$

Define recursively for $n \geq 1$

$$B_n = \{(x, y) \in \mathbb{R}_+^2 \, ; \, x > S_{n-1}, \, y \leq r(x - S_{n-1})\}$$

and

$$S_n = \tau_{B_n}.$$

For clarity, we should perhaps point out that $r(x) = 0$ for all $x < 0$. Let

$$Y_0 = S_0,$$

and for $n \geq 1$

$$Y_n = S_n - S_{n-1}.$$

This gives us a renewal process $S = (S_n)_{n=0}^{\infty}$, with lifelengths $Y_0, Y_1 \ldots$.

It is obvious from the construction that $Y_0 \stackrel{\mathcal{D}}{=} F_a$ and $Y_n \stackrel{\mathcal{D}}{=} F$, for $n \geq 1$, and since they are generated by disjoint parts of the Poisson process ξ, they are independent. It should be noted that the initial age a can be randomized.

2.2 Lifelengths in a random environment

We will now describe how the embedding technique can be used to generate a random lifelength Y in a random environment $Z = (Z_t)_0^{\infty}$. That will, by extension, give us a way of defining more general renewal processes.

Let Z be a birth and death process on $\mathsf{S} = \{0, \ldots, M\}$, for some $M \in \mathbb{Z}_+$. Suppose that $r^{(0)}, \ldots, r^{(M)}$ are failure rate functions satisfying

$$r^{(i)}(x) \leq r^{(j)}(x)$$

for all $0 \leq i \leq j \leq M$ and all $x \geq 0$. Define the stochastic failure rate r_Y by

$$r_Y(t) = r^{(Z_t)}(t) = \begin{cases} r^{(0)}(t) & \text{if } Z_t = 0 \\ \vdots & \vdots \\ r^{(M)}(t) & \text{if } Z_t = M \end{cases}$$

Then, Y is defined through Poisson embedding of r_Y:

$$B = \{(x, y) \in \mathbb{R}_+^2 \,;\, y \leq r_Y(x)\}$$

and

$$Y = \tau_B.$$

Although we will not try to model any real phenomena in this paper, we may think of lifelengths of some kind of components. Suppose that these imaginary items has a lifelength distribution given by

$$F^{(i)}(x) = 1 - e^{-\int_0^t r^{(i)}(x)dx}$$

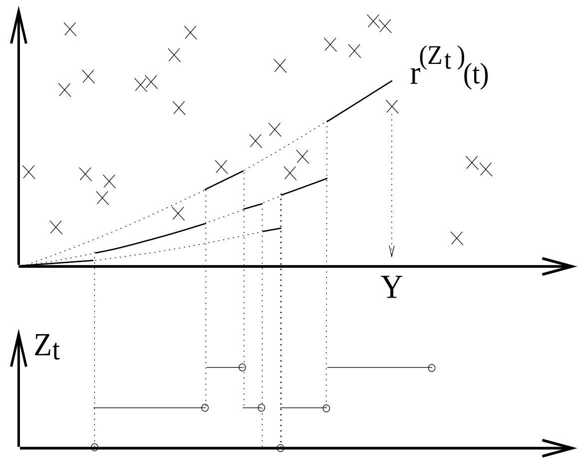

Figure 2: *Poisson embedding of the stochastic failure rate function* $r^{(Z_t)}(t)$.

if we use them in an environment which is constantly equal to i. ($i \in \mathsf{S}$.) Since we assumed that $r^{(i)}(x) \leq r^{(j)}(x)$ for $0 \leq i \leq j \leq M$ and for all x, it follows that

$$F^{(i)}(x) \leq F^{(j)}(x)$$

for all x. That is, long time survival is more likely in an environment that constantly equals i, than in one that equals j. Therefore we regard i as *better* than j, and j as *worse* than i. One way of modelling the lifelength of such items used in an environment which fluctuates randomly between states $0, \ldots, M$, is by using the Poisson embedding described above; Y is then the lifelength.

2.3 Renewal processes in a random environment

Renewal processes in the random environment Z will now be defined, in a way analogous to how we generated the standard renewal processes. However, we have to define stochastic failure rate functions r_{Y_i} for each of the lifelengths Y_i, $i \geq 0$. The lifelength sequence $(Y_n)_{n=0}^{\infty}$ is then defined through Poisson embedding. Let $a \geq 0$ be the initial age, and define

$$r_{Y_0}(t) = r_a^{(Z_t)}(t) = \begin{cases} r_a^{(0)}(t) & \text{if } Z_t = 0 \\ \vdots & \vdots \\ r_a^{(M)}(t) & \text{if } Z_t = M, \end{cases}$$

$$B_0 = \{(x,y) \in \mathbb{R}_+^2\, ;\, y \le r_{Y_0}(x)\}$$

and

$$S_0 = \begin{cases} 0 & \text{if } a = 0 \\ \tau_{B_0} & \text{if } a > 0. \end{cases}$$

Let $Y_0 = S_0$. Define recursively, for $n \ge 1$,

$$r_{Y_n}(t) = r^{(Z_t)}(t - S_{n-1}) = \begin{cases} r^{(0)}(t - S_{n-1}) & \text{if } Z_t = 0 \\ \vdots & \vdots \\ r^{(M)}(t - S_{n-1}) & \text{if } Z_t = M, \end{cases}$$

$$B_n = \{(x,y) \in \mathbb{R}_+^2\, ;\, y \le r_{Y_n}(x - S_{(n-1)})\},$$

and

$$S_n = \tau_{B_n}.$$

The lifelengths $Y_n, n \ge 1$, are defined through

$$Y_n = S_n - S_{n-1}.$$

We obtain a sequence $S = (S_n)_{n=0}^{\infty}$ which gives rise to a point process N giving mass 1 to each of the S_n-points:

$$N(B) = \sharp\{n; S_n \in B\}$$

for all $B \in \mathcal{B}(\mathbb{R}_+)$. In more compact form,

$$N = \sum_{i=0}^{\infty} \delta_{S_i}$$

We call such a point process N (and the sequence S) a *Renewal Process in a Random Environment*, abbreviated to *RPRE*. We use the term *renewals* for the S_n-points, though it is a slight abuse of terminology.

We will refer to the construction described above, by saying that N is *generated* by ξ, Z and a, or simply by ξ, when it is understood which environment and initial age we use. If $a = 0$ we say that the RPRE process is *zero-delayed.*

2.4 Further definitions and notation

We define V to be the pair

$$V = (N, Z)$$

and when there is no risk for confusion, we also term V an RPRE process and that V is generated by ξ. Furthermore, let

$$V_t = (N_t, Z_t)$$

where $N_t = N([0, t])$.

We assume throughout the paper that the random environment Z is an irreducible and positive recurrent birth and death process on a finite state space $\mathsf{S} = \{0, \dots, M\}$. (And hence that an unique stationary distribution $\pi = (\pi_i)_0^M$ exists). We often use $\lambda = (\lambda_i)_0^M$ and $\lambda' = (\lambda_i')_0^M$ to denote start distributions on S.

We say that $r^{(i)}$ and the distribution $F^{(i)}$ are *associated to the state* i, and that $r^{(0)}, \dots, r^{(M)}$ and $F^{(0)}, \dots, F^{(M)}$ *are associated with the environment* Z.

The term *increasing* (*decreasing*) is used for monotone nondecreasing (nonincreasing). In order to avoid some trite technicalities, we assume throughout the paper that whenever monotone failure rates are used, they are defined for all arguments, i.e., if $r(x)$ is increasing (decreasing) then

$$r(y) \leq (\geq)\ r(x)$$

for all $x \geq 0$ and $y \geq 0$ satisfying $x \geq y$. A distribution F is said to be of IFR (DFR) type if it has an increasing (decreasing) failure rate.

We say that the RPRE process is *within the IFR (DFR) class* if the failure rate functions $r^{(i)}$ are increasing (decreasing) for all $i \in \mathsf{S}$.

Given two RPRE processes $V = (N, Z)$ and $V' = (N', Z')$, we say that they are of *the same type* if

(i) the environment processes Z and Z' are governed by the same birth and death intensities, on the same state space S, and if

(ii) $r^{(i)}(x) = r'^{(i)}(x)$ for all $i \in \mathsf{S}$, for all $x \geq 0$, where $\{r^{(0)}, \dots, r^{(M)}\}$ and $\{r'^{(0)}, \dots, r'^{(M)}\}$ are the sets of failure rates associated to Z and Z', respectively.

Processes of the same type may be distributionally different since they can have different initial distributions.

Let $\mathcal{N}_+$ denote the class of integer valued measures on $\mathbb{R}_+$ giving finite mass to bounded sets, and $\mathcal{B}_+$ the standard σ-field on $\mathcal{N}_+$, generated by the vague topology. The point process $N = \sum_0^\infty \delta_{S_n}$ associated with a RPRE process $S = (S_n)_0^\infty$ is a random element in $(\mathcal{N}_+, \mathcal{B}_+)$.

The space $\mathbb{D}_E[0,\infty)$ (abbreviated with $\mathbb{D}_E$) is the space of functions with values in the space E and defined on $[0,\infty)$, which are right-continuous and have left-hand limits at all arguments t. (E is assumed to be Polish, if not explicitly defined.) The $\sigma-$field on $\mathbb{D}_E$ (generated by the Skorohod topology) will be denoted by $\mathcal{D}_E$. A birth and death process $Z = (Z_t)_0^\infty$ is a random element in $(\mathbb{D}_{\mathbb{Z}_+}, \mathcal{D}_{\mathbb{Z}_+})$.

For measures $\nu \in \mathcal{N}_+$, define the *shift operator*

$$\theta_t \nu(\,\cdot\,) = \nu(\,t + \,\cdot\,),$$

and for $z \in \mathbb{D}_E$

$$\theta_t z = (z_{t+s})_{s=0}^\infty.$$

Then we define the shifted RPRE process

$$\theta_t V = (\,\theta_t N, \theta_t Z\,)\,.$$

An RPRE process V' is *stationary* if

$$\theta_t V' \stackrel{\mathcal{D}}{=} V'$$

for all $t \geq 0$. If V and V' of the same type and V' is stationary, then we say that V' is a *stationary version* of V. We observe that a stationary RPRE process has initial environment distribution equal to π, the stationary distribution for the environment process.

If an RPRE process V has a random initial age and a random initial environment state, with probability distributions H and λ, respectively, then we abbreviate that by simply saying that V *has initial distributions* (H, λ).

We use the notation $\mathbb{P}_{(a,i)}$ or $\mathbb{P}_{(H,\lambda)}$ when there is reason to emphasize the initial conditions; the former notation is used when the initial age is a and the initial environment state is i, and the latter one is used when the start conditions are randomized. Also, $\mathbb{E}_{(a,i)}$ and $\mathbb{E}_{(H,\lambda)}$ are used for expectations w.r.t $\mathbb{P}_{(a,i)}$, $\mathbb{P}_{(H,\lambda)}$.

For distribution functions F we denote the expectation $\int xF(dx)$ by $m(F)$, and we use the notation

$$\mu_\alpha = m_\alpha(F) = \int x^\alpha F(dx)$$

for the α moments of F. Furthermore, we denote the support of F with $s(F)$.

2.5 The coupling method

The coupling method will be an indispensable tool for us, and is defined and briefly summarized in the following: we follow Lindvall [6], Chapter I. Let X and X' be two random elements defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with state space $(E, \mathcal{E})$. Let $P = \mathbb{P}X^{-1}$ and $P' = \mathbb{P}X'^{-1}$ be the distributions of X and X' respectively. Furthermore, assume that E is Polish (metric, separable and complete).

Definition 1 *A coupling of P and P' is a probability measure $\tilde{P}$ on $(E^2, \mathcal{E}^2)$ such that $P(A) = \tilde{P}(A \times E)$ and $P'(A) = \tilde{P}(E \times A)$ for all $A \in \mathcal{E}$.*

One equivalent way of defining coupling is in terms of random elements: a random element $(\tilde{X}, \tilde{X}') \in (E^2, \mathcal{E}^2)$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is said to be a coupling of X and X' if $X \stackrel{\mathcal{D}}{=} \tilde{X}$ and $X' \stackrel{\mathcal{D}}{=} \tilde{X}'$.

In Section 6 we will study asymptotic properties of RPRE processes, and among other things prove convergence towards stationarity. In doing so, we will compare the distribution of a shifted non-stationary process with the distribution of a stationary one, and therefore we need a distance between probability measures. A common way of measuring such distances is in terms of *the total variation norm.*

Definition 2 *The total variation norm of $P - P'$ is given by*

$$\| P - P' \| = 2 \sup_{A \in \mathcal{E}} \left(\mathbb{P}(X \in A) - \mathbb{P}(X' \in A) \right).$$

By choosing an appropriate coupling $(\tilde{X}, \tilde{X}')$ of X and X', it is often possible to estimate $\mathbb{P}(\tilde{X} \neq \tilde{X}')$. The total variation norm of $P - P'$ can then be bounded using *the coupling inequality:*

$$\|\mathbb{P}(X \in \cdot) - \mathbb{P}(X' \in \cdot)\| \leq 2\mathbb{P}(\tilde{X} \neq \tilde{X}'). \tag{1}$$

Consider now the case when the stochastic elements under study are stochastic processes: let $Z = (Z_t)_0^\infty$ and $Z' = (Z'_t)_0^\infty$ be the processes, and let $(\tilde{Z}, \tilde{Z}')$ be a coupling of Z and Z'. Suppose that there is a random time $T \in \mathbb{R}_+$ such that

$$\tilde{Z}_t = \tilde{Z}'_t \quad \text{for all } t \geq T. \tag{2}$$

We call T a *coupling time* for $\tilde{Z}$ and $\tilde{Z}'$ (and when there is no risk for confusion, for Z and Z') and obtain from the coupling inequality that

$$\|\mathbb{P}(Z_t \in \cdot) - \mathbb{P}(Z'_t \in \cdot)\| \leq 2\,\mathbb{P}(\tilde{Z}_t \neq \tilde{Z}_t{}') \leq 2\,\mathbb{P}(T > t). \tag{3}$$

If T is finite the coupling is said to be *successful*, and then (3) gives

$$\lim_{t\to\infty} \|\mathbb{P}(Z_t \in \ \cdot\ ) - \mathbb{P}(Z'_t \in \ \cdot\ )\| = 0.$$

Since T is also a coupling time for $(\theta_t \tilde{Z})_0^\infty$ and $(\theta_t \tilde{Z}')_0^\infty$, the coupling inequality yields that

$$\|\mathbb{P}(\theta_t Z \in \cdot) - \mathbb{P}(\theta_t Z' \in \cdot)\| \leq 2\,\mathbb{P}(T > t) \tag{4}$$

which implies

$$\lim_{t\to\infty} \|\mathbb{P}(\theta_t Z \in \ \cdot\ ) - \mathbb{P}(\theta_t Z' \in \ \cdot\ )\| = 0$$

if T is finite. In Section 7 we investigate the speed of convergence towards stationary, by means of establishing finite moments of the coupling time, among other things. If $\mathbb{E}[T^\alpha] < \infty$ for some $\alpha > 0$ then

$$t^\alpha\,\mathbb{P}(T > t) \leq \mathbb{E}[T^\alpha \cdot I(T > t)] \leq \mathbb{E}[T^\alpha] < \infty.$$

By dominated convergence, we obtain $\mathbb{P}(T > t) = o(t^{-\alpha})$ and (4) yields

$$\|\mathbb{P}(\theta_t Z \in \cdot) - \mathbb{P}(\theta_t Z' \in \cdot)\| = o(t^{-\alpha}). \tag{5}$$

Moreover, if $\mathbb{E}[e^{\rho T}] < \infty$ for some $\rho > 0$ we obtain exponential convergence, i.e.,

$$\|\mathbb{P}(\theta_t Z \in \cdot) - \mathbb{P}(\theta_t Z' \in \cdot)\| = o(e^{-\rho t}). \tag{6}$$

2.6 Three domination lemmas

In this section we gather three observations which will be useful later on. Although they are quite obvious (and at least two of them are probably well known), we present proofs.

The first result concerns stochastic domination of lifelengths in a random environment (defined as before).

Lemma 1 *Suppose that Y is a lifelength in the random environment Z, possessing a stochastic failure rate $r_Y(\cdot)$, and let W be a random variable with a failure rate $\rho(\cdot)$. If $\rho(x) \le r_Y(x)$ a.s. for all $x \ge 0$ then*

$$W \overset{\mathcal{D}}{\geq} Y$$

Proof. Let ξ be a bivariate Poisson process in $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$ with expectation measure ℓ_+ and define

$$B = \{(x,y) \in \mathbb{R}_+^2 : y \le r_Y(x)\},$$
$$B^* = \{(x,y) \in \mathbb{R}_+^2 : y \le \rho(x)\},$$
$$Y = \tau_B(\xi)$$

and

$$W' = \tau_{B^*}(\xi).$$

We have achieved a coupling (W', Y) of (W, Y) such that

$$W' \ge Y \text{ a.s.},$$

and the result follows. □.

Lemma 2 *Suppose that F is an absolutely continuous DFR-distribution on $[0,\infty)$, with failure rate function $r(\cdot)$ satisfying $r(x) \ge \lambda$ for some $\lambda > 0$ and for all $x \ge 0$. Then, there exists random variables X and Y such that*

(i) $X \overset{\mathcal{D}}{=} Exp(\lambda)$,

(ii) $Y \overset{\mathcal{D}}{=} F$, *and*

(iii) $Y \le X$ *a.s.*

Proof. Let $r^*(x) = \lambda$ for all $x \ge 0$. A Poisson embedding of $r(\cdot)$ and $r^*(\cdot)$ results in two random variables Y and X such that (i) and (ii) holds, and since $r(x) \ge r^*(x)$ for all $x \ge 0$, (iii) holds. □

So if F is a distribution satisfying the conditions in Lemma 2 then F is stochastically dominated by some exponential distribution. An analogous result holds in the IFR case:

Lemma 3 *Suppose that F is an absolutely continuous IFR-distribution on $[0, \infty)$, with failure rate function $r(\cdot)$. Then, there exist finite constants $\lambda > 0$ and $x_\lambda \geq 0$ and random variables X and Y such that*

(i) $X \stackrel{\mathcal{D}}{=} Exp(\lambda)$,

(ii) $Y \stackrel{\mathcal{D}}{=} F$, *and*

(iii) $Y \leq x_\lambda + X$ *a.s.*

Proof. Since F is not degenerate there exists $x_\lambda \geq 0$ such that $0 < r(x_\lambda) < \infty$. Let $\lambda = r(x_\lambda)$ and define $r^*(x) = \mathbb{1}_{\{x \geq x_\lambda\}}(x)$ for all $x \geq 0$. Define

$$B = \{(x, y) \in \mathbb{R}_+^2 : y \leq r(x)\}$$

and

$$B^* = \{(x, y) \in \mathbb{R}_+^2 : y \leq r^*(x)\}.$$

Let ξ be a bivariate Poisson process as before and define $Y = \tau_B$, giving (ii), and define

$$X = \tau_{B^*} - x_\lambda.$$

Obviously, X is exponentially distributed with parameter λ, and (iii) holds. □

3 Domination

Consider two RPRE process (N, Z) and (N', Z') with the same state space S for the environment processes, with the same set of failure rate functions $\{r^{(0)}, \dots, r^{(M)}\}$, but with Z' is stochastically larger than Z. Then it is quite natural to expect some kind of domination results between the point processes N and N'. It turns out to be rather easy to establish such results under monotonicity conditions on the failure rates; then couplings can be carried out by using the Poisson embedding technique. We can not hope for similar results with general failure rates.

