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Approach regions for the square root of the Poisson kernel and weak L^p boundary functions

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APPROACH REGIONS FOR THE SQUARE ROOT OF THE POISSON KERNEL AND WEAK L^p BOUNDARY FUNCTIONS.

MARTIN BRUNDIN

ABSTRACT. Let $P_0 f(z) = \int_{\mathbb{T}} \sqrt{P(z,\beta)} f(e^{i\beta})\, d\beta$ for $f \in L^1(\mathbb{T})$, where $P(z,\beta)$ is the Poisson kernel in the unit disc. In this paper we consider the convergence properties of the normalised operator $P_0 f / P_0 1$. We give a complete characterisation of the natural approach regions along which one has almost everywhere convergence for weak L^p boundary functions, $1 < p < \infty$.

1. INTRODUCTION

This paper is divided into four main sections. In this one, Introduction, we introduce the problem and discuss related results. The essence of the second section, Preliminaries, is to prove a converse Banach principle for $L^{p,\infty}$ needed to prove the main result, Theorem 2, which is done in Section 3. Section 4 concludes the paper with a brief discussion of open questions.

Let $P(z,\beta)$ be the standard Poisson kernel in the unit disc U,

$$P(z,\beta) = \frac{1}{2\pi} \cdot \frac{1 - |z|^2}{|z - e^{i\beta}|^2}$$

where $z \in U$ and $\beta \in \mathbb{T} = \partial U \cong \mathbb{R}/2\pi\mathbb{Z}$. Note that

$$P(z,\beta) = \frac{1}{2\pi} \cdot \mathrm{Re} \left(\frac{e^{i\beta} + z}{e^{i\beta} - z} \right),$$

so the mapping $z \mapsto P(z,\beta)$, being the real part of a holomorphic function, is harmonic.

We shall in what follows consider boundary functions, i.e. functions with domain $\mathbb{T}$. In that context we identify $\mathbb{T}$ with $(-\pi, \pi]$, i.e. we write $f(\theta)$ instead of $f(e^{i\theta})$.

Now, let

$$Pf(z) = \int_{\mathbb{T}} P(z,\beta) f(\beta)\, d\beta,$$

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the Poisson integral of $f \in C(\mathbb{T})$. Then $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$. This was shown in 1872 by Schwarz [HAS], and it is considered a well known result today.

For any function $h : \mathbb{R}_+ \to \mathbb{R}_+$ let

$$\mathcal{A}_h(\theta) = \{z \in U : |\arg z - \theta| \le h(1 - |z|)\}. \tag{1}$$

We refer to $\mathcal{A}_h(\theta)$ as the (natural) approach region determined by h at $\theta \in \mathbb{T}$. This is the only form of approach regions that we will be concerned with, throughout the thesis. However, we point out that there are other approach regions, defined in different manners. The Nagel-Stein approach regions are examples of this (see [NS]). The word "region" is usually used only when the set (region) in question is open. However, we shall use it in a wider sense, with no openness assumptions.

Now, if we only assume that $f \in L^1(\mathbb{T})$, the convergence properties are different than in the case of continuous functions. If $h(t) = \alpha t$, $\alpha > 0$, then $Pf(z) \to f(\theta)$ a.e. as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, i.e. the convergence is non-tangential. This is proved by showing that the corresponding maximal operator is dominated by the Hardy-Littlewood maximal operator, which is of weak typ $(1, 1)$. Then the result follows via approximation with continuous functions. This result was first shown by Fatou [F] in 1906. Littlewood [L] proved that the theorem, in a certain sense, is best possible.

For a more complete treatise on the concepts and theorems mentioned so far, see Di Biase [DB].

For $z = x + iy$ let

$$L_z = \frac{1}{4}(1 - |z|^2)(\partial_x^2 + \partial_y^2),$$

the hyperbolic Laplacian. Then

$$u(z) = P_\lambda f(z) = \int_{\mathbb{T}} P(z, \beta)^{\lambda + 1/2} f(\beta)\, d\beta, \text{ for } \lambda \ge 0,$$

defines a solution of the equation

$$L_z u = (\lambda^2 - 1/4)u.$$

The powers $P(z, \beta)^{\lambda+1/2}$, $\lambda \ge 0$, of the Poisson kernel are used in connection with representation theory of the group $SL(2, \mathbb{R})$.

One can show that

$$P_\lambda 1(z) \sim (1 - |z|)^{1/2 - \lambda}$$

if $\lambda > 0$, and that

$$P_0 1(z) \sim (1 - |z|)^{1/2} \log \frac{1}{1 - |z|},$$

where $f \sim g$ means that there exists a constant $c > 0$ such that $c^{-1} \leq f/g \leq c$. This thesis is concerned with convergence properties of the square root of the Poisson kernel ($\lambda = 0$) and boundary functions $f \in L^{p,\infty}$ (weak L^p). To get boundary convergence we have to normalise P_0, since $P_0 1(z)$ does not converge to 1:

$$\mathcal{P}_0 f(z) = \frac{P_0 f(z)}{P_0 1(z)}.$$

We point out that if one considers normalised $\lambda-$Poisson integrals for $\lambda > 0$, i.e. $\mathcal{P}_\lambda f(z) = P_\lambda f(z)/P_\lambda 1(z)$, the convergence properties are the same as for the ordinary Poisson integral. This is because the kernels behave essentially in the same way.

If $f \in C(\mathbb{T})$ then $\mathcal{P}_0 f(z) \to f(\theta)$ unrestrictedly as $z \to e^{i\theta}$ for all $\theta \in \mathbb{T}$, just as in the case of the Poisson integral itself. This is because $\mathcal{P}_0$ is a convolution operator with a kernel being an approximate identity in $\mathbb{T}$. Moreover, convergence results are known for $f \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$. The case $p = 1$ was solved by Sjögren [PS1]:

Theorem (Sjögren, 1984)**.** *Let $f \in L^1(\mathbb{T})$. For a.e. $\theta \in \mathbb{T}$ one has $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, where $h(t) = \alpha t |\log t|$ for any fixed $\alpha > 0$.*

This result was generalised to L^p, $1 \leq p < \infty$, by Rönning [JOR]:

Theorem (Rönning, 1992)**.** *Let $1 \leq p < \infty$ be given and let $f \in L^p(\mathbb{T})$. For a.e. $\theta \in \mathbb{T}$ one has $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, where $h(t) = \alpha t |\log t|^p$ for any fixed $\alpha > 0$.*

Rönning also proved that Sjögren's result is best possible, when the approach regions are given in the form (1) and h is increasing, and that in his own theorem for L^p the exponent p in $h(t) = \alpha t (\log 1/t)^p$ cannot be improved.

The method used in the proof of Rönning's result was a weak type estimate for the corresponding maximal operator. The continuous functions, for which convergence is known to hold, are dense in L^p, and a standard approximation argument together with the weak type estimate then proves the theorem.

