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ELASTIC SH WAVE PROPAGATION IN A LAYERED ANISOTROPIC PLATE WITH INTERFACE DAMAGE MODELED BY SPRING BOUNDARY CONDITIONS

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Summary

Elastic SH wave propagation in a layered anisotropic plate with interface damage is modeled in several steps. A single interface crack between two half-spaces is first studied and an explicit solution for the crack-opening displacement is obtained at low frequencies. This is then generalized to a random distribution of cracks at the interface and the result is reformulated as a spring boundary condition. This boundary condition is then used in the derivation of a plate equation by expanding the displacements in power series in the thickness coordinate.

1 Introduction

In laminated composites a common failure mode is the occurence of microcracks at the interface between the plies. Such damage should be possible to detect by ultrasonic nondestructive testing, and the present contribution aims at making a model of this situation. There are many benefits with a good model of the testing situation, in particular it is easy to perform parametric studies and investigate the influence of the various parameters of the situation. A good source of knowledge in the area of wave propagation and scattering in the presence of cracks is the monograph by Zhang and Gross (**1**), which also contains numerous references to earlier work.

The modeling is only performed in the simple setting of 2D SH waves with the polarization perpendicular to the plane of wave propagation. However, the anisotropy that occurs in most composites is taken into account. The model is built in several steps. First a single interface crack between two anisotropic half-spaces is solved for perpendicular incidence of a plane wave. The solution procedure is only sketched as this problem has essentially been solved before, by Ohyoshi (**2**) (for an interior crack) by dual integral equations and by Boström (**3**) (for isotropic media) and Zhang and Gross (**1**) (for an interior crack) by a hypersingular integral equation for the crack-opening displacement. In the low frequency limit (small cracks) the same explicit solution is obtained by both methods.

The next step is to look at a random distribution of cracks at the interface between two anisotropic half-spaces. SH wave propagation in isotropic space with an interface containing a periodic array of cracks has been considered by Achenbach and Li (**4**, **5**), where the reflection coefficient is defined in terms of the coefficient of the zeroth-order scattered wave. The exact reflection coefficient is successfully compared in (**4**) to approximate theories including replacement of the interface by a spring interface. The scattering of an SH wave by equally sized but randomly oriented cracks is considered by Achenbach and Mikata (**7**), where it is shown that the reflection coefficient for the periodic array of coplanar cracks at normal incidence is approximately twice that for an array of arbitrarily oriented cracks. The reflection coefficient for a disordered periodic array of equally sized cracks is calculated by Mikata (**8**), the reflection coefficient differs slightly from the periodic array case at low frequencies.

Following the work by Boström and Wickham (**9**), which is similar to earlier work by Sotiropoulos and Achenbach (**10**), the total reflection and transmission is obtained using a reciprocal theorem and ensemble averaging. Assuming that the extra scattering due to the distribution of microcracks is small, the interface with microcracks is then modeled as a (distributed) spring boundary condition, again following Boström and Wickham (**9**), see also Baik and Thompson (**11**).

The last step is to insert the spring boundary condition into some plate equation. This is performed with the method of Boström et al. (**12**), where the displacement in each layer is expanded into a power series in the thickness coordinate. From the wave equation in each layer recursion relations for the expansion coefficients follow and the interface and boundary conditions finally give the plate equations. In this simple case it is easy to eliminate among the equations to obtain a single plate equation for the displacement in the middle of the plate. It is noted that the method can in principle be used to any order (in the plate thickness) and that it is believed to be asymptotically exact (**12**). Numerical results for the dispersion relation show the influence of the spring boundary condition.

2 Single interface crack

In this section an interface crack between two anisotropic half-spaces is considered. A coordinate system with origin at the interface and the z axis perpendicular to the interface is introduced. The (strip-like) crack is situated at the interface for $|x| < l$. Only SH wave propagation in the xz plane is considered, and the displacement field thus only have a y component that is denoted u. Time harmonic waves are assumed and the factor $\exp(-i\omega t)$ is suppressed throughout, where ω is the angular frequency and t is time.

