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An Alternative Derivation of the Gaussian Mixture Cardinalized Probability Hypothesis Density Filter

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Abstract

In this report, an alternative approach to derive the Gaussian mixture cardinalized probability hypothesis density (GM-CPHD) filter is presented. The derivations differ in that the presented ones are based on "ordinary" statistics, while the original GM-CPHD derivation started from the finite set statistics (FISST) description of the CPHD filter. The results of the derivations are compared with filter update equations presented in another paper. The sets of equations are not completely equivalent. However, initial performance evaluations of the approaches indicate similar performance. Future work is needed to understand the differences between different GM-CPHD filter equations.

CONTENTS

1 INTRODUCTION

The purpose of this report is to derive the Cardinalized Probability Hypothesis Density (CPHD) filter, using ordinary (Bayesian) statistics. More precisely, we are interested in the Gaussian Mixture version of the algorithm, called GM-CPHD. Since previous derivations rely on higher-order mathematical concepts, we believe that a derivation using more common mathematical statistics is of interest to more people than the authors of this report.

CPHD is a recursive algorithm that updates both the so-called probability hypothesis density, or intensity function, and the full cardinality distribution in each step. The cardinality distribution is a representation of the number of targets in the scene, while the intensity function represents the intensity of targets in each volume element of the single-target state space. The integral of the intensity function over a volume, gives the expected number of targets within that volume.

The original derivations of CPHD (Mahler 2006), (Mahler 2007) used Finite Set Statistics (FISST), which is a higher-order mathematical concept invented by Mahler. It has relations to point process theory (Daley 1988). A CPHD derivation using infinitesimal bins is given in (Erdinc *et al.* 2006). The first derivation of the gaussian mixture version is in (Vo *et al.* 2006), with a more thorough discussion in (Vo *et al.* 2007). The GM-CPHD algorithm is also summarized in (Ulmke *et al.* 2007).

The equations derived in this report regards the update step of the GM-CPHD filter. The derived equations are compared to the equations in (Ulmke *et al.* 2007). The update step has two parts, viz. the update of the cardinality distribution, and the update of the mixture components of the intensity function representation. For the cardinality distribution, the update equation derived in this report is equivalent to the corresponding equation in (Erdinc *et al.* 2006). For the mixture components, there is however a difference. Both equations have been used for tracking in a ground target tracking scenario, for which the two alternative updates performs equally well.

1.1 Outline

The report is structured in the following way. The remaining parts of Section 1 is devoted to notation used in the report and to the assumptions made in the CPHD derivation. Section 2 regards the update of the cardinality distribution. In Section 3, we derive the update step for the intensity function. The results are summarized in Section 4.

1.2 Notation

The following list summarizes some of the notation used in this report. An RFS is a random finite set. That is, the number of elements is a discrete random variable, and each element is a random variable (often continuous).

$\mathbf{X}_k$	Target RFS at time k
$\mathbf{Z}_k$	Measurement RFS (including both true and clutter measurements)
$v_{k\|k-1}$	Predicted intensity function at time k
$\Pr\{n\|\mathbf{Z}_{1:k-1}\}$	Predicted cardinality distribution at time k
$N_{k\|k-1}$	Expected number of targets at time k, given old data.
$v_{k\|k}$	Posterior intensity function at time index k
$\Pr\{n\|\mathbf{Z}_{1:k}\}$	Posterior cardinality distribution at time k
$N_{k\|k}$	Expected number of targets at time k, given data up to time k
n_k	Number of targets at time k
m_k	The cardinality of the measurement set at time k, $m_k = \|\mathbf{Z}_k\|$
j_k	Number of target-generated measurements at time k
P_c	The clutter cardinality distribution
$\mathbf{d}$	Vector that states which measurements in a set that are target-generated, and which are clutter
$\mathbf{a}$	Vector that associates measurements to targets

1.3 Assumptions

The CPHD relies on the following assumptions.

- Clutter is a cluster RFS, which implies that the number of elements in the set is described by an arbitrary cardinality mass function and that the elements of the set are independent and identically distributed. Further, the clutter RFS is independent of the object generated RFS.
- Predicted and posterior target RFSs are approximated as cluster RFSs.
- The birth RFS is a cluster RFS, and independent of the surviving target RFS.
- Each target evolves and generates measurements independently of each other.

In the derivations, we also utilize the following assumptions

- Motion and measurement models are linear and Gaussian.
- The detection probability is constant over the single-target state space.
- The clutter RFS homogeneous, which means that the clutter detections are spatially homogeneous.

2 CARDINALITY UPDATE

Using Bayes' rule, the posterior cardinality mass function is expressed as

$$\Pr\{n_k|\mathbf{Z}_{1:k}\} = \frac{p(\mathbf{Z}_k|n_k, \mathbf{Z}_{1:k-1}) \Pr\{n_k|\mathbf{Z}_{1:k-1}\}}{p(\mathbf{Z}_k|\mathbf{Z}_{1:k-1})} \tag{2.1}$$

where the first factor $p(\mathbf{Z}_k|n_k, \mathbf{Z}_{1:k-1})$ is the density of the measurement set, given n_k targets and old data. As a function of n_k it is a likelihood. The second numerator factor is the cardinality mass function at the previous time instant $k-1$, which is a prior on n_k at time k.

2.1 The likelihood $p(\mathbf{Z}_k|n_k, \mathbf{Z}_{1:k-1})$

We start off by deriving the likelihood. Since $\mathbf{Z}_k$ is an RFS, it is described by both the probability of its cardinality $m_k = |\mathbf{Z}_k|$ and the density of those m_k elements. Thus,

$$p(\mathbf{Z}_k|n_k, \mathbf{Z}_{1:k-1}) = p(\mathbf{Z}_k, m_k|n_k, \mathbf{Z}_{1:k-1}), \tag{2.2}$$

since m_k is an intrinsic property of $\mathbf{Z}_k$. We introduce the variable j_k of target-generated measurements at time k, as it provides a way of expressing the above density. We marginalize over j_k,

$$\begin{aligned}
p(\mathbf{Z}_k, m|n) &= \sum_{j_k=0}^{\min\{m_k,n_k\}} p(\mathbf{Z}_k, m_k, j_k|n_k, \mathbf{Z}_{1:k-1}) && (2.3)\\
&= \sum_{j_k=0}^{\min\{m_k,n_k\}} p(\mathbf{Z}_k|m_k, j_k, n_k, \mathbf{Z}_{1:k-1}) \Pr\{m_k, j_k|n_k, \mathbf{Z}_{1:k-1}\} \\
&= \sum_{j_k=0}^{\min\{m_k,n_k\}} p(\mathbf{Z}_k|m_k, j_k, n_k, \mathbf{Z}_{1:k-1}) \Pr\{m_k|j_k, n_k, \mathbf{Z}_{1:k-1}\} && (2.4)\\
&\qquad \cdot \Pr\{j_k|n_k, \mathbf{Z}_{1:k-1}\}.
\end{aligned}$$

Naturally, j_k cannot be larger than the minimum of the number of targets and the total number of measurements, hence the min function in the summation.

For notational simplicity, we introduce the notation

$$\mathbf{Z}_k^p \equiv \mathbf{Z}_{1:k-1}. \tag{2.5}$$

We now derive the latter two probabilities in the sum of (2.4). First, the probability of receiving m_k measurements, given that the true number of measurements is j_k

and the number of targets is n_k. Here, n_k and old data are uninformative given m_k and j_k, so

$$\Pr\{m_k|j_k, n_k, \mathbf{Z}_k^p\} = \Pr\{m_k - j_k \text{ clutter detections}\} = P_c(m_k - j_k), \tag{2.6}$$

where P_c is the cardinality distribution of the clutter RFS, often assumed Poisson. The third factor in the sum of (2.4) is the probability of obtaining j_k true measurements given that the number of targets is n_k, and given old data. The j_k true detections can be drawn from the n_k true ones in $\binom{n_k}{j_k}$ ways, hence

$$\Pr\{j_k|n_k, \mathbf{Z}_k^p\} = \binom{n_k}{j_k} P_d^{j_k}(1-P_d)^{n_k-j_k} = \frac{n_k!}{j_k!(n_k - j_k!)} P_d^{j_k}(1-P_d)^{n_k-j_k}, \tag{2.7}$$

where P_d is the detection probability, assumed constant over the measurement space (if not, the average detection probability can be used).

The remaining factor $p(\mathbf{Z}_k|m_k, j_k, n_k, \mathbf{Z}_k^p)$ in (2.4) is the most complicated one. In order to find an expression for it, we start by transforming the measurement set into a matrix, meaning that we introduce an ordering of the measurements. There are $m_k!$ ways of ordering a set of m_k elements, and all of them are equally likely. Since a point in the m_k-dimensional set space represents $m_k!$ different points in the m_k-dimensional vector space that all have the same interpretation, the set density can be written as

$$p(\mathbf{Z}_k|m_k, j_k, n_k) = m_k! p(\overline{\mathbf{Z}}_k|m_k, j_k, n_k, \mathbf{Z}_k^p), \tag{2.8}$$

where $\overline{\mathbf{Z}}_k$ represents any of the $m_k!$ set-to-matrix transformations of the set. The mapping from set to matrix is many-to-one, while the reverse mapping is one-to-one. However, it does not matter which of the set-to-matrix transformations that are chosen, since the density $p(\overline{\mathbf{Z}}_k|m_k, j_k, n_k, \mathbf{Z}_k^p)$ is equal regardless of how the columns of $\overline{\mathbf{Z}}_k$ are ordered. Note that even though we order the set, we still do not know which of the detections that are true and which are clutter. So to express the density function, we want to marginalize over a variable that provides information on if a measurement in $\overline{\mathbf{Z}}_k$ is clutter or target-generated. Out of the m_k received measurements, j_k are target-generated. These j_k observations can be drawn in $\binom{m_k}{j_k}$ different ways from the set of observations. For one selection of target-generated measurements (the rest being clutter), we introduce the vector $\mathbf{d}$ with the property

$$d(i) = \begin{cases} 1 & \text{if measurement } i \text{ is true} \\ 0 & \text{if it is clutter.} \end{cases} \tag{2.9}$$

The marginalization over $\mathbf{d}$ is

$$\begin{aligned} &\sum_{\mathbf{d}} p(\overline{\mathbf{Z}}_k, \mathbf{d}|m_k, j_k, n_k, \mathbf{Z}_k^p) \\ &\quad = \sum_{\mathbf{d}} p(\overline{\mathbf{Z}}_k|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) \Pr\{\mathbf{d}|m_k, j_k, n_k, \mathbf{Z}_k^p\}. \end{aligned} \tag{2.10}$$

The factor $\Pr\{\mathbf{d}|m_k, j_k, n_k, \mathbf{Z}_k^p\}$ is interpreted as the prior on $\mathbf{d}$, which is uniform, hence

$$\Pr\{\mathbf{d}|m_k, j_k, n_k, \mathbf{Z}_k^p\} = \frac{1}{\binom{m_k}{j_k}} = \frac{1}{\frac{m_k!}{j_k!(m_k-j_k)!}} = \frac{j_k!(m_k-j_k)!}{m_k!}. \tag{2.11}$$

What now remains is an expression for the likelihood of the measurement matrix $\overline{\mathbf{Z}}_k$, given that we know which of the measurements are true and which are clutter. First of all, we can separate the true detections from clutter, since they are independent

$$\begin{aligned} p(\overline{\mathbf{Z}}_k|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = p(&\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) \\ &\cdot p(\overline{\mathbf{Z}}_k^{\text{clut}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p). \end{aligned} \tag{2.12}$$

The conditioning on m_k, j_k, n is uninformative when also conditioning on $\mathbf{d}$, but is kept for clarity. The clutter detections are independent, with densities $c\,(\mathbf{z}_k)$. Assuming uniformly distributed clutter in the measurement space, we get

$$p(\overline{\mathbf{Z}}_k^{\text{clut}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = c^{m_k-j_k} \tag{2.13}$$

where c is the clutter density in the measurement space. Conversion of density from target state-space to measurement state-space is through a Jacobian. If one cannot assume uniform distribution, the expression will get just a bit more complicated, where the density at each clutter measurement has to be used instead of the constant density.