The case of $f \in L^\infty$ turned out to be different. Since the continuous functions are not dense in this space, the weak type estimate approach would be inadequate. However, using a result by Bellow and Jones [B-J], Sjögren [PS2] managed to determine the approach regions:

Theorem (Sjögren, 1997)**.** *The following are equivalent for any increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any* $f \in L^\infty(\mathbb{T})$ *one has for almost all* $\theta \in \mathbb{T}$ *that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $h(t) = O(t^{1-\varepsilon})$ *as* $t \to 0$ *for any* $\varepsilon > 0$.

Actually, the assumption that h should be increasing is never invoked in the proof. Thus, this result determines *all* admissible approach regions for L^∞ when given in the form (1). Note also that these approach regions are strictly wider than the ones in the case of finite p (as anticipated, since $L^\infty \subset L^p$ for all $p \geq 1$).

Basically, the Bellow-Jones result for L^∞ states that a.e. convergence is equivalent to continuity of the maximal operator at 0, when restricted to the unit ball in L^∞, in the topology of convergence in measure. Thus, what Sjögren had to show was that if $\|f\|_\infty \leq 1$ then for all $\varepsilon > 0$ and all $\kappa > 0$ there exists $\delta > 0$ such that

$$\|f\|_1 < \delta \Rightarrow |\{\theta \in \mathbb{T} : Mf(\theta) > \varepsilon\}| < \kappa,$$

where M denotes the relevant maximal operator. (It is easy to see that, in the unit ball in L^∞, the topology of convergence in measure is equivalent with the L^1-topology.) This led to a kind of optimization problem, where the constraint basically was $\|f\|_\infty \leq 1$.

As in the case of L^∞, the continuous functions are not dense in $L^{p,\infty}$. To solve this, we shall extend the Bellow-Jones result to cover functions in $L^{p,\infty}$, and by doing so we may adopt the approach used by Sjögren. The significant difference, of course, is that $L^{p,\infty}$ contains significantly "wilder" functions than L^∞ does.

Our main result, Theorem 2, shows that a convergence result similar to Sjögren's holds in case of $L^{p,\infty}$ boundary functions. We shall prove that $\mathcal{A}_h(\theta)$ is an admissible approach region for almost every $\theta \in \mathbb{T}$ if, and only if,

$$\sum_{k=0}^{\infty} c_k 2^{(1-p)k} < \infty, \tag{2}$$

where $c_k = \sup_{2^{k-1} \leq s \leq 2^k} h(2^{-s}) 2^s s^{-1}$.

We conclude this section with a couple of equivalent ways of writing the series condition (2).

First of all, for $2^{k-1} \leq s \leq 2^k$ we have $2^{(1-p)k} \sim s^{1-p}$. In other words, if we substitute s^{1-p} for $2^{(1-p)k}$ we may move it inside the supremum defining c_k. In that way we get an equivalent condition for the admissible approach regions:

$$\sum_{k=0}^{\infty} d_k < \infty,$$

where

$$d_k = \sup_{2^{k-1} \le s \le 2^k} h(2^{-s}) 2^s s^{-p}.$$

Secondly, if we let $\sigma = 2^{-s}$ we may rewrite d_k as

$$d_k \sim \sup_{2^{-2^k} \le \sigma \le 2^{-2^{k-1}}} \frac{h(\sigma)}{\sigma(\log 1/\sigma)^p}.$$

With this at hand it is easy to see that if we let $h(t) = t(\log 1/t)^q$ we have a.e. convergence if, and only if, $q < p$. The interesting feature here is the strict inequality, since it reveals that we do not have convergence along Rönning's L^p-regions. Note that convergence, when $q < p$, follows directly from the inclusion $L^{p,\infty} \subset L^q$ together with Rönning's result.

Finally, note that, as in the case of L^∞ boundary functions, there is no best possible (i.e. largest) approach region in case of boundary functions in $L^{p,\infty}$.

2. Preliminaries

We denote weak $L^p(\mathbb{T})$ by $L^{p,\infty}$, $1 \le p < \infty$, with quasi-norm

$$[f]_p = \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{1/p},$$

where $\lambda_f(\alpha) = |\{x : |f(x)| > \alpha\}|$, $|\cdot|$ denoting Lebesgue measure. It follows that on $\mathbb{T}$, endowed with Lebesgue measure, we have the inclusions

$$L^p \subset L^{p,\infty} \subset L^q$$

for $1 \le q < p < \infty$. However, $L^{1,\infty} \not\subset L^1$. That is, $L^{1,\infty}$ contains functions which are not integrable, and thus we assume that $1 < p < \infty$ in what follows.

Note also that $L^\infty \subset L^{p,\infty}$. This means that we can expect smaller approach regions for $L^{p,\infty}$ than for L^∞.

We point out that for $f, g \in L^{p,\infty}$ we have $[f+g]_p \le 2([f]_p + [g]_p)$. The constant 2 cannot be replaced by 1, so the ordinary triangle inequality fails.

Let $B_{p,\infty} = \{f \in L^{p,\infty} : [f]_p \le 1\}$, the unit ball in $L^{p,\infty}$, and let M denote the set of all measurable functions on $\mathbb{T}$. Endow $B_{p,\infty}$ and M with the topology of convergence in measure, given by the metric

$$d(f,g) = \int_{\mathbb{T}} \frac{|f(\beta) - g(\beta)|}{1 + |f(\beta) - g(\beta)|} \, d\beta,$$

$f, g \in M$. The metric d is induced by the "norm" ρ defined by

$$\rho(f) = \int_{\mathbb{T}} \frac{|f(\beta)|}{1+|f(\beta)|}\, d\beta,$$

$f \in M$ (ρ is not a norm, since it fails to be homogeneous, but we still refer to it in this way in lack of better terminology).

Lemma 1. *For $f, g \in B_{p,\infty}$ we have $d(f,g) \leq \|f-g\|_1$. Moreover, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(f,g) < \delta \Rightarrow \|f-g\|_1 < \varepsilon$.*

Proof. The inequality $d(f,g) \leq \|f-g\|_1$ is trivially true.
To prove the second statement let $\varepsilon > 0$ be given and let $\varphi = f - g$. For $A > 0$ fixed we have that

$$\{x : |\varphi(x)| > \alpha\} = \{x : \alpha < |\varphi(x)| \leq A\} \cup \{x : |\varphi(x)| > A\} \subset$$

$$\subset \{x : |\varphi(x)|/(1+|\varphi(x)|) > \alpha/(1+A)\} \cup \{x : |\varphi(x)| > A\}.$$

Since $[\varphi]_p \leq 2([f]_p + [g]_p) \leq 4$, we get the estimate

$$\begin{aligned}
\int |\varphi(x)|\, dx &= \int_0^\infty \lambda_\varphi(\alpha)\, d\alpha \\
&\leq \int_0^A \lambda_\varphi(\alpha)\, d\alpha + C \cdot A^{1-p} \\
&\leq \int_0^A \lambda_{\frac{\varphi}{1+|\varphi|}}\left(\frac{\alpha}{1+A}\right) d\alpha + A\lambda_\varphi(A) + C \cdot A^{1-p} \\
&\leq (1+A) \int_0^\infty \lambda_{\frac{\varphi}{1+|\varphi|}}(\alpha)\, d\alpha + C \cdot A^{1-p} \\
&= (1+A) d(f,g) + C \cdot A^{1-p}.
\end{aligned}$$

Now take A such that $C \cdot A^{1-p} < \varepsilon/2$, and then take $\delta = \frac{\varepsilon}{2(1+A)}$.