The material constants and the fields are denoted by an upper index $j = 1$ for the lower half-space and $j = 2$ for the upper one. The densities are ρ^j and the only relevant stiffness constants are c_{44}^j and c_{66}^j. Note that the anisotropy is assumed to have the principal axes parallel to the coordinate axes and that thus $c_{46}^j = 0$. The wave motion in the two half-spaces is governed by the equation

$$c_{66}^j \frac{\partial^2 u^j}{\partial x^2} + c_{44}^j \frac{\partial^2 u^j}{\partial z^2} = \rho^j \omega^2 u^j, \qquad j = 1, 2. \tag{2.1}$$

The two relevant stress components are

$$\sigma_{yx}^j = c_{66}^j \frac{\partial u^j}{\partial x},$$

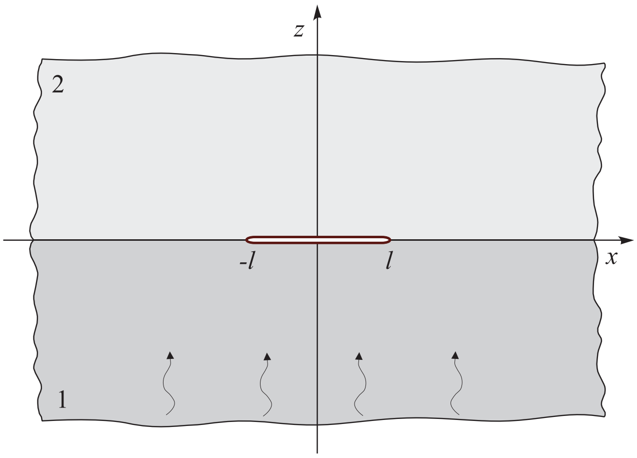

Fig. 1 Geometry for single interface crack.

$$\sigma_{yz}^j = c_{44}^j \frac{\partial u^j}{\partial z}.$$

For notational ease one of the stress components is hereafter renamed: $\tau^j = \sigma_{yz}^j$.

Consider now the scattering problem for a plane wave propagating from below in the half-space $z < 0$ and being transmitted and reflected by the interface and scattered by the crack. It is useful to divide the total field into the field $u^{j,\mathrm{in}}$ in the absence of the crack and an extra, scattered, part $u^{j,\mathrm{sc}}$ due to the presence of the crack. The field in the absence of the crack is easily calculated

$$u^{\mathrm{in}} = \begin{cases} u^{1,\mathrm{in}} = e^{ik_4^1 z} + R^- e^{-ik_4^1 z}, & z < 0, \\ u^{2,\mathrm{in}} = T^- e^{ik_4^2 z}, & z > 0, \end{cases} \tag{2.2}$$

where the reflection and transmission coefficients are

$$R^- = \frac{c_{44}^1 k_4^1 - c_{44}^2 k_4^2}{c_{44}^1 k_4^1 + c_{44}^2 k_4^2},$$

$$T^- = \frac{2c_{44}^1 k_4^1}{c_{44}^1 k_4^1 + c_{44}^2 k_4^2}.$$

Here $k_m^j = \omega\sqrt{\rho^j / c_{mm}^j} = \omega / s_m^j$, $m = 4, 6$, are the wave numbers and s_m^j are SH wave velocities along the z and x axes, respectively, in the two half-spaces ($j = 1, 2$).

The scattered waves satisfy the same wave equation as the incoming waves, of course, and the following boundary conditions along the interface between the two half-spaces

$$\begin{cases} u^{1,\mathrm{sc}} = u^{2,\mathrm{sc}}, & |x| > l \\ \tau^{1,\mathrm{sc}} = \tau^{2,\mathrm{sc}}, & |x| > l \\ \tau^{1,\mathrm{sc}} = \tau^{2,\mathrm{sc}} = -\tau^{1,\mathrm{in}}, & |x| < l. \end{cases} \tag{2.3}$$

As the incoming stress field is continuous at the interface the stress field $-\tau^{2,\mathrm{in}}$ could equally well be used in the last equation.

The scattered waves can be written in terms of Fourier integrals as follows

$$u^{\mathrm{sc}} = \begin{cases} \int\limits_{-\infty}^{\infty} U^1(\alpha) e^{i(\alpha x - \sigma^1 z)} d\alpha, & z < 0 \\ \int\limits_{-\infty}^{\infty} U^2(\alpha) e^{i(\alpha x + \sigma^2 z)} d\alpha, & z > 0 \end{cases}$$

where

$$\sigma^j = \sqrt{(k_4^j)^2 - \alpha^2 c_{66}^j / c_{44}^j}, \quad \mathrm{Im}\,\sigma^j \geqslant 0.$$

As the traction is continuous for all x it follows that $c_{44}^2 \sigma^2 U^2 = -c_{44}^1 \sigma^1 U^1$, so U^2 can easily be eliminated. The two remaining boundary conditions then give

$$\int\limits_{-\infty}^{\infty} U^1(\alpha) \left(1 + \frac{c_{44}^1 \sigma^1}{c_{44}^2 \sigma^2}\right) e^{i\alpha x}\, d\alpha = \begin{cases} v(x), & |x| < l \\ 0, & |x| > l, \end{cases} \tag{2.4}$$

$$\int\limits_{-\infty}^{\infty} i c_{44}^1 \sigma^1 U^1(\alpha) e^{i\alpha x}\, d\alpha = -i c_{44}^1 k_4^1 (1 - R^-), \quad |x| < l. \tag{2.5}$$

Here the crack opening displacement $v(x) = u^{1,\mathrm{sc}}(x, 0^-) - u^{2,\mathrm{sc}}(x, 0^+)$ is introduced, which is used as the primary unknown in one of the two methods to be used for solving the integral equations.