3.1 A partial ordering on $\mathcal{N}_+$

To establish inequalities we need a partial ordering on $\mathcal{N}_+$. For measures $\nu, \nu' \in \mathcal{N}_+$ we define $\nu \preceq \nu'$ if

$$\nu(A) \leq \nu'(A) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}_+)$$

We will also consider mappings Ψ from $(\mathcal{N}_+, \mathcal{B}(\mathcal{N}_+))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that

$$\Psi(\nu) \leq \Psi(\nu') \text{ for all } \nu, \nu' \in \mathcal{N}_+ \text{ satisfying } \nu \preceq \nu'$$

or

$$\Psi(\nu) \geq \Psi(\nu') \text{ for all } \nu, \nu' \in \mathcal{N}_+ \text{ satisfying } \nu \preceq \nu'$$

In the first case we say that Ψ is *increasing* and in the second case that Ψ is *decreasing.*

3.2 Good and bad random environment

We saw in Section 2 that it seemed reasonable to regard i as a better environment state than j if $i \leq j$, with the interpretation of the renewals as failures of certain components. In the light of that, the following definition should not come as a surprise. Suppose that Z and Z' are random environments on the same state space S , and that

$$Z \overset{\mathcal{D}}{\leq} Z'$$

holds. Then Z is said to be *better* than Z'. The following well-known result will be useful for us: $Z \overset{\mathcal{D}}{\leq} Z'$ if and only if there exists a coupling $(\tilde{Z}, \tilde{Z}')$ of (Z, Z') such that

$$\tilde{Z}_t \leq \tilde{Z}_t' \quad \text{for all } t \geq 0 \text{ a.s.}$$

3.3 The DFR case

Theorem 1 *Suppose that Z is a better environment than Z'. Then we may construct RPRE processes $\tilde{N}$ and $\tilde{N}'$ within the DFR class, with the same initial age $a \geq 0$, and with random environments distributed as Z and Z', respectively, such that*

$$\tilde{N} \preceq \tilde{N}' \quad a.s. \tag{7}$$

Proof. It is well known how to establish a coupling $(\tilde{Z}, \tilde{Z}')$ such that $\tilde{Z}_t \leq \tilde{Z}'_t$ for all $t \geq 0$ a.s. (see [6], Chapter 5), under the condition that Z is better than Z'. Use now the two-dimensional Poisson process ξ in $(\mathbb{R}^2_+, \mathcal{R}^2_+)$ with expectation measure = the Lebesgue measure ℓ_+, together with the environments $\tilde{Z}$ and $\tilde{Z}'$ to obtain RPRE-processes $\tilde{N}$ and $\tilde{N}'$. We get $Y_0 \geq Y'_0$ a.s., and proceed recursively (see Figure ?) to find that we have established (7). □

The rather intuitive result that if $V = (N, Z)$ and $V' = (N', Z')$ are RPRE processes with common initial age a, with Z better than Z', then

$$N \overset{\mathcal{D}}{\leq} N', \tag{8}$$

follows of course from Theorem 1, and gives even more sense to the definition of *better* (with the interpretation of long lifelengths as something good, e.g., periods between failures of expensive components or between earthquakes, etc.) The result (8) is a special case of the following stronger result:

Proposition 1 *Let N and N' be RPRE processes within the DFR class, with random environments Z and Z' and with initial age distributions H, H'. Suppose that Z is a better environment than Z', and that $H' \overset{\mathcal{D}}{\leq} H$. Then*

$$\Psi(\theta_t N) \overset{\mathcal{D}}{\leq} (\overset{\mathcal{D}}{\geq}) \Psi(\theta_t N')$$

for all increasing (decreasing) mappings $\Psi : \mathcal{N}_+ \longrightarrow \mathbb{R}_+$.

Proof. Strassen's Theorem (see [6], Chapter 4) implies that there exists a probability measure $\tilde{H}$ on $(\mathbb{R}^2_+, \mathcal{R}^2_+)$ with marginals H resp H' such that $\tilde{H}\left(\{(x, y) \in \mathbb{R}^2_+ : x \leq y\}\right) = 1$, hence also random initial ages a, a' with distributions H, H' respectively, satisfying $a' \leq a$ a.s. Repeating the proof of Theorem 1 with these randomized ages renders a coupling $(\tilde{N}, \tilde{N}')$ of N

and N' satisfying (7). Using this coupling together with the fact that $\Psi \circ \theta_t$ is increasing if Ψ is, we conclude that

$$\Psi(\theta_t N') \stackrel{\mathcal{D}}{=} \Psi(\theta_t \tilde{N}') \geq \Psi(\theta_t \tilde{N}) \stackrel{\mathcal{D}}{=} \Psi(\theta_t N).$$

Decreasing mappings are treated analogously. □

3.4 The IFR case

The case with increasing failure rates is not so rewarding as the DFR case. The IFR property makes it impossible to achieve domination in the sense of (7) in Theorem 1. We have to content ourselves with a weaker form of domination.

Theorem 2 *Suppose that the random environment Z is better than Z'. Then we may construct RPRE processes $\tilde{N}$ and $\tilde{N}'$ within the IFR class, with any initial age a and with environments distributed as Z and Z', respectively, and such that*

$$\tilde{N}_t \leq \tilde{N}'_t \quad \textit{for all } t \geq 0 \quad \textit{a.s.} \tag{9}$$

Proof. The proof is basically the same as Theorem 2: establish the coupling $(\tilde{Z}, \tilde{Z}')$ of Z and Z', and use the Poisson process ξ to generate two point processes $\tilde{N}$ and $\tilde{N}'$. The stochastic failure rate functions belonging to the two point processes shows an alternating structure, which renders

$$0 \leq S'_0 \leq S_0 \leq S'_1 \leq S_1 \leq S'_2 \leq S_2 \ldots \quad \text{a.s.} \quad \square$$

4 Monotonicity in the DFR case

In this section we study some stochastical monotonicity properties. Our main concern will be to generalize a monotonicity result from the classical DFR theory which states that if N is a zero-delayed DFR renewal process and if $\Psi : \mathcal{N}_+ \longrightarrow \mathbb{R}_+$ is increasing (decreasing), then $\Psi(\theta_t N)$ is stochastically decreasing (increasing) in t. The result is proved in [6], Section V.22, p 196 and has many consequences, such as

(i) $N(t+B)$ is stochastically decreasing in t for all $B \in \mathcal{B}(\mathbb{R}_+)$,
(ii) the renewal function $\mathbb{E}[N_t]$ is concave,
(iii) the delay D_t is stochastically increasing in t, and
(iv) the age A_t is stochastically increasing in t.

It will be seen that these results also hold in the random environment case, if the start is zero-delayed, in the worst possible environment state. We use proof methods similar to those found in [6]. To illuminate differences between RPRE processes and standard renewal processes, we show that concavity does not hold for all zero-delayed RPRE processes within the DFR class. In addition, it is proved that concavity does not always hold when $Z_0 = M$.

4.1 Monotonicity properties of functionals of the process

Once again we consider monotone mappings $\Psi : \mathcal{N}_+ \to \mathbb{R}_+$ (monotone with respect to the partial ordering on $\mathcal{N}_+$ defined in the last section).

Theorem 3 *Let N be a zero-delayed RPRE process within the DFR class, in the environment Z with $Z_0 = M$, and suppose Ψ is increasing (decreasing). Then $\Psi(\theta_t N)$ is stochastically decreasing (stochastically increasing) in t.*

Proof. We must show that $\Psi(\theta_s N) \overset{\mathcal{D}}{\geq} \Psi(\theta_t N)$ for $s \leq t$. Let $\Psi^0 = \Psi \circ \theta_s$, which is increasing if Ψ is, and let $V' = (N', Z') := \theta_{t-s} V = (\theta_{t-s} N, \theta_{t-s} Z)$. Then N' has some initial age distribution $H' \overset{\mathcal{D}}{\geq} \delta_0$, the initial age distribution of N. Due to the fact that $Z_0 = M$ we have $Z_0 \geq Z_0'$ and therefore we can apply Proposition 1 which gives:

$$\begin{aligned}\Psi(\theta_s N) &= \Psi^0(N) \overset{\mathcal{D}}{\geq} \Psi^0(N') \\ &= \Psi(\theta_s N') = \Psi(\theta_s \theta_{t-s} N) \\ &= \Psi(\theta_t N) \qquad \square\end{aligned}$$

Corollary 1 *Suppose that $Z_0 = M$ and that N zero-delayed. Then $N(\,t+B\,)$ is stochastically decreasing in t, for all $B \in \mathcal{B}(\mathbb{R}_+)$*

Proof. Use Theorem 3 together with the fact that $\Psi(\nu) := \nu(B)$ is increasing. $\square$.

Corollary 2 *If $Z_0 = M$ and N is zero-delayed, then $m(t) = \mathbb{E}_{(0,M)}[N_t]$ is concave in t.*

Proof. From Corollary 1 we have that $N(t, t+h] \overset{\mathcal{D}}{\leq} N(t-h, t]$. Therefore

$$\begin{aligned} m(t+h) - m(t) &= \mathbb{E}_{(0,M)}[N(t,t+h]] \\ &\leq \mathbb{E}_{(0,M)}[N(t-h,t]] \\ &= m(t) - m(t-h) \end{aligned}$$

holds. Hence $m(t) \geq (m(t-h) + m(t+h))/2$ for all $t \geq 0$ and $0 \leq h \leq t$, which is sufficient for concavity. $\square$

Corollary 3 *If $Z_0 = M$ and N is zero-delayed then*

(i) *the delay D_t is stochastically increasing in t, and*
(ii) *the age A_t is stochastically increasing in t.*

Proof. The mapping $\Psi(\nu) = \inf\{s > 0; \nu(0,s] > 0\}$ is decreasing, so (i) follows from Theorem 3. For (ii) KOLLA!!! $\square$.

Proposition 2 *The condition that N is zero-delayed is not sufficient for concavity; there exist zero-delayed RPRE processes (N, Z) within the DFR class, with $Z_0 = i < M$ such that $m(t) = \mathbb{E}_{(0,i)}[N_t]$ is not concave in t.*

Proof. We consider a certain RPRE process N with constant failure rates, i.e., we have a Cox process. Let $\mathsf{S} = \{0, 1\}$, and let the failure rates be $r^{(0)}(t) = \lambda_0$ and $r^{(1)}(t) = \lambda_1$ for all $t \geq 0$, where $0 < \lambda_0 < \lambda_1$. Let $Z_0 = 0$. If we can prove that $\mathbb{E}_{(0,0)}[N(0,1]] < \mathbb{E}_{(0,0)}[N(1,2]]$ holds, then

$$\begin{aligned} m(1) - m(0) &= \mathbb{E}_{(0,0)}[N(0,1]] \\ &< \mathbb{E}_{(0,0)}[N(1,2]] \\ &= m(2) - m(1) \end{aligned}$$

follows. That implies $m(1) < (m(2) + m(0))/2$, which is sufficient for non-concavity of $m(\cdot)$.

With constant failure rates, we have

$$\mathbb{E}_{(0,0)}[N(1,2] \,\|\, Z_1 = 0] \;=\; \mathbb{E}_{(0,0)}[N(0,1]]$$

and

$$\mathbb{E}_{(0,0)}[N(1,2] \,\|\, Z_1 = 1] \;=\; \mathbb{E}_{(0,1)}[N(0,1]].$$

Therefore we have

$$\begin{aligned} \mathbb{E}_{(0,0)}[N(1,2]] &= \mathbb{E}_{(0,0)}[N(1,2] \,\|\, Z_1 = 0] \cdot \mathbb{P}_{(0,0)}(Z_1 = 0) \\ &+ \;\mathbb{E}_{(0,0)}[N(1,2] \,\|\, Z_1 = 1] \cdot \mathbb{P}_{(0,0)}(Z_1 = 0) \\ &= \mathbb{E}_{(0,0)}[N(0,1]] \cdot \mathbb{P}_{(0,0)}(Z_1 = 0)] \\ &+ \;\mathbb{E}_{(0,1)}[N(0,1]] \cdot \mathbb{P}_{(0,0)}(Z_1 = 1). \end{aligned}$$

So it suffices to prove that $\mathbb{E}_{(0,1)}[N(0,1]] > \mathbb{E}_{(0,0)}[N(0,1]]$. A coupling argument will do it: let (N', Z') and (N'', Z'') be RPRE processes of the same type as (N, Z), with $Z_0^{'} = 0$ and $Z_0^{''} = 1$. Suppose that N' and N'' are generated by the same Poisson process ξ. Define

$$T_{env} = \inf\{t \geq 0; Z_t^{'} = Z_t^{''}\}$$

and let

$$\tilde{Z}_t = \begin{cases} Z_t^{'} & \text{if } t \leq T_{env} \\ Z_t^{''} & \text{if } t \geq T_{env}. \end{cases}$$

Then $(\tilde{Z}, Z'')$ is a coupling of (Z', Z'') satisfying

$$\tilde{Z}_t \leq Z_t^{''} \text{ for all } t \geq 0 \text{ a.s.}$$

Now, use ξ to generate $\tilde{N}$ in the environment $\tilde{Z}$. Since $\tilde{Z} \stackrel{\mathcal{D}}{=} Z'$, we have a coupling $(\tilde{N}, N'')$ of (N', N''). We have

$$\begin{aligned}
\mathbb{E}_{(0,0)}[N(0,1]] &= \mathbb{E}_{(0,0)}[N'(0,1]] \\
&= \mathbb{E}_{(0,0)}[\tilde{N}(0,1]] \\
&= \int_0^1 \mathbb{E}_{(0,0)}[\tilde{N}(0,1] \| T_{env} = u](\beta_0 + \delta_1)e^{-(\beta_0+\delta_1)u} du \\
&+ \int_1^\infty \mathbb{E}_{(0,0)}[\tilde{N}(0,1] \| T_{env} = u](\beta_0 + \delta_1)e^{-(\beta_0+\delta_1)u} du \\
&= \int_0^1 (\lambda_0 \cdot u + \mathbb{E}_{(0,0)}[N''(u,1] \| T_{env} = u])(\beta_0 + \delta_1)e^{-(\beta_0+\delta_1)u} du \\
&+ \int_1^\infty \lambda_0(\beta_0 + \delta_1)e^{-(\beta_0+\delta_1)u} du \\
&< \int_0^1 (\lambda_1 \cdot u + \mathbb{E}_{(0,0)}[N''(u,1] \| T_{env} = u])(\beta_0 + \delta_1)e^{-(\beta_0+\delta_1)u} du \\
&+ \int_1^\infty \lambda_1(\beta_0 + \delta_1)e^{-(\beta_0+\delta_1)u} du \\
&= \int_0^1 \mathbb{E}_{(0,1)}[N''(0,1] \| T_{env} = u](\beta_0 + \delta_1)e^{-(\beta_0+\delta_1)u} du \\
&+ \int_1^\infty \mathbb{E}_{(0,0)}[N''(0,1] \| T_{env} = u](\beta_0 + \delta_1)e^{-(\beta_0+\delta_1)u} du \\
&= \mathbb{E}_{(0,1)}[N''(0,1]] \\
&= \mathbb{E}_{(0,1)}[N(0,1]] \qquad \square
\end{aligned}$$

Proposition 3 *The condition that $Z_0 = M$ is not sufficient for concavity; there exist delayed RPRE processes (N, Z) within the DFR class with $Z_0 = M$, such that $m(t) = \mathbb{E}_{(a,M)}[N_t]$ is not concave in t.*

Proof: We consider a certain delayed renewal process, i.e., an RPRE process with $Z_t \equiv M = 0$ for all $t \geq 0$. Let $0 < \lambda_0 < \lambda_1$ and let the lifelength failure rate function be

$$r(t) = \begin{cases} \lambda_1 & \text{if } 0 \leq t \leq 2 \\ \lambda_0 & \text{if } t > 2. \end{cases}$$

Suppose that N has initial age $a > 2$. The failure rate function is choosed so that

$$\begin{aligned}
\mathbb{E}_a[N(1,2] \, \| N(0,1] = 0] &= \mathbb{E}_{a+1}[N(0,1]] \\
&= \mathbb{E}_a[N(0,1]
\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_a[N(1,2] \,\|(A_1\|N(0,1] > 0) = s] &= \mathbb{E}_s[N(0,1]] \\ &> \mathbb{E}_a[N(0,1]]\end{aligned}$$

holds, for $0 \le s \le 1$. Now, let G be the distribution of $(A_1\|N(0,1] > 0)$. Therefore

$$\begin{aligned}\mathbb{E}_a[N(1,2]\,] &= \mathbb{E}_a[N(1,2] \,\|N(0,1] = 0]\,\mathbb{P}_a(N(0,1] = 0) \\ &\quad + \mathbb{E}_a[N(1,2] \,\|N(0,1] > 0]\,\mathbb{P}_a(N(0,1] > 0) \\ &= \mathbb{E}_{a+1}[N(0,1]\,]\,\mathbb{P}_a(N(0,1] = 0) \\ &\quad + \int_0^1 \mathbb{E}_a[N(1,2] \,\|(A_1\|N(0,1] > 0) = s]G(ds)\mathbb{P}_a(N(0,1] > 0) \\ &> \mathbb{E}_a[N(0,1]\,]\,\mathbb{P}_a(N(0,1] = 0) + \mathbb{E}_a[N(0,1]]\mathbb{P}_a(N(0,1] > 0) \\ &= \mathbb{E}_a[N(0,1]\,]\end{aligned}$$

which implies non-concavity of $m(\cdot)$. □

5 Stationarity

Suppose that N is a standard renewal process with lifelengths $Y_0, Y_1, \ldots$, where $Y_1, Y_2, \ldots$ are distributed accordingly to the lifelength distribution F. (The delay Y_0 may have a different distribution.) It is well known that if $\mu = m(F) < \infty$, then there exists a stationary version N' of N, i.e., a renewal process with the same lifelength distribution as N, but such that $\theta_t N' \stackrel{\mathcal{D}}{=} N'$, for all $t \geq 0$. In the case with standard renewal processes, it is easy to construct the stationary version given N: let N' have the delay $Y_0' \stackrel{\mathcal{D}}{=} G_s$, where G_s is given by

$$dG_s(x) = (1/\mu) \cdot \bar{F}(x)dx.$$

In the case with renewal processes in random environments, it is not obvious if a stationary version exist or how to construct it. However, each (non-degenerate) RPRE process $V = (N, Z)$ inheres an regenerative process. We can then rely on a well-known result concerning the existence of stationary regenerative processes.