To express the joint density $p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p)$ of the target-generated detections in (2.13), we need to marginalize over the target RFS $\mathbf{X}_k$, which has n_k elements

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = \int p(\overline{\mathbf{Z}}_k^{\text{det}}, \mathbf{X}_k|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\mathbf{X}_k. \tag{2.14}$$

To proceed further, we transform $\mathbf{X}_k$ into a matrix $\overline{\mathbf{X}}_k$, just as we did for the measurement set previously. There are $n_k!$ different transformations into a matrix, for the set $\mathbf{X}_k$. The resulting density is however the same, since the elements of the set are unordered. So,

$$\begin{aligned} p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = n_k! \int &p(\overline{\mathbf{Z}}_k^{\text{det}}|\overline{\mathbf{X}}_k, \mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) \\ &\cdot p(\overline{\mathbf{X}}_k|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\overline{\mathbf{X}}_k. \end{aligned} \tag{2.15}$$

We start with the second factor. An assumption of the CPHD algorithm is that targets are independent, which implies that the density can be split up into a product of n_k single-target densities $p(\overline{\mathbf{x}}_k^{(q)}|\mathbf{d}, m_k, j_k, n_k)$ according to

$$p(\overline{\mathbf{X}}_k|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = \prod_{q=1}^{n_k} p(\overline{\mathbf{x}}_k^{(q)}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p). \tag{2.16}$$

We now want to find an expression for $p(\overline{\mathbf{Z}}_k^{\text{det}}|\overline{\mathbf{X}}_k, \mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p)$ in (2.15). The conditioning on the matrix of state vectors does not provide information about which measurement that belongs to which target (if necessary). To be formal and concise, we thus introduce an assignment vector $\mathbf{a}$, which associates measurements to targets. Then we marginalize over all such vectors, yielding

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\overline{\mathbf{X}}_k, \mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = \sum_{\mathbf{a}} p(\overline{\mathbf{Z}}_k^{\text{det}}, \mathbf{a}|\overline{\mathbf{X}}_k, \mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) \tag{2.17}$$

$$= \sum_{\mathbf{a}} p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{a}, \overline{\mathbf{X}}_k, \mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) \Pr\{\mathbf{a}|\overline{\mathbf{X}}_k, \mathbf{d}, m_k, j_k, n, \mathbf{Z}_k^p\}. \tag{2.18}$$

As stated several times, the targets are assumed identically distributed. This means that all assignment vectors have the same probability

$$\Pr\{\mathbf{a}|\overline{\mathbf{X}}_k, \mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p\} = \frac{1}{\frac{n_k!}{j_k!}}, \tag{2.19}$$

where $\frac{n_k!}{j_k!}$ is the number of possible association vectors. For the likelihood $p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{a}, \overline{\mathbf{X}}_k, \mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p)$ we know that given the associations, the measurements are independent. The joint density is thus split up into a product of the j_k single-detection likelihoods

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{a}, \overline{\mathbf{X}}_k, \mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = \prod_{s \in \mathcal{D}} p(\overline{\mathbf{z}}_k^{(s)}|\overline{\mathbf{x}}_k^{(s)}), \tag{2.20}$$

where we assume that the index of measurements and their associated targets are the same, i.e., measurement s is associated with target s. The set $\mathcal{D}$ is introduced to represent the indices of the target-generated detections, i.e., it is the set of all indexes i for which $d(i) = 1$.

We now insert the results of (2.16) and (2.20) into (2.15),

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = n_k! \int \sum_{\mathbf{a}} \prod_{s \in \mathcal{D}} p(\overline{\mathbf{z}}_k^{(s)}|\overline{\mathbf{x}}_k^{(s)}) \frac{1}{\frac{n_k!}{j_k!}} \tag{2.21}$$
$$\cdot \prod_{q=1}^{n_k} p(\overline{\mathbf{x}}_k^{(q)}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\overline{\mathbf{x}}_k^{(1)} \cdots d\overline{\mathbf{x}}_k^{(n_k)}.$$

Reordering the multiplications, we obtain

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = n_k! \int \frac{1}{\frac{n_k!}{j_k!}} \sum_{\mathbf{a}} \tag{2.22}$$
$$\prod_{s \in \mathcal{D}} p(\overline{\mathbf{z}}_k^{(s)}|\overline{\mathbf{x}}_k^{(s)}) p(\overline{\mathbf{x}}_k^{(s)}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\overline{\mathbf{x}}_k^{(s)} \prod_{\substack{q=1 \\ q \neq s}}^{n_k} p(\overline{\mathbf{x}}_k^{(q)}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\overline{\mathbf{x}}_k^{(q)}.$$

The multi-dimensional integral can hence be separated into a product of single-dimensional integrals

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = n_k! \frac{1}{\frac{n_k!}{j_k!}} \sum_{\mathbf{a}} \prod_{s\in\mathcal{D}} \int p(\overline{\mathbf{z}}_k^{(s)}|\overline{\mathbf{x}}_k^{(s)}) p(\overline{\mathbf{x}}_k^{(s)}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\overline{\mathbf{x}}_k^{(s)} \tag{2.23}$$

$$\cdot \prod_{\substack{q=1\\q\neq s}}^{n_k} \int p(\overline{\mathbf{x}}_k^{(q)}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\overline{\mathbf{x}}_k^{(q)}.$$

The integral of the single-target density $p(\overline{\mathbf{x}}_k^{(q)}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p)$ is one, so the last factor will be equal to one, whereafter we have

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = n_k! \frac{1}{\frac{n_k!}{j_k!}} \sum_{\mathbf{a}} \prod_{s\in\mathcal{D}} \int p(\overline{\mathbf{z}}_k^{(s)}|\overline{\mathbf{x}}_k^{(s)}) p(\overline{\mathbf{x}}_k^{(s)}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\overline{\mathbf{x}}_k^{(s)}. \tag{2.24}$$

We change the integration variable from $\overline{\mathbf{x}}_k^{(s)}$ to $\overline{\mathbf{x}}_k$, which gives

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = n_k! \frac{1}{\frac{n_k!}{j_k!}} \sum_{\mathbf{a}} \prod_{s\in\mathcal{D}} \int p(\overline{\mathbf{z}}_k^{(s)}|\overline{\mathbf{x}}_k) p(\overline{\mathbf{x}}_k|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\overline{\mathbf{x}}_k. \tag{2.25}$$

As we can see, no terms in the summation depend on the summation variable. The summation is from 1 to $\frac{n_k!}{j_k!}$, so the expression can be simplified

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = n_k! \prod_{s\in\mathcal{D}} \int p(\overline{\mathbf{z}}_k^{(s)}|\overline{\mathbf{x}}_k) p(\overline{\mathbf{x}}_k|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\overline{\mathbf{x}}_k. \tag{2.26}$$

Finally, we transform the ordered state vector into a state vector in the set, yielding

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) = \prod_{s\in\mathcal{D}} \int p(\overline{\mathbf{z}}_k^{(s)}|\mathbf{x}_k) p(\mathbf{x}_k|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_k^p) d\mathbf{x}_k. \tag{2.27}$$

Replacing $\mathbf{Z}_k^p$ with its equivalent $\mathbf{Z}_{1:k-1}$ and removing uninformative variables for clarity, we obtain

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_{1:k-1}) = \prod_{s\in\mathcal{D}} \int p(\overline{\mathbf{z}}_k^{(s)}|\mathbf{x}_k) p(\mathbf{x}_k|\mathbf{Z}_{1:k-1}) d\mathbf{x}_k. \tag{2.28}$$

Again, we now need to utilize the assumptions of CPHD. All targets are assumed identically distributed according to $v(\mathbf{x})/N$ where N is the expected number of targets. Hence, the density

$$p(\mathbf{x}_k|\mathbf{Z}_{1:k-1}) = \frac{v_{k|k-1}(\mathbf{x}_k)}{N_{k|k-1}}, \tag{2.29}$$

where $v_{k|k-1}(\mathbf{x}_k)$ is the predicted PHD and where $N_{k|k-1}$ is the expected number of targets after prediction. Inserted into (2.28)

$$p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_{1:k-1}) = \prod_{s\in\mathcal{D}} \frac{1}{N_{k|k-1}} \int p(\overline{\mathbf{z}}_k^{(s)}|\mathbf{x}_k) v_{k|k-1}(\mathbf{x}_k) d\mathbf{x}_k. \tag{2.30}$$

In the Gaussian-mixture CPHD, the predicted PHD is approximated as

$$v_{k|k-1}(\mathbf{x}_k) \cong \sum_{q=1}^{J_{k|k-1}} w_{k|k-1}^{(q)} \mathcal{N}\left(\mathbf{x}_k; \mathbf{m}_{k|k-1}^{(q)}, \mathbf{P}_{k|k-1}^{(q)}\right), \tag{2.31}$$

in which $w_{k|k-1}^{(q)}$ are the weights, $\mathbf{m}_{k|k-1}^{(q)}$ the expected values and $\mathbf{P}_{k|k-1}^{(q)}$ the covariance matrices of the $J_{k|k-1}$ mixture components of the predicted PHD. Furthermore, the single-detection likelihood is given by the measurement model

$$p(\overline{\mathbf{z}}_k^{(s)}|\mathbf{x}_k) = \mathcal{N}\left(\overline{\mathbf{z}}_k^{(s)}; \mathbf{H}\mathbf{x}_k, \mathbf{R}^{(s)}\right) \tag{2.32}$$

in the linear-Gaussian case, where $\mathbf{H}$ is the measurement matrix and $\mathbf{R}^{(s)}$ the measurement uncertainties in the region of detection s. We obtain

$$\begin{aligned} p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_{1:k-1}) = \prod_{s\in\mathcal{D}} \frac{1}{N_{k|k-1}} \sum_{q=1}^{J_{k|k-1}} w_{k|k-1}^{(q)} \\ \cdot \int \mathcal{N}\left(\overline{\mathbf{z}}_k^{(s)}; \mathbf{H}\mathbf{x}_k, \mathbf{R}^{(s)}\right) \mathcal{N}\left(\mathbf{x}_k; \mathbf{m}_{k|k-1}^{(q)}, \mathbf{P}_{k|k-1}^{(q)}\right) d\mathbf{x}_k. \end{aligned} \tag{2.33}$$

Using standard results from multiplication of normal densities, we get

$$\begin{aligned} \mathcal{N}\left(\overline{\mathbf{z}}_k^{(s)}; \mathbf{H}\mathbf{x}_k, \mathbf{R}^{(s)}\right) \mathcal{N}\left(\mathbf{x}_k; \mathbf{m}_{k|k-1}^{(q)}, \mathbf{P}_{k|k-1}^{(q)}\right) = \mathcal{N}\left(\overline{\mathbf{z}}_k^{(s)}; \mathbf{z}_{k|k-1}^{(q)}, \mathbf{S}_{k|k-1}^{(q,s)}\right) \\ \cdot \mathcal{N}\left(\mathbf{x}_k; \mathbf{m}_{k|k}^{(q,s)}, \mathbf{P}_{k|k}^{(q,s)}\right), \end{aligned} \tag{2.34}$$

where

$$\mathbf{S}_{k|k-1}^{(q)} = \mathbf{R} + \mathbf{H}\mathbf{P}_{k|k-1}^{(q)}\mathbf{H}^T \tag{2.35}$$

$$\mathbf{m}_{k|k}^{(q,s)} = \mathbf{m}_{k|k-1}^{(q)} + \mathbf{W}\left(\mathbf{z}_k^{(s)} - \mathbf{H}\mathbf{m}_{k|k-1}^{(q)}\right) \tag{2.36}$$

$$\mathbf{P}_{k|k}^{(q,s)} = (\mathbf{1} - \mathbf{W}\mathbf{H})\,\mathbf{P}_{k|k-1}^{(q)}. \tag{2.37}$$

As we see, the first Gaussian in (2.34) does not depend on $\mathbf{x}_k$ and can be moved out of the integral in (2.33), leaving only the integral of a Gaussian, which equals one.