□

Lemma 2. *$C(\mathbb{T})$ is dense in $(B_{p,\infty}, d)$.*

Proof. By Lemma 2, we have that $d(f,g) \leq \|f-g\|_1$ for any $f, g \in B_{p,\infty}$. Since $C(\mathbb{T})$ is dense in L^1, the lemma follows. □

Theorem 1 below is a slightly modified version of Theorem 1 in [B-J]. The main difference is that here L^∞ is replaced by $L^{p,\infty}$.

Theorem 1. *Assume that we are given a sequence of operators $\{S_n\}_{n=1}^\infty$, $S_n : L^{p,\infty} \to M$, such that* (*i*) *each* $S_n : L^{p,\infty} \to M$ *is linear,*

(*ii*) *the maximal operator is well defined, that is* $S^*f(x) = \sup_{n\geq 1} |S_n f(x)|$ *is finite a.e. for* $f \in L^{p,\infty}$*, and*

(*iii*) $S^* : (B_{p,\infty}, d) \to (M, d)$ *is continuous at 0.*

Then the set E *of elements* $f \in B_{p,\infty}$ *for which* $(S_n f)$ *converges a.e. is closed in* $(B_{p,\infty}, d)$*.*

Proof. Let $\overline{E}$ be the closure of E in $(B_{p,\infty}, d)$. It follows that $\overline{E} \subset B_{p,\infty}$. Let $f \in \overline{E}$. We want to show that for all $\lambda > 0$

$$|\{\beta \in \mathbb{T} : \limsup_{m,n} |S_m f(\beta) - S_n f(\beta)| > \lambda\}| = 0.$$

We know that $|S_n v(\beta) - S_m v(\beta)| \leq |S_n v(\beta)| + |S_m v(\beta)| \leq 2S^* v(\beta)$ and hence, for any $g \in E$, we have

$$|\{\beta \in \mathbb{T} : \limsup_{m,n} |S_m f(\beta) - S_n f(\beta)| > \lambda\}|$$

$$= |\{\beta \in \mathbb{T} : \limsup_{m,n} |S_m (f-g)(\beta) - S_n (f-g)(\beta)| > \lambda\}|$$

$$\leq |\{\beta \in \mathbb{T} : 2S^*(f-g)(\beta) > \lambda\}|$$

$$= \left|\left\{\beta \in \mathbb{T} : S^*(\frac{1}{4}f - \frac{1}{4}g)(\beta) > \frac{\lambda}{8}\right\}\right|.$$

Let $\varepsilon > 0$ be given. For $u \in B_{p,\infty}$ we clearly have

$$\frac{\lambda/8}{1+\lambda/8} |\{\beta \in \mathbb{T} : S^*u(\beta) > \lambda/8\}| \leq \int_{\{\theta \in \mathbb{T} : S^*u(\theta) > \lambda/8\}} \frac{S^*u(\beta)}{1+S^*u(\beta)}\, d\beta \leq \rho(S^*u).$$

By the continuity of S^* at 0 in $(B_{p,\infty}, d)$ and by Lemma 1, we can choose a $\delta = \delta(\varepsilon, \lambda/8)$ such that $u \in B_{p,\infty}$ and $\|u\|_1 \leq \delta$ implies $\rho(S^*u) \leq \lambda\varepsilon/(\lambda+8)$, and therefore $|\{\beta \in \mathbb{T} : S^*u(\beta) > \lambda/8\}| < \varepsilon$.

By Lemma 1 again we can choose $g \in E$ such that $\|f-g\|_1 \leq \delta$. Then $(f-g)/4 \in B_{p,\infty}$ and of course $\|(f-g)/4\|_1 \leq \delta$. Thus

$$\left|\left\{\beta \in \mathbb{T} : S^*(\frac{1}{4}f - \frac{1}{4}g)(\beta) > \frac{\lambda}{8}\right\}\right| \leq \varepsilon.$$

□

The following trivial consequence of Theorem 1 is what we shall make use of:

Corollary 1. *Assume that in addition to the hypotheses on* $\{S_n\}_{n=1}^{\infty}$ *in Theorem 1, we also have that there is a set* $D \subset B_{p,\infty}$*, dense in* $(B_{p,\infty}, d)$*, such that* $\lim_{n\to\infty} S_n f(x)$ *exists a.e. for each* $f \in D$.
Then $\lim_{n\to\infty} S_n f(x)$ *exists a.e. for all* $f \in L^{p,\infty}$.

Proof. Let E be the set of elements $f \in B_{p,\infty}$ for which $(S_n f)$ converges a.e. By assumption $D \subset E$, $\overline{D} = B_{p,\infty}$, and by Theorem 1 we have $\overline{E} = E$. Thus $E = B_{p,\infty}$. By normalising $g \in L^{p,\infty}$, the corollary follows. □

Lemma 3. *Let* $\lambda_0 > 0$ *and* $1 < p < \infty$ *be given. For* $g \in B_{p,\infty}$ *let* $f(x) = g(x)\chi_{\{g>\lambda_0\}}$. *Then* $\|f\|_1 \le C\lambda_0^{1-p}$*, where* C *only depends on* p.

Proof. Note that $\lambda_g(\alpha) \le \alpha^{-p}[g]_p^p \le \alpha^{-p}$ for all $\alpha > 0$. Consequently $\lambda_f(\alpha) = |\{x : g(x) > \lambda_0\}| \le \lambda_0^{-p}$ for $0 \le \alpha \le \lambda_0$ and $\lambda_f(\alpha) = \lambda_g(\alpha) \le \alpha^{-p}$ for $\alpha > \lambda_0$. This yields

$$\begin{aligned} \|f\|_1 &= \int_0^{\infty} \lambda_f(\alpha)\, d\alpha \\ &= \int_0^{\lambda_0} \lambda_f(\alpha)\, d\alpha + \int_{\lambda_0}^{\infty} \lambda_f(\alpha)\, d\alpha \\ &\le \lambda_0^{1-p} + \int_{\lambda_0}^{\infty} \frac{1}{\alpha^p}\, d\alpha \\ &\le \frac{p}{p-1}\lambda_0^{1-p}. \end{aligned}$$

□

3. The main theorem

This last part of the thesis is entirely devoted to the proof and some of the consequences of the main result, Theorem 2.