One way to solve the integral equations (2.4) and (2.5) is to employ traditional dual integral equations methods in the same way as Ohyoshi (**2**), who solves the 2D SH wave scattering by a crack in an anisotropic space. To this end the Fourier transform of the crack opening displacement $V(\alpha)$ is introduced as the primary unknown and the equations are rewritten as

$$\frac{2}{\pi} \int\limits_0^{\infty} V(\alpha) \cos(\alpha x) d\alpha = 0, \quad |x| > l \tag{2.6}$$

$$\frac{2}{\pi} \int\limits_0^{\infty} \alpha V(\alpha) \cos(\alpha x) d\alpha = h(x), \quad |x| < l \tag{2.7}$$

The right-hand side here also contains the unknown

$$h(x) = -\frac{k_4^1 (1 - R^-)}{i\beta\pi} - \int\limits_0^{\infty} \frac{2G(\alpha)}{i\beta\pi} V(\alpha) \cos(\alpha x) d\alpha.$$

The kernel that appears in the traction boundary condition

$$L(\alpha) = \frac{c_{44}^2 \sigma^1 \sigma^2}{c_{44}^1 \sigma^1 + c_{44}^2 \sigma^2}$$

has here been split according to

$$L(\alpha) = i\beta\alpha + G(\alpha), \quad \beta = \frac{\sqrt{c_{66}^1 c_{66}^2 c_{44}^2 / c_{44}^1}}{\sqrt{c_{66}^2 c_{44}^2} + \sqrt{c_{66}^1 c_{44}^1}},$$

where the term containing β is the dominant one for large α, and where $G(\alpha) = \mathrm{O}(1/\alpha)$.

The dual equations (2.6) and (2.7) have exactly the same form as in Ohyoshi (**2**). By introducing the new unknown $\Lambda(\eta)$ according to

$$V(\alpha) = \frac{ik_4^1 l^2}{2\beta}\left(1 - R^-\right)\int_0^1 \eta\Lambda(\eta) J_0(\alpha l \eta)\, d\eta, \tag{2.8}$$

the dual equations thus reduce to a single integral equation of the second kind

$$\Lambda(\eta) = 1 + \int_0^1 \eta' \Lambda(\eta') F(\eta, \eta') d\eta', \tag{2.9}$$

where the kernel is

$$F(\eta, \eta') = \frac{il^2}{\beta}\int_0^\infty G(\alpha) J_0(\alpha l \eta) J_0(\alpha l \eta') d\alpha. \tag{2.10}$$

At low frequencies the solution is simply

$$\Lambda = 1,$$

which gives

$$V(\alpha) = \frac{ik_4^1 l J_1(\alpha l)}{2\alpha\beta}\left(1 - R^-\right)$$

and subsequently the crack-opening displacement at low frequencies becomes (after some simplifications)

$$v_0(x) = 2K\sqrt{l^2 - x^2}, \tag{2.11}$$

where

$$K = i\frac{c_{44}^1 k_4^1 k_6^2 + c_{44}^2 k_6^1 k_4^2}{c_{44}^1 k_4^1 + c_{44}^2 k_4^2}. \tag{2.12}$$

In the second method to solve Eqs. (2.4) and (2.5) the Fourier transform is first inverted in Eq. (2.4) to yield

$$U_1(\alpha) = \frac{c_{44}^2 \sigma^2}{2\pi(c_{44}^1 \sigma^1 + c_{44}^2 \sigma^2)}\int_{-\infty}^{\infty} v(x) e^{-i\alpha x} dx.$$

Here the crack-opening displacement is expanded as

$$v(x) = \sum_{n=1}^{\infty} \alpha_n \psi_n(x/l) \tag{2.13}$$

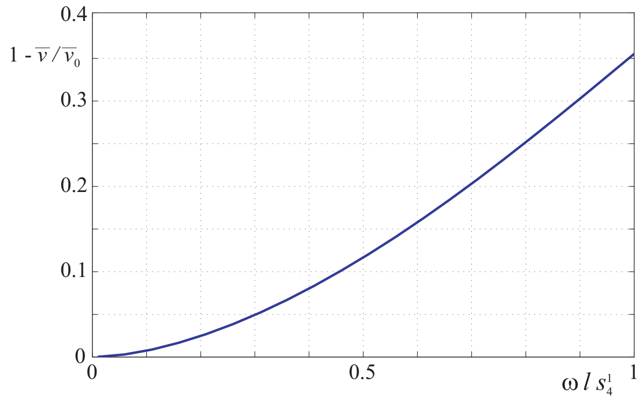

Fig. 2 The relative difference $1-|\overline{v}/\overline{v_0}|$ as a function of frequency.