Finally, we can express the density of target-generated measurements as

$$\begin{aligned} p(\overline{\mathbf{Z}}_k^{\text{det}}|\mathbf{d}, m_k, j_k, n_k, \mathbf{Z}_{1:k-1}) \\ \cong \prod_{s\in\mathcal{D}} \frac{1}{N_{k|k-1}} \sum_{q=1}^{J_{k|k-1}} w_{k|k-1}^{(q)} \mathcal{N}\left(\overline{\mathbf{z}}_k^{(s)}; \mathbf{z}_{k|k-1}^{(q)}, \mathbf{S}_{k|k-1}^{(q,s)}\right). \end{aligned} \tag{2.38}$$

For notational simplicity we introduce the weighted single-detection likelihood

$$L_k^{(s)} = \frac{1}{N_{k|k-1}} \sum_{q=1}^{J_{k|k-1}} w_{k|k-1}^{(q)} \mathcal{N}\left(\overline{\mathbf{z}}_k^{(s)}; \mathbf{z}_{k|k-1}^{(q)}, \mathbf{S}_{k|k-1}^{(q,s)}\right). \tag{2.39}$$

We are now ready to express the density $p(\mathbf{Z}_k|m_k, j_k, n_k, \mathbf{Z}_{1:k-1})$ in (2.8) by going back to set notation (see (2.8)) and by inserting the results in (2.10), (2.11), (2.12), (2.13), (2.38) and (2.39)

$$\begin{aligned} p(\mathbf{Z}_k|m_k, j_k, n_k, \mathbf{Z}_{1:k-1}) &\cong m_k! \sum_{\mathbf{d}} \prod_{s\in\mathcal{D}} L_k^{(s)} c^{m_k-j_k} \frac{j_k!(m_k-j_k)!}{m_k!} \\ &= j_k!(m_k-j_k)! \left(\frac{1}{V_z}\right)^{m_k-j_k} \sum_{\mathbf{d}} \prod_{s\in\mathcal{D}} L_k^{(s)}. \end{aligned} \tag{2.40}$$

The last sum and product factor of the expression can be expressed using a so-called elementary symmetric function σ_{j_k}

$$\sigma_{j_k}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\}\right) \triangleq \sum_{\mathbf{d}} \prod_{s\in\mathcal{D}} L_k^{(s)}. \tag{2.41}$$

The elementary symmetric function does just what we express with the sum and the product symbols in (2.40), viz. it summarizes over all possible combinations of j_k measurements, and for each combination evaluates the product of the weighted single-detection likelihoods of those j_k measurements. Seen in a different way, if $\{L_k^{(1)}, \ldots, L_k^{(m_k)}\}$ are the roots of a polynomial of order m_k, then σ_{j_k} gives the corresponding polynomial coefficient of order j_k ($j_k = 0, \ldots, m_k$), where $\sigma_0 = 1$ by convention. The density $p(\mathbf{Z}_k|m_k, j_k, n_k)$ in (2.8) is now given by

$$p(\mathbf{Z}_k|m_k, j_k, n_k, \mathbf{Z}_{1:k-1}) \cong j_k!(m_k-j_k)! c^{m_k-j_k} \sigma_{j_k}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\}\right). \tag{2.42}$$

By this we are ready to express the likelihood $p(\mathbf{Z}_k|n_k, \mathbf{Z}_{1:k-1})$ in (2.1). Combining (2.4), (2.6), (2.7) and (2.42) we get

$$\begin{aligned} p(\mathbf{Z}_k, m_k|n_k, \mathbf{Z}_{1:k-1}) \cong &\sum_{j_k=0}^{\min\{m_k,n_k\}} \frac{n!}{j_k!(n_k-j_k!)} P_d^{j_k}(1-P_d)^{n_k-j_k} P_c(m_k-j_k) j_k! \\ &\cdot (m_k-j_k)! c^{m_k-j_k} \sigma_{j_k}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\}\right). \end{aligned} \tag{2.43}$$

Simplifying and rearranging,

$$\begin{aligned} p(\mathbf{Z}_k, m_k|n_k, \mathbf{Z}_{1:k-1}) \cong &\sum_{j_k=0}^{\min\{m_k,n_k\}} (m_k-j_k)! P_c(m_k-j_k) \frac{n_k!}{(n_k-j_k!)} P_d^{j_k} \\ &\cdot (1-P_d)^{n_k-j_k} c^{m_k-j_k} \sigma_{j_k}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\}\right). \end{aligned} \tag{2.44}$$

Since we have a normalization term in the update of the cardinality distribution, constant factors, i.e., those that do not depend on n_k or j_k, can be removed from the above expression, since they will be canceled in the normalization. The factor c^{m_k} is constant and can be removed. We then obtain

$$\begin{aligned} p(\mathbf{Z}_k|n_k, \mathbf{Z}_{1:k-1}) \cong \sum_{j_k=0}^{\min\{m_k,n_k\}} & (m_k - j_k)! P_c(m_k - j_k) \frac{n_k!}{(n_k - j_k!)} P_d^{j_k} \\ & \cdot (1 - P_d)^{n_k - j_k} c^{-j_k} \sigma_{j_k} \left(\left\{ L_k^{(1)}, \ldots, L_k^{(m_k)} \right\} . \right), \end{aligned} \tag{2.45}$$

which is the equation we were after in this section. By comparing the expression in (2.45) with the corresponding expression in (Ulmke *et al.* 2007), we note that they have a factor $\frac{1}{m_k!}$ in the sum, which makes the expressions different. That is, however, a constant, so it can be removed without effect, if it is also removed in the normalization. Then, the above expression is equal to the one in (Ulmke *et al.* 2007). Notice that c in this report equals λ in (Ulmke *et al.* 2007).

A fundamental assumption in CPHD is that target prior and predicted RFSs are cluster-RFSs. That implies that the target are assumed independent and identically distributed. Often, this is not a good assumption, which indicates that there are room for improvements of the algorithm.

2.2 The prior $\Pr\{n_k|\mathbf{Z}_{1:k-1}\}$

The prior $\Pr\{n_k|\mathbf{Z}_{1:k-1}\}$ in the Bayes' update of the cardinality distribution is the predicted cardinality mass function, which states the probability that there at time k are n_k targets in the scene, given data up to the previous time $k-1$. By marginalizing over n_{k-1},

$$\begin{aligned} \Pr\{n_k|\mathbf{Z}_{1:k-1}\} &= \sum_{n_{k-1}=0}^{\infty} \Pr\{n_k, n_{k-1}|\mathbf{Z}_{1:k-1}\} \\ &= \sum_{n_{k-1}=0}^{\infty} \Pr\{n_k|n_{k-1}, \mathbf{Z}_{1:k-1}\} \Pr\{n_{k-1}|\mathbf{Z}_{1:k-1}\}. \end{aligned} \tag{2.46}$$

We identify the latter factor $\Pr\{n_{k-1}|\mathbf{Z}_{1:k-1}\}$ as the cardinality distribution at the previous time instant, which is known at the current instance. We thus only need to find an expression for the first factor.

From the birth process, we have the probability

$$P_{\text{birth}}(i_k) = \Pr\{i_k \text{ new targets appear between time } k-1 \text{ and time } k\}, \tag{2.47}$$

and the death process provides information on

$$P_S = \Pr\{\text{An existing target at time } k-1 \text{ survives to time } k\}. \tag{2.48}$$

We define i_k as the number of targets that survive from the previous time instant, and marginalize over that variable. Naturally, i_k is not larger than the minimum of n_k and n_{k-1}. Using the probabilities of the birth and death processes, the sought probability is expressed as

$$\Pr\{n_k|n_{k-1}, \mathbf{Z}_{1:k-1}\} = \Pr\{n_k|n_{k-1}\} = \sum_{i_k=0}^{\min\{n_k,n_{k-1}\}} \Pr\{n_k, i_k|n_{k-1}\} \tag{2.49}$$

$$= \sum_{i_k=0}^{\min\{n_k,n_{k-1}\}} \Pr\{n_k|i_k, n_{k-1}\} \Pr\{i_k|n_{k-1}\}. \tag{2.50}$$

The first factor is the probability that $n_k - i_k$ targets appear between time instants $k-1$ and k, expressed as

$$\Pr\{n_k|i, n_{k-1}\} = \Pr\{n_k|i_k\} = P_{\text{birth}}(n_k - i_k). \tag{2.51}$$

The second factor is the probability that i_k targets survive. Of the n_{k-1} targets at $k-1$, the i_k surviving ones can be drawn in $\binom{n_{k-1}}{i_k}$ ways, so

$$\Pr\{i_k|n_{k-1}\} = \binom{n_{k-1}}{i_k} P_S^{i_k}(1-P_S)^{n_{k-1}-i_k}. \tag{2.52}$$

The total expression for the prior is hence

$$\Pr\{n_k|n_{k-1}|\mathbf{Z}_{1:k-1}\} = \sum_{n_{k-1}=0}^{\infty} \Pr\{n_{k-1}|\mathbf{Z}_{1:k-1}\} \tag{2.53}$$
$$\cdot \sum_{i_k=0}^{\min\{n_k,n_{k-1}\}} P_{\text{birth}}(n_k - i_k) \binom{n_{k-1}}{i_k} P_S^{i_k}(1-P_S)^{n_{k-1}-i_k}.$$

In short form, we write this as

$$\Pr\{n_k|\mathbf{Z}_{1:k-1}\} = \sum_{n_{k-1}=0}^{\infty} \Pr\{n_k|n_{k-1}\} \Pr\{n_{k-1}|\mathbf{Z}_{1:k-1}\}, \tag{2.54}$$

where the so-called transfer matrix is defined as

$$\Pr\{n_k|n_{k-1}\} = \sum_{i_k=0}^{\min\{n_k,n_{k-1}\}} P_{\text{birth}}(n_k - i_k) \binom{n_{k-1}}{i_k} P_S^{i_k}(1-P_S)^{n_{k-1}-i_k}. \tag{2.55}$$