Theorem 2. *Let* $1 < p < \infty$ *be given. Then the following are equivalent for any function* $h : \mathbb{R}_+ \to \mathbb{R}_+$*:*

(*i*) *For any* $f \in L^{p,\infty}(\mathbb{T})$ *one has for almost all* $\theta \in \mathbb{T}$ *that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $\sum_{k=0}^{\infty} c_k 2^{(1-p)k} < \infty$ *where* $c_k = \sup_{2^{k-1} \le s \le 2^k} h(2^{-s}) 2^s s^{-1}$.

3.1. **Proof of Theorem 2.** We shall prove that (ii) implies (i) via Proposition 1 below, and that (i) implies (ii) via contraposition. First we introduce a suitable notation.

If we write $t = 1 - |z|$, then $z = (1-t)e^{i\theta}$ and
$$\mathcal{P}_0 f(z) = R_t * f(\theta),$$
where the convolution is taken in $\mathbb{T}$ and
$$R_t(\theta) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t(2-t)}}{|(1-t)e^{i\theta} - 1|} \frac{1}{P_0 1(1-t)}.$$
Here $\theta \in \mathbb{T} = (-\pi, \pi]$. Since $P_0 1(1-t) \sim \sqrt{t} \log 1/t$, the order of magnitude of R_t is given by
$$R_t(\theta) \sim Q_t(\theta) = \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta|}.$$
Since we are only interested in small t, we might as well assume that $t < 1/2$. Now let τ_η denote the translation (or rotation, rather) $\tau_\eta f(\theta) = f(\theta - \eta)$. Then the convergence condition (i) in Theorem 2 above means
$$\lim_{\substack{t \to 0 \\ |\eta| < h(t)}} \tau_\eta R_t * f(\theta) = f(\theta).$$
The relevant maximal operator for our problem is
$$M_0 f(\theta) = \sup_{\substack{|\arg z - \theta| < h(1-|z|) \\ |z| > 1/2}} |\mathcal{P}_0 f(z)|.$$

Notice that $M_0 f(\theta)$ is dominated by a constant times

$$M f(\theta) = \sup_{\substack{|\eta| < h(t) \\ t < 1/2}} \tau_\eta Q_t * |f|(\theta). \tag{3}$$

Proposition 1. *Assume that condition* (ii) *in Theorem 2 holds and let* $\varepsilon > 0$. *Given* $\kappa > 0$ *there exists* $\delta > 0$ *such that for* $f \in B_{p,\infty}$
$$\|f\|_1 < \delta \Rightarrow |\{\theta \in \mathbb{T} : Mf(\theta) > \varepsilon\}| < \kappa.$$

Note that Proposition 1 precisely means that M is continuous at 0 in the topology of convergence in measure, when restricted to $B_{p,\infty}$. We can then apply Corollary 1 to the family of operators $f \mapsto \tau_\eta R_t * f$, $|\eta| < h(t)$, $t \in (0, 1/2)$. This is not a sequence of operators, but one can easily reduce it to a sequence. Thus, the implication $(ii) \Rightarrow (i)$ in Theorem 2 is a consequence of the proposition.

Proof. (*Proposition 1*) We may assume that $f \geq 0$, without loss of generality. Write
$$Q_t(\theta) = Q_t(\theta)\chi_{\{|\theta| \leq 2h(t)\}} + Q_t(\theta)\chi_{\{|\theta| > 2h(t)\}} = Q_t^1(\theta) + Q_t^2(\theta).$$

By letting

$$M_j f(\theta) = \sup_{\substack{|\eta|<h(t)\\ 0<t<1/2}} \tau_\eta Q_t^j * f(\theta) \ , j \in \{1,2\},$$

we get $Mf \le M_1 f + M_2 f$ and hence

$$\{Mf > \varepsilon\} \subset \{M_1 f > \varepsilon/2\} \cup \{M_2 f > \varepsilon/2\}.$$

To deal with $M_2 f$ we observe that when $|\eta| < h(t)$

$$\tau_\eta Q_t^2(\theta) \le \frac{1}{\log 1/t} \cdot \frac{1}{t+|\theta-\eta|} \chi_{\{|\theta-\eta|>2h(t)\}} \le \frac{2}{\log 1/t} \cdot \frac{1}{t+|\theta|}.$$

The last expression is a decreasing function of $|\theta|$, whose integral in $\mathbb{T}$ is bounded uniformly in t. It is well known that convolution by such a function is controlled by the Hardy-Littlewood maximal operator M_{HL}, so that $M_2 f \le C M_{HL} f$. Since M_{HL} is of weak type $(1,1)$, we obtain

$$|\{M_2 f > \varepsilon/2\}| \le C\varepsilon^{-1} \|f\|_1.$$

Finally, we consider $M_1 f$. If $M_1 f(\theta) > \varepsilon$, there exists $t \in (0,1/2)$ and $|\eta| \le h(t)$ such that $Q_t^1 * f(\theta - \eta) > \varepsilon$. This means that

$$\int_{|\varphi|<2h(t)} \frac{1}{\log 1/t} \cdot \frac{1}{t+|\varphi|} f(\theta-\eta-\varphi)\, d\varphi > \varepsilon.$$

We decompose the kernel $(t+|\varphi|)^{-1}$ as

$$\begin{aligned}\frac{1}{t+|\varphi|} &= \frac{1}{t+|\varphi|}\chi_{\{\varphi:|\varphi|<t\}}(\varphi) + \sum_{m=1}^{\infty} \frac{1}{t+|\varphi|}\chi_{\{\varphi:2^{m-1}t\le|\varphi|<2^m t\}}(\varphi)\\ &\le \frac{\chi_{(-t,t)}(\varphi)}{t} + \sum_{m=1}^{\infty} \frac{1}{2^{m-1}t}\chi_{\{\varphi:|\varphi|<2^m t\}}(\varphi).\end{aligned}$$

Now define

$$K_t^0(\varphi) = \frac{1}{\log 1/t} \cdot \frac{\chi_{(-t,t)}(\varphi)}{t},$$

and for $m \in \mathbb{N}$ define

$$K_t^m(\varphi) = \frac{1}{\log 1/t} \cdot \frac{1}{2^{m-1}t}\chi_{\{\varphi:|\varphi|<2^m t\}}(\varphi),$$

so that

$$\frac{1}{\log 1/t} \cdot \frac{1}{t+|\theta|} \le \sum_{m=0}^{\infty} K_t^m(\varphi).$$

We then have

$$M_1 f \leq \sum_{m=0}^{\infty} M^{(m)} f,$$

where

$$M^{(m)} f(\theta) = \sup_{\substack{|\eta|<h(t) \\ 0<t<1/2}} \tau_\eta K_t^m * f(\theta).$$

For some suitable sequence $\{\mu_m\}_0^\infty$ of positive numbers, with $\sum_m \mu_m = 1$, we intend to use the inequality

$$\lambda_{M_1 f}(\varepsilon) \leq \sum_{m=0}^{\infty} \lambda_{M^{(m)} f}(\mu_m \varepsilon), \tag{4}$$

in order to show that M_1 is continuous at 0 in the topology of convergence in measure.