where the Chebyshev functions are

$$\psi_n(s)=\begin{cases}\frac{1}{\pi}\cos\left(n\arcsin s\right), & n=1,3,\ldots\\ \frac{i}{\pi}\sin\left(n\arcsin s\right), & n=2,4,\ldots\end{cases}$$

These functions form a complete set on the interval and also have the correct square root behaviour at the crack edges. Inserting everything into Eq. (2.5) and projecting on the Chebyshev functions gives the following discretized form of the integral equation

$$\sum_{n'=1}^{\infty}Q_{nn'}\alpha_{n'}=-\frac{lc_{44}^2k_4^1k_4^2}{c_{44}^1k_4^1+c_{44}^2k_4^2}\,\delta_{n1}, \tag{2.14}$$

where δ_{n1} is the Kronecker delta. In the equation

$$Q_{nn'}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{c_{44}^2\sigma^1\sigma^2}{c_{44}^1\sigma^1+c_{44}^2\sigma^2}J_n(\alpha l)J_{n'}(\alpha l)\frac{d\alpha}{\alpha},$$

which can be computed exactly in the low frequency limit

$$Q_{nn'}=\frac{i\sqrt{c_{44}^2c_{66}^2c_{66}^2/c_{44}^1}}{2\pi\left(\sqrt{c_{44}^1c_{66}^1}+\sqrt{c_{44}^2c_{66}^2}\right)}\int_{-\infty}^{\infty}J_n(\alpha l)J_{n'}(\alpha l)\frac{d\alpha}{\alpha}=\frac{i\sqrt{c_{44}^2c_{66}^2c_{66}^2/c_{44}^1}}{2\pi n\left(\sqrt{c_{44}^1c_{66}^1}+\sqrt{c_{44}^2c_{66}^2}\right)}\delta_{nn'}.$$

Using this to compute the crack-opening displacement gives exactly the result Eqs. (2.11) and (2.12) once again.

To investigate the accuracy of the low frequency approximation Eq. (2.11), the average of the crack-opening displacement is computed from Eq. (2.11) as $\overline{v_0}=\pi Kl/2$ which is compared to the average computed from Eq. (2.13) (because of the orthogonality of the Chebyshev polynomials with square root weight)

$$\overline{v}=\pi\alpha_1 l/2.$$

Here α_1 is computed by solving the system Eq. (2.14). As an example consider two fiber-reinforced graphite-epoxy half-spaces, the lower half-space has density $1578\,\mathrm{kg/m^3}$ and stiffness constants $c^1_{44} = 3.50\,\mathrm{GPa}$ and $c^1_{66} = 7.07\,\mathrm{GPa}$. The uppper half-space is of the same material but the fibers are rotated by $90°$ relatively to the first one. The relative difference $1 - |\overline{v}/\overline{v_0}|$ is shown in Fig. 2 and demonstrates the error introduced by the low frequency approximation. It is seen that the low frequency approximation is valid for $\omega l/s^1_4 << 1$ with an error that is only about 10% at $\omega l/s^1_4 = 0.5$. As the interest is here focussed on small interface cracks due to damage, the low frequency approximation is henceforth applied.

3 Distribution of interface cracks

Consider next the same situation as in the previous section except that the single interface crack is replaced by a random distribution of cracks at the interface. The distribution is assumed to be translationally invariant, and it is also assumed that all cracks have the same size, although a generalization to cracks of different sizes is readily performed. The interaction between the cracks is neglected, but as shown by Sotiropoulos and Achenbach (**10**) this is not a very restrictive assumption at low frequencies.

The incoming field is still the field in the absence of the cracks and is thus given by Eq. (2.2). The total field is still written as the sum $u = u^{\mathrm{in}} + u^{\mathrm{sc}}$, and the scattered field u^{sc} is of course stochastic in nature due to the randomness of the crack distribution. The exact scattered field is impossible to determine and is in fact of no interest. Instead the ensemble average of the scattered field is calculated and some distance away from the interface this should correspond closely to the field found in practice.