5.1 The embedded regenerative process

Suppose that an RPRE process $V = (N, Z)$ is given to us, with a random initial age a distributed according to the probability distribution H. Define

$$x^* = \sup\{x \geq 0;\ x \in s(F^{(i)}) \text{ for all } i \in \mathsf{S}\},$$

allowing $x^* = \infty$ if all the distributions $F^{(i)}$ have unbounded support. We will, with exception for the IFR case, restrict attention to initial distributions H satisfying

$$s(H) \subseteq [0, x^*)$$

to avoid some difficulties with the delay S_0. (Here is an example which illuminates why a restriction is needed; suppose that the failure rates are general, $s(F^{(i)}) = [0, x_0]$ for some $x_0 > 0$, for all $i \in \mathsf{S}$, and that $s(H) = (x_0, \infty)$. Then the RPRE is degenerate in the sense that $S_0 = \infty$ a.s., since $r_a^{(i)}(x) = 0$ for all $x \geq 0$ and all $i \in \mathsf{S}$ if $a > x_0$.) Define the *age process* $A = (A_t)_0^\infty$ by letting

$$A_t = \begin{cases} t + a & \text{if } t < S_0 \\ \min\{t - S_n; t - S_n \geq 0\} & \text{if } t \geq S_0 \end{cases}$$

where S_n is the location of the n:th point as before. Define the two-dimensional process $X = (X_t)_0^\infty$ by

$$X_t = (\, A_t, Z_t \,)$$

and the sequence $T = (T_n)_{n=0}^\infty$ recursively through

$$T_0 = \inf\{\, t \geq 0;\; X_t = (0,0)\}$$

and

$$T_n = \inf\{\, t > T_{n-1};\; X_t = (0,0)\}.$$

Define also

$$C_0 = T_0$$

and, for $n \geq 1$,

$$C_n = T_n - T_{n-1}.$$

$(T_n)_{n=0}^\infty$ is indeed a (zero-delayed or delayed) renewal process. Obviously the C_n are independent, and to see that $\lim_{n\to\infty} T_n = \infty$ a.s, we may argue as follows: the distributions $F^{(0)}, \dots, F^{(M)}$ are absolute continuous, and therefore we have

$$C_n \overset{\mathcal{D}}{\geq} \tilde{C} > 0 \quad \text{a.s.}$$

if $\tilde{C} \overset{\mathcal{D}}{=} F^{(M)}$, since C_n includes at least one lifelength. (INTE HELT BRA) Hence $T_n = \sum_{i=0}^n C_n \to \infty$ a.s. From now and onwards we assume that C is a random variable, independent of $C_0, C_1, \dots$ and distributed as C_1.

The key to the existence of a stationarity process is the observation that $X = (X_t)_0^\infty$ is a regenerative process with respect to the renewal process $(T_n)_{n=0}^\infty$. Recall that a stochastic process $X = (X_t)_0^\infty$ is a *regenerative* process with respect to the renewal process $(T_n)_{n=0}^\infty$ if

(i) the distributions of $\theta_{T_n} X$ are equal for all $n \geq 0$, and
(ii) $\theta_{T_n} X$ is independent of $\{X_t;\, t < T_n\}$ and $\{\, T_0, \dots, T_n \,\}$ for each $n \geq 0$.

The C_n-variables are called the *cycle lengths*, and we call $(T_n)_{n=0}^\infty$ the *embedded* renewal process.

A moment of thought reveals that we have (i) satisfied; the processes

$$\theta_{T_0} X,\, \theta_{T_1} X, \dots$$

have the same start value $(0,0)$, and they are all of the same type. It should be obvious from the definition of RPRE processes that *(ii)* holds; $\theta_{T_n} X$

is defined through Poisson embedding, and the environment is Markovian. (EJ HELT BRA) (There are other possible embedded renewal processes, exchange for instance the state $(0,0)$ to $(0,i)$ in the definition of the T_n-variables above, for any $i \in \mathsf{S}$.)

5.2 The existence of a stationary RPRE process

If X is a $\mathbb{D}_E$-valued regenerative process (where E is a Polish space) with respect to a renewal process $S = (S_n)$, then there exist a stationary version X' of X, if the cycle lengths has finite moments of the first order, i.e., if $m(F) < \infty$ where F is the distribution of C. (See [9].)

Our process X is $\mathbb{D}_{(\mathbb{R}_+ \times \mathcal{S})}$-valued. So we need to investigate under which conditions we have $\mathbb{E}[C] < \infty$. The following result, which is the main result in this section, implies that a stationary version exists whenever $m(F^{(0)}) < \infty$.

Theorem 4 *If* $m(F^{(0)}) < \infty$ *then* $\mathbb{E}[C] < \infty$.

Proof. Suppose that $V = (N, Z)$ is an RPRE process with the embedded renewal process $(T_k)_{k=0}^{\infty}$ and cycle lengths $(C_k)_{k=0}^{\infty}$. As before, denote the locations of the points associated with $S_0, S_1, \ldots$. Since we only are interested in the cycles, we can assume V is zero-delayed and that $Z_0 = 0$, which implies that $(T_k)_{k=0}^{\infty}$ is zero-delayed, i.e., $T_0 = S_0 = 0$. Then $C \overset{\mathcal{D}}{=} T_1$ holds, so it is sufficient to prove $\mathbb{E}[\,T_1\,] < \infty$. Let

$$\eta = \min\{k \geq 1;\ Z_{S_k} = 0\}.$$

Then

$$T_1 = \sum_{k=1}^{\eta} Y_k$$

where $Y_n = S_n - S_{n-1}$, for $n \geq 1$. Define

$$\mathcal{G}_n = \sigma(S_0, \ldots, S_n) \bigvee \sigma(Z_t; t \leq S_n)$$

for $n \geq 1$. Observe that η is a stopping time w.r.t. the filtration $(\mathcal{G}_n)_{n=0}^{\infty}$.

Therefore we have

$$\begin{aligned} \mathbb{E}[\sum_{k=1}^{\eta} Y_k] &= \mathbb{E}[\sum_{k=1}^{\infty} Y_k \mathbb{1}_{\{\eta \geq k\}}] \\ &= \sum_{k=1}^{\infty} \mathbb{E}[Y_k \mathbb{1}_{\{\eta \geq k\}}] \\ &= \sum_{k=1}^{\infty} \mathbb{E}[\mathbb{E}[Y_k \| \mathcal{G}_{k-1}] \mathbb{1}_{\{\eta \geq k\}}] \end{aligned}$$

It should be quite obvious that

$$\begin{aligned} \mathbb{E}[Y_k \| \mathcal{G}_{k-1}] &= \mathbb{E}[Y_k \| S_{k-1}, Z_{S_{k-1}}] \\ &\leq m(F^{(0)}). \end{aligned}$$

Now, in order to prove that $\sum_{k=1}^{\infty} \mathbb{P}(\eta \geq k) < \infty$, we prove that $\eta \overset{\mathcal{D}}{\leq} \zeta$, where ζ is a geometrically distributed random variable. We may argue as follows: η is the number of points $S_1, S_2, \ldots$ that we have to inspect, until we find one, S_k say, in the best environment state, i.e., $Z_{S_k} = 0$. So we can think of η as the number of required trials until the first successful one. The first trial is performed in $(0, S_1]$, the second one in $(S_1, S_2]$, and so on. Trial nr i is successful if $Z_{S_i} = 0$. Let

$$p = \inf_{0 \leq i \leq M} \mathbb{P}_{(0,i)}(Z_{S_1} = 0).$$

The probability for a trial to be successful is obviously at least p. It is quite clear that p>0; first, observe that

$$p = \mathbb{P}_{(0,M)}(Z_{S_1} = 0),$$

(which can be proved by a coupling argument using Poisson embedding). Define $\tau = \inf\{t \geq 0; Z_t = 0\}$, and let F_τ be the distribution of τ when $Z_0 = M$. The environment process hits the state 0 for the first time at τ; the probability that $Z_{S_1} = 0$ is at least the probability that S_1 occurs under the first visit to the state 0. Therefore

$$\begin{aligned} \mathbb{P}_{(0,M)}(Z_{S_1} = 0) &\geq \int_{u=0}^{\infty} \int_{v=0}^{\infty} \mathbb{P}_{(0,M)}(S_1 \in [u, u+v)) \, \beta_0 \cdot e^{-\beta_0 \cdot v} dv \, F_\tau(du) \\ &\geq \int_{u=0}^{\infty} \int_{v=0}^{\infty} e^{-R^{(M)}(u)} \cdot (1 - e^{-R^{(0)}(u+v)}) \, \beta_0 \cdot e^{-\beta_0 \cdot v} dv \, F_\tau(du) \\ &> 0. \end{aligned}$$

We conclude that $\mathbb{E}[T_1] < \infty$ since

$$\sum_{k=1}^{\infty} \mathbb{P}(\eta \geq k) = \mathbb{E}[\eta] \leq \mathbb{E}[\zeta] < \infty.; \qquad \square$$

The condition $m(F^{(0)}) < \infty$ implies that $m(F^{(j)}) < \infty$ for all $j \in \mathsf{S}$. The next result shows that it is possible to have $\mathbb{E}[C] < \infty$ even though $m(F^{(i)}) = \infty$ for all $i < M$, if we can stochastically dominate the distribution $F^{(M)}$ by an exponential distribution. Here we consider the regenerative process defined through

$$T_0 = \inf\{\, t \geq 0;\ X_t = (0, M)\}$$

and

$$T_n = \inf\{\, t > T_{n-1};\ X_t = (0, M)\},$$

with cycles

$$C_0 = T_0$$

and, for $n \geq 1$,

$$C_n = T_n - T_{n-1}.$$

Proposition 4 *Suppose that there exists $\gamma > 0$ such that $r^{(M)}(x) \geq \gamma$ for all $x \geq 0$. Then $\mathbb{E}[C] < \infty$.*

Proof. The idea is that the environment process will have infinitely many 'long' visits to state M,; the probability for a point under such a visit is then proved to be uniformly > 0.

Let η_i be the ith holding time of the Z process in the state M. The holding times $\eta_1, \eta_2, \ldots$ are exponentially distributed with parameter δ_M (since Z is a birth and death process) and therefore

$$\mathbb{P}(\eta_i > s) = e^{-\delta_M \cdot s} > 0$$

for all $i \geq 0$. Furthermore, $\eta_0, \eta_1, \ldots$ are independent, so the Second Borel-Cantelli Lemma gives

$$\mathbb{P}(\eta_i > s,\ i.o.) = 1$$

We conclude that there will be infinitely many visits to the state M, which lasts for at least s time-units (abbreviated s-visits). Since the stochastic failure rate is at least γ at any time instant regardless of the history of the process, the probability of having renewals in a s-visit is at least $p = 1 - e^{-\delta s} > 0$. Therefore, the required number of such $s-$visits are stochastically dominated by a geometrical distribution. $\square$

Conjecture 1 *If $m(F^{(M)}) < \infty$ then $\mathbb{E}[C] < \infty$.*

5.3 The stationary age distribution in the DFR case

In this section we establish a stochastic domination result concerning the distribution of the initial age (or the age at any time instant) of a stationary RPRE process N'. Suppose that N'' is a stationary standard renewal process with lifelength distribution $F^{(0)}$, the distribution associated with the environment state 0. It is then rather intuitive that N'' has a stochastically larger age (at any time instant) than N'. This is proved by establishing a coupling such that all points of the standard renewal process which occur after a random time T also are points of the stationary RPRE process. The result is useful for us in Section 7, in the proof of Theorem 9, where we need to prove that stationary RPRE processes within the DFR class with finite $(\alpha+1)-$moment on the distribution $F^{(0)}$ has an age distribution with finite $\alpha-$moment (for $\alpha > 0$).

Lemma 4 *Let $V' = (N', Z')$ be a stationary RPRE process within the DFR class, with the distribution $F^{(0)}$ associated to the state 0. Let N'' be a stationary standard renewal process with lifelength distribution $F^{(0)}$. Then*

$$H' \overset{\mathcal{D}}{\leq} H''$$

holds, where H' and H'' are the initial age distributions of N' and N'', respectively.

Proof. Let ξ be a two-dimensional Poisson process in $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$ with expectation measure ℓ_+. Let $a' \overset{\mathcal{D}}{=} H'$ and $a'' \overset{\mathcal{D}}{=} H''$. We may now generate a stationary RPRE process $V^* = (N^*, Z')$ and a stationary renewal process N^{**} in the standard way with Poisson embedding; we generate N^* by using ξ, Z' and a', and N^{**} by using ξ, and a''. Denote the renewals of N^* and N^{**} with $(S_n^*)_{n=0}^{\infty}$ and $(S_n^{**})_{n=0}^{\infty}$, respectively. Observe that (N^*, N^{**}) is a coupling of (N', N''). Define $T = S_0^*$, and let

$$\tau = \min\{n \geq 0; S_n^{**} \geq T, S_n^{**} = S_m^* \text{ for some } m \geq 0\}.$$

Certainly, $T < \infty$ a.s. We observe that on $\{n \geq \tau\}$, there exist an $m \geq 0$ such that

$$S_n^{**} = S_m^* \text{ a.s.},$$

due to the DFR property. Loosely speaking, all N^{**}-renewals occuring after S_0^* are also N^*-renewals. Therefore

$$A_t^* \leq A_t^{**}$$

on $\{t \geq T\}$. Since N^* is stationary,

$$A_t^* \stackrel{\mathcal{D}}{=} A_0^*$$

for all $t \geq 0$, and the same holds for A_t^{**}. In particular,

$$\mathbb{P}(A_0^* > s) = \lim_{t\to\infty} \mathbb{P}(A_t^* > s),$$

and

$$\mathbb{P}(A_0^{**} > s) = \lim_{t\to\infty} \mathbb{P}(A_t^{**} > s),$$

holds for all $s \geq 0$. It follows that

$$\begin{aligned}
\mathbb{P}(A_0^* > s) &= \lim_{t\to\infty} \mathbb{P}(A_t^* > s) \\
&= \lim_{t\to\infty} \big(\mathbb{P}(A_t^* > s, t < T) + \mathbb{P}(A_t^* > s, t \geq T)\big) \\
&\leq \lim_{t\to\infty} \big(\mathbb{P}(t < T) + \mathbb{P}(A_t^{**} > s, t \geq T\big) \\
&= \lim_{t\to\infty} \mathbb{P}(A_t^{**} > s, t \geq T) \\
&= \lim_{t\to\infty} \mathbb{P}(A_t^{**} > s) \\
&= \mathbb{P}(A_0^{**} > s),
\end{aligned}$$

i.e., we have proved that

$$H' \stackrel{\mathcal{D}}{=} \mathbb{P}(A_0^* \in \cdot) \stackrel{\mathcal{D}}{\leq} \mathbb{P}(A_0^{**} \in \cdot) \stackrel{\mathcal{D}}{=} H''. \quad \square$$

6 Asymptotics

In this section we investigate the asymptotical behaviour of the RPRE processes. We prove that the processes 'forget' their initial conditions, and, under certain conditions, become stationary in the limit, just like the case with standard renewal processes. We first consider the case with general failure rates. The Poisson embedding technique seems to be quite useless without some monotonicity conditions on the failure rates, and therefore the proof relies on a coupling result on regenerative processes. This proof covers of course also the DFR- and IFR cases, but we present two additional proofs, based on the Poisson embedding technique. This has the advantage of making the arguments rather transparent because of the monotonicity of the failure rates, and in extension it also gives the rate results in Section 7.

6.1 Asymptotics in the general case

Let $\|\cdot\|$ be the total variation norm. As before, we denote the distribution associated to the state 0 with $F^{(0)}$. Let

$$x^* = \sup\{x \geq 0; x \in s(F^{(i)}) \text{ for all } i \in \mathsf{S}\}.$$

Theorem 5 *Suppose that $V = (N, Z)$ is an RPRE process with initial distribution (H, λ), and where H is a probability distribution with support $s(H) \subseteq [0, x^*)$. If $m(F^{(0)}) < \infty$ then*

$$\| \mathbb{P}_{(H,\lambda)}(\theta_t V \in \cdot\,) - \mathbb{P}(V' \in \cdot\,) \| \longrightarrow 0, \quad t \to \infty, \tag{10}$$

where V' is the stationary version of V.

Proof. The condition on the initial distribution guarantees that an embedded regenerative process exists, which we denote by X, defined as in Section 5. Denote the cycle length distribution with F and the delay distribution with G, i.e., $T_0 \stackrel{\mathcal{D}}{=} G$. We use a result from [8], Corollary 1.1. It states that if $m(F) < \infty$ and F non-singular then

$$\| \mathbb{P}_G(\theta_t X \in \cdot\,) - \mathbb{P}(X' \in \cdot\,) \| \longrightarrow 0, \quad t \to \infty,$$

where X' denotes the stationary version of X. Since we have assumed that $m(F^{(0)}) < \infty$, Theorem 4 in Section 5 implies that the cycle lengths has finite expectation. It is quite obvious that F has an absolutely continuous component, i.e., that there exists a subprobability measure $F^* \neq 0$ with a density f^* such that

$$F(B) \geq F^*(B) = \int_B f^*(x)dx$$

for all $B \in \mathcal{B}(\mathbb{R}_+)$: let $\eta = \inf\{t \geq 0; Z_t > 0\}$ and observe that

$$F(x) = \mathbb{P}_{(0,0)}(T_1 \leq x) \geq \mathbb{P}_{(0,0)}(T_1 \leq x, \eta > x) = (1 - e^{-R^{(0)}(x)}) \cdot e^{-\beta_0 x}.$$

Therefore (10) holds. □

6.2 Asymptotics in the DFR case

Theorem 6 *Let $V = (N, Z)$ and $V' = (N', Z')$ be RPRE-processes of the same type within the DFR class, with initial distributions (H, λ) and (H', λ'), respectively. Suppose that the initial ages are finite a.s. Then*

$$\| \mathbb{P}_{(H,\lambda)}(\theta_t V \in \cdot) - \mathbb{P}_{(H',\lambda')}(\theta_t V' \in \cdot) \| \longrightarrow 0, \quad \text{as } t \to \infty. \tag{11}$$

When a stationary version V' of V exists we have

$$\| \mathbb{P}_{(H,\lambda)}(\theta_t V \in \cdot) - \mathbb{P}(V' \in \cdot) \| \longrightarrow 0, \quad \text{as } t \to \infty \tag{12}$$

Proof. We shall construct a coupling (V'', V') of (V, V'), such that

$$\theta_t V'' = \theta_t V' \text{ a.s. for all } t \geq T, \tag{13}$$

where T is a finite coupling time. Then (11) follows, and if V' is stationary, then $\theta_t V' \stackrel{\mathcal{D}}{=} V'$ for all t, giving (12). (A stationary RPRE process has a.s. finite initial age, due to the construction of stationary regenerative processes, see ([9])) Let ξ be a Poisson process in $(\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+))$, with expectation measure ℓ_+, and let V and V' be generated by ξ in the standard way. We will construct V'', once again using ξ, in the following way: we let V'' be equal to V up to the time T_{env}, where the environment processes of V and V' first meet. Thereafter, we let V'' evolve in the same environment process as V'. If N'' and N' ever have a common renewal occuring after T_{env}, at T say, then they will be identical from T and onwards, giving (13). That is due to the fact that they are both generated by the same Poisson process ξ, in identical environments.