2.3 The normalization $p(\mathbf{Z}_k|\mathbf{Z}_{1:k-1})$

What remains in the update of the cardinality distribution $\Pr\{n_k|\mathbf{Z}_{1:k-1}\}$ is the normalization factor in the Bayes' update (see (2.1)). It is given as the sum over n_k

from 0 to infinity of the numerator, assuring that the result after the Bayes' update is a probability mass function. Utilizing the results of (2.45), we obtain

$$
\begin{aligned}
p(\mathbf{Z}_k|\mathbf{Z}_{1:k-1}) &= \sum_{n_k=0}^{\infty} p(\mathbf{Z}_k|n_k)\Pr\{n_k|\mathbf{Z}_{1:k-1}\} && (2.56)\\
&\cong \sum_{n_k=0}^{\infty}\left[\sum_{j_k=0}^{\min\{m_k,n_k\}}\left[(m_k-j_k)!P_c(m_k-j_k)\frac{n_k!}{(n_k-j_k!)}P_d^{j_k}(1-P_d)^{n_k-j_k}\right.\right. && (2.57)\\
&\qquad\left.\left.\cdot\, c^{-j_k}\sigma_{j_k}\left(\left\{L_k^{(1)},\ldots,L_k^{(m_k)}\right\}\right)\right]\Pr\{n_k|\mathbf{Z}_{1:k-1}\}\right].
\end{aligned}
$$

The summations are absolutely convergent, which implies that the order of summation can be switched. By doing so, we obtain the normalization expression used in (Ulmke *et al.* 2007) (except for a constant $\frac{1}{m_k!}$)

$$
\begin{aligned}
p(\mathbf{Z}_k|\mathbf{Z}_{1:k-1}) &= \sum_{n_k=0}^{\infty} p(\mathbf{Z}_k|n_k)\Pr\{n_k|\mathbf{Z}_{1:k-1}\} && (2.58)\\
&\cong \sum_{j_k=0}^{m_k}\left[\sum_{n_k=j_k}^{\infty}\frac{n_k!}{(n_k-j_k!)}(1-P_d)^{n_k-j_k}\Pr\{n_k|\mathbf{Z}_{1:k-1}\}\right] && (2.59)\\
&\qquad(m_k-j_k)!P_c(m_k-j_k)c^{-j_k}P_d^{j_k}\sigma_{j_k}\left(\left\{L_k^{(1)},\ldots,L_k^{(m_k)}\right\}\right).
\end{aligned}
$$

3 PHD UPDATE

In this section, we derive the update equations for the mixture weights of the posterior intensity function, in the Gaussian mixture version of the CPHD algorithm. We assume that we have information on the predicted cardinality $\Pr\{n_k|\mathbf{Z}_{1:k-1}\}$ and the predicted PHD $v_{k|k-1}(\mathbf{x}_k)$, and seek an update equation for the posterior PHD $v_{k|k}(\mathbf{x}_k)$. To do so, we start by expressing the posterior multi-target density $p(\mathbf{X}_k|\mathbf{Z}_{1:k})$. Then, we introduce the CPHD assumptions and necessary simplifications to finally reach the CPHD update for the posterior intensity function.

Using Bayes' rule, we get

$$p(\mathbf{X}_k|\mathbf{Z}_{1:k}) = \frac{p(\mathbf{Z}_k|\mathbf{X}_k, \mathbf{Z}_{1:k-1})p(\mathbf{X}_k|\mathbf{Z}_{1:k-1})}{p(\mathbf{Z}_k|\mathbf{Z}_{1:k-1})}. \tag{3.1}$$

We note that the denominator in (3.1) is equal to the one in (2.1), which we gave expressions for in (2.57) and (2.59). To express the posterior multi-target density $p(\mathbf{X}_k|\mathbf{Z}_{1:k})$, we thus need to find equations that describe the multi-target prior $p(\mathbf{X}_k|\mathbf{Z}_{1:k-1})$ and the multi-target likelihood $p(\mathbf{Z}_k|\mathbf{X}_k, \mathbf{Z}_{1:k-1})$. This is an intricate issue, which is combinatorial in nature and infeasible in practice. Approximations are therefore a necessity. The CPHD update equation is an example of such an approximation.

Instead of working with the posterior RFS $\mathbf{X}_k|\mathbf{Z}_{1:k}$, we turn to its first-order moment (in the FISST sense): the posterior PHD $v_{k|k}(\mathbf{x}_k)$. In the approximation, we lose information on the relationship between the position of the targets and on the number of targets. In conjunction to the intensity function $v_{k|k}(\mathbf{x}_k)$, we also propagate the cardinality distribution $\Pr\{n_k|\mathbf{Z}_{1:k}\}$ in time, from which an expected value on the number of targets is found.

An assumption in CPHD is that targets are independent and identically distributed. Suppose that we randomly select one of the targets $\mathbf{x}_k$ within the set $\mathbf{X}_k$. According to the assumptions, that target is then distributed according to

$$p(\mathbf{x}_k|\mathbf{Z}_{1:k}) = \frac{v_{k|k}(\mathbf{x}_k)}{N_{k|k}}. \tag{3.2}$$

So, in order to update the posterior intensity $v_{k|k}(\mathbf{x}_k)$, we choose a random target and then calculate

$$v_{k|k}(\mathbf{x}_k) = p(\mathbf{x}_k|\mathbf{Z}_{1:k})N_{k|k}. \tag{3.3}$$

Rewriting, using Bayes' rule,

$$v_{k|k}(\mathbf{x}_k) = \frac{p(\mathbf{Z}_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1})p(\mathbf{x}_k|\mathbf{Z}_{1:k-1})}{p(\mathbf{Z}_k|\mathbf{Z}_{1:k-1})}N_{k|k}. \tag{3.4}$$

With the CPHD assumptions of independent and identically distributed targets at prediction, $p(\mathbf{x}_k|\mathbf{Z}_{1:k-1}) = \frac{v_{k|k-1}(\mathbf{x}_k)}{N_{k|k-1}}$, where $v_{k|k-1}(\mathbf{x}_k)$ is the first-order FISST approximation of the prior RFS $\mathbf{X}_k|\mathbf{Z}_{1:k-1}$, and where $N_{k|k-1}$ is the predicted number of targets at time k. Since the denominator of (3.4) is the same as for (3.1), which we already have an expression for (see (2.57)), we only need to find an equation describing $p(\mathbf{Z}_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1})$, i.e., the density of the current measurement set, given that there is at least one target with state $\mathbf{x}_k$, and that we have observed the old data $\mathbf{Z}_{1:k-1}$.

The PHD update can be written as

$$v_{k|k}(\mathbf{x}_k) \propto p(\mathbf{Z}_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1}) v_{k|k-1}(\mathbf{x}_k), \tag{3.5}$$

where we need to find an expression for $p(\mathbf{Z}_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1})$. The update is then completed by normalizing $v_{k|k}(\mathbf{x}_k)$ such that $\int v_{k|k}(\mathbf{x}_k)\, d\mathbf{x}_k = N_{k|k}$, where $N_{k|k}$ is the expected number of targets a posteriori, given by the updated cardinality distribution. We thus need to derive the likelihood $p(\mathbf{Z}_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1})$ to be able to express the CPHD update step for the intensity function.

We start by marginalizing over n_k — the true number of targets at time k. Since we have chosen one randomly selected target, $n_k > 0$, so

$$\begin{aligned} p(\mathbf{Z}_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1}) &= \sum_{n_k=1}^{\infty} p(\mathbf{Z}_k, n_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1}) & (3.6)\\ &= \sum_{n_k=1}^{\infty} p(\mathbf{Z}_k|\mathbf{x}_k, n_k, \mathbf{Z}_{1:k-1}) \Pr\{n_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1}\}. & (3.7) \end{aligned}$$

With the approximation

$$\begin{aligned} \Pr\{n_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1}\} &\cong \Pr\{n_k|n_k > 0, \mathbf{Z}_{1:k-1}\} = \{n_k \neq 0\} & (3.8)\\ &= \frac{\Pr\{n_k \big| \mathbf{Z}_{1:k-1}\}}{1 - \Pr\{n_k = 0 \big| \mathbf{Z}_{1:k-1}\}}, \end{aligned}$$

we identify $\Pr\{n_k \big| \mathbf{Z}_{1:k-1}\}$ as the predicted cardinality distribution, and $1 - \Pr\{n_k = 0 \big| \mathbf{Z}_{1:k-1}\}$ as a constant, given by the predicted cardinality. The constant can be moved out of the summation, and need not be calculated, since it can be added to a large constant term which also includes the constant in (3.4). The constant factors vanish in the normalization of $v_{k|k}$. So,

$$\Pr\{n_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1}\} \cong \frac{1}{c_{n_k=0}} \Pr\{n_k|\mathbf{Z}_{1:k-1}\}, \tag{3.9}$$

where $c_{n_k=0}$ is the constant. What remains to derive is the first factor in (3.6).

We first note that the distribution of the measurement set depends on if the target $\mathbf{x}_k$ under consideration is detected or not. By introducing the information sets

$$\begin{aligned} \mathcal{I}_{\mathbf{x}_k}^{\text{det}} &= \text{The target } \mathbf{x}_k \text{ is detected} & (3.10)\\ \mathcal{I}_{\mathbf{x}_k}^{\text{miss}} &= \text{The target } \mathbf{x}_k \text{ is not detected,} & (3.11) \end{aligned}$$

we can marginalize over these two events

$$p(\mathbf{Z}_k|n_k,\mathbf{x}_k,\mathbf{Z}_{1:k-1})=p(\mathbf{Z}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k}|n_k,\mathbf{x}_k,\mathbf{Z}_{1:k-1})+p(\mathbf{Z}_k,\mathcal{I}^{\text{miss}}_{\mathbf{x}_k}|n_k,\mathbf{x}_k,\mathbf{Z}_{1:k-1}) \tag{3.12}$$

$$= p(\mathbf{Z}_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1})\Pr\{\mathcal{I}^{\text{det}}_{\mathbf{x}_k}|n_k,\mathbf{x}_k,\mathbf{Z}_{1:k-1}\} \tag{3.13}$$
$$+ p(\mathbf{Z}_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{miss}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1})\Pr\{\mathcal{I}^{\text{miss}}_{\mathbf{x}_k}|n_k,\mathbf{x}_k,\mathbf{Z}_{1:k-1}\}.$$

The probability that the target is detected depends on the detection probability, which is independent of the number of targets and the previous measurements, given the target state. Hence,

$$p(\mathbf{Z}_k|n_k,\mathbf{x}_k,\mathbf{Z}_{1:k-1}) = p(\mathbf{Z}_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1})\Pr\{\mathcal{I}^{\text{det}}_{\mathbf{x}_k}|\mathbf{x}_k\} \tag{3.14}$$
$$+ p(\mathbf{Z}_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{miss}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1})\Pr\{\mathcal{I}^{\text{miss}}_{\mathbf{x}_k}|\mathbf{x}_k\}$$

$$= p(\mathbf{Z}_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1})P_d(\mathbf{x}_k) \tag{3.15}$$
$$+ p(\mathbf{Z}_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{miss}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1})\left(1-P_d(\mathbf{x}_k)\right).$$

We will treat the detection and missed detection likelihoods separately, starting with the detection likelihood.