To determine the scattered field the Betti-Rayleigh reciprocal theorem is applied to the elastodynamic states u^{in} and u^{sc}

$$\int_{S^-} \left(u^{\mathrm{in}} \sigma^{\mathrm{sc}}_{yj} - u^{\mathrm{sc}} \sigma^{\mathrm{in}}_{yj} \right) n_j dS = 0,$$

with n_j the outward-pointing normal and an implied summation over $j = 1, 2$. The contour S^- consists of lines along $x = \pm x_0$, $z = -z_0$, and $z = 0^-$, see Fig. 3. Add to this equation a second application of the reciprocal theorem to the same states but with the symmetric contour S^+ in the upper half-plane, see Fig. 3. Along the uncracked part of the interface the integrals then cancel and along the cracked parts the crack-opening displacement v appears. Taking the ensemble average of the sum of the reciprocal theorems gives

$$\int_{z=z_0} \left(u^{\mathrm{in}} \langle \sigma^{\mathrm{sc}}_{yz} \rangle - \langle u^{\mathrm{sc}} \rangle \sigma^{\mathrm{in}}_{yz} \right) dx - \int_{z=-z_0} \left(u^{\mathrm{in}} \langle \sigma^{\mathrm{sc}}_{yz} \rangle - \langle u^{\mathrm{sc}} \rangle \sigma^{\mathrm{in}}_{yz} \right) dx + \left\langle \int_D v\, \sigma^{\mathrm{in}}_{yz} dx \right\rangle = 0. \quad (3.1)$$

Here D is the cracked part of the interface and the brackets denote the ensemble average. The integrals along the lines at $x = \pm x_0$ vanish due to the ensemble average.

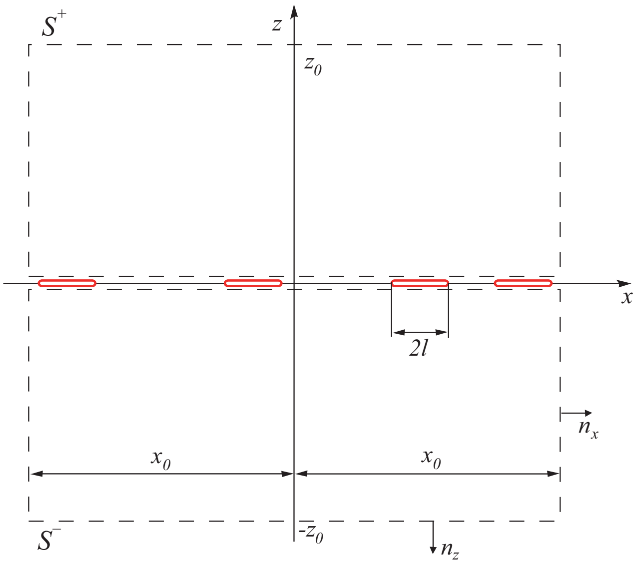

Fig. 3 Integral contours for a distribution of interface cracks.

The ensemble average of the scattered field can only consist of outgoing plane waves propagating in the $\pm z$ direction

$$\langle u^{\mathrm{sc}} \rangle = \begin{cases} P^- e^{-ik_4^1 z}, & z < 0 \\ P^+ e^{ik_4^2 z}, & z > 0 \end{cases} \tag{3.2}$$

The first term in Eq. (3.1) then vanishes because both waves are propagating upwards

$$\int_{z=z_0} \left(u^{\mathrm{in}} \langle \sigma_{yz}^{\mathrm{sc}} \rangle - \langle u^{\mathrm{sc}} \rangle \sigma_{yz}^{\mathrm{in}} \right) dx = 0.$$

The presence of both upgoing and downgoing waves gives a contribution in the second term

$$\int_{z=-z_0} \left(u^{\mathrm{in}} \langle \sigma_{yz}^{\mathrm{sc}} \rangle - \langle u^{\mathrm{sc}} \rangle \sigma_{yz}^{\mathrm{in}} \right) dx = -2ik_4^1 c_{44}^1 P^- (2x_0).$$

As the incoming field contains no stochastic part, the last term in Eq. (3.1) can be written

$$\left\langle \int_D v\, \sigma_{yz}^{\mathrm{in}} dx \right\rangle = 2ik_4^1 c_{44}^1 (1 - R^-)(2x_0) C \overline{v}.$$

Here C is the relative part of the interface that is cracked, which can be written as $C = N_c l / x_0$, where N_c is the number of cracks along the part of the interface of length $2x_0$ (assuming that no crack is cut at the ends). The quantity C can be regarded as a damage parameter that measures the degree to which the interface is cracked. The ensemble average of the crack opening displacement, conditional on the point of observation being a crack point, is

$$\overline{v} = \frac{1}{2l} \int_{-l}^{l} v(x) dx.$$

Thus Eq. (3.1) determines the reflection coefficient

$$P^{-} = -\frac{1}{2}(1 - R^{-}) C \overline{v}. \tag{3.3}$$

Returning again to the reciprocal relation, but now with an incoming wave from above

$$u^{\text{in}} = \begin{cases} T^{+} e^{-ik_4^2 z}, & z < 0 \\ e^{-ik_4^1 z} + R^{+} e^{ik_4^1 z}, & z > 0 \end{cases} \tag{3.4}$$