More formally, define

$$T_{env} = \inf\{t \geq 0 : Z_t = Z_t^{'}\}$$

and let

$$Z''_t = \begin{cases} Z_t & \text{if } t \le T_{env} \\ Z'_t & \text{if } t \ge T_{env}. \end{cases}$$

Then (Z'', Z') is a coupling of (Z, Z'), (see Lindvall [6]) and it is well known that $T_{env} < \infty$ a.s. By letting V'' be generated by ξ, with initial age a'' equal to the one of V, i.e., $a'' = a$, and with environment Z'', it follows that $V'' \stackrel{\mathcal{D}}{=} V$, since $Z'' \stackrel{\mathcal{D}}{=} Z$ and since the initial ages are the same. Suppose $(S''_i)_0^\infty$ and $(S'_i)_0^\infty$ are the points associated with N'' and N', respectively. Define $D'_{T_{env}}$ and $D''_{T_{env}}$, the overshoots at T_{env},

$$D''_{T_{env}} = \min_{i \ge 0}\{S''_i - T_{env}; S''_i - T_{env} \ge 0\}$$

and

$$D'_{T_{env}} = \min_{i \ge 0}\{S'_i - T_{env}; S'_i - T_{env} \ge 0\}.$$

It may be helpful to consult Figure 6.2 to see that the first common renewal of N'' and N' occurs at

$$T = T_{env} + \max\{D'_{T_{env}}, D''_{T_{env}}\}.$$

So T is the coupling time of V'' and V'. We must prove that $T < \infty$ a.s. Let H and H' be the initial age distributions of V and V', respectively, and let

$$W = \max\{D'_{T_{env}}, D''_{T_{env}}\}.$$

We have

$$\mathbb{P}(T < \infty) = \mathbb{P}(T_{env} + W < \infty) =$$

$$\sum_{i \in \mathsf{S}} \sum_{j \in \mathsf{S}} \int_{s \ge 0} \int_{s' \ge 0} \mathbb{P}_{(s,s'),(i,j)}(T_{env} + W < \infty) H(ds) H'(ds') \lambda_i \lambda'_j =$$

$$\sum_{i \in \mathsf{S}} \sum_{j \in \mathsf{S}} \int_{s \ge 0} \int_{s' \ge 0} \int_{u=0}^{\infty} \mathbb{P}_{(s,s'),(i,j)}(W < \infty \| T_{env} = u) F_{ij}(du) \times$$

$$\times H(ds) H'(ds') \lambda_i \lambda'_j$$

where F_{ij} is the distribution of T_{env} when $Z_0 \equiv i$ and $Z'_0 \equiv j$. We now argue that

$$\mathbb{P}_{(s,s'),(i,j)}(W < \infty \| T_{env} = u) = 1 \tag{14}$$

for all fixed $s, s', u < \infty$. Given the initial ages s, s' and $T_{env} = u$, define $A = \max(A''_u, A'_u)$, the largest age at u of the two RPRE processes considered. It satisfies

$$A \le \max(s, s') + u, \tag{15}$$

where the right hand expression is the largest age at u if there are no renewals in $[0, u]$. The failure rate of $(W \| T_{env} = u, a'' = s, a' = s')$ (from now abbreviated $(W\|u, s, s')$) is equal to

$$r_A^{(Z''_{u+x})}(x)$$

for $x \ge 0$ by definition, and satisfies

$$r_A^{(Z''_{u+x})}(x) \ge r^{(0)}_{\max(s,s')+u}(x) \text{ a.s.}$$

for all $x \ge 0$, due to (15) and that the failure rates $r^{(0)}, \ldots r^{(M)}$ are decreasing. Let $b = \max(s, s') + u$ and let W_b denote a random variable distributed accordingly to $F_b^{(0)}$. Then Lemma 1 gives that

$$(W\|u, s, s') \overset{\mathcal{D}}{\le} W_b.$$

Then (14) follows if we, for all $b < \infty$, can prove that $W_b < \infty$ a.s. An easy argument concerning the distribution functions $F^{(0)}$ and $F_b^{(0)}$ shows this. Since $F^{(0)}$ is absolute continuous, it has no mass in infinity, giving

$$1 = \lim_{u \to \infty} F^{(0)}(u) = 1 - \exp\left(-\int_0^\infty r^{(0)}(t)\, dt\right).$$

Therefore

$$\int_0^\infty r^{(0)}(t)\, dt = \infty.$$

Since $F^{(0)}$ is DFR, it has unbounded support, so

$$\int_b^\infty r^{(0)}(t)\, dt = \infty$$

for all $b < \infty$, or, equivalently,

$$\int_0^\infty r_b^{(0)}(t)\, dt = \infty.$$

This gives

$$\mathbb{P}(W_b < \infty) = 1 - \exp\left(-\int_0^\infty r_b^{(0)}(t)\, dt\right) = 1$$

We conclude that $T < \infty$ a.s.$\square$.

6.3 Asymptotics in the IFR case

Theorem 7 *Let $V = (N, Z)$ be an RPRE-processes within the IFR class, with initial distribution (H, λ), and suppose that V' is a stationary version of V. Then*

$$\| \mathbb{P}_{(H,\lambda)}(\theta_t V \in \cdot) - \mathbb{P}(V' \in \cdot) \| \longrightarrow 0, \quad t \to \infty. \tag{16}$$

Proof. We shall construct a coupling (V'', V') of (V, V'), such that

$$\theta_t V'' = \theta_t V' \quad \text{a.s.},$$

for all $t \geq T$, where T is a finite coupling time. We construct V' and V'' by proceeding as in the DFR case: let ξ be a Poisson process on $\mathbb{R}_+^2$, with expectation measure ℓ_+. Let H' be the age distribution of V'. Define Z'' as in the DFR proof, by carrying out a coupling of Z and Z' at T_{env}, and generate V' and V'', with initial age distributions H and H', respectively, using ξ in the standard way. Let $(S_i^{'})$ and $(S_i^{''})$ be the (sequences of) locations of renewals associated with N' and N'', respectively. Define

$$T = \min_i \{S_i''; \; S_i'' \geq T_{env} \text{ and } S_i'' = S_j^{'} \text{ for some } j \geq 0\}.$$

The random time T is the first time, after T_{env}, where N' and N'' have a common renewal. Since N' and N'' evolve in the same environment after T_{env}, their futures are identical from T and onwards. Therefore T is our coupling time. We must prove that $T < \infty$ a.s. Define

$$J_0 = \min\Big(\min_i \{S_i' : S_i' \geq T_{env}\}, \, \min_i \{S_i'' : S_i'' \geq T_{env}\} \Big).$$

On $\{J_0 = S_m^{'} \text{ for some } m \geq 0\}$, define the sequences $(I_k)_1^\infty$, $(J_k)_1^\infty$ and $(\nu_k)_1^\infty$ recursively through

$$I_k = \min_i \{S_i^{''} : S_i^{''} > J_{k-1}\},$$

$$J_k = \min_i \{S_i^{'} : S_i^{'} > J_{k-1}\},$$

and

$$\nu_k = \begin{cases} 1 & \text{if } N''[\{J_k\}] = 1 \\ 0 & \text{otherwise.} \end{cases}$$

On $\{J_0 = S'_m \text{ for some } m \geq 0\}^C$, swap the primes and the double-prims in the definitions above. The coupling time T is then satisfying

$$T \leq T_{env} + U_0 + \sum_{i=1}^{\tau} U_i,$$

where

$$\tau = \min_{k \geq 1}\{k;\ \nu_k = 1\},$$
$$U_0 = J_0 - T_{env}$$

and

$$U_k = J_k - J_{k-1},$$

for $k \geq 1$. It may be helpful to think of the coupling in the following way: we let N'' and N' evolve until their environments coincide, occuring at T_{env}. Then, one of the processes has a renewal at J_0 and hence an age equal to 0. Thereafter, we carry out a series of trials, coupling attempts, until N' and N'' have a common renewal. We make the k:th attempt in the interval $(J_{k-1}, J_k]$, successfully if $\nu_k = 1$, for $k \geq 1$.

Each of these $(J_{k-1}, J_k]$-attempts can be thought of as consisting of two sub-trials: the first at I_k and the second at J_k. These coincide if N' and N'' have a common renewal at I_k, i.e., then $I_k = J_k$, and otherwise, $I_k < J_k$. At J_{k-1}, one of the two processes has a renewal, whereas the other one had the last renewal at I_{k-1}. Assume for simplicity that the renewal at J_k was a N'-point. Then, $A'_{J_k} = 0$, whereas $A''_{J_k} = J_k - I_{k-1} = a$, for some $a > 0$ and $Z''_{J_k} = j$, for some $j \in \mathsf{S}$. Define

$$\gamma := \inf_{a \geq 0,\, i \in \mathsf{S}} \mathbb{P}_{(0,a,i)}(\nu_k = 1) \tag{17}$$

where the notation $\mathbb{P}_{(0,a,i)}$ is supposed to be self-explanatory. Let $p_{a,i}(k)$ be a short for $\mathbb{P}_{(0,a,i)}(\nu_k = 1)$. We will prove that $\gamma > 0$, and hence that the required number of coupling attempts is stochastically dominated by a geometrical distribution with parameter γ. In order to estimate the probability $p_{a,i}(k)$, let Z^* be a birth and death process on S, governed by the same intensities as Z and with $Z^* = i$. Define

$$B^{(a)} = \{(x, y) \in \mathbb{R}^2_+ : y \leq r_a^{(Z^*_x)}(x)\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2_+ : y \leq r^{(Z^*_x)}(x)\}.$$

Let ξ be a two-dimensional Poisson process as before, and let

$$I = \tau_{B^{(a)}}(\xi),$$

$$J = \tau_B(\xi),$$

$$B^{(I)} = \{(x,y) \in \mathbb{R}_+^2 : y \le r^{(Z_x^*)}(x - I)\}$$

and

$$K = \tau_{B^{(I)}}(\xi).$$

Recall that if $x < 0$, then $r^{(j)}(x) = 0$, for all $j \in \mathsf{S}$. Let

$$\nu = \begin{cases} 1 & \text{if } J = I \text{ or } J = K \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$p_{a,i}(k) = \mathbb{P}_{(a,i)}(\nu = 1),$$

Now, define

$$B' = \{(x,y) \in \mathbb{R}_+^2 : y \le r^{(Z_x^*)}(x) \cdot \mathbb{1}_{\{x>I\}}(x)\}$$

$$J' = \tau_{B'}(\xi)$$

and

$$\nu' = \begin{cases} 1 & \text{if } J' = K \\ 0 & \text{otherwise.} \end{cases}$$

When defining J' the truncated failure rate $r^{(Z_x^*)}(x) \cdot \mathbb{1}_{\{x>I\}}(x)$ is used, i.e., we 'ignore' the possibility that there is a common ξ-points under the failure rates at I. Therefore, $\{J' = I\}$ is an impossible event, and moreover

$$\{\nu' = 1\} \subseteq \{\nu = 1\},$$

which gives

$$\mathbb{P}_{(a,i)}(\nu = 1) \ge \mathbb{P}_{(a,i)}(\nu' = 1).$$

We continue our estimations, now by 'ignoring' the evolution of the environment process in (I, ∞): define

$$B''_M = \{(x,y) \in \mathbb{R}_+^2 : y \le r^{(M)}(x) \cdot \mathbb{1}_{\{x>I\}}(x)\}$$

$$B''_0 = \{(x,y) \in \mathbb{R}_+^2 : y \le r^{(0)}(x) \cdot \mathbb{1}_{\{x>I\}}(x)\}$$

$$J'' = \tau_{B''_M}(\xi),$$

$$K'' = \tau_{B_0''}(\xi)$$

and

$$\nu'' = \begin{cases} 1 & \text{if } J'' = K'' \\ 0 & \text{otherwise.} \end{cases}$$

Certainly,

$$\{\nu'' = 1\} \subseteq \{\nu' = 1\},$$

so we have

$$\mathbb{P}_{(a,i)}(\nu' = 1) \geq \mathbb{P}_{(a,i)}(\nu'' = 1).$$

In order to estimate the latter probability, introduce another Poisson process ξ' in $\mathbb{R}_+^2$, with expectation measure ℓ_+. Define

$$A^{(s)} = \{(x,y) \in \mathbb{R}_+^2 : y \leq r_s^{(M)}(x)\}$$

and

$$A = \{(x,y) \in \mathbb{R}_+^2 : y \leq r^{(0)}(x)\},$$

for $s \geq 0$. Also, define

$$J^*(s) = \tau_{A^{(s)}}(\xi')$$

and

$$K^* = \tau_A(\xi').$$

Certainly,

$$\{J^*(s') = K^*\} \subseteq \{J^*(s) = K^*\}$$

for all $s' \geq s$, i.e., the probability $\mathbb{P}(J^*(s) = K^*)$ is decreasing in s. We observe that

$$\mathbb{P}_{(a,i)}(\nu'' = 1) = \mathbb{P}(J^*(I) = K^*),$$

where I is defined as before. Choose a constant $C > 0$ large enough to ensure that $F^{(0)}(C) > 0$ holds. Since $I \overset{\mathcal{D}}{\leq} F^{(0)}$, we have

$$\begin{aligned} \mathbb{P}(J^*(I) = K^*) &\geq \int_{s \geq 0} \mathbb{P}(J^*(s) = K^*) dF^{(0)}(s) \\ &\geq \int_{s=0}^{C} \mathbb{P}(J^*(s) = K^*) dF^{(0)}(s) \\ &\geq \int_{s=0}^{C} \mathbb{P}(J^*(C) = K^*) dF^{(0)}(s) \\ &= \mathbb{P}(J^*(C) = K^*) F^{(0)}(C). \end{aligned}$$

It should be quite obvious that $\mathbb{P}(J^*(C) = K^*) > 0$, since $C > 0$. Furthermore, we note that $\mathbb{P}(J^*(C) = K^*)F^{(0)}(C)$ is not a function of a or i. Therefore, we have proved that

$$\inf_{a\geq 0,\, i\in \mathsf{S}} p_{(a,i)}(k) \geq \delta > 0,$$

where

$$\delta = \mathbb{P}(J^*(C) = K^*)F^{(0)}(C).$$

We have proved that the number of required coupling attempts τ is dominated by a geometrically distributed variable with parameter γ. It remains to prove that

$$\sum_{i=1}^{\tau} U_i < \infty \text{ a.s.}$$

We postpone the arguments for that until Section 7.3, Theorem 10, where the coupling achieved here is used to prove a stronger result. □

Remark. The reason for the 'dubble chance' structure of the coupling attempts is perhaps not apparent. If we instead would use a 'one chance' structure then

$$\inf_{a\geq 0,\, i\in \mathsf{S}} \mathbb{P}_{(0,a,i)}(\text{ successful attempt }) = 0,$$

since

$$\lim_{a\to\infty} \frac{r_0^{(Z_t)}(t)}{r_a^{(Z_t)}(t)} = 0 \text{ a.s.}$$

for all $t \geq 0$.

7 Rate of Convergence

In this section we continue the study initiated in the last section, where convergence towards stationarity was proved. We consider the couplings carried out there, and prove finite $\alpha-$ and exponential moments of the coupling times under some additional conditions. That yields rate of convergence results, due to (5) and (6) in Section 2. We keep to the lines laid down in the last section and concentrate us on the DFR and IFR cases; we content ourselves with some brief comments in the general case. It will be seen that processes within the IFR class have exponential rate of convergence, since we can prove that the lifelengths have finite exponential moments without further conditions on the failure rates. The DFR case is slightly more complicated since DFR distributions not always have finite moments. The condition $m(F^{(0)}) < \infty$ turns out to be sufficient for proving a rate of order t^{α} $(\alpha > 1)$, and exponential rate follows under the condition that $F^{(0)}$ has finite exponential moments. We point out that we have no ambition to find the exact rates, e.g., if a rate of order $e^{\alpha t}$ is settled, we do not try to find the best possible exponent α.

A distribution F (and random variables with distribution F) is said to have finite exponential moment, if there exists an $\alpha > 0$ such that

$$\varphi(\alpha) = \int_0^\infty e^{\alpha x} F(dx) < \infty. \tag{18}$$

7.1 Convergence of the environment process

We first need a well known result concerning the rate of convergence of the environment processes. It states that the the coupling time of two environment processes has a finite exponential moment.

Lemma 5 *Let Z and Z' be two independent random environments on* S. *Then, the coupling time defined by*

$$T_{env} = \inf\{t \geq 0; Z_t = Z_t'\}$$

has finite exponential moments.

Proof. It is well known that irreducible and positive recurrent time-continuous Markov chains on a finite state space converges exponentially fast towards stationarity. REFERENS !!!! □

7.2 The DFR case

In order to investigate the rate of convergence, we first state a result, which is a special case of a result (Theorem 6.3) proved in [2]

Lemma 6 *Suppose that F is a DFR distribution on $[0,\infty)$, with failure rate function r. If $\varphi(\gamma) < \infty$ for an $\gamma > 0$ then $r(x) \geq \gamma$ for all $x \geq 0$.*

7.2.1 Finite exponential moments

Theorem 8 *Let V be an RPRE process of within the DFR class with initial distribution (H,λ). Suppose that the distribution $F^{(0)}$ has finite exponential moment. Then there exists an $\alpha > 0$ such that*

$$\| \mathbb{P}_{(H,\lambda)}(\theta_t V \in \cdot) - \mathbb{P}(V' \in \cdot) \| = o(e^{-\alpha t}) \tag{19}$$

where V' is the stationary version of V.