3.1 Detection likelihood

To find the detection likelihood $p(\mathbf{Z}_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1})$, we start by marginalizing over the number of target-generated detections j_k, which under the detection hypothesis ranges from 1 to the minimum of the number of targets n_k and the number of received measurements m_k. The number of measurements m_k is inherent in the measurement set, and can be introduced without changing the pdf.

$$p(\mathbf{Z}_k,m_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1}) = \sum_{j_k=1}^{\min\{m_k,n_k\}} p(\mathbf{Z}_k,m_k,j_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1}) \tag{3.16}$$

$$= \sum_{j_k=1}^{\min\{m_k,n_k\}} p(\mathbf{Z}_k|m_k,n_k,j_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1})\Pr\{m_k,j_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1}\} \tag{3.17}$$

$$= \sum_{j_k=1}^{\min\{m_k,n_k\}} p(\mathbf{Z}_k|m_k,n_k,j_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1}) \tag{3.18}$$
$$\cdot\Pr\{m_k|j_k,n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1}\}\Pr\{j_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1}\}$$

The second factor is

$$\Pr\{m_k|j_k,n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1}\} = \Pr\{m_k|j_k\} = P_c(m_k-j_k). \tag{3.19}$$

The third factor is

$$\Pr\{j_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k},\mathbf{Z}_{1:k-1}\} = \Pr\{j_k|n_k,\mathbf{x}_k,\mathcal{I}^{\text{det}}_{\mathbf{x}_k}\}. \tag{3.20}$$

Equation (3.20) states the probability that, given n_k targets, one of which is the randomly selected and detected target $\mathbf{x}_k$, we receive j_k target-generated measurements. One of the targets have already been detected, so there are $j_k - 1$ targets more to be detected. The number of missed detections is $(n_k - 1) - (j_k - 1) = n_k - j_k$. The remaining $j_k - 1$ targets that give rise to a detection, can be chosen in $\binom{n_k-1}{j_k-1}$ ways, hence

$$\Pr\{j_k | n_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}\} = \binom{n_k - 1}{j_k - 1} P_d^{j_k - 1} (1 - P_d)^{n_k - j_k}. \tag{3.21}$$

We now turn to the remaining pdf. As a start, we transform the set $\mathbf{Z}_k$ into an ordered matrix $\overline{\mathbf{Z}}_k$,

$$p(\mathbf{Z}_k | m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) = m_k! p(\overline{\mathbf{Z}}_k | m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}). \tag{3.22}$$

Next, we introduce the association variable a which connects a measurement in $\overline{\mathbf{Z}}_k$ to the randomly selected target $\mathbf{x}_k$, and marginalize over the m_k different a scalars

$$p(\overline{\mathbf{Z}}_k | m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) = \sum_a p(\overline{\mathbf{Z}}_k, a | m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \tag{3.23}$$

$$= \sum_a p(\overline{\mathbf{Z}}_k | a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \Pr\{a | m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\}. \tag{3.24}$$

The probability that a certain measurement is associated to the randomly selected target is uniform a priori. Thus,

$$\Pr\{a | m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\} = \frac{1}{m_k}. \tag{3.25}$$

The measurement associated to $\mathbf{x}_k$ is independent of the rest of the measurements in the set. We call that measurement $\mathbf{z}_k^{(s)}$. Then,

$$p(\overline{\mathbf{Z}}_k | a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) = p\left(\mathbf{z}_k^{(s)} | a, \mathbf{x}_k, m_k, n_k, j_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) \cdot p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right). \tag{3.26}$$

We leave the first factor in (3.26) for now, and consider the second factor.

To find an expression for the second pdf in (3.26), we marginalize over the remaining targets in the target RFS $\mathbf{X}_k$,

$$p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) \tag{3.27}$$

$$= \int p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)}, \mathbf{X}_k \setminus \mathbf{x}_k | a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) d\mathbf{X}_k \setminus \mathbf{x}_k$$

$$= \int p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | \mathbf{X}_k, a, m_k, n_k, j_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) \cdot p\left(\mathbf{X}_k \setminus \mathbf{x}_k | a, m_k, n_k, j_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) d\mathbf{X}_k \setminus \mathbf{x}_k. \tag{3.28}$$

We then transform the target set into an ordered matrix, whose pdf is equal for any ordering. Thus,

$$\begin{aligned} &p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\ &\quad = (n_k - 1)! \int p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\ &\qquad \cdot p\left(\overline{\mathbf{X}}_k \setminus \mathbf{x}_k | a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) d\overline{\mathbf{X}}_k \setminus \mathbf{x}_k. \end{aligned} \tag{3.29}$$

By indexing the target under consideration $\overline{\mathbf{x}}_k^{(n)} = \mathbf{x}_k$, we note that

$$p\left(\overline{\mathbf{X}}_k \setminus \mathbf{x}_k | a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) = \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(1)}\right)}{N_{k|k-1}} \cdots \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(n_k-1)}\right)}{N_{k|k-1}}, \tag{3.30}$$

since all targets in $\overline{\mathbf{X}}_k \setminus \mathbf{x}_k^{(n_k)}$ are independent and identically distributed according to the predicted PHD divided by the expected number of targets $N_{k|k-1}$ at prediction. The matrix $\overline{\mathbf{X}}_k$ has always its last column $\mathbf{x}_k^{(n_k)} = \mathbf{x}_k$.

To differ between detection types, we introduce the vector $\mathbf{d}$, which states which detections in $\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)}$ that are target-generated and which are clutter. We marginalize over that vector

$$\begin{aligned} &p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\ &\quad = \sum_{\mathbf{d}} p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | \mathbf{d}, \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\ &\qquad \cdot \Pr\{\mathbf{d} | a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\}. \end{aligned} \tag{3.31}$$

All measurement classifications are equally probable, so

$$\Pr\{\mathbf{d} | a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\} = \frac{1}{\binom{m_k-1}{j_k-1}} = \frac{(j_k-1)!(m_k-j_k)!}{(m_k-1)!}. \tag{3.32}$$

Further, the clutter measurements are independent from the target-generated measurements, by which

$$\begin{aligned} &p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\ &\quad = \sum_{\mathbf{d}} p\left(\overline{\mathbf{Z}}_k^{\text{clut}} | \mathbf{d}, \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\ &\qquad \cdot p\left(\overline{\mathbf{Z}}_k^{\text{det}} \setminus \mathbf{z}_k^{(s)} | \mathbf{d}, \mathbf{X}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \frac{(j_k-1)!(m_k-j_k)!}{(m_k-1)!}. \end{aligned} \tag{3.33}$$

The clutter detections are independent, with densities $c(\mathbf{z}_k)$. Assuming uniformly distributed clutter in the target state-space, we get

$$p\left(\overline{\mathbf{Z}}_k^{\text{clut}} | \mathbf{d}, \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) = c^{m_k-j_k} \tag{3.34}$$

where c is the clutter density in the measurement space, assumed uniform. Conversion of density from target state-space to measurement state-space is through a Jacobian. If one cannot assume uniform distribution, the expression will get just a bit more complicated, where the density at each clutter measurement has to be used instead of the constant density.

To express the joint target-generated detections density, we introduce the association vector $\mathbf{a}^{\text{det}}$, which states the association between measurements in $\overline{\mathbf{Z}}_k^{\text{det}} \backslash \mathbf{z}_k^{(s)}$ to targets in $\overline{\mathbf{X}}_k \setminus \mathbf{x}_k^{(n_k)}$. It does not include the association of $\mathbf{z}_k^{(s)}$ to $\mathbf{x}_k^{(n_k)}$, since it is given by a. There are $j_k - 1$ measurements to associate, so $n_k - j_k$ targets will be without detection. We marginalize

$$
\begin{aligned}
&p\left(\overline{\mathbf{Z}}_k^{\text{det}} \setminus \mathbf{z}_k^{(s)} | \mathbf{d}, \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) && (3.35) \\
&= \sum_{\mathbf{a}^{\text{det}}} p\left(\overline{\mathbf{Z}}_k^{\text{det}} \setminus \mathbf{z}_k^{(s)}, \mathbf{a}^{\text{det}} | \mathbf{d}, \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\
&= \sum_{\mathbf{a}^{\text{det}}} p\left(\overline{\mathbf{Z}}_k^{\text{det}} \setminus \mathbf{z}_k^{(s)} \middle| \mathbf{a}^{\text{det}}, \mathbf{d}, \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) && (3.36) \\
&\qquad \cdot \Pr\left\{\mathbf{a}^{\text{det}} \middle| \mathbf{d}, \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right\}.
\end{aligned}
$$

The measurements are independent, given their associations, so

$$
\begin{aligned}
&p\left(\overline{\mathbf{Z}}_k^{\text{det}} \setminus \mathbf{z}_k^{(s)} | \mathbf{d}, \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) && (3.37) \\
&= \sum_{\mathbf{a}^{\text{det}}} \Pr\{\mathbf{a}^{\text{det}} | a, \overline{\mathbf{X}}_k, \mathbf{Z}_{1:k-1}\} \prod_{i \in \mathcal{D}} p\left(\overline{\mathbf{z}}_k^{(i)} | \overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1}\right),
\end{aligned}
$$

where uninformative variables have been removed for clarity. For notational simplicity, we use the same superscript for associated measurements and targets. In (3.37), $\mathcal{D}$ represents the target-generated detections in $\overline{\mathbf{Z}}_k^{\text{det}} \setminus \mathbf{z}_k^{(s)}$, i.e., it is the set of indices i for which $d(i) = 1$. We do not express the association probability at the current stage. For notational simplicity, we use the short-form

$$
P_{\mathbf{a}} = \Pr\{\mathbf{a}^{\text{det}} | a, \overline{\mathbf{X}}_k, \mathbf{Z}_{1:k-1}\}. \tag{3.38}
$$

We are now ready to turn back to (3.28), inserting the expressions derived thereafter,

$$
\begin{aligned}
&p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) && (3.39) \\
&= \frac{(j_k - 1)!(m_k - j_k)!}{(m_k - 1)!} (n_k - 1)! \int \sum_{\mathbf{d}} c^{m_k - j_k} \\
&\quad \cdot \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \prod_{i \in \mathcal{D}} p\left(\overline{\mathbf{z}}_k^{(i)} | \overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1}\right) \\
&\quad \cdot \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(1)}\right)}{N_{k|k-1}} \cdots \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(n_k-1)}\right)}{N_{k|k-1}} d\overline{\mathbf{x}}_k^{(1)} \cdots d\overline{\mathbf{x}}_k^{(n_k-1)}.
\end{aligned}
$$

Combining likelihoods $p(\overline{\mathbf{z}}_k^{(i)}|\overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1})$ with densities $v_{k|k-1}(\overline{\mathbf{x}}_k^{(i)})$ we can write

$$\begin{aligned} & p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)}|a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\ &= \frac{(j_k-1)!(m_k-j_k)!}{(m_k-1)!}(n_k-1)! \int \sum_{\mathbf{d}} c^{m_k-j_k} \\ &\cdot \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \prod_{i\in\mathcal{D}} p\left(\overline{\mathbf{z}}_k^{(i)}|\overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1}\right) \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(i)}\right)}{N_{k|k-1}} \\ &\cdot \prod_{l\notin\mathcal{D}} \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(l)}\right)}{N_{k|k-1}} d\mathbf{x}_k^{(1)} \cdots d\mathbf{x}_k^{(n-1)}, \end{aligned} \tag{3.40}$$

where $l \notin \mathcal{D}$ points out the clutter detections in $\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)}$, given by $\mathbf{d}$. The multi-dimensional integral can be split up into a product of single-dimensional (in terms of $\mathbf{x}_k$) integrals

$$\begin{aligned} & p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)}|a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\ &= \frac{(j_k-1)!(m_k-j_k)!}{(m_k-1)!}(n_k-1)! \sum_{\mathbf{d}} c^{m_k-j_k} \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \\ &\cdot \left[\prod_{i\in\mathcal{D}} \int p\left(\overline{\mathbf{z}}_k^{(i)}|\overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1}\right) \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(i)}\right)}{N_{k|k-1}} d\overline{\mathbf{x}}_k^{(i)}\right] \\ &\cdot \left[\prod_{l\notin\mathcal{D}} \int \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(l)}\right)}{N_{k|k-1}} d\overline{\mathbf{x}}_k^{(l)}\right]. \end{aligned} \tag{3.41}$$