Let $m \in \mathbb{N}$ be given and assume that $M^{(m)} f(\theta) > \varepsilon$. Then there exists $t \in (0, 1/2)$ and $|\eta| < h(t)$ such that

$$\int_{|\varphi|<2h(t)} \frac{\chi_{(-2^m t, 2^m t)}(\varphi)}{2^{m-1} t \log 1/t} f(\theta - \eta - \varphi)\, d\varphi > \varepsilon.$$

If we let $A(t) = \min\{2h(t), 2^m t\}$ this yields

$$\int_{-A(t)}^{A(t)} f(\theta - \eta - \varphi)\, d\varphi > \varepsilon 2^{m-1} t \log 1/t,$$

which is equivalent to

$$\int_{[\theta-\eta-A(t), \theta-\eta+A(t)]} f(\varphi)\, d\varphi > \varepsilon 2^{m-1} t \log 1/t. \tag{5}$$

Let $A_\nu = \sup_{2^{-\nu} \leq t < 2^{-\nu+1}} A(t)$, $\nu \geq 2$, and let $A = \sup_{\nu \geq 2} A_\nu$. For $j \in \mathbb{N}$ let i_j be the number of ν such that $2^{-j} A < A_\nu \leq 2^{-j+1} A$. Note that, since $\lim_{k\to\infty} A_k = 0$, we have that i_j is finite for all j. Let $\gamma(0) = 1$ and for $j \geq 1$ let $\gamma(j) = \sum_{k=1}^{j} i_k$.

We write

$$2^{-j} A < A_{\nu_{\gamma(j-1)+1}}, \ldots, A_{\nu_{\gamma(j)}} \leq 2^{-j+1} A,$$

i.e. ν_k, for $\gamma(j-1)+1 \leq k \leq \gamma(j)$, denotes precisely those ν for which $2^{-j} A < A_\nu \leq 2^{-j+1} A$.

Choose a maximal family of mutually disjoint open intervals I of lengths $2A(t)$, with $t \in [2^{-\nu_1}, 2^{-\nu_1+1}) \cup \ldots \cup [2^{-\nu_{\gamma(1)}}, 2^{-\nu_{\gamma(1)}+1})$, such that $\int_I f(\varphi)\, \varphi > \varepsilon 2^{m-1} t \log 1/t$. Denote the union of these intervals J_1.

We now construct J_j recursively: Given J_k for $k \leq j-1$ choose a maximal family of mutually disjoint open intervals I of lengths $2A(t)$, with

$$t \in [2^{-\nu_{\gamma(j-1)+1}}, 2^{-\nu_{\gamma(j-1)+1}+1}) \cup \ldots \cup [2^{-\nu_{\gamma(j)}}, 2^{-\nu_{\gamma(j)}+1}),$$

disjoint also with $\cup_{i=1}^{j-1} J_i$, such that $\int_I f(\varphi)\,\varphi > \varepsilon 2^{m-1} t \log 1/t$. Denote the union of these intervals J_j.

Note that each chosen interval is of length $2A(t)$ for some t. Let N_ν, $\nu \geq 2$, denote the number of chosen intervals with corresponding values of t in the interval $[2^{-\nu}, 2^{-\nu+1})$. Denote their union by I_ν.

Let $\nu' \in \mathbb{N}$ be arbitrary and define $\tilde{f} = f\chi_{\{f > \varepsilon\nu'(\log 2)/16\}}$.

For $2^{-\nu} \leq t < 2^{-\nu+1}$ we have $A(t) \leq 2^m t \leq 2^{m-\nu+1}$, so we get

$$\begin{aligned} \|(\tilde{f} - f)\chi_{I_\nu}\|_1 &= \int_{\mathbb{T}} f(\varphi)\chi_{\{f \leq \varepsilon\nu'(\log 2)/16\}}(\varphi)\chi_{I_\nu}(\varphi)\,d\varphi \\ &\leq (\varepsilon\nu'(\log 2)/16)|I_\nu| \\ &\leq (\log 2)N_\nu 2^{m-\nu+2}\varepsilon\nu' 2^{-4} \\ &= (\log 2)N_\nu 2^{m-\nu-2}\varepsilon\nu'. \end{aligned}$$

Furthermore, by (5), we get

$$\|f\chi_{I_\nu}\|_1 \geq N_\nu \varepsilon 2^{m-1} 2^{-\nu} \log 2^\nu = (\log 2) N_\nu 2^{m-\nu-1}\varepsilon\nu.$$

Combining these two estimates, we get

$$\begin{aligned} \|\tilde{f}\|_1 &\geq \sum_{\nu \geq \nu'} \|\tilde{f}\chi_{I_\nu}\|_1 \\ &\geq \sum_{\nu \geq \nu'} \Big(\|f\chi_{I_\nu}\|_1 - \|(\tilde{f} - f)\chi_{I_\nu}\|_1\Big) \\ &\geq C\sum_{\nu \geq \nu'} \left(N_\nu \varepsilon 2^{m-\nu-1}\nu - N_\nu 2^{m-\nu-2}\varepsilon\nu'\right) \\ &= C\sum_{\nu \geq \nu'} N_\nu 2^{m-\nu-2}\varepsilon(2\nu - \nu') \\ &\geq C\sum_{\nu \geq \nu'} N_\nu 2^{m-\nu}\varepsilon\nu. \end{aligned}$$

This together with Lemma 3 gives

$$C\sum_{\nu \geq \nu'} N_\nu 2^{m-\nu}\varepsilon\nu \leq \|\tilde{f}\|_1 \leq C(\varepsilon\nu')^{1-p}. \tag{6}$$

Upon dividing the left- and right-hand sides of (6) by $\varepsilon\nu'$ and estimating in the obvious way, we get

$$\sum_{\nu\geq\nu'} N_\nu 2^{m-\nu} \leq \frac{C}{(\varepsilon\nu')^p} \tag{7}$$

for all $\nu' \in \mathbb{N}$.