Proof. Recall the definitions in Section 6.2. The coupling epoch T satisfies $T = T_{env} + W$. Let (H',π) denote the initial distribution of V'. We have

$$\begin{aligned} \mathbb{E}_{(H,\lambda),(H',\pi)}[e^{\alpha T}] &= \mathbb{E}_{(H,\lambda),(H',\pi)}[e^{\alpha(T_{env}+W)}] \\ &= \mathbb{E}_{(\lambda,\pi)}[\mathbb{E}_{(H,H')}[e^{\alpha(T_{env}+W)} \| T_{env}]] \\ &= \mathbb{E}_{(\lambda,\pi)}[e^{\alpha T_{env}} \mathbb{E}_{(H,H')}[e^{\alpha W} \| T_{env}]], \end{aligned}$$

and

$$\mathbb{E}_{(H,H')}[e^{\alpha W} \| T_{env}] = \int_u \int_s \int_{s'} \mathbb{E}_{(s,s')}[e^{\alpha W} \| T_{env} = u]\, H(ds) H'(ds') F_{T_{env}}(du).$$

Since $F^{(0)}$ has finite exponential moments, there exists a $\gamma > 0$ such that $r^{(0)}(x) \geq \gamma$ for all $x \geq 0$, due to Lemma 6. Therefore, the overshot W has a failure rate which certainly is at least γ, i.e., we can stochastically dominate W by a random variable X, which is exponentially distributed with parameter γ (by Lemma 2.) It follows that

$$\mathbb{E}_{(s,s')}[e^{\alpha W} \| T_{env} = u] \leq \mathbb{E}[e^{\alpha X}]$$

which is finite (uniformly in s, s' and u) if $\alpha < \gamma$. By Lemma 5 there exists a $\beta > 0$ such that

$$\mathbb{E}_{(\lambda,\pi)}[e^{\beta T_{env}}] < \infty$$

and therefore the result follows with $\alpha = \min(\beta,\gamma)$ $\square$.

7.2.2 Finite α moments

Now we weaken the conditions a bit and allow the failure rate $r^{(0)}(x)$ to converge to zero, as $x \to \infty$. Then, due to Lemma 6, we can not hope for exponential rate of convergence. However, under certain conditions, we can establish finite α moments of the coupling time T, and hence that the rate of convergence towards stationarity is of order t^{α}.

In the standard renewal theory, an analogue result is obtained under the conditions that the initial age distributions and the lifelength distribution have finite $\alpha-$moments. If F is the lifelength distribution and G_s is the stationary initial age distribution, one have to require that $m_{\alpha+1}(F) < \infty$ to ensure that $m_{\alpha}(G_s) < \infty$ holds, i.e., one moment is 'lost' in the stationary initial distribution. (REF?)

In the case with a random environment, it is quite intuitive that some analogue conditions on the distributions $F^{(i)}, i \in \mathsf{S}$, is needed. The natural condition

$$m_{\alpha+1}(F^{(0)}) < \infty$$

turns out to be sufficient, due to Lemma 7. That certainly implies that the stationary initial age distribution has finite α moment, due to Theorem 4.

Lemma 7 *Let $\alpha > 0$ and let F be a DFR distribution on $[0, \infty)$ with failure rate r, and suppose that $F(0) = 0$. If $\mu_{\alpha} < \infty$ then $\lim_{x\to\infty} x \cdot r(x) \geq \alpha$.*

Proof. The Lemma is a special case of Theorem 6.2. in [2]. □.

Theorem 9 *Let $\alpha > 1$. Suppose that V is an RPRE processes within the DFR class, with initial distribution (H, λ), and that $m_{\alpha}(H) < \infty$. If $m_{\alpha+1}(F^{(0)}) < \infty$ then*

$$\| \mathbb{P}_{(H,\lambda)}(\theta_t V \in \cdot) - \mathbb{P}(V' \in \cdot) \| = o(t^{-\alpha})$$

holds, where V' is the stationary version of V.

Proof of Theorem 9. As mentioned above, $m_{\alpha+1}(F^{(0)}) < \infty$ implies that the distribution G_s given by

$$dG_s(x) = \lambda(1 - F^{(0)}(x))dx$$

has a finite α moment. G_s is the age distribution for a stationary standard renewal process with lifelength distribution $F^{(0)}$. Now, use Lemma 4 to see that $m_{\alpha}(H') < m_{\alpha}(G_s) < \infty$, where H' is the initial age distribution of V'.

Once again we consider the coupling carried out in the proof of Theorem 6. Let $F_{i,j}$ denote the distribution of T_{env} when $Z_0 = i$ and $Z'_0 = j$. Then

$$\begin{aligned} \mathbb{E}_{(0,\lambda),(H',\pi)}[T^\alpha] &= \mathbb{E}_{(\lambda,\pi)}[\mathbb{E}_{(H,H')}[(T_{env}+W)^\alpha \| T_{env}]] \\ &\le C \cdot \mathbb{E}_{(\lambda,\pi)}[T_{env}^\alpha + \mathbb{E}_{(H,H')}[W^\alpha \| T_{env}]] \\ &= C \cdot \mathbb{E}_{(\lambda,\pi)}[T_{env}^\alpha] + C \cdot \mathbb{E}_{(\lambda,\pi)}[\mathbb{E}_{(H,H')}[W^\alpha \| T_{env}]] \end{aligned}$$

for some constant $C > 0$. (REFERERA TILL Lp olikheten?) The first term in the last expression is finite, since $F_{i,j}$ has finite exponential moments, and hence has finite $\alpha-$moments of all orders. When trying to estimate the second term we use the same idea as in the proof of Theorem 6. We want to stochastically dominate the overshoot W in some useful way. We saw in the proof of Theorem 6 that given that $T_{env} = u$ and that the initial ages are s, s', the smallest possible failure rate for $(W\|s, s', u)$ is

$$r_{\max(s,s')+u}^{(Z''_{u+x})}(x)$$

for $x \ge 0$, the failure rate if no renewals occur in $[0, u]$. (Recall that (Z'', Z') is a coupling of (Z, Z') and that $Z''_t = Z'_t$ for all $t \ge T_{env}$). Now, for $a \ge 0$, let W_a be a random variable independent of W, and distributed according to $F_a^{(0)}$, i.e., with failure rate function $r_a^{(0)}(x)$ for $x \ge 0$. Certainly

$$r_{\max(s,s')+u}^{(Z''_{u+x})}(x) \ge r_{s+s'+u}^{(0)}(x)$$

holds and therefore

$$(W\|s, s', u) \overset{\mathcal{D}}{\le} W_{s+s'+u},$$

by Lemma 1. We get

$$\begin{aligned} &\mathbb{E}_{(\lambda,\pi)}[\mathbb{E}_{(H,H')}[W^\alpha \| T_{env}]] = \\ &= \sum_{i,j\in\mathsf{S}} \int_s \int_{s'} \int_u \mathbb{E}_{(s,i),(s',j)}[W^\alpha \| T_{env} = u]\, dF_{i,j}(u) dH(s) dH'(s') \lambda_i \pi_j \\ &\le \sum_{i,j\in\mathsf{S}} \int_s \int_{s'} \int_u \mathbb{E}[W_{s+s'+u}^\alpha]\, dF_{i,j}(u) dH(s) dH'(s') \lambda_i \pi_j \end{aligned}$$

Denote the right-hand tripple integral with $I_{i,j}$, for later purposes. Now, use Lemma 7 to see that

$$\lim_{x\to\infty} x \cdot r^{(0)}(x) \ge (\alpha + 1),$$

i.e., that there exists a $x_0 \geq 0$ such that

$$r^{(0)}(x) \geq \frac{\alpha+1}{x}$$

for all $x \geq x_0$ (since $r^{(0)}(\cdot)$ is decreasing). We split $I_{i,j}$ into two terms, denoted with $I_{i,j}^{(1)}$ and $I_{i,j}^{(2)}$ respectively:

$$I_{i,j} = \int_{s=0}^{x_0} \int_{s'=0}^{x_0} \int_{u=0}^{x_0} \mathbb{E}[\, W_{s+s'+u}^{\alpha} \,]\, dF_{i,j}(u) dH(s) dH'(s')$$
$$+ \int_{s=x_0}^{\infty} \int_{s'=x_0}^{\infty} \int_{u=x_0}^{\infty} \mathbb{E}[\, W_{s+s'+u}^{\alpha} \,]\, dF_{i,j}(u) dH(s) dH'(s').$$

It should be quite obvious that $I_{i,j}^{(1)}$ is finite; under the restriction $s+s'+u \leq 3x_0$ we observe that

$$\mathbb{E}[W_{s+s'+u}^{\alpha}] \leq \mathbb{E}[W_{3x_0}^{\alpha}] = \mathbb{E}[W_0^{\alpha} \| W_0 > 3x_0].$$

Since $m_{\alpha+1}(F^{(0)}) < \infty$ holds, we see that

$$\begin{aligned} \mathbb{E}[W_0^{\alpha} \| W_0 > 3x_0]\, \mathbb{P}(W_0 > 3x_0) \;&\leq\; \mathbb{E}[W_0^{\alpha} \| W_0 > 3x_0]\, \mathbb{P}(W_0 > 3x_0) \\ &\quad + \mathbb{E}[W_0^{\alpha} \| W_0 \leq 3x_0]\, \mathbb{P}(W_0 \leq 3x_0) \\ &= \mathbb{E}[W_0^{\alpha}] < \infty. \end{aligned}$$

Since $\mathbb{P}(W_0 > 3x_0) > 0$, we conclude that $\mathbb{E}[W_{s+s'+u}^{\alpha}] < \infty$ and that $I_{i,j}^{(1)} < \infty$.

Too see that $I_{i,j}^{(2)} < \infty$, observe that under the restriction $s+s'+u \geq 3x_0$ it holds that

$$r_{s+s'+u}(x) \geq \frac{\alpha+1}{s+s'+u+x}$$

for all $x \geq 0$. Therefore

$$\begin{aligned} R_{s+s'+u}(x) &= \int_{t=0}^{x} r_{s+s'+u}(t) dt \\ &\geq \int_{t=0}^{x} \frac{\alpha+1}{s+s'+u+t} dt \\ &= (\alpha+1) \cdot \log\left(\frac{s+s'+u+x}{s+s'+u}\right) \end{aligned}$$

yielding

$$e^{-R_{s+s'+u}(x)} \leq \left(\frac{s+s'+u}{s+s'+u+x}\right)^{\alpha+1}.$$

We use that in the next estimation.

$$\begin{aligned}\mathbb{E}[W^{\alpha}_{s+s'+u}] &= \alpha \int_{x=0}^{\infty} x^{\alpha-1}\mathbb{P}(W_{s+s'+u} > x)dx \\ &= \alpha \int_{x=0}^{\infty} x^{\alpha-1} e^{-R_{s+s'+u}(x)} dx \\ &\leq \alpha \int_{x=0}^{\infty} x^{\alpha-1} (\frac{s+s'+u}{s+s'+u+x})^{\alpha+1} dx \\ &= \alpha(s+s'+u)^{\alpha} \cdot \int_{x=0}^{\infty} \frac{x^{\alpha-1}(s+s'+u)}{(s+s'+u+x)^{\alpha+1}} dx\end{aligned}$$

The integral in the last expression is uniformly bounded in s, s' and u; to see that, first rewrite it as

$$\int_{x=0}^{\infty} \frac{x^{\alpha-1}c}{(c+x)^{\alpha+1}} dx,$$

where $c = s + s' + u$. Then, the variable substitution $x = c \cdot z$ gives

$$\begin{aligned}\int_{x=0}^{\infty} \frac{x^{\alpha-1}c}{(c+x)^{\alpha+1}} dx &= c \cdot \int_{z=0}^{\infty} \frac{(cz)^{\alpha-1}c}{(c+cz)^{\alpha+1}} dz \\ &= \int_{z=0}^{\infty} \frac{z^{\alpha-1}}{(1+z)^{\alpha+1}} dz = C < \infty,\end{aligned}$$

for some constant $C > 0$. Therefore

$$I^{(2)}_{i,j} \leq \int_{s=x_0}^{\infty} \int_{s'=x_0}^{\infty} \int_{u=x_0}^{\infty} C\alpha(s+s'+u)^{\alpha}\, dF_{i,j}(u)dH(s)dH'(s')$$

which is finite by the assumptions.

□

7.3 The IFR case

In this section we prove that IFR processes have exponential rate of convergence towards stationarity. The proof is uses a method found in [6], p $30 - 31$.

Theorem 10 *Suppose that V is an RPRE process within the IFR class, with initial distribution (H, λ), and suppose that V' is a stationary version of V. Then, there exists $\alpha > 0$ such that*

$$\| \mathbb{P}_{(H,\lambda)}(\theta_t V \in \cdot) - \mathbb{P}(V' \in \cdot) \| = o(e^{-\alpha t}).$$

Proof. Denote the stationary initial distribution by (H', π). We prove that the coupling time T defined in Section 6.3 has exponential moment, i.e., that there exists $\alpha > 0$ such that

$$\mathbb{E}_{(H,\lambda),(H',\pi)}[\, e^{\alpha T}\,] < \infty.$$

Then, the theorem follows from (6) in Section 2.

Recall the definitions of the sequences $(I_k)_1^\infty$, $(J_k)_0^\infty$ and $(\nu_k)_1^\infty$. Let

$$U_0 = J_0 - T_{env}$$

and

$$U_k = J_k - J_{k-1}$$

for $k \geq 1$. With $\tau = \min\{k : \nu_k = 1\}$ we get

$$T \leq T_{env} + U_0 + \sum_{i=1}^{\infty} U_i \cdot \mathbb{1}_{\{\tau \geq i\}}.$$

With $\mathbb{E}[\cdot]$ as a shorthand for $\mathbb{E}_{(H,\lambda),(H',\pi)}[\cdot]$, we have

$$\mathbb{E}[\, e^{\alpha T}] = \lim_{n\to\infty} \mathbb{E}\Big[e^{\alpha(T_{env} + U_0 + \sum_{i=1}^{n} U_i \cdot \mathbb{1}_{\{\tau\geq i\}})}\Big]$$

for $\alpha > 0$. Denote that right-hand expectation with $I_1(n)$ and notice that

$$\begin{aligned} I_1(n) &= \mathbb{E}\Big[\, e^{\alpha(T_{env}+U_0)}\Big] \\ &\quad + \sum_{j=0}^{n-1} \mathbb{E}\Big[e^{\alpha(T_{env}+U_0)} \Big(e^{\alpha \sum_{i=1}^{j+1} U_i \cdot \mathbb{1}_{\{\tau\geq i\}}} - e^{\alpha \sum_{i=1}^{j} U_i \cdot \mathbb{1}_{\{\tau\geq i\}}}\Big)\Big] \\ &= \mathbb{E}\Big[\, e^{\alpha(T_{env}+U_0)}\Big] + \sum_{j=0}^{n-1} I_2(j) \end{aligned}$$

(using the convention $\sum_{i=1}^{0}(\,\cdot\,) = 0$.) We may estimate $I_2(j)$ by

$$\begin{aligned} I_{(2)}(j) &\leq \mathbb{E}\Big[\,(\, e^{\alpha(T_{env}+U_0)} e^{\alpha \sum_{i=1}^{j+1} U_i}\,) \cdot \mathbb{1}_{\{\tau \geq j+1\}}\Big] \\ &\leq \mathbb{E}\Big[\, e^{4\alpha(T_{env}+U_0)}\Big]^{1/4} \mathbb{E}\Big[\, e^{4\alpha \sum_{i=1}^{j+1} U_i}\Big]^{1/4} \mathbb{P}\,(\tau \geq j+1)^{1/2} \end{aligned}$$

where the last inequality is due to Schwarz' Inequality. Hence

$$\begin{aligned} \mathbb{E}[\, e^{\alpha T}] &\leq \mathbb{E}\Big[\, e^{\alpha(T_{env}+U_0)}\Big] \\ &\quad + \lim_{n\to\infty} \sum_{j=0}^{n-1} \mathbb{E}\Big[\, e^{4\alpha(T_{env}+U_0)}\Big]^{1/4} \mathbb{E}\Big[\, e^{4\alpha \sum_{i=1}^{j+1} U_i}\Big]^{1/4} \mathbb{P}\,(\tau \geq j+1)^{1/2}. \end{aligned}$$

We need now some estimates. We start with $\mathbb{E}[\, e^{4\alpha \sum_{i=1}^{j+1} U_i}]$. Define

$$\mathcal{F}_0 = \sigma(V_t^{'}, V_t^{''}, 0 \le t \le J_0)$$

and

$$\mathcal{F}_i = \sigma(V_t^{'}, V_t^{''}, 0 \le t \le J_i),$$

for $i \ge 1$. The $\sigma-$field $\mathcal{F}_j$ contains all information concerning the point processes under the j first coupling attempts. The variables $U_0, U_1, \dots, U_j$ is $\mathcal{F}_j-$measurable, and therefore

$$\begin{aligned}
\mathbb{E}\Big[\, e^{4\alpha \sum_{i=1}^{j+1} U_i}\Big] &= \mathbb{E}\Big[\, \mathbb{E}\Big[\, e^{4\alpha \sum_{i=1}^{j+1} U_i} \| \, \mathcal{F}_j \Big]\, \Big] \\
&= \mathbb{E}\Big[\, (e^{4\alpha \sum_{i=1}^{j} U_i}) \cdot \mathbb{E}\Big[\, e^{4\alpha U_{j+1}} \| \, \mathcal{F}_j \Big]\, \Big] \\
&= \mathbb{E}\Big[\, (e^{4 \sum_{i=1}^{j} U_i}) \cdot \mathbb{E}\Big[\, e^{4\, U_{j+1}} \| \, A''_{J_j}, A'_{J_j}, Z_{J_j} \Big]\, \Big].
\end{aligned}$$