The integral of a pdf is one, so the last factor is equal to one. Thus,

$$\begin{aligned} & p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)}|a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\ &= \frac{(j_k-1)!(m_k-j_k)!}{(m_k-1)!}(n_k-1)! \sum_{\mathbf{d}} c^{m_k-j_k} \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \\ &\cdot \left[\prod_{i\in\mathcal{D}} \int p\left(\overline{\mathbf{z}}_k^{(i)}|\overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1}\right) \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(i)}\right)}{N_{k|k-1}} d\overline{\mathbf{x}}_k^{(i)}\right]. \end{aligned} \tag{3.42}$$

By changing integration variable from $\overline{\mathbf{x}}_k^{(i)}$ to $\overline{\mathbf{x}}_k$ and by going back to set notation for $\mathbf{X}_k \setminus \mathbf{x}_k$ (where $\mathbf{x}_k$ is again a vector in the unordered set), we obtain

$$\begin{aligned} & p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)}|a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}\right) \\ &= \frac{(j_k-1)!(m_k-j_k)!}{(m_k-1)!} \sum_{\mathbf{d}} c^{m_k-j_k} \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \\ &\cdot \left[\prod_{i\in\mathcal{D}} \int p\left(\overline{\mathbf{z}}_k^{(i)}|\mathbf{x}_k, \mathbf{Z}_{1:k-1}\right) \frac{v_{k|k-1}\left(\mathbf{x}_k\right)}{N_{k|k-1}} d\mathbf{x}_k\right]. \end{aligned} \tag{3.43}$$

In the Gaussian-mixture CPHD, the predicted intensity function is a sum of Gaussian components,

$$v_{k|k-1}\left(\mathbf{x}_k\right) = \sum_{q=1}^{J_{k|k-1}} w_{k|k-1}^{(q)} \mathcal{N}\left(\mathbf{x}_k; \mathbf{m}_{k|k-1}^{(q)}, \mathbf{P}_{k|k-1}^{(q)}\right).$$

Furthermore, the measurement models gives

$$p\left(\overline{\mathbf{z}}_k^{(i)} | \mathbf{x}_k, \mathbf{Z}_{1:k-1}\right) = \mathcal{N}\left(\overline{\mathbf{z}}_k^{(i)}; \mathbf{H}\mathbf{x}_k, \mathbf{R}_k\right). \tag{3.44}$$

Using the product rule for Gaussian densities in (2.34), (2.35)–(2.37), we get

$$\int p\left(\overline{\mathbf{z}}_k^{(i)} | \mathbf{x}_k, \mathbf{Z}_{1:k-1}\right) \frac{v_{k|k-1}\left(\mathbf{x}_k\right)}{N_{k|k-1}} d\mathbf{x}_k \tag{3.45}$$

$$= \frac{1}{N_{k|k-1}} \sum_{q=1}^{J_{k|k-1}} w_{k|k-1}^{(q)} \int \mathcal{N}\left(\overline{\mathbf{z}}_k^{(i)}; \mathbf{z}_{k|k-1}^{(q)}, \mathbf{S}_{k|k-1}^{(q,s)}\right) \mathcal{N}\left(\mathbf{x}_k; \mathbf{m}_{k|k-1}^{(q,s)}, \mathbf{P}_{k|k-1}^{(q,s)}\right) d\mathbf{x}_k$$

$$= \frac{1}{N_{k|k-1}} \sum_{q=1}^{J_{k|k-1}} w_{k|k-1}^{(q)} \mathcal{N}\left(\overline{\mathbf{z}}_k^{(i)}; \mathbf{z}_{k|k-1}^{(q)}, \mathbf{S}_{k|k-1}^{(q,s)}\right) = L_k^{(i)}, \tag{3.46}$$

where $L_k^{(i)}$ is the weighted single-detection likelihood, introduced in (2.39).

We thus have

$$\begin{aligned} & p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\det}, \mathbf{Z}_{1:k-1}\right) \\ & \quad = \frac{(j_k-1)!(m_k-j_k)!}{(m_k-1)!} \sum_{\mathbf{d}} c^{m_k-j_k} \sum_{\mathbf{a}^{\det}} P_{\mathbf{a}} \prod_{i \in \mathcal{D}} L_k^{(i)}. \end{aligned} \tag{3.47}$$

As we see, there is no $\mathbf{a}^{\det}$ dependency in the sum over $\mathbf{a}^{\det}$, since $\mathcal{D}$ is determined only by $\mathbf{d}$. So, since $P_{\mathbf{a}}$ sums to one, we can remove the sum over the association vector $\mathbf{a}^{\det}$. Hence,

$$\begin{aligned} & p\left(\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)} | a, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\det}, \mathbf{Z}_{1:k-1}\right) \\ & \quad = \frac{(j_k-1)!(m_k-j_k)!}{(m_k-1)!} \sum_{\mathbf{d}} c^{m_k-j_k} \prod_{i \in \mathcal{D}} L_k^{(i)}. \end{aligned} \tag{3.48}$$

The sum over $\mathbf{d}$ is the permutation of all combinations of $j_k - 1$ detections from the set $\overline{\mathbf{Z}}_k \setminus \mathbf{z}_k^{(s)}$, so it never includes measurement s. The sum-multiplication of weighted single-detection likelihoods can be expressed as the elementary symmetric function (cf. (2.41))

$$\sigma_{j_k-1}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\} \setminus L_k^{(s)}\right) \triangleq \sum_{\mathbf{d}} \prod_{i \in \mathcal{D}} L_k^{(i)}. \tag{3.49}$$

We are now ready to express the detection likelihood in (3.16), where we transform back to set notation of measurements (cf. (3.62)). We also reintroduce the pdf

$p\left(\mathbf{z}_k^{(s)}|\mathbf{x}_k, \mathbf{Z}_{1:k-1}\right)$ that we left unconsidered in (3.26), where measurement $\mathbf{z}_k^{(s)}$ is determined by the association variable a. We obtain

$$\begin{aligned}
&p(\mathbf{Z}_k, m_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}) \qquad (3.50)\\
&= \sum_{j_k=1}^{\min\{m_k,n_k\}} \binom{n_k-1}{j_k-1} P_d^{j-1} \left(1-P_d\right)^{n_k-j_k} P_c\left(m_k-j_k\right)\\
&\cdot m_k! \sum_a \frac{1}{m_k} p\left(\mathbf{z}_k^{(s)}|\mathbf{x}_k, \mathbf{Z}_{1:k-1}\right) \frac{(j_k-1)!(m_k-j_k)!}{(m_k-1)!}\\
&\cdot c^{m_k-j_k} \sigma_{j_k-1}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\} \setminus L_k^{(s)}\right).
\end{aligned}$$

Expanding the binomial coefficient, and using that $m_k(m_k-1)! = m_k!$, we get

$$\begin{aligned}
&p(\mathbf{Z}_k, m_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}) \qquad (3.51)\\
&= \sum_{j_k=1}^{\min\{m_k,n_k\}} \frac{(n_k-1)!}{(n_k-j_k)!(j_k-1)!} P_d^{j_k-1} \left(1-P_d\right)^{n_k-j_k} P_c\left(m_k-j_k\right)\\
&\cdot m_k! \sum_a p\left(\mathbf{z}_k^{(s)}|\mathbf{x}_k, \mathbf{Z}_{1:k-1}\right) \frac{(j_k-1)!(m_k-j_k)!}{m_k!}\\
&\cdot c^{m_k-j_k} \sigma_{j_k-1}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\} \setminus L_k^{(s)}\right).
\end{aligned}$$

Simplification yields

$$\begin{aligned}
&p(\mathbf{Z}_k, m_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{det}}, \mathbf{Z}_{1:k-1}) \qquad (3.52)\\
&= \sum_{j_k=1}^{\min\{m_k,n_k\}} (m_k-j_k)! \frac{(n_k-1)!}{(n_k-j_k)!} P_d^{j_k-1} \left(1-P_d\right)^{n_k-j_k} P_c\left(m_k-j_k\right)\\
&\cdot \sum_a p\left(\mathbf{z}_k^{(s)}|\mathbf{x}_k, \mathbf{Z}_{1:k-1}\right) \cdot c^{m_k-j_k} \sigma_{j_k-1}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\} \setminus L_k^{(s)}\right).
\end{aligned}$$

Note that summing over a implies summing over all measurements $\mathbf{z}_k^{(s)} \in \mathbf{Z}_k$.

There is still one unknown factor, the likelihood $p\left(\mathbf{z}_k^{(s)}|\mathbf{x}_k, \mathbf{Z}_{1:k-1}\right)$. This pdf is given by the measurement model,

$$p\left(\mathbf{z}_k^{(s)}|\mathbf{x}_k, \mathbf{Z}_{1:k-1}\right) = \mathcal{N}\left(\mathbf{z}_k^{(s)}; \mathbf{H}\mathbf{x}_k, \mathbf{R}\right). \qquad (3.53)$$

After multiplication with the predicted intensity function $v_{k|k-1}\left(\mathbf{x}_k|\mathbf{Z}_{1:k-1}\right)$ (cf. (3.5)), we will obtain components

$$\mathcal{N}\left(\mathbf{z}_k^{(s)}; \mathbf{H}\mathbf{x}_k, \mathbf{R}\right) \frac{1}{N_{k|k-1}} \sum_{q=1}^{J_{k|k-1}} w_{k|k-1}^{(q)} \mathcal{N}\left(\mathbf{x}_k; \mathbf{m}_{k|k-1}^{(q)}, \mathbf{P}_{k|k-1}^{(1)}\right) \qquad (3.54)$$

$$= \sum_{q=1}^{J_{k|k-1}} w_{k|k-1}^{(q)} \mathcal{N}\left(\mathbf{z}_k^{(s)}; \mathbf{z}_{k|k-1}^{(q)}, \mathbf{S}_{k|k-1}^{(q,s)}\right) \mathcal{N}\left(\mathbf{x}_k; \mathbf{m}_{k|k}^{(q,s)}, \mathbf{P}_{k|k}^{(q,s)}\right). \qquad (3.55)$$

The latter Gaussian is the updated component of the Gaussian mixture, when measurement $\mathbf{z}_k^{(s)}$ has been declared a target measurement. The first Gaussian is the likelihood of mixture component q, given measurement $\mathbf{z}_k^{(s)}$. For each component in the above sum, we will obtain m_k components, when multiplying with (3.52) (excluding $p(\mathbf{z}_k^{s)}|\mathbf{x}_k))$. Thus, in the detection update, *we will obtain one posterior Gaussian mixture component for each component in the prediction and for each measurement in* $\mathbf{Z}_k$, in total $J_{k|k-1}m_k$ components.

When comparing the expression in (3.52) with the corresponding expression in (Ulmke *et al.* 2007), it is observed that the expressions differ by a factor n_k. We will discuss more about the differences between the equation after the subsequent section.