Let $\tilde{h}_\nu = \sup_{2^{-\nu}\leq s\leq 2^{-\nu+1}} h(s)$ and note that for $2^{-\nu} \leq t < 2^{-\nu+1}$ we have $A(t) \leq 2\tilde{h}_\nu$. Recall that I_ν is of the form

$$I_\nu = \cup_{i\in E(\nu)}[\theta_i - A(t_i), \theta_i + A(t_i)],$$

for some index set $E(\nu)$ and some mapping $i \mapsto (\theta_i, t_i)$. For each interval in I_ν we have

$$[\theta_i - A(t_i), \theta_i + A(t_i)] \subset [\theta_i - 10\tilde{h}_\nu, \theta_i + 10\tilde{h}_\nu].$$

Let

$$\tilde{I}_\nu = \cup_{i\in E(\nu)}[\theta_i - 10\tilde{h}_\nu, \theta_i + 10\tilde{h}_\nu].$$

We claim that $\{\theta \in \mathbb{T} : M^{(m)}f(\theta) > \varepsilon\} \subset \cup_{\nu\geq 2}\tilde{I}_\nu$. To prove this, assume that $M^{(m)}f(\theta) > \varepsilon$. Then there is a $t \in (0, 1/2)$ and $|\eta| < h(t)$ such that (5) holds. Assume that $2^{-\nu} \leq t < 2^{-\nu+1}$. If $\theta \in \tilde{I}_\nu$ we are done. If not, the reason must be that $[\theta - \eta - A(t), \theta - \eta + A(t)]$ intersects with some interval I in J_k, for some k. The point is, however, that I, which is of the form $I = [\theta_{i'} - A(t_{i'}), \theta_{i'} + A(t_{i'})]$ for some i', by maximality must have been chosen *before* the intervals in I_ν. Thus, by construction, when J_k is scaled as above it contains θ.

It follows that

$$|\{M^{(m)}f > \varepsilon\}| \leq C\sum_\nu N_\nu \tilde{h}_\nu, \tag{8}$$

for some positive constant C.

We want to show that the right-hand side of (8) tends to zero as $\|f\|_1 \to 0$. To that end, note first that the assumption $\|f\|_1 < \delta$, by the definition of N_ν and by (5), forces N_ν to be zero for $\nu \leq 2^{k_0}$, where $k_0 = k_0(\delta) \to \infty$ as $\delta \to 0$.

If we let $a_\nu = N_\nu 2^{m-\nu}\nu\varepsilon$ and $b_\nu = 2^{\nu-m}(\nu\varepsilon)^{-1}\tilde{h}_\nu$, we get

$$\sum_{\nu>2^{k_0}} N_\nu \tilde{h}_\nu = \sum_{\nu>2^{k_0}} a_\nu b_\nu. \tag{9}$$

The definition of a_ν and condition (7) immediately yield that for $k = k_0+1, k_0+2, ..$ we have

$$\sum_{\nu=2^{k-1}+1}^{2^k} a_\nu \le 2^k\varepsilon \sum_{\nu=2^{k-1}+1}^{2^k} N_\nu 2^{m-\nu} \le 2^k\varepsilon C(2^{k-1}\varepsilon)^{-p} = C(2^k\varepsilon)^{1-p}. \tag{10}$$

Furthermore,

$$\begin{aligned}
\max_{2^{k-1}+1\le\nu\le 2^k} b_\nu &\le 2^{-m}\varepsilon^{-1} \sup_{2^{k-1}+1\le s\le 2^k}\left(2^s s^{-1} \sup_{2^{-s}\le t\le 2^{-s+1}} h(t)\right)\\
&\le C2^{-m}\varepsilon^{-1} \sup_{2^{k-1}+1\le s\le 2^k}\left(\sup_{2^{-s}\le t\le 2^{-s+1}} \frac{h(t)}{t\log 1/t}\right)\\
&\le C2^{-m}\varepsilon^{-1} \sup_{2^{k-1}+1\le s\le 2^k}\left(\sup_{2^{s-1}\le x\le 2^{s}} \frac{h(x^{-1})x}{\log x}\right)\\
&\le C2^{-m}\varepsilon^{-1} \sup_{2^{2^{k-1}}\le x\le 2^{2^k}} \frac{h(x^{-1})x}{\log x}\\
&\le C2^{-m}\varepsilon^{-1} \sup_{2^{k-1}\le t\le 2^k} h(2^{-t})2^t t^{-1}\\
&\le C2^{-m}\varepsilon^{-1} c_k.
\end{aligned} \tag{11}$$

By (8), (9), (11) and (10), in that order, we get

$$\begin{aligned}
|\{M^{(m)}f > \varepsilon\}| &\le C\sum_\nu a_\nu b_\nu\\
&\le C \sum_{k=k_0+1}^{\infty} \sum_{\nu=2^{k-1}+1}^{2^k} a_\nu b_\nu\\
&\le C \sum_{k=k_0+1}^{\infty} 2^{-m}\varepsilon^{-1} c_k \sum_{\nu=2^{k-1}+1}^{2^k} a_\nu\\
&\le C2^{-m}\varepsilon^{-p} \sum_{k=k_0}^{\infty} c_k 2^{(1-p)k}\\
&= C2^{-m}\varepsilon^{-p} S(k_0),
\end{aligned}$$

where $S(k_0) = \sum_{k=k_0}^{\infty} c_k 2^{(1-p)k}$.

By invoking (4) with

$$\mu_m = \frac{2^{-m/(2p)}}{\sum_{i=0}^{\infty} 2^{-i/(2p)}},$$

we have that

$$\lambda_{M_1 f}(\varepsilon) \leq C \cdot S(k_0)\varepsilon^{-p} \sum_{m=0}^{\infty} 2^{-m}\mu_m^{-p} \leq C\varepsilon^{-p} \cdot S(k_0).$$

By assumption $S(k_0) \to 0$ as $k_0 \to \infty$ (i.e. as $\delta \to 0$). To sum up, we have shown that

$$\lambda_{Mf}(\varepsilon) \leq \lambda_{M_1 f}(\varepsilon/2) + \lambda_{M_2 f}(\varepsilon/2) \leq C\varepsilon^{-1}\delta + C\varepsilon^{-p}S(k_0) \to 0$$

as $\delta \to 0$. That concludes the proof of Proposition 1. □

Proof. (Theorem 2) We have already shown that (ii) implies (i), as a consequence of Proposition 1.

To prove that (i) implies (ii), assume that (ii) is false, i.e. that

$$\sum_{k=l}^{\infty} c_k 2^{(1-p)k} = \infty \tag{12}$$

for all l. We shall now construct a function $f \in L^{p,\infty}$ that violates (i).

Let $\varepsilon > 0$ be given. Assume for the moment that

$$\lim_{k\to\infty} c_k 2^{(1-p)k} = 0. \tag{13}$$

Recall that $c_k = \sup_{2^{k-1} \leq s \leq 2^k} h(2^{-s})2^s s^{-1}$. Thus, for all $k \in \mathbb{N}$, we can find an $s_k \in [2^{k-1}, 2^k]$, such that

$$c_k/2 < h(2^{-s_k})2^{s_k} s_k^{-1} \leq c_k. \tag{14}$$

Let $t_k = 2^{-s_k}$. We shall now construct a subset of $\mathbb{T}$ consisting of a number, n_k, of intervals, each of length t_k and with gaps $h(t_k)$, the first of these intervals starting at $\theta = 0$. Then we do this again, starting from the point where we last stopped, but this time with n_{k+1}, t_{k+1} and $h(t_{k+1})$. We shall proceed in this way and show that sooner or later this process yields a subset whose endpoint exceeds $\theta = \pi$. Then we start over again, i.e. at $\theta = 0$, with the next n_k, t_k and $h(t_k)$.