When trying to establish an upper bound for $\mathbb{E}[\, e^{4\, U_{j+1}} \| \, A''_{J_j}, A'_{J_j}, Z_{J_j}\,]$, we notice that

$$U_{j+1} = J_{j+1} - J_j = (J_{j+1} - I_{j+1}) + (I_{j+1} - J_j).$$

Define $A_j = \max(A''_{J_j}, A'_{J_j})$, the maximal age at J_j. Furthermore, let W and W' be independent random variables, independent of the RPRE processes considered, and with common distribution function $F^{(0)}$. The random variable $(I_{j+1} - J_j)$ has a failure rate

$$r_{A_j}^{(Z_{J_j+x})}(x)$$

for $x \ge 0$, obviousy satisfying

$$r_{A_j}^{(Z_{J_j+x})}(x) \ge r^{(0)}(x)$$

for all $x \ge 0$. Therefore

$$(I_{j+1} - J_j) \overset{\mathcal{D}}{\le} W$$

by Lemma 1 in Section 2. A similar argument applied on $(J_{j+1} - I_{j+1})$ shows that

$$J_{j+1} - I_{j+1} \overset{\mathcal{D}}{\le} W'$$

holds, and therefore

$$(U_{j+1} \| \mathcal{F}_j) \overset{\mathcal{D}}{\le} W + W'$$

holds, giving the following estimation:

$$\mathbb{E}[\, e^{4\alpha U_{j+1}} \,\|\, A''_{J_j}, A'_{J_j}, Z_{J_j}\,] \leq \mathbb{E}[\, e^{4\alpha(W+W')}\,] \leq \mathbb{E}[e^{4\alpha W}\,]^2$$

due to the independence of W and W'. Due to Lemma 3 in Section 2, there exists finite constants $t_\lambda \geq 0$ and $\lambda > 0$ and a random variable X such that

$$W \overset{\mathcal{D}}{\leq} t_\lambda + X$$

where $X \overset{\mathcal{D}}{=} Exp(\lambda)$, and therefore we have

$$\begin{aligned} \mathbb{E}[\, e^{4\alpha W}\,]^2 &\leq \mathbb{E}[\, e^{4\alpha(t_\lambda+X)}\,]^2 \\ &\leq e^{8\alpha\, t_\lambda}\, \mathbb{E}[\, e^{4\alpha X}]^2. \end{aligned}$$

Denote the last expression with $K(\alpha)$, and observe that it is finite for sufficiently small $\alpha > 0$. By conditioning recursively on $\mathcal{F}_{j-1}, \mathcal{F}_{j-2}, \dots, \mathcal{F}_0$ and reasoning as above, we realize that

$$\mathbb{E}\Big[\, e^{4\alpha \sum_{i=1}^{j+1} U_i}\Big] \leq K(\alpha)^{j+1}$$

The estimation of $\mathbb{E}[\, e^{\alpha(T_{env}+U_0)})\,]$ is straigthforward. Denote the initial distribution of N' with H'.

$$\begin{aligned} \mathbb{E}[\, e^{4\alpha(\,T_{env}+U_0\,)}] &= \mathbb{E}_{(\lambda,\pi)}\Big[e^{4\alpha\, T_{env}}\; \mathbb{E}_{(H,H')}\Big[\, e^{4\alpha\, U_0} \,\|\, T_{env}\Big]\,\Big] \\ &= \mathbb{E}_{(\lambda,\pi)}\Big[e^{4\alpha\, T_{env}} \mathbb{E}_{(H,H')}\Big[\; \mathbb{E}_{(H,H')}\Big[\, e^{4\alpha U_0} \,\|\, A_{T_{env}}, A_{T_{env}}\Big] \,\|\, T_{env}\;\Big]\,\Big] \\ &\leq \mathbb{E}_{(\lambda,\pi)}\Big[e^{4\alpha\, T_{env}}\; \mathbb{E}\Big[e^{\,4\alpha(t_\lambda+X)}\Big]\Big] \\ &\leq K(\alpha)\mathbb{E}_{(\lambda,\pi)}\Big[e^{4\alpha T_{env}}\Big] \end{aligned}$$

We proved in Section 6.3 that the number of required coupling attempts is dominated by a $Geo(\gamma)$-distribution. Therefore $\mathbb{P}(\tau \geq i) \leq (1-\gamma)^{i-1}$. To sum up, we have

$$\begin{aligned} &\mathbb{E}[\, e^{\alpha T}] \leq \mathbb{E}\Big[\, e^{\alpha(T_{env}+U_0)}\Big] + \\ &\quad + \lim_{n\to\infty} \sum_{j=0}^{n-1} \mathbb{E}\Big[\, e^{4\alpha(T_{env}+U_0)}\Big]^{1/4} \mathbb{E}\Big[\, e^{4\alpha \sum_{i=1}^{j+1} U_i}\Big]^{1/4} \mathbb{P}\,(\tau \geq j+1)^{1/2} \\ &\quad \leq K(\alpha)\, \mathbb{E}_{(\lambda,\pi)}\Big[e^{4\alpha T_{env}}\Big] + K(\alpha)\, \mathbb{E}_{(\lambda,\pi)}\Big[e^{4\alpha T_{env}}\Big] \sum_{j=0}^{\infty} K(\alpha)^{(j+1)/4}\,(1-\gamma)^{j/2} \\ &\quad \leq K(\alpha)\, \mathbb{E}_{(\lambda,\pi)}\Big[e^{4\alpha T_{env}}\Big] + K(\alpha)^2\, \mathbb{E}_{(\lambda,\pi)}\Big[e^{4\alpha T_{env}}\Big] \sum_{j=0}^{\infty} \big(K(\alpha)\,(1-\gamma)\big)^{j/4}. \end{aligned}$$

To see that there exists $\alpha > 0$ such that the right hand expression is finite, first observe that by Lemma 5 there exists $\beta > 0$ such that $\mathbb{E}_{(\lambda,\pi)}[e^{4\alpha T_{env}}] < \infty$ for all $0 < \alpha \leq \beta$. Since $K(\alpha) \searrow 1$ when $\alpha \searrow 0$, we can choose $\alpha > 0$ such that $\alpha < \beta$ and such that

$$K(\alpha)(1-\gamma) < 1$$

and the result follows. $\square$

8 Blackwell's theorem in a random environment

Let $\| \nu \|_B$ denote the total variation norm of ν restricted to B, where $\nu \in \mathcal{N}_+$ and $B \in \mathcal{B}(\mathbb{R}_+)$.

The basic result in classical renewal theory is Blackwell's Theorem, which states that

$$\mathbb{E}\Big[N(\,[\,t, t+A\,]\,)\Big] \to A \cdot \lambda \qquad \text{as } t \to \infty \tag{20}$$

for all $A > 0$, where N is a (possible delayed) renewal process with nonlattice lifelength distribution F, and with $\lambda = 1/\mu$, where $\mu = \mathbb{E}[Y_1]$ and $Y_1 \overset{\mathcal{D}}{=} F$. Under a stronger condition, we have

$$\| \, \theta_t \, M - \lambda \cdot \ell \, \|_{[\,0,A\,]} \to 0 \qquad \text{as } t \to \infty \tag{21}$$

where M is the expectation measure giving mass $\mathbb{E}[N(B)]$ to $B \in \mathcal{B}(\mathbb{R}_+)$, and ℓ is the Lebesgue measure.

We prove similar results for RPRE processes, using proof techniques and results taken from [8], in particular from the section 1.6 concerning regenerative random measures. The notation M for the expectation measure of an RPRE process $V = (N, Z)$ will be used. In this section we say that a coupling of two RPRE processes exists, if a coupling exists in the sense of (2) in Section 2.5.

Theorem 11 *Let $V = (N, Z)$ and $V' = (N', Z')$ be RPRE process of the same type, such that a successful coupling exists, with coupling time T. Then*

$$\| \, \theta_t \, M - \theta_t \, M' \, \|_{[0,A]} \to 0 \;\; \textit{as } t \to \infty,$$

for some finite constant $C > 0$ and for all $A > 0$. If $\mathbb{E}[T] < \infty$ then

$$\| \, \theta_t \, M - \theta_t \, M' \, \| \to 0 \;\; \textit{as } t \to \infty$$

Proof: The essential parts of the proof can be found in [8]. Fix a constant $A > 0$ and let $B \in \mathcal{B}([\,0, A\,])$. Observe that $N(t + B) \cdot I(T \leq t) \overset{\mathcal{D}}{=} N'(t + B) \cdot I(T \leq t)$. Hence

$$\Big| \, \mathbb{E}[\,N(t+B)\,] - \mathbb{E}[\,N'(t+B)\,] \, \Big| =$$

$$\Big| \mathbb{E}[\, N(t+B) \cdot I(T>t)] - \mathbb{E}[\, N'(t+B) \cdot I(T>t)] \Big| \leq$$

$$\Big| \mathbb{E}[\, (N(t+B) + N'(t+B)) \cdot I(T>t)] \Big| \leq$$

$$2 \cdot \sup_{H,\lambda} \mathbb{E}_{(H,\lambda)}[\, N([\,0,A\,])\,] \cdot \mathbb{P}(T>t)$$

where the supremum is taken over all possible initial age- and environment-state distributions. But

$$\begin{aligned} \sup_{H,\lambda} \mathbb{E}_{(H,\lambda)}[\, N([\,0,A\,])\,] &\leq \sup_{H} \mathbb{E}_H[\, \tilde{N}([\,0,A\,])\,] \\ &= \mathbb{E}_0[\, \tilde{N}([\,0,A\,])\,] < \infty \end{aligned}$$

where $\tilde{N}$ is a standard renewal process with lifelength distribution $F = F^{(M)}$, the distribution associated to the environment state M as before. (The inequality can be proved by a coupling argument using Poisson embedding, and the finiteness of the right-hand expectation should be obvious.) This yields

$$\| \, \theta_t \, M - \theta_t \, M' \, \|_{[\,0,A\,]} \leq 2 \cdot C \cdot \mathbb{P}(T>t) \to 0,$$

as $t \to \infty$, since $T < \infty$ a.s. Now, suppose that $\mathbb{E}[T] < \infty$. Then

$$\begin{aligned} \| \, \theta_t \, M - \theta_t \, M \, \|_{[0,A]} &\leq \sum_{n=0}^{[A]} \| \, \theta_{t+n} \, M - \theta_{t+n} \, M' \, \|_{[0,1]} \\ &\leq 2 \cdot C \sum_{n=0}^{[A]} \mathbb{P}(T > t+n) \end{aligned}$$

which gives

$$\begin{aligned} \| \, \theta_t \, M - \theta_t \, M' \, \| &= \lim_{A \to \infty} \| \, \theta_t \, M - \theta_t \, M' \, \|_{[0,A]} \leq \\ &= 2 \cdot C \sum_{n=0}^{\infty} \mathbb{P}(T > t+n) \\ &\leq 2 \cdot C \sum_{n=[t]}^{\infty} \mathbb{P}(T > n) \to 0 \text{ as } t \to \infty \end{aligned}$$

since $\sum_{n=0}^{\infty} \mathbb{P}(T>n) < \infty$. □

In order to see that random environment versions of (21) holds, we once again use theory for regenerative processes. As before, for an RPRE process

$V = (N, Z)$, denote the embedded regenerative process with $X_t = (A_t, Z_t)$, the embedded renewal process with $(T_n)_{n=0}^{\infty}$ and the cycle length distribution with F. If we let

$$V^0 = (N^0, Z^0) = (\theta_{T_0} N, \theta_{T_0} Z)$$

so that the embedded regenerative process is zero-delayed, the following holds:

Proposition 5 *If $V' = (N', Z')$ is a stationary version of V, then*

$$M' = c \cdot \ell_+$$

where c, the equilibrium intensity, is given by

$$c = \frac{1}{m(F)} \mathbb{E}\Big[\int_0^{T_0} N^0[t, t+1) dt\Big]$$

Proof. A proof is given in [8]. □

By using our rate results from Section 7 we come to the following conclusions.

Corollary 4 *If $V = (N, Z)$ is within the IFR class, then there exists an $\alpha > 0$ such that*

$$\| \theta_t M - c_1 \cdot \ell \| = o(e^{-\alpha t}). \tag{22}$$

If $V = (N, Z)$ is within the DFR class such that $m_{\alpha+1}(F^{(0)}) < \infty$ holds for an $\alpha > 1$, then

$$\| \theta_t M - c_2 \cdot \ell \| = o(t^{-(\alpha-1)}), \tag{23}$$

and if $F^{(0)}$ has exponential moments then

$$\| \theta_t M - c_3 \cdot \ell \| = o(e^{-\alpha t}) \tag{24}$$

for some $\alpha > 0$. The constants c_1, c_2 and c_3 is determined by Proposition 5.

Proof. Once again, the essential steps in the proof is taken from [8]. Under the assumptions, $\mathbb{E}[T] < \infty$ holds, proved in Section 7. The proof of the second statement in Theorem 11 reveals that

$$\| \theta_t M - \theta_t M' \| \leq 2 \cdot C \sum_{n=[t]}^{\infty} \mathbb{P}(T > n).$$

To estimate the rate of convergence towards zero of the last term, we use the following result. Suppose that $g(\cdot)$ is a non-negative function with an increasing derivative, and that $\mathbb{E}[\,g(T)\,] < \infty$. Then $\mathbb{E}[\,g(T+2)\,] < \infty$ holds, which is equivalent to

$$\int_0^\infty g'(x)\mathbb{P}(T > x-2)\,dx < \infty$$

by Fubini's Theorem. Hence

$$\begin{aligned} g'(t)\sum_{n=[t]}^{\infty}\mathbb{P}(T>n) &\le g'(t)\cdot\int_t^\infty \mathbb{P}(T>x-2)dx \\ &\le \int_t^\infty g'(x)\mathbb{P}(T>x-2)dx \to 0 \end{aligned}$$

as $t \to \infty$, i.e.,

$$\sum_{n=[t]}^{\infty}\mathbb{P}(T>n) = o(\frac{1}{g'(t)}).$$

To see that (22) holds, let $g(x) = e^{\alpha x}$ and recall Theorem 10 where exponential moments of T in the IFR case is proved. The statement (24) follows in the same way from Theorem 8 and (23) follows from Theorem 9 with $g(x) = x^\alpha$ $\Box$.

Remark 1 *Theorem (11) implies of course a RPRE version of (20), i.e., that* $\mathbb{E}[N([t, t+a])] \to c\cdot a$ *when* $t \to \infty$.

9 Simulations

In this section we describe two techniques which enable us to perform exact simulations of RPRE processes with unbounded failure rates. Furthermore, we present some simulations. As before we concentrate ourselves to the IFR and DFR case, though the case with general failure rates should cause no further complications. (?)

9.1 Exact simulations in the IFR case

Consider first the case when the failure rates are bounded by a constant $C > 0$. A simulation is then straightforward; it is only to implement the Poisson embedding technique described in Section 2. As well known, one can simulate a two dimensional Poisson process ξ in $[0,t]\times[0,C]$ with expectation measure ℓ_+ by first simulating a one-dimensional Poisson process on $[0,t]$ with intensity C (with points at $S_0, S_1, \dots, S_K$, for some $K \geq 0$) and then assigning to each point S_n a random variable I_n uniformly distributed on $[0,C]$. The I_n variables must of course be independent of the S_n's. Then

$$\{(S_n, I_n);\ 0 \leq n \leq K\}$$

is the desired two dimensional Poisson process.

Now, suppose that at least one of the failure rates are unbounded. Then the method described above has to be modified, since if one of the lifelengths survives sufficiently long, its failure rate will exceed C and causes an error. Even though such a simulation probably can be done good enough for the most practical purposes, it is not an exact simulation. One way of handle this problem is to successively 'enlarge' the state space of ξ until the failure rate 'collapses', i.e., until we encounter a Poisson point under the failure rate. This can be carried out step by step, with the state space enlarged in each step until the failure rate collapses. The number of required steps is dominated by a geometrically distributed random variable.

To be more exact, pick a number q satisfying $0 < q < 1$, and let W_a denote a random variable distributed according to $F_a^{(0)}$. We want to simulate a lifelength Y with initial age a, in the random environment Z. Define

$$x_a = \inf\{x \geq 0;\ R_a^{(0)}(x) \geq \log(1/q)\,\}, \tag{25}$$

and let

$$C_a = r_a^{(M)}(x_a). \tag{26}$$

Then

$$\mathbb{P}(W_a > x_a) = e^{-R_a^{(0)}(x_a)} \leq q.$$

The lifelenght Y has failure rate $r_a^{(Z_x)}(x)$ for $x \geq 0$, which certainly satisfies

$$r_a^{(0)}(x) \leq r_a^{(Z_x)}(x) \leq r_a^{(M)}(x) \text{ a.s.}$$

for all $x \geq 0$, and hence

$$\mathbb{P}(Y > x_a) \leq \mathbb{P}(W_a > x_a) \leq q.$$

We use the Poisson embedding algorithm described above, using a two-dimensional Poisson process in $[0, x_a] \times [0, C_a]$. Two things can happen. Either we encounter a point (S_n, I_n) in $[0, x_a] \times [0, C_a]$ such that

$$I_n \leq r_a^{(Z_{S_n})}(S_n)$$

and hence $Y_0 \leq x_a$. Then we are finished. Or the lifelength survives the first x_a time units, forcing us to take measures if we want to avoid an error. With a new initial age $b = a + x_a$ in the definitions (25) and (26) above, we get x_b and C_b satisfying

$$C_a \leq C_b$$

and

$$\mathbb{P}(Y > x_a + x_b \| Y > x_a) \leq \mathbb{P}(W_b > x_b) \leq q.$$

We continue the embedding, now using a Poisson process in $(x_a, x_a + x_b] \times [0, C_b]$, i.e., we use now an 'enlarged' Poisson process. Once again, the failure rate will collapse in $(x_a, x_a + x_b]$ with a probability at least q, and if not, continue as described above until it happens.

After this procedure, the next lifelength can be generated in the same way; then, in the first step, we use x_0 and C_0 defined as in (25) and (26) with initial age 0, and a Poisson process in $(Y, Y + x_0] \times [0, C_0]$.

The following simulations are done with this algorithm, implemented in Matlab.

9.2 Exact simulations in the DFR case

We saw that when the failure rates are increasing and bounded, we still can implement the Poisson embedding algorithm. That is not possible in the case with unbounded decreasing failure rates. To overcome this problem simulations can be made by using the distributions $F^{(0)}, \dots, F^{(M)}$.

We want to simulate a lifelength Y with no initial age, in the environment Z starting with $Z_0 = i$, say. The problem is to handle the possible singularity of $r^{(i)}(x)$ at $x = 0$. Since Z is a birth and death process, we know that the times between jumps are exponentially distributed. Therefore, we can first simulate an exponential holding time η_0, the time until the next environment change. Then we use the distribution function

$$F^{(i)}(x) = 1 - e^{-R^{(i)}(x)}$$

together with some standard algorithm to generate an observation W_1 on $F^{(i)}$. If $W_1 < \eta_0$ then we are finished: let

$$Y = \eta_0.$$

Otherwise, we have two choices. Either we simulate the next environment state, j say, a new holding time η_1 and an observation W_2 of the distribution $F^{(j)}_{\eta_0}$. We let

$$Y = \eta_0 + W_2$$

if $W_2 < \eta_1$, and if not, we continue with the next holding time and so on. Our other choice is to continue with Poisson embedding in the usual way since all the failure rates are bounded from η_0 and onwards. (But then, if we want to generate another lifelength after Y, we must of course once again handle the singularities by repeating the method above.)

Below we present some simulations. Here we have used the straightforward simulation technique since we only simulated RPRE processes with bounded failure rates. The programs are written in Matlab.