Here the reflection and transmission coefficients are related through

$$R^{+} = -R^{-},$$

$$T^{+} = 1 + R^{+}.$$

However, the scattered wave is kept the same, i.e. it is not the scattered wave emanating from the new incoming wave. (This is of course possible, because the reciprocal relation is valid for any two states that satisfy the correct wave equations.) Similar operations now give

$$P^{+} = -\frac{1}{2}(1 + R^{-}) C \overline{v}.$$

Subsequently the ensemble average of the total transmission coefficient for the cracked interface can be given as

$$\widetilde{T} = T^{-} + P^{+} = T^{-} \left(1 - \frac{1}{2} C \overline{v} \right). \tag{3.5}$$

The total transmission by the cracked interface is thus obtained in terms of the material constants, the length of the small cracks, and the damage parameter C.

4 The spring boundary condition

The next step is to transform the above transmission coefficient into a (distributed) spring boundary condition. This cannot be done exactly but in the low frequency limit (or for small cracks) this is readily performed. Consider again the reflection and transmission at an interface between two half-spaces using the same ansatz Eq. (2.2) as in Sec. 2, but with the reflection and transmission coefficients denoted by $\widehat{R}^{-}$ and $\widehat{T}^{-}$. The boundary conditions between the half-spaces are no longer continuity of displacement and stress, but rather continuity of stress and proportionality between stress and the crack opening displacement

$$\sigma_{yz}^{1} = \sigma_{yz}^{2} = \kappa \left(u^{1} - u^{2} \right), \tag{4.1}$$

where κ is the (distributed) spring constant. The reflection and transmission coefficients are obtained as

$$\widehat{R}^{-} = \frac{ic_{44}^1 k_4^1 c_{44}^2 k_4^2 + \kappa(c_{44}^1 k_4^1 - c_{44}^2 k_4^2)}{ic_{44}^1 k_4^1 c_{44}^2 k_4^2 + \kappa(c_{44}^1 k_4^1 + c_{44}^2 k_4^2)}, \tag{4.2}$$

$$\widehat{T}^{-} = \frac{2\kappa c_{44}^1 k_4^1}{ic_{44}^1 k_4^1 c_{44}^2 k_4^2 + \kappa(c_{44}^1 k_4^1 + c_{44}^2 k_4^2)}. \tag{4.3}$$

In the limit $\kappa \to \infty$ the interface becomes infinitely stiff and the reflection and transmission coefficients from Sect. 2 are obtained. When $\kappa \to 0$ total reflection results, of course.

The expression for $\widehat{T}^{-}$ cannot be equated directly with Eq. (3.5), because the functional forms do not agree. To this end Eq (4.3) is first expanded for large κ

$$\widehat{T}^{-} = \frac{2c_{44}^1 k_4^1}{c_{44}^1 k_4^1 + c_{44}^2 k_4^2}\left(1 - \frac{ic_{44}^1 k_4^1 c_{44}^2 k_4^2}{c_{44}^1 k_4^1 + c_{44}^2 k_4^2} \cdot \kappa^{-1}\right) + O(\kappa^{-2}).$$

Equating this with Eq. (3.5) and using the expression for K from Eq. (2.12) gives the spring constant

$$\kappa = \frac{4}{C\pi l} \cdot \frac{c_{44}^1 c_{44}^2 k_4^1 k_4^2}{c_{44}^1 k_4^1 k_6^2 + c_{44}^2 k_6^1 k_4^2}. \tag{4.4}$$

It is worth remarking on the condition that κ must be large. As κ is not dimensionless it is first necessary to introduce a dimensionless quantity and the natural choice is to put

$$\kappa = c_{44}^1 k_4^1 \gamma,$$

where γ is the dimensionless parameter that determines the convergence of the expansion of $\widehat{T}^{-}$ above. Assuming that the quotients between stiffnesses and wave speeds in the two materials are not large or small, γ is large if the product between the damage parameter C and the dimensionless crack length $k_4^1 l$ is small. The expression (4.4) for the spring stiffness is approximately equal to the more rough estimate for the spring stiffness obtained by Achenbach and Li (**5**).