We shall do this infinitely many times, and along the way we construct the function f, which, as we finally shall show, disproves a.e. convergence.

The construction of the subsets of $\mathbb{T}$ mentioned above is done recursively:

Construction on level l:

Let integers $l < m$ be given. For $l \leq k \leq m$, let $s_k \in [2^{k-1}, 2^k]$ be such that (14) holds.

Let $n_k = [2^{s_k}(s_k\varepsilon)^{-1}(2^k\varepsilon)^{1-p}]$ and recall that $t_k = 2^{-s_k}$.

Define τ_k, for $l \leq k \leq m$, recursively as $\tau_l = 0$ and $\tau_{k+1} = \tau_k + n_k(h(t_k) + t_k)$. Now, let

$$E_k^j = [\tau_k + (j-1)(h(t_k) + t_k), \tau_k + (j-1)(h(t_k) + t_k) + t_k], \tag{15}$$

$1 \leq j \leq n_k$ and $l \leq k \leq m$. Note that $|E_k^j| = t_k$ and that the distance between E_k^j and E_k^{j+1} is $h(t_k) = h(2^{-s_k})$. The length (on $\mathbb{T}$) required for this construction is therefore $\sum_{k=l}^m n_k(h(t_k) + t_k)$. By (14) we have

$$\begin{aligned} n_k(h(t_k) + t_k) &\leq 2^{s_k}(s_k\varepsilon)^{-1}(2^k\varepsilon)^{1-p}(h(2^{-s_k}) + 2^{-s_k}) \\ &\leq \varepsilon^{-p}2^{(1-p)k}(c_k + s_k^{-1}), \end{aligned} \tag{16}$$

and by (13) it follows that the terms in (16) tend to zero as $k \to \infty$.

Hence, there exists an $L \in \mathbb{N}$ such that for $k \geq L$ we have $n_k(h(t_k) + t_k) < \pi$. We assume from now on that $l \geq L$.

Note that condition (13) implies that $\sum_{k=1}^\infty h(2^{-s_k})$ converges. Combining this with (12) and (14), we get

$$\begin{aligned} \sum_{k=l}^m n_k(h(t_k) + t_k) &\geq \sum_{k=l}^m (2^{s_k}(s_k\varepsilon)^{-1}(2^k\varepsilon)^{1-p} - 1)(h(2^{-s_k}) + 2^{-s_k}) \\ &\geq \varepsilon^{-p}\sum_{k=l}^m 2^{(1-p)k}(\frac{c_k}{2} + s_k^{-1}) - \sum_{k=l}^m (h(2^{-s_k}) + 2^{-s_k}), \end{aligned}$$

so that

$$\sum_{k=l}^{m} n_k(h(t_k)+t_k) \to \infty \tag{17}$$

as $m \to \infty$. Now, (17) and the fact that $n_k(h(t_k+t_k)) < \pi$ yield the existence of an $m = m(l)$ such that

$$\pi < \sum_{k=l}^{m(l)} n_k(h(2^{-s_k})+2^{-s_k}) < 2\pi. \tag{18}$$

This concludes the construction on level l.

The first time we make this construction we get an $l_1 \in \mathbb{N}$, $l_1 \geq L$, and an $m_1 = m(l_1) \in \mathbb{N}$ such that (18) holds. Let $E_k = \cup_{j=1}^{n_k} E_k^j$, $l_1 \leq k \leq m_1$. Take $f_k(\theta) = s_k \varepsilon \chi_{E_k}(\theta)$ and let

$$f^{(1)}(\theta) = \sum_{k=l_1}^{m_1} f_k(\theta).$$

Given $f^{(j)}$, with corresponding values of l_j and m_j, we now describe how to construct $f^{(j+1)}$. Since $m_j \geq L$, we can make a new construction on level $l_{j+1} = m_j + 1$. This yields an $m_{j+1} = m(l_{j+1})$ and a sequence of functions $\{f_k\}_{k=l_{j+1}}^{m_{j+1}}$. Let $f^{(j+1)}(\theta) = \sum_{k=l_{j+1}}^{m_{j+1}} f_k(\theta)$.

Proceeding inductively, we get a sequence of functions $\{f^{(j)}\}_{j=1}^{\infty}$, $f^{(j)} : \mathbb{T} \to \mathbb{R}_+$. Let

$$f(\theta) = \sum_{j=1}^{\infty} f^{(j)}(\theta) = \sum_{k=l_1}^{\infty} f_k(\theta).$$

We shall show that f is an element in $L^{p,\infty}$ violating condition (i) in Theorem 2. For any $k' \in \mathbb{N}$ we have $\sum_{k \leq k'-2} s_k \varepsilon \leq \varepsilon \sum_{k \leq k'-2} 2^k \leq \varepsilon 2^{k'-1} \leq \varepsilon s_{k'}$, so that

$$\begin{aligned}
\lambda_f(s_{k'}\varepsilon) &\leq \sum_{k \geq k'-1} |E_k| \\
&\leq \varepsilon^{-p} \sum_{k \geq k'-1} s_k^{-1} 2^{(1-p)k} \\
&\leq C\varepsilon^{-p} \sum_{k \geq k'-1} 2^{-pk} \\
&\leq C\varepsilon^{-p} 2^{-p(k'-1)} \\
&\leq C(s_{k'}\varepsilon)^{-p}.
\end{aligned}$$

For small $\alpha > 0$, $\alpha \leq C$ say, it is clear that $\alpha^p \lambda_f(\alpha)$ is bounded. If α is large, take $k' \in \mathbb{N}$ such that $2^{k'}\varepsilon \leq \alpha < 2^{k'+1}\varepsilon$. Then, by what we have just shown, $\lambda_f(\alpha) \leq \lambda_f(2^{k'}\varepsilon) \leq C(2^{k'}\varepsilon)^{-p} \leq C\alpha^{-p}$. It follows that $f \in L^{p,\infty}$.

Furthermore, we have

$$|\{\theta \in \mathbb{T} : f(\theta) \neq 0\}| \leq \sum_{j=1}^{\infty} \sum_{k=l_j}^{m_j} n_k t_k \leq \varepsilon^{-p} \sum_{k=l_1}^{\infty} s_k^{-1} 2^{(1-p)k},$$

which can be taken arbitrarily small by just taking l_1 sufficiently large, $|\{\theta \in \mathbb{T} : f(\theta) \neq 0\}| < \pi/2$ say.