3.2 Missed detection likelihood

To express the missed detection likelihood $p(\mathbf{Z}_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}, \mathbf{Z}_{1:k-1})$ in (3.14), we start by marginalizing over the true number of target-generated detection j_k. Since we know that target $\mathbf{x}_k$ is not detected, the true number of detections is somewhere between 0 and $\min\{m_k, n_k-1\}$, hence

$$p(\mathbf{Z}_k, m_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}, \mathbf{Z}_{1:k-1}) = \sum_{j_k=0}^{\min\{m_k,n_k-1\}} p(\mathbf{Z}_k, m_k, j_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}, \mathbf{Z}_{1:k-1}) \quad (3.56)$$

$$= \sum_{j_k=0}^{\min\{m_k,n_k-1\}} p(\mathbf{Z}_k|m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}, \mathbf{Z}_{1:k-1}) \cdot \Pr\{m_k, j_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}, \mathbf{Z}_{1:k-1}\} \quad (3.57)$$

$$= \sum_{j_k=0}^{\min\{m_k,n_k-1\}} p(\mathbf{Z}_k|m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}, \mathbf{Z}_{1:k-1}) \cdot \Pr\{m_k|j_k, n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}, \mathbf{Z}_{1:k-1}\} \Pr\{j_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}, \mathbf{Z}_{1:k-1}\}. \quad (3.58)$$

In (3.58), the second factor is

$$\Pr\{m_k|j_k, n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}, \mathbf{Z}_{1:k-1}\} = \Pr\{m_k|j_k\} = P_c\left(m_k - j_k\right), \quad (3.59)$$

and the third factor is

$$\Pr\{j_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}, \mathbf{Z}_{1:k-1}\} = \Pr\{j_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}\}. \quad (3.60)$$

Equation (3.60) states the probability that, given n_k targets, one of which is the randomly selected and *not* detected target $\mathbf{x}_k$, we receive j_k target-generated measurements. Of the $n_k - 1$ targets which could have given rise to a measurement, j_k of them actually has. The number of ways in which j_k detections can be chosen from $n_k - 1$ targets is $\binom{n_k-1}{j_k}$, hence

$$\Pr\{j_k|n_k, \mathbf{x}_k, \mathcal{I}_{\mathbf{x}_k}^{\text{miss}}\} = \binom{n_k-1}{j_k} P_d^{j_k-1}\left(1-P_d\right)^{n_k-1-j_k}. \quad (3.61)$$

To express the density $p(\mathbf{Z}_k|m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1})$ in (3.60), we transform the RFS $\mathbf{Z}_k$ into an ordered matrix

$$p(\mathbf{Z}_k|m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) = m_k! p(\overline{\mathbf{Z}}_k|m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}). \tag{3.62}$$

We then marginalize over the remaining targets $\mathbf{X}_k \setminus \mathbf{x}_k$

$$\begin{aligned} &p(\overline{\mathbf{Z}}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) & (3.63)\\ &\quad = \int p\left(\overline{\mathbf{Z}}_k, \mathbf{X}_k \setminus \mathbf{x}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) d\mathbf{X}_k \setminus \mathbf{x}_k \\ &\quad = \int p\left(\overline{\mathbf{Z}}_k \big| \mathbf{X}_k, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) & (3.64)\\ &\qquad \cdot p\left(\mathbf{X}_k \setminus \mathbf{x}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) d\mathbf{X}_k \setminus \mathbf{x}_k. \end{aligned}$$

By transforming the RFS $\mathbf{X}_k \setminus \mathbf{x}_k$ into an ordered matrix, we obtain

$$\begin{aligned} &p(\overline{\mathbf{Z}}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) & (3.65)\\ &\quad = (n_k - 1)! \int p\left(\overline{\mathbf{Z}}_k \big| \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) \\ &\qquad \cdot p\left(\overline{\mathbf{X}}_k \setminus \mathbf{x}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) d\overline{\mathbf{X}}_k \setminus \mathbf{x}_k. \end{aligned}$$

By indexing the target under consideration $\overline{\mathbf{x}}_k^{(n)} = \mathbf{x}_k$, we note that

$$p\left(\overline{\mathbf{X}}_k \setminus \mathbf{x}_k | m_k, n_k, j_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) = \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(1)}\right)}{N_{k|k-1}} \cdots \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(n_k-1)}\right)}{N_{k|k-1}}, \tag{3.66}$$

since all targets in $\overline{\mathbf{X}}_k \setminus \mathbf{x}_k^{(n_k)}$ are independent and identically distributed according to the predicted PHD divided by the expected number of targets $N_{k|k-1}$ at prediction. The matrix $\overline{\mathbf{X}}_k$ has always its last column $\mathbf{x}_k^{(n_k)} = \mathbf{x}_k$.

To describe the density $p\left(\overline{\mathbf{Z}}_k \big| \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right)$, we introduce the vector $\mathbf{d}$, which states which detections in $\overline{\mathbf{Z}}_k$ that are target-generated and which are clutter. We marginalize over that vector

$$\begin{aligned} &p\left(\overline{\mathbf{Z}}_k | \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) & (3.67)\\ &\quad = \sum_{\mathbf{d}} p\left(\overline{\mathbf{Z}}_k | \mathbf{d}, \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) \\ &\qquad \cdot \Pr\{\mathbf{d} | \mathbf{X}_k, m_k, n_k, j_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\}. \end{aligned}$$

All measurement classifications are equally probable, so

$$\Pr\{\mathbf{d} | \mathbf{X}_k, m_k, n_k, j_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\} = \frac{1}{\binom{m_k}{j_k}} = \frac{j_k!(m_k - j_k)!}{m_k!}. \tag{3.68}$$

Further, the clutter measurements are independent from the target-generated measurements, by which

$$
\begin{aligned}
& p\left(\overline{\mathbf{Z}}_k | \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathcal{I}^{\mathrm{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) \\
&= \sum_{\mathbf{d}} p\left(\overline{\mathbf{Z}}^{\mathrm{clut}}_k | \mathbf{d}, \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathcal{I}^{\mathrm{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) \\
&\quad \cdot p\left(\overline{\mathbf{Z}}^{\mathrm{det}}_k | \mathbf{d}, \mathbf{X}_k, m_k, n_k, j_k, \mathcal{I}^{\mathrm{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) \frac{j_k!(m_k - j_k)!}{m_k!}.
\end{aligned} \tag{3.69}
$$

The clutter detections are independent, with densities $c\left(\mathbf{z}_k\right)$. Assuming uniformly distributed clutter in the measurement space, we get

$$
p\left(\overline{\mathbf{Z}}^{\mathrm{clut}}_k | \mathbf{d}, \overline{\mathbf{X}}_k, a, m_k, n_k, j_k, \mathcal{I}^{\mathrm{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) = c^{m_k - j_k} \tag{3.70}
$$

where c is the clutter density in the measurement space.

To express the joint target-generated detections density, we introduce the association vector $\mathbf{a}^{\mathrm{det}}$, which states the association between measurements in $\overline{\mathbf{Z}}^{\mathrm{det}}_k$ to targets in $\overline{\mathbf{X}}_k \setminus \mathbf{x}^{(n_k)}_k$. There are j_k measurements to associate, so $n_k - 1 - j_k$ targets will be without detection, since target $\mathbf{x}^{(n_k)}_k$ is undetected. We marginalize

$$
\begin{aligned}
& p\left(\overline{\mathbf{Z}}^{\mathrm{det}}_k | \mathbf{d}, \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathcal{I}^{\mathrm{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) && (3.71) \\
&= \sum_{\mathbf{a}^{\mathrm{det}}} p\left(\overline{\mathbf{Z}}^{\mathrm{det}}_k, \mathbf{a}^{\mathrm{det}} | \mathbf{d}, \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathcal{I}^{\mathrm{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) \\
&= \sum_{\mathbf{a}^{\mathrm{det}}} p\left(\overline{\mathbf{Z}}^{\mathrm{det}}_k \middle| \mathbf{a}^{\mathrm{det}}, \mathbf{d}, \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathcal{I}^{\mathrm{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) && (3.72) \\
&\quad \cdot \Pr\left\{\mathbf{a}^{\mathrm{det}} \middle| \mathbf{d}, \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathcal{I}^{\mathrm{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right\}.
\end{aligned}
$$

The measurements are independent, given their associations, so

$$
\begin{aligned}
& p\left(\overline{\mathbf{Z}}^{\mathrm{det}}_k | \mathbf{d}, \overline{\mathbf{X}}_k, m_k, n_k, j_k, \mathcal{I}^{\mathrm{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}\right) \\
&= \sum_{\mathbf{a}^{\mathrm{det}}} \Pr\{\mathbf{a}^{\mathrm{det}} | \overline{\mathbf{X}}_k, \mathbf{Z}_{1:k-1}\} \prod_{i \in \mathcal{D}} p\left(\overline{\mathbf{z}}^{(i)}_k | \overline{\mathbf{x}}^{(i)}_k, \mathbf{Z}_{1:k-1}\right),
\end{aligned} \tag{3.73}
$$

where uninformative variables have been removed for clarity. For notational simplicity, we use the same superscript for associated measurements and targets. In (3.73), $\mathcal{D}$ represents the target-generated detections in $\overline{\mathbf{Z}}^{\mathrm{det}}_k$, i.e., it is the set of indices i for which $d(i) = 1$. We do not express the association probability at the current stage. For notational simplicity, we use the short-form

$$
P_{\mathbf{a}} = \Pr\{\mathbf{a}^{\mathrm{det}} | \overline{\mathbf{X}}_k, \mathbf{Z}_{1:k-1}\}. \tag{3.74}
$$

We can now express (3.65), by using the expressions derived thereafter,

$$\begin{aligned}
&p(\overline{\mathbf{Z}}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\
&\quad = (n_k - 1)! \int \sum_{\mathbf{d}} \frac{j_k!(m_k - j_k)!}{m_k!} c^{m_k - j_k} \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \prod_{i \in \mathcal{D}} p\left(\overline{\mathbf{z}}_k^{(i)} | \overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1}\right) \\
&\qquad \cdot \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(1)}\right)}{N_{k|k-1}} \cdots \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(n_k-1)}\right)}{N_{k|k-1}} d\overline{\mathbf{x}}_k^{(1)} \cdots d\overline{\mathbf{x}}_k^{(n_k-1)}.
\end{aligned} \tag{3.75}$$

Combining likelihoods $p(\overline{\mathbf{z}}_k^{(i)} | \overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1})$ with densities $v_{k|k-1}(\overline{\mathbf{x}}_k^{(i)})$ we can write

$$\begin{aligned}
&p(\overline{\mathbf{Z}}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\
&\quad = (n_k - 1)! \int \sum_{\mathbf{d}} \frac{j_k!(m_k - j_k)!}{m_k!} c^{m_k - j_k} \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \\
&\qquad \cdot \prod_{i \in \mathcal{D}} p\left(\overline{\mathbf{z}}_k^{(i)} | \overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1}\right) \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(i)}\right)}{N_{k|k-1}} \\
&\qquad \cdot \prod_{l \notin \mathcal{D}} \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(l)}\right)}{N_{k|k-1}} d\mathbf{x}_k^{(1)} \cdots d\mathbf{x}_k^{(n-1)},
\end{aligned} \tag{3.76}$$

where $l \notin \mathcal{D}$ points out the clutter detections in $\overline{\mathbf{Z}}_k$, given by $\mathbf{d}$. The multi-dimensional integral can be split up into a product of single-dimensional (in terms of $\mathbf{x}_k$) integrals

$$\begin{aligned}
&p(\overline{\mathbf{Z}}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\
&\quad = (n_k - 1)! \int \sum_{\mathbf{d}} \frac{j_k!(m_k - j_k)!}{m_k!} c^{m_k - j_k} \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \\
&\qquad \cdot \left[\prod_{i \in \mathcal{D}} \int p\left(\overline{\mathbf{z}}_k^{(i)} | \overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1}\right) \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(i)}\right)}{N_{k|k-1}} d\overline{\mathbf{x}}_k^{(i)} \right] \\
&\qquad \cdot \left[\prod_{l \notin \mathcal{D}} \int \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(l)}\right)}{N_{k|k-1}} d\overline{\mathbf{x}}_k^{(l)} \right].
\end{aligned} \tag{3.77}$$