10 Some comments

We have so far investigated aspects like asymptotics, stochastic domination and stochastic monotonicity, under (for us) suitable model assumptions. There are however other aspects and problems not mentioned in this paper. Furthermore, the model has (of course) limitations, which raise the question of possible generalizations of the model. These aspects are briefly discussed in the following.

We have chosen to work entirely with a random enviroment on a *finite* state space S, a model assumption keeping certain technicalities on a moderate level. Obvious extentions would be to have a infinite S state space, countable or uncountable.

One other aspect is the following. Suppose that we model the failure process of a certain component of a machine with a RPRE process. Given that a component has survived s time-units (at time t_0 say), its failure intensity is determined by s and the random enviroment state Z_{t_0}, but it will be totally independent of $\{Z_t\,;\,(t_0 - s) \leq t < t_0\}$. Whether that component was exposed for the worst or the best possible enviroment state under its first s time-units is irrelevant, i.e., the model exhibits no *wear*.

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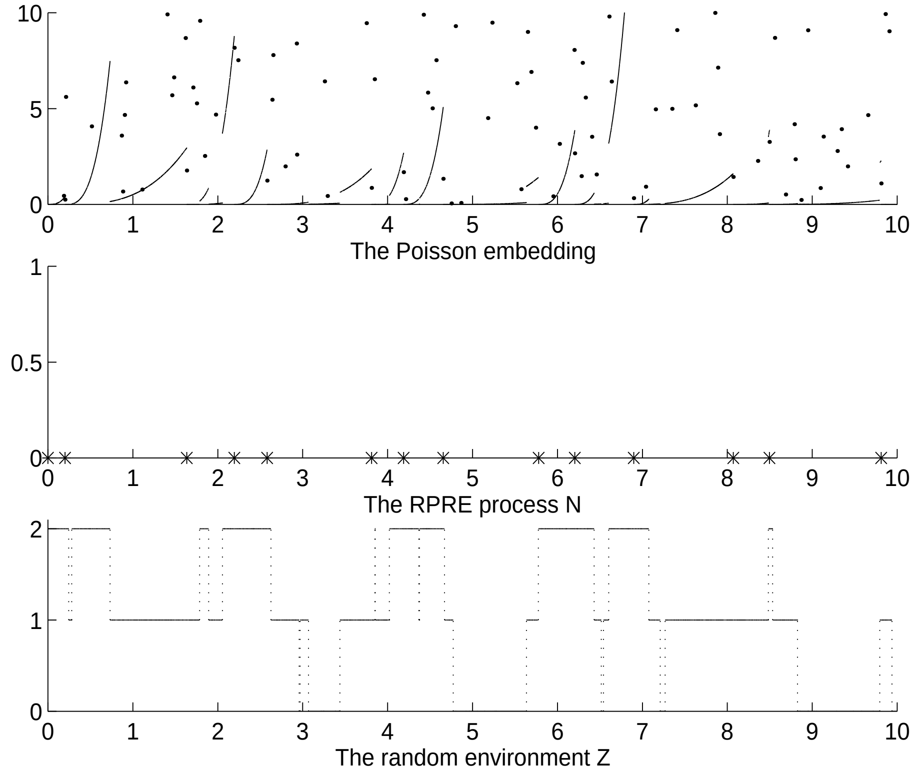
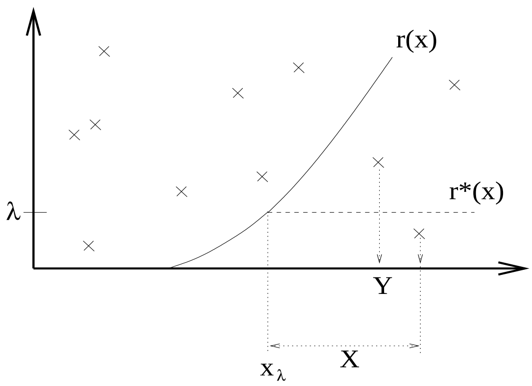


Figure 3: *A Matlab simulation of a three state RPRE process within the IFR class. The failure rates:* $r^{(0)}(x) = 0.01{\cdot}x^3$, $r^{(1)}(x) = {\cdot}x^3$ *and* $r^{(2)}(x) = 50{\cdot}x^3$, *for* $x \geq 0$. *The initial age was* 0, *and* $Z_0 = 2$, *the birth intensities:* $(2, 1, 0)$; *the death intensities:* $(0, 1, 2)$.



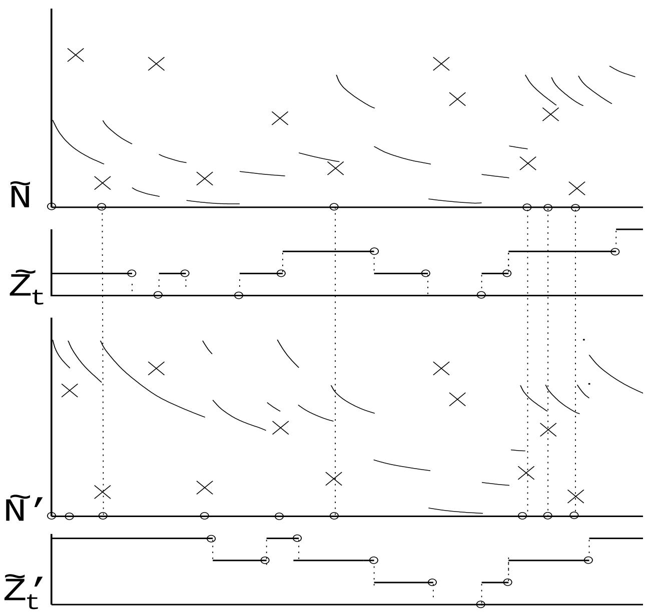


Figure 4: *The construction in the proof of Theorem 1 renders* $\tilde{N} \preceq \tilde{N}'$ *a.s.*

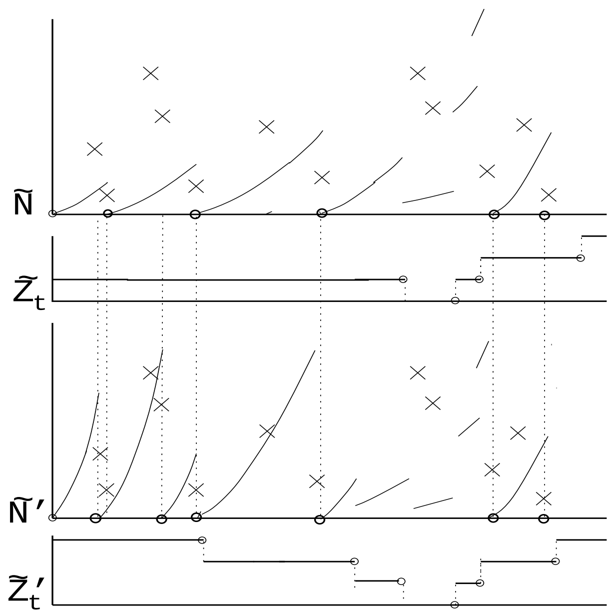


Figure 5: *The alternating structure in the IFR case, which gives* $\tilde{N}_t \leq \tilde{N}'_t$ *for all* $t \geq 0$ *a.s.*

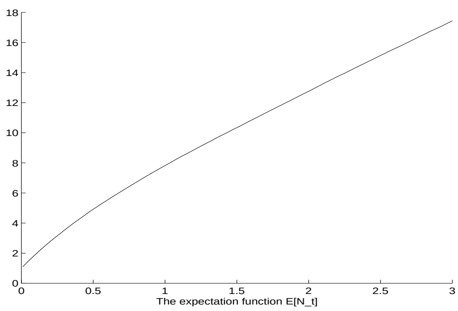


Figure 6: *A Matlab simulation of* $m(t)$*, when* N *is zero-delayed, with a two state environment process starting with* $Z_0 = 1$*. The birth and death intensities are* $\beta_0 = 1, \beta_1 = 0, \delta_0 = 0$ *and* $\delta_1 = 1$*, and the associated failure rates are* $r^{(0)}(x) = 0.01 \cdot (1+x)^{-1}$ *and* $r^{(1)}(x) = 10 \cdot (1+x)^{-0.5}$ *for all* $x \geq 0$*. The simulation is the average of 10000 realizations of* N_t*, for* $0 \leq t \leq 3$*.*

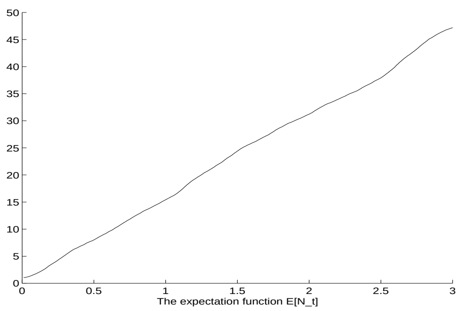


Figure 7: *A Matlab simulation of $m(t)$, when N is zero-delayed, with a two state environment process starting with $Z_0 = 0$. The birth and death intensities are $\beta_0 = 0.125, \beta_1 = 0, \delta_0 = 0$ and $\delta_1 = 8$, and the associated failure rates are $r^{(0)}(x) = 1$ and $r^{(1)}(x) = 1000$ for all $x \geq 0$. The simulation is the average of 1000 realizations of N_t, for $0 \leq t \leq 3$.*

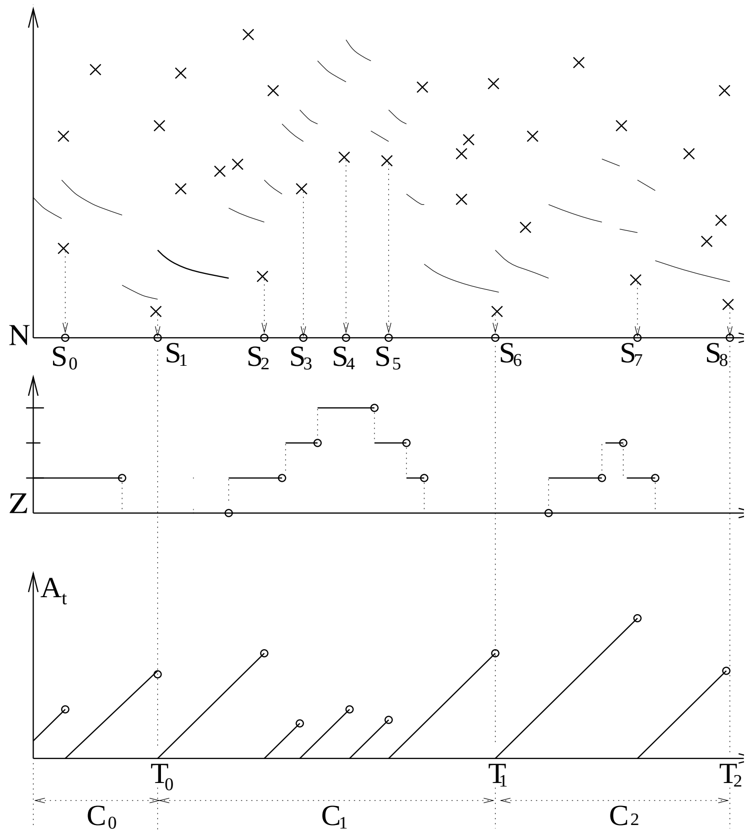


Figure 8: *The embedded regenerative process $X = (A, Z)$ and the embedded renewal process $(T_n)_{n=0}^{\infty}$.*

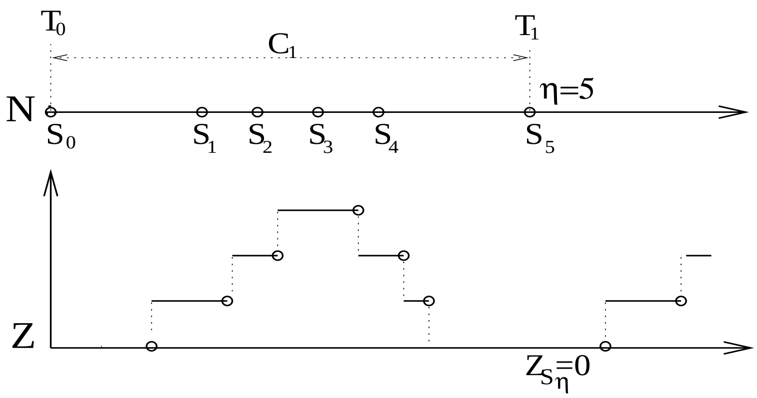


Figure 9:

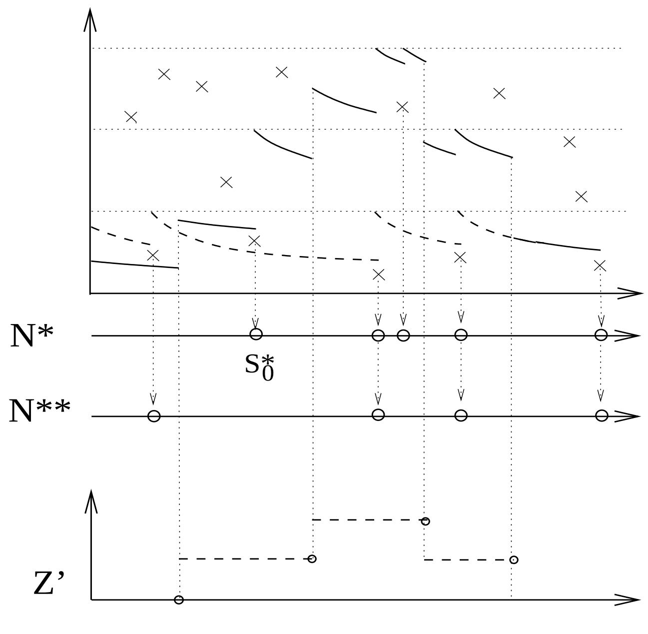


Figure 10: *The coupling* (N^*, N^{**}) *of* (N', N'').

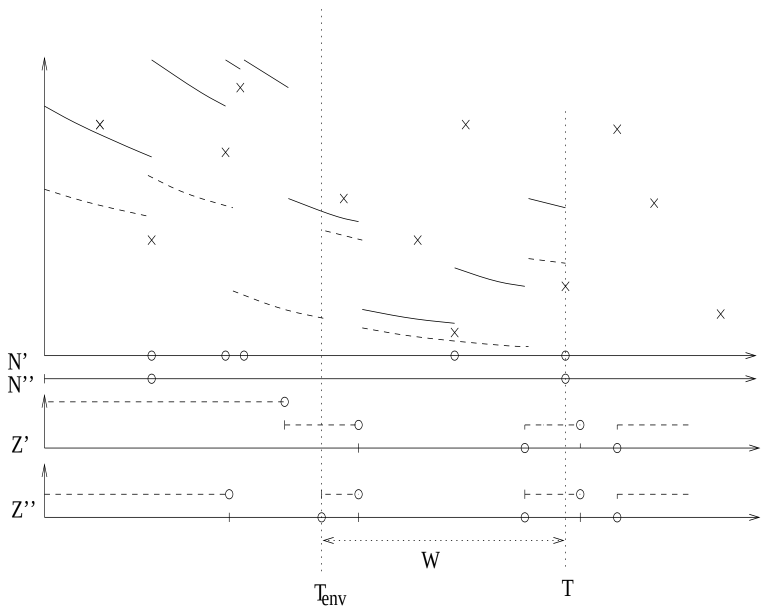


Figure 11: *The DFR case. The coupling time* $T = T_{env} + \max\{D'_{T_{env}}, D''_{T_{env}}\}$

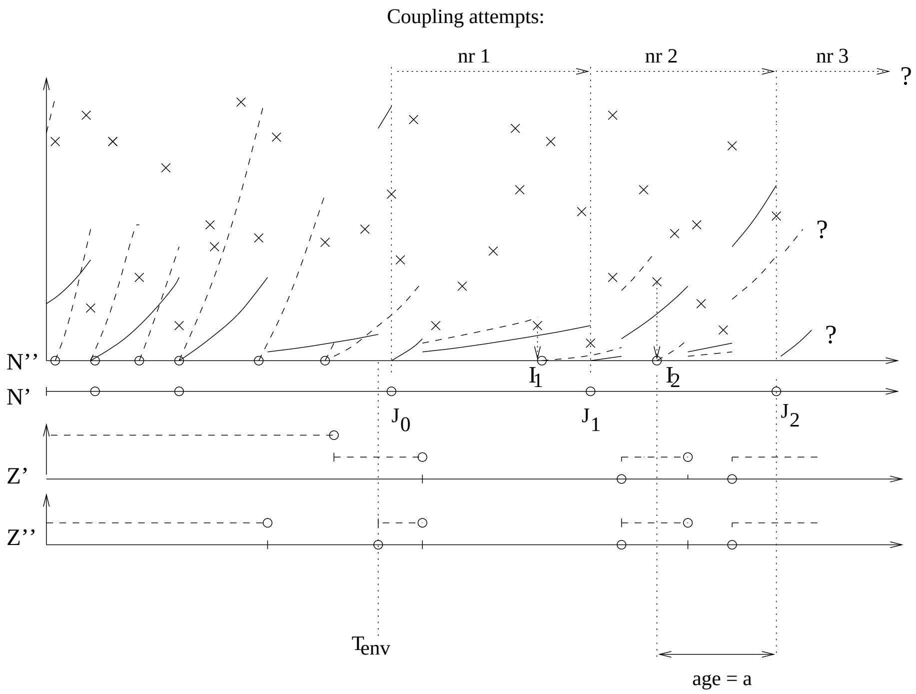


Figure 12: *Coupling in the IFR case.*

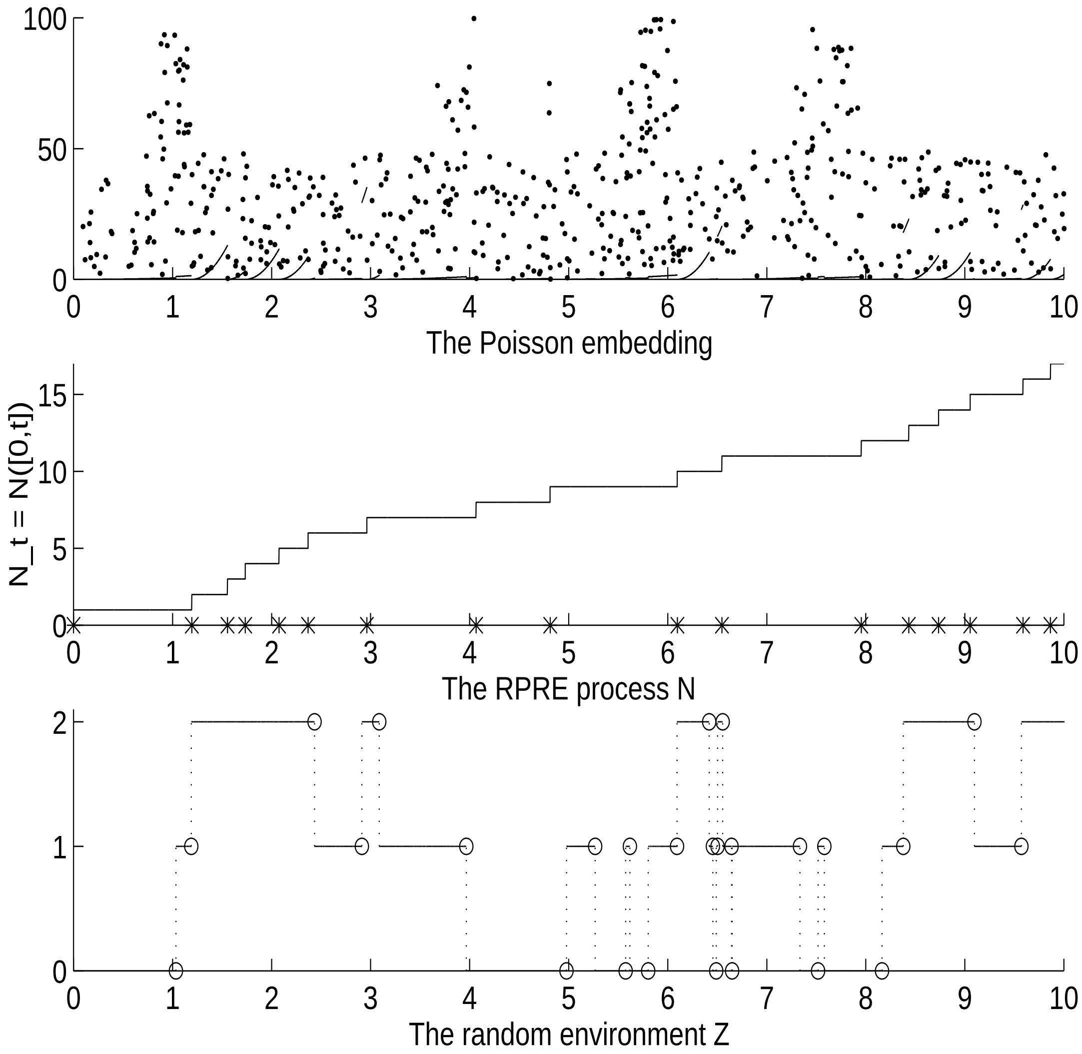


Figure 13: *A exact Matlab simulation of a three state RPRE process of IFR type. The failure rates were* $r^{(0)}(x) = \frac{1}{2} \cdot x^{1.1}$, $r^{(1)}(x) = 5 \cdot x^2$, *and* $r^{(1)}(x) = 100 \cdot x^3$, *and* $Z_0 = 1$.