Let $\theta \in (0, \pi) \subset \mathbb{T}$ be given. The claim is that there exists a subsequence $\{t_{k_i}\}_{i=1}^{\infty}$ of $\{t_k\}$ such that for each t_{k_i} there is a $z_{k_i} \in \mathcal{A}_h(\theta)$ with $|z_{k_i}| = 1 - t_{k_i}$, and $\mathcal{P}_0 f(z_{k_i}) > \tilde{\varepsilon} > 0$, uniformly in i. When we have shown this we are done, since then it follows that

$$\limsup_{\substack{z \to e^{i\theta} \\ z \in \mathcal{A}_h(\theta)}} \mathcal{P}_0 f(z) > \tilde{\varepsilon}$$

for all $\theta \in (0, \pi)$. But that in turn, compared with $|\{\theta \in \mathbb{T} : f(\theta) = 0\}| > \pi/2$, disproves a.e. convergence.

Consider the construction done on level l_i, $i \in \mathbb{N}$, above. The set obtained there was

$$\cup_{k=l_i}^{m_i} E_k = \cup_{k=l_i}^{m_i} \cup_{j=1}^{n_k} E_k^j.$$

Let $k_i, j_i \in \mathbb{N}$, $l_i \leq k_i \leq m_i$ and $1 \leq j_i \leq n_{k_i}$ be such that

$$\operatorname{dist}(\theta, E_{k_i}^{j_i}) = \min_{l_i \leq k \leq m_i} \min_{1 \leq j \leq n_k} \operatorname{dist}(\theta, E_k^j), \tag{19}$$

i.e. $E_{k_i}^{j_i}$ is the interval closest to θ, among all intervals constructed on level l_i. Note that $\operatorname{dist}(\theta, E_{k_i}^{j_i}) \leq h(t_{k_i})/2$.

By (19) and (15) it follows that there exists η, $|\eta| < h(t_{k_i})/2$, such that either $(\theta - \eta, \theta - \eta + t_{k_i}/2)$ or $(\theta - \eta - t_{k_i}/2, \theta - \eta)$ lies completely within $E_{k_i}^{j_i}$. Assume, without loss of generality, that $(\theta - \eta, \theta - \eta + t_{k_i}/2) \subset E_{k_i}^{j_i}$, and let $E(\theta) = (\theta - \eta, \theta - \eta + t_{k_i}/2)$.

Let $z_{k_i} = (1 - t_{k_i})e^{i(\theta - \eta)}$. Then $|z_{k_i}| = 1 - t_{k_i}$, and trivially $|\theta - (\theta - \eta)| < h(t_{k_i})$, so $z_{k_i} \in \mathcal{A}_h(\theta)$.

We now have

$$\begin{aligned}
\mathcal{P}_0 f(z_{k_i}) &\geq \mathcal{P}_0 f_{k_i}(z_{k_i}) \\
&\geq \mathcal{P}_0\left(s_{k_i}\varepsilon\chi_{E^{j_i}_{k_i}}\right)(z_{k_i}) \\
&\geq \mathcal{P}_0\left(s_{k_i}\varepsilon\chi_{E(\theta)}\right)(z_{k_i}) \\
&\geq C s_{k_i}\varepsilon \int_{\mathbb{T}} \frac{\chi_{E(\theta)}(\theta-\eta-\varphi)}{(\log 1/t_{k_i})(t_{k_i}+|\varphi|)}\, d\varphi \\
&= C s_{k_i}(\log 1/t_{k_i})^{-1}\varepsilon \int_{-t_{k_i}/2}^{0} \frac{d\varphi}{t_{k_i}+|\varphi|} \\
&= C\varepsilon.
\end{aligned}$$

This concludes the proof in the case when (13) holds. Note that if (13) does not hold, we should intuitively face an easier task than above, since then the divergence of the series $\sum_{k=0}^{\infty} c_k 2^{(1-p)k}$ is even worse than before, meaning that h is larger. We shall construct a function $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $\theta \in \mathbb{T}$ we have $\mathcal{A}_H(\theta) \subset \mathcal{A}_h(\theta)$, and such that (12) and (13) holds, thus disproving convergence (as above).

To make this precise let $\alpha_k = c_k 2^{(1-p)k}$ and assume that (13) does not hold. That is, we have

$$\alpha_k \not\to 0,$$

as $k \to \infty$. Now, there exists $\kappa > 0$ and a subsequence $\{\alpha_{k_j}\}_{j=j_0}^{\infty} \subset \{\alpha_k\}_{k=1}^{\infty}$, where $j_0 > \kappa^{-1}$, such that $\alpha_{k_j} > \kappa$ for all $j \geq j_0$. Let

$$\beta_i = \begin{cases} 0 & \text{if } i \neq k_j \text{ for all } j \geq j_0 \\ \frac{1}{j} & \text{if } i = k_j \text{ for some } j \geq j_0 \end{cases}$$

It is clear that $\beta_k \leq \alpha_k$ for all $k \in \mathbb{N}$. Furthermore, $\lim_{k\to\infty} \beta_k = 0$ and $\sum_{k=0}^{\infty} \beta_k = \infty$.

For each $k \in \mathbb{N}$ fix an $s_k \in [2^{k-1}, 2^k]$ such that (14) holds. Let

$$H(t) = \begin{cases} \beta_k 2^{(p-1)k} 2^{-s_k-1} s_k \text{ if } t = 2^{-s_k} \text{ for some } k \in \mathbb{N} \\ 0 \text{ otherwise} \end{cases}.$$

Then $H(2^{-s_k}) \leq \alpha_k 2^{(p-1)k} 2^{-s_k-1} s_k = (c_k/2) 2^{-s_k} s_k \leq h(2^{-s_k})$, the last inequality by (14), so that $H(t) \leq h(t)$ for all (relevant) t. It follows that, for all $\theta \in \mathbb{T}$, we have $\mathcal{A}_H(\theta) \subset \mathcal{A}_h(\theta)$.

However, by construction, H satisfies (12) and (13) so we do not have convergence along $\mathcal{A}_H(\theta)$, and consequently not along $\mathcal{A}_h(\theta)$ either. This concludes the proof of Theorem 2. □

4. Open questions

It is easy to see that the sequence of functions one gets by letting l_1 increase in the definition of f, defined above in order to disprove convergence, also disproves continuity of the maximal operator at 0 in the topology of convergence in measure. It is reasonable to believe that this is not a coincidence. As mentioned in section 1, the Bellow-Jones result [B-J] for L^∞ basically shows that a.e. convergence is equivalent to continuity of the maximal operator at 0. A similar result for $L^{p,\infty}$ could very well hold and would be interesting in its own right.

To understand better the significant difference between the approach regions for L^p and the ones for L^∞ one could investigate "intermediate spaces". Of course $L^{p,\infty}$, via the inclusions $L^\infty \subset L^{p,\infty} \subset L^q$ for $q < p$, is an example of such a one. Convergence results for boundary functions in BMO($\mathbb{T}$) or suitable Orlicz spaces would certainly be a step along the very same lines.

The author intends to investigate these questions further.

Another generalisation would be to find the Nagel-Stein approach regions for $L^{p,\infty}$. Yet another would be to leave $\mathbb{T}$ and consider general symmetric spaces of rank 1.

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