5 Plate theory with spring boundary conditions

As an application of the spring boundary condition, the plate equation for the symmetric waves in a symmetric three-layered plate as shown in Fig. 4 is considered. For a plate equation to be meaningful it is of course assumed that the total thickness of the plate is (much) smaller than the wavelength. It may seem questionable to use the spring boundary condition in this context as it was derived via the transmission coefficient for the distribution of cracks, which implicitly seems to require a certain thickness. As long as the microcracks that constitute the damage at the interface are smaller than the layer thicknesses, it might still be a reasonable assumption. In this section it is no longer assumed that the waves are time harmonic.

The material parameters in the middle layer are denoted by an upper index $j = 1$ and the two surrounding layers by $j = 2$. Still only SH waves in the xz plane are considered. The middle layer is situated at $|z| < d_1$ and the outer layers at $d_1 < |z| < d_2$. Due to symmetry

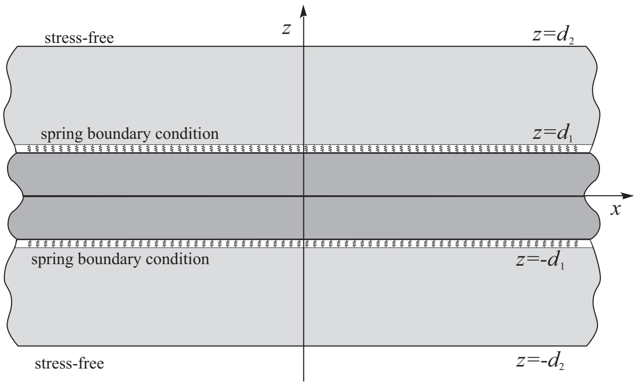

Fig. 4 Three-layered symmetric plate with spring boundary conditions.

only the top half of the plate is considered. The boundary conditions are that the upper surface is stress-free and that the interface is described by the spring boundary conditions

$$\begin{gathered} \tau^2 = 0, \quad z = d_2, \\ \tau^1 = \tau^2 = \kappa(u^1 - u^2), \quad z = d_1 \end{gathered} \tag{5.1}$$

Due to the symmetry assumption the fields in the two layers are expanded as

$$u(x,z,t) = \begin{cases} u_0^1(x,t) + z^2 u_2^1(x,t) + z^4 u_4^1(x,t) + \dots, & 0 < z < d_1 \\ u_0^2(x,t) + d_2 z u_1^2(x,t) + z^2 u_2^2(x,t) + d_2 z^3 u_3^2(x,t) + \dots, & d_1 < z < d_2 \end{cases} \tag{5.2}$$

The extra d_2 in the odd powers is introduced for convenience.

Inserting the expansion (5.2) into the equation of motion (2.1) and equating equal powers of z gives

$$c_{66}^j \frac{\partial^2 u_n^j}{\partial x^2} + (n+2)(n+1) c_{44}^j u_{n+2}^j = \rho^j \frac{\partial^2 u_n^j}{\partial t^2}, \qquad j = 1, 2; \quad n = 0, 1, ..$$

Introduce the wave operator

$$D_j u_n^j = \left(\rho^j \frac{\partial^2 u_n^j}{\partial t^2} - c_{66}^j \frac{\partial^2 u_n^j}{\partial x^2} \right), \quad j = 1, 2,$$

and write this as a recursion relation

$$u_n^j = \frac{1}{n!} \cdot \frac{1}{(c_{44}^j)^{n/2}} \cdot D_j^{n/2} u_0^j, \quad n = 2, 4, \dots,$$

$$u_n^j = \frac{1}{n!} \cdot \frac{1}{(c_{44}^j)^{(n-1)/2}} \cdot D_j^{(n-1)/2} u_1^j, \quad n = 3, 5, \dots.$$

This can be used to eliminate all the expansion functions except u_0^1, u_0^2, and u_1^2. The boundary condition on $z = d_2$ gives

$$c_{44}^2 u_1^2 = -D_2 u_0^2 + O(d_2^2),$$

keeping only the two terms of lowest order. Keeping terms to linear order, the spring boundary condition on $z = d_1$ gives

$$d_1 D_1 u_0^1 - c_{44}^2 d_2 u_1^2 - d_1 D_2 u_0^2 + O(d_2^2) = 0,$$

$$d_1 D_1 u_0^1 = \kappa c_{44}^1 (u_0^1 - u_0^2) + O(d_2^2).$$

If u_0^2 and u_2^2 are eliminated a single plate equation for the displacement u_0^1 of the centre of the plate is obtained

$$\left[d_1 D_1 + (d_2 - d_1) D_2 + \frac{d_1(d_2 - d_1)}{\kappa} D_2 D_1 \right] u_0^1 = 0. \tag{5.3}$$

This equation is written to second order in thickness, but it is straightforward to go to higher orders. The first two terms are just the thickness weighted average of the the two wave operators, as should be expected. The last term describes the influence of the spring boundary condition, i.e. of the damage. In the limit $\kappa \to 0$ the last term dominates and gives two uncoupled wave equations for each layer, exactly as it should.