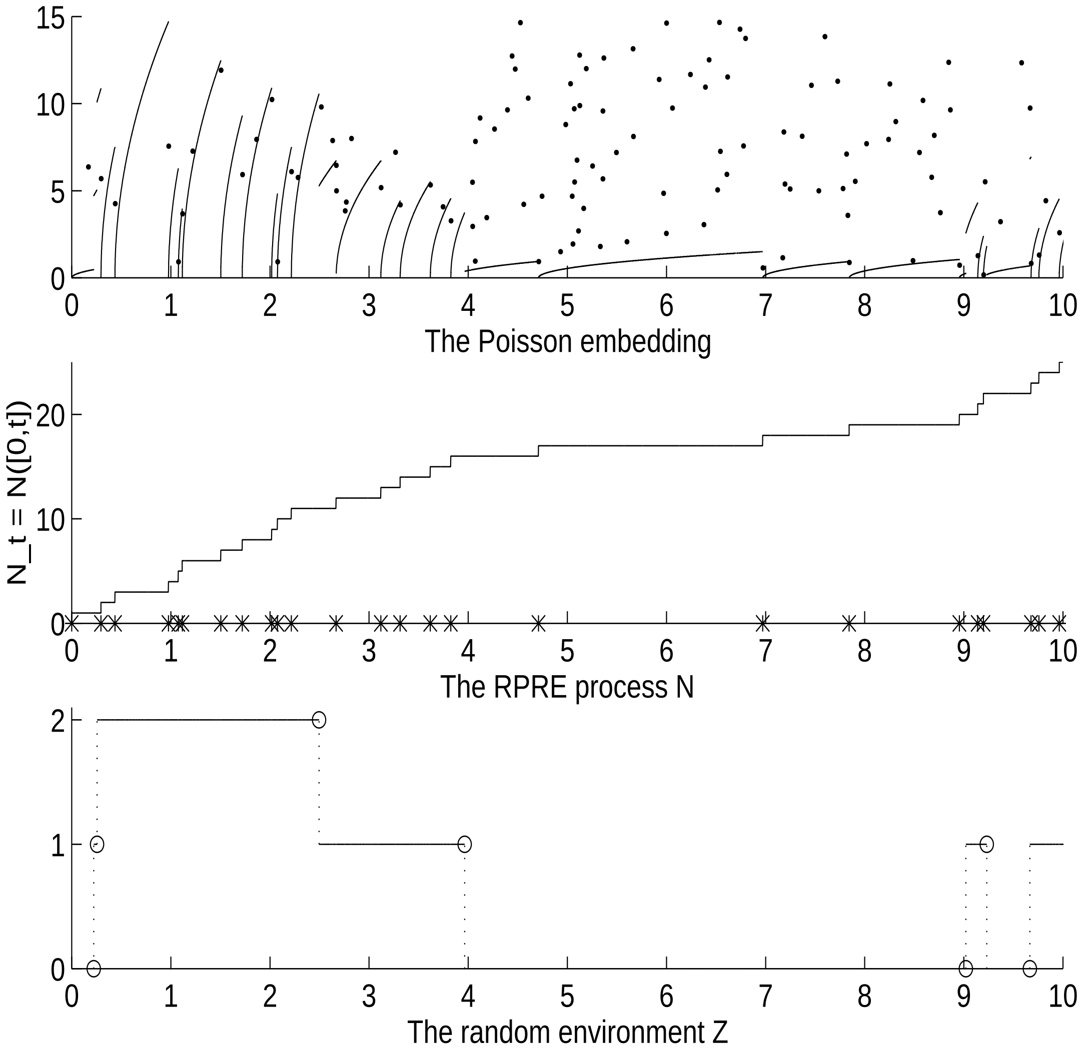


Figure 14: *A two state RPRE process. The failure rates were* $r^{(0)}(x) = \frac{1}{2} \cdot x^{1.1}$, $r^{(1)}(x) = 5 \cdot x^3$. *The initial age was* 2.5, *and* $Z_0 = 1$.

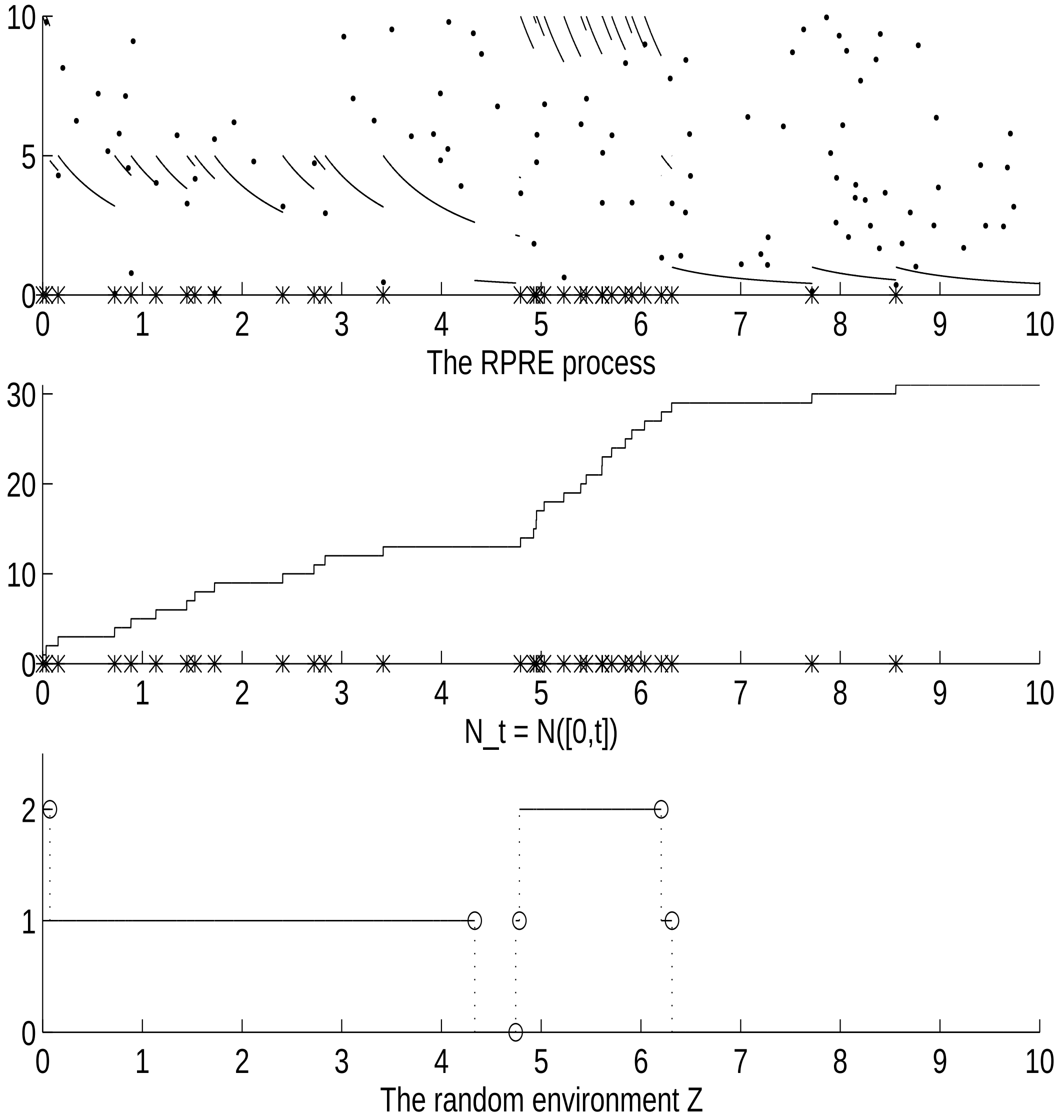


Figure 15: *A three state RPRE process of DFR type. The failure rates were* $r^{(0)}(x) = \frac{1}{1+x}$, $r^{(1)}(x) = \frac{5}{1+x}$, *and* $r^{(2)} = \frac{10}{1+x}$. *The initial age was* 0, *and* $Z_0 = 2$.

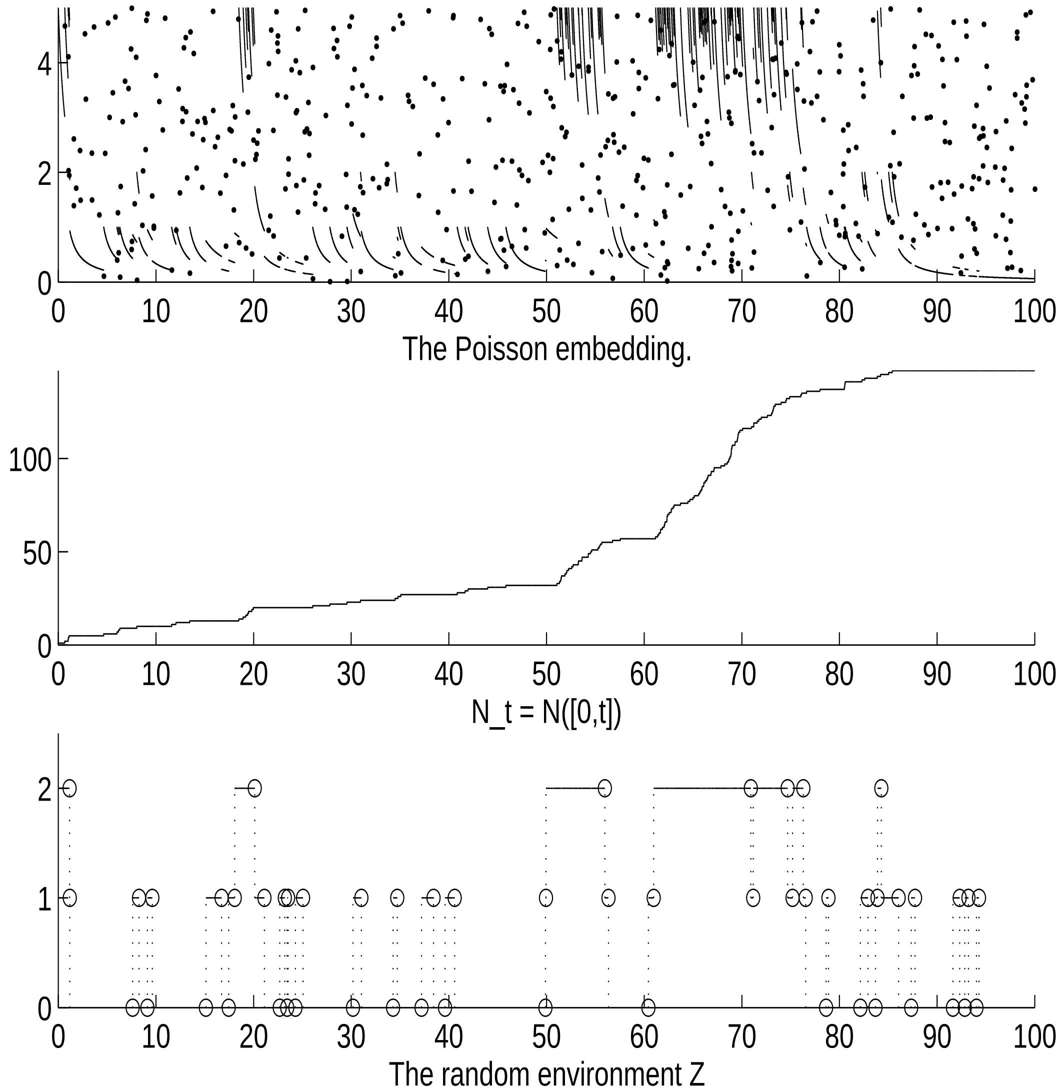


Figure 16: *A three state RPRE process of DFR type. The failure rates were* $r^{(0)}(x) = \frac{1}{1+x}$, $r^{(1)}(x) = \frac{2}{1+x}$, *and* $r^{(2)} = \frac{5}{1+x}$. *The initial age was* 0, *and* $Z_0 = 2$.

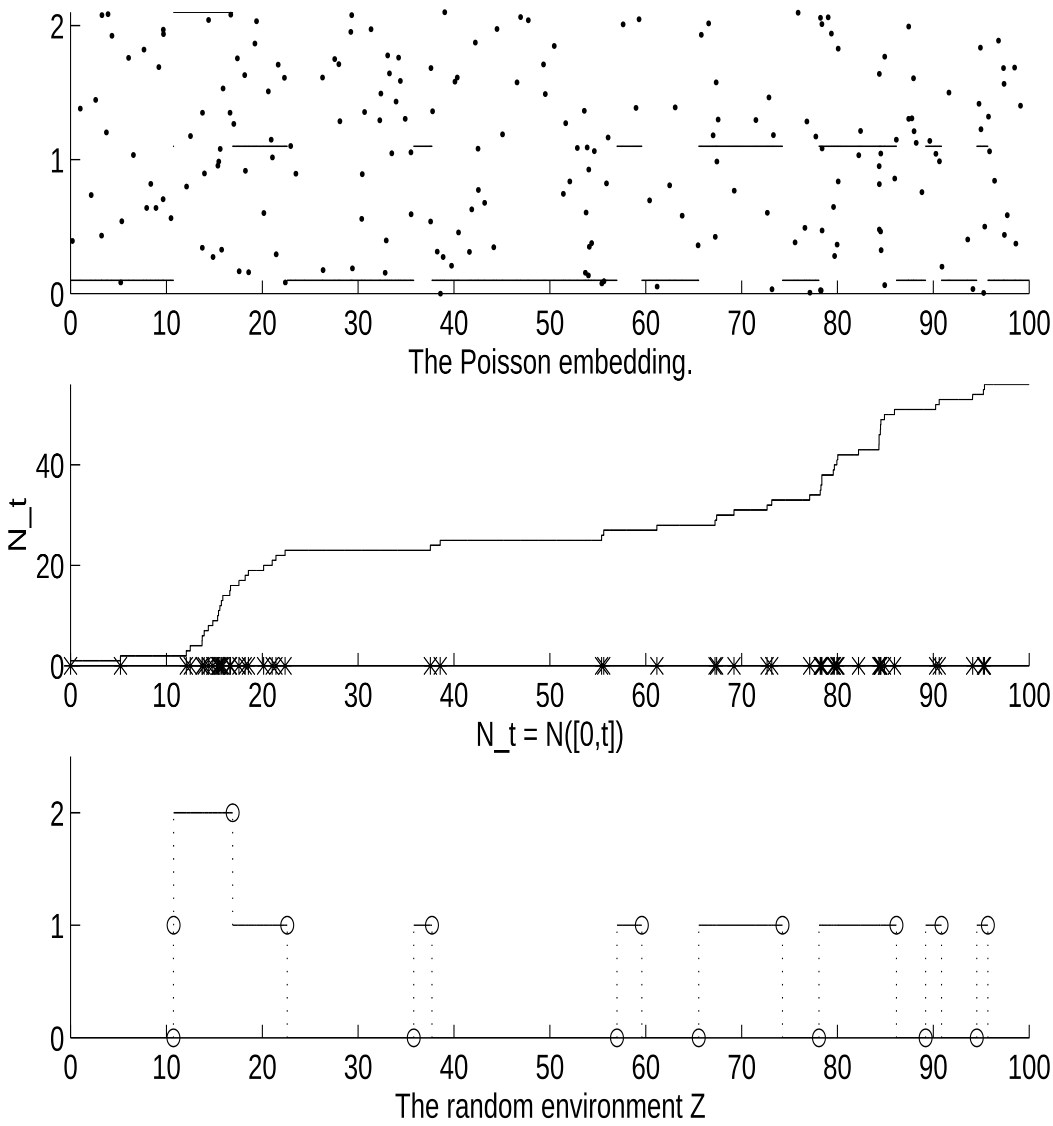


Figure 17: *A Cox process. The failure rates were* $r^{(0)}(x) = 0.1$*,* $r^{(1)}(x) = 1.1$*, and* $r^{(2)} = 2.1$*. The initial age was* 0*, and* $Z_0 = 0$*, the birth intensities was* $(0.1, 0.05, 0)$ *and the death intensities* $(0, 0.05, 0.1)$