Assume now a propagating mode in the plate, which can be assumed to be given by

$$u_0^1 = U_0 \exp(i(kx - \omega t)).$$

As before ω is the angular frequency and k is the wave number in the propagation direction. Insertion into the equation (5.3) gives a quadratic equation in k^2 which gives one real root for k. As an example with strong anisotropy, consider a fiber-reinforced graphite-epoxy composite which has density 1578 kg/m^3 and stiffness constants $c_{44}^1 = 3.50$ GPa and $c_{66}^1 = 7.07$ GPa. The lay-up is assumed to be $0°/90°/0°$ so that the material in the middle layer is rotated $90°$ relative the top and bottom layers. The quotient between the layer thicknesses is $d_2/d_1 = 2$, so that there are as much fibers in the two perpendicular directions.

The dimensionless phase velocity v_{ph} as a function of frequency for the first symmetric mode for this plate is shown in Fig. 5 for different values of the parameter combination Cl/d (which determines the spring constant). Here the dimensionless frequency $\omega d\sqrt{\rho/c_{44}^1}$ and dimensionless phase velocity $v_{ph} = kd/\omega$ are used, where $d = 2(d_1 + d_2)$ is the total plate thickness. In Fig. 5(a) the phase velocity from the plate theory is shown and in Fig. 5(b) the phase velocity due to an exact calculation of the dispersion relation (solving the wave equation in each layer and matching the boundary conditions). Note that small values of Cl/d correspond to large values of the spring constant. It is to be expected that the plate theory is valid only for low frequencies and small Cl/d and that is exactly what Fig. 5 shows.

6 Concluding remarks

A model for a layered anisotropic plate with interface damage has been constructed and a few numerical results given to show the influence of the damage. A key feature of the method is the modeling of the damage as a (distributed) spring boundary condition. A way to check the validity of using this boundary condition to model damage in a plate theory

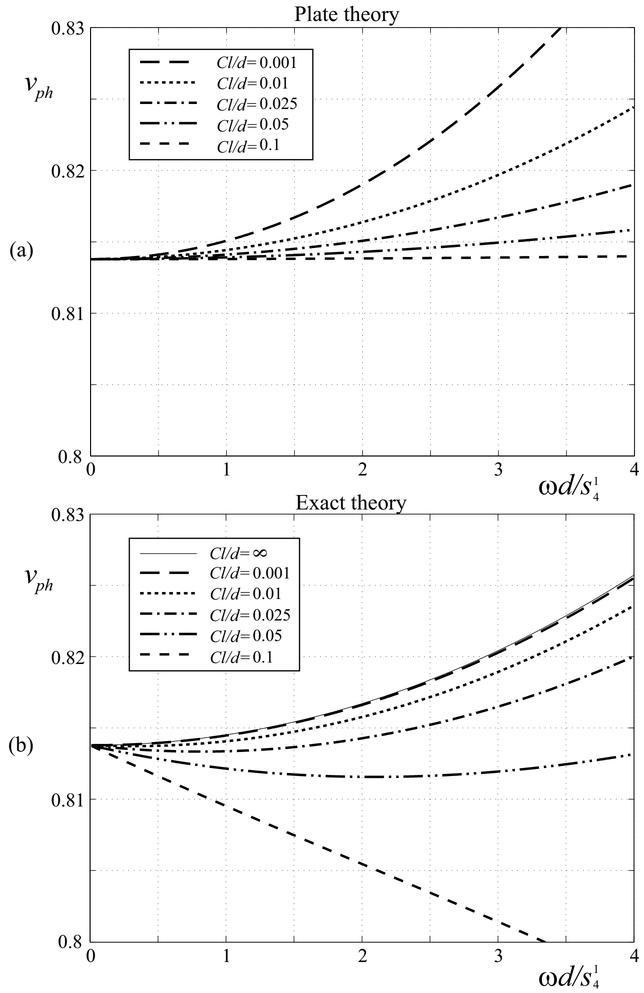

Fig. 5 Phase velocity as a function of frequency: (a) plate theory, (b) exact theory.

would be to compare with a periodic distribution of microcracks at the interface, as this is a problem that is possible to solve exactly and at low frequencies should correspond closely to a random distribution, cf. Sotiropoulos and Achenbach (**10**).

In this paper only the 2D case with SH waves is investigated, but it should be possible to generalize to 2D in-plane waves and to 3D problems (with rectangular cracks). Other problems of interest include plates with localized damage and plates that are excited by reasonable models of ultrasonic probes.

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