The integral of a pdf is one, so the last factor is equal to one. Thus,

$$\begin{aligned}
&p(\overline{\mathbf{Z}}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\
&\quad = (n_k - 1)! \int \sum_{\mathbf{d}} \frac{j_k!(m_k - j_k)!}{m_k!} c^{m_k - j_k} \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \\
&\qquad \cdot \left[\prod_{i \in \mathcal{D}} \int p\left(\overline{\mathbf{z}}_k^{(i)} | \overline{\mathbf{x}}_k^{(i)}, \mathbf{Z}_{1:k-1}\right) \frac{v_{k|k-1}\left(\overline{\mathbf{x}}_k^{(i)}\right)}{N_{k|k-1}} d\overline{\mathbf{x}}_k^{(i)} \right].
\end{aligned} \tag{3.78}$$

By changing integration variable from $\overline{\mathbf{x}}_k^{(i)}$ to $\overline{\mathbf{x}}_k$ and by going back to set notation for $\mathbf{X}_k \setminus \mathbf{x}_k$ (where $\mathbf{x}_k$ is again a vector in the unordered set), we obtain

$$\begin{aligned} &p(\overline{\mathbf{Z}}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\ &= \int \sum_{\mathbf{d}} \frac{j_k!(m_k-j_k)!}{m_k!} c^{m_k-j_k} \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \\ &\quad \cdot \left[\prod_{i\in\mathcal{D}} \int p\left(\overline{\mathbf{z}}_k^{(i)} | \mathbf{x}_k, \mathbf{Z}_{1:k-1}\right) \frac{v_{k|k-1}(\mathbf{x}_k)}{N_{k|k-1}} d\mathbf{x}_k \right]. \end{aligned} \tag{3.79}$$

Using the results from (3.45) and (3.46), we get

$$\begin{aligned} &p(\overline{\mathbf{Z}}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\ &= \sum_{\mathbf{d}} \frac{j_k!(m_k-j_k)!}{m_k!} c^{m_k-j_k} \sum_{\mathbf{a}^{\text{det}}} P_{\mathbf{a}} \prod_{i\in\mathcal{D}} L_k^{(i)}. \end{aligned} \tag{3.80}$$

As we see, there is no $\mathbf{a}^{\text{det}}$ dependency in the sum over $\mathbf{a}^{\text{det}}$, since $\mathcal{D}$ is determined only by $\mathbf{d}$. So, since $P_{\mathbf{a}}$ sums to one, we can remove the sum over the association vector $\mathbf{a}^{\text{det}}$. Hence,

$$\begin{aligned} &p(\overline{\mathbf{Z}}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\ &= \sum_{\mathbf{d}} \frac{j_k!(m_k-j_k)!}{m_k!} c^{m_k-j_k} \prod_{i\in\mathcal{D}} L_k^{(i)}. \end{aligned} \tag{3.81}$$

Using the elementary symmetric function, defined in (2.41), we write

$$\begin{aligned} &p(\overline{\mathbf{Z}}_k \big| m_k, n_k, j_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\ &= \frac{j_k!(m_k-j_k)!}{m_k!} c^{m_k-j_k} \sigma_{j_k}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\}\right). \end{aligned} \tag{3.82}$$

We are now ready to express the missed-detection likelihood $p(\mathbf{Z}_k | n_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1})$ in (3.56)

$$\begin{aligned} &p(\mathbf{Z}_k, m_k | n_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\ &= \sum_{j_k=0}^{\min\{m_k, n_k-1\}} m_k! \frac{j_k!(m_k-j_k)!}{m_k!} c^{m_k-j_k} \sigma_{j_k}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\}\right) \\ &\quad \cdot P_c(m_k-j_k) \binom{n_k-1}{j_k} P_d^{j_k-1} (1-P_d)^{n_k-1-j_k}. \end{aligned} \tag{3.83}$$

By expanding the binomial factor $\binom{n_k-1}{j_k}$, we obtain

$$\begin{aligned} &p(\mathbf{Z}_k, m_k | n_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\ &= \sum_{j_k=0}^{\min\{m_k, n_k-1\}} m_k! \frac{j_k!(m_k-j_k)!}{m_k!} c^{m_k-j_k} \sigma_{j_k}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\}\right) \\ &\quad \cdot P_c(m_k-j_k) \frac{(n_k-1)!}{j_k!(n_k-1-j_k)!} P_d^{j_k-1} (1-P_d)^{n_k-1-j_k}. \end{aligned} \tag{3.84}$$

By simplifying and rearranging, we get the final result

$$\begin{aligned}
&p(\mathbf{Z}_k, m_k | n_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) \\
&= \sum_{j_k=0}^{\min\{m_k, n_k-1\}} \frac{(n_k-1)!}{(n_k-1-j_k)!} (m_k - j_k)! P_c\left(m_k - j_k\right) \\
&\qquad \cdot P_d^{j_k-1}\left(1-P_d\right)^{n_k-1-j_k} \sigma_{j_k}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\}\right).
\end{aligned} \tag{3.85}$$

The update under missed detection described by (3.85), differs from the numerator of the corresponding expression in (Ulmke *et al.* 2007) by a factor n_k, which was also the case for the detection update in section (3.1). However, the weights calculated according to the equations in this section are to be normalized, so they do not describe the final weight after missed detection update.

3.3 Update equations

In order to update the intensity function of the GM-CPHD algorithm, we saw previously that it can be performed by multiplying the predicted intensity function with the density $p(\mathbf{Z}_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1})$, followed by a normalization. The density function is first expressed in (3.6). Using the results from Sections 3.1 and 3.2, we can explicitly express the density.

$$\begin{aligned}
p(\mathbf{Z}_k|\mathbf{x}_k, \mathbf{Z}_{1:k-1}) \cong &\sum_{n_k=1}^{\infty} \Pr\{n_k | \mathbf{Z}_{1:k-1}\} \\
&p(\mathbf{Z}_k | n_k, \mathbf{x}_k, \mathcal{I}^{\text{det}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1}) P_d\left(\mathbf{x}_k\right) \\
&\quad + p(\mathbf{Z}_k | n_k, \mathbf{x}_k, \mathcal{I}^{\text{miss}}_{\mathbf{x}_k}, \mathbf{Z}_{1:k-1})\left(1 - P_d\left(\mathbf{x}_k\right)\right)
\end{aligned} \tag{3.86}$$

For a missed detection, each component q in the predicted intensity function becomes a component in the posterior intensity function. The mean and covariance of the updated component are equal to the predicted ones. The unnormalized weight $\overline{w}^{(q,0)}_{k|k}$, where 0 represents missed detection, is given by

$$\begin{aligned}
\overline{w}^{(q,0)}_{k|k} = w^{(q)}_{k|k-1} &\left(1 - P_d^{(q)}\right) \sum_{n_k=1}^{\infty} \Pr\{n_k | \mathbf{Z}_{1:k-1}\} \frac{(n_k-1)!}{(n_k-1-j_k)!} \\
\cdot &\sum_{j_k=0}^{\min\{m_k, n_k-1\}} (m_k - j_k)! P_c\left(m_k - j_k\right) P_d^{j_k-1}\left(1-P_d\right)^{n_k-1-j_k} \\
&\qquad \cdot \sigma_{j_k}\left(\left\{L_k^{(1)}, \ldots, L_k^{(m_k)}\right\}\right),
\end{aligned} \tag{3.87}$$

where $P_d^{(q)}$ is a state-dependent detection probability, associated with component q. Often, a constant detection probability $P_d = P_d^{(q)}$ is used.

For each detection $\mathbf{z}_k^{(s)}$ and each predicted mixture component q, a new mixture component (q, s) is created after the measurement update. The mean and covariance matrix of this component is given by

$$\mathbf{m}_{k|k}^{(q,s)} = \mathbf{m}_{k|k-1}^{(q)} + \mathbf{W}_k \left(\mathbf{z}_k^{(s)} - \mathbf{H}\mathbf{m}_{k|k-1}^{(q)} \right) \tag{3.88}$$

$$\mathbf{P}_{k|k}^{(q,s)} = \left(\mathbf{1} - \mathbf{W}_k \mathbf{H} \right) \mathbf{P}_{k|k-1}^{(q)}, \tag{3.89}$$

where

$$\mathbf{W}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1} \tag{3.90}$$

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1}^{(q)} \mathbf{H}_k^T + \mathbf{R}_k. \tag{3.91}$$

The matrix $\mathbf{H}_k$ is the measurement model, $\mathbf{P}_{k|k-1}^{(q)}$ is the covariance matrix of the predicted mixture component q, and $\mathbf{m}_{k|k-1}^{(q)}$ is its mean value. The unnormalized weight $\overline{w}_{k|k}^{(q,s)}$ of the mixture component is given by

$$\begin{aligned}
\overline{w}_{k|k}^{(q,s)} = {} & w_{k|k-1}^{(q)} P_d^{(q)} \mathcal{N}\left(\mathbf{z}_k^{(s)}; \mathbf{z}_{k|k-1}^{(q)}, \mathbf{S}_{k|k-1}^{(q,s)} \right) \\
& \cdot \sum_{n_k=1}^{\infty} \Pr\{n_k \big| \mathbf{Z}_{1:k-1}\} \frac{(n_k-1)!}{(n_k-j_k)!} \sum_{j_k=1}^{\min\{m_k,n_k\}} (m_k - j_k)! \\
& \cdot P_d^{j_k-1} \left(1 - P_d\right)^{n_k - j_k} P_c\left(m_k - j_k\right) \\
& \cdot c^{m_k - j_k} \sigma_{j_k-1} \left(\left\{ L_k^{(1)}, \ldots, L_k^{(m_k)} \right\} \setminus L_k^{(s)} \right),
\end{aligned} \tag{3.92}$$

where $P_d^{(q)}$ is the component-dependent detection probability, which is often assumed equal for all components, i.e., $P_d^{(q)} = P_d$.

The update equations for both detection and missed detection differs from the numerator equations in the corresponding expressions in (Ulmke *et al.* 2007). However, in the procedure of this report, the calculated weights of the mixture components are to be normalized such that they sum to the expected number of targets. That is, the weights are normalized such that the sum equals $N_{k|k}^{\text{MAP}}$. After normalization, the sum of the weight will thus be equal to the sum of the weights obtained by the approach in (Ulmke *et al.* 2007). It is hence clear that the cardinality estimates will not differ between the approaches. The mean and covariance of the mixture components propagate in the same manner for both descriptions. The only thing that can differ between the presented result and the one in (Ulmke *et al.* 2007) is then the distribution of the component weights among the mixture components. In order to determine if there is a difference in practise between the two GM-CPHD update equations, the two sets of expressions were implemented. The alternative filters were run on a ground target tracking scenario, described in (Svensson *et al.* 2009). For the specific scenario tested, no difference in performance was observed. As measure of performance the Optimal Subpattern Assignment (OSPA) measure (Schuhmacher *et al.* 2008) was used.

4 CONCLUSION

In this report, an alternative derivation of the update equations of the Gaussian mixture cardinalized probability hypothesis density (GM-CPHD) filter has been presented. The equations have been compared with the equations in (Ulmke *et al.* 2007). However, since the update procedures are a bit different, the comparison is not direct. It does however appear to be a slight difference in the update of the weights of the mixture components between the two sets of expressions. To study if any of the sets of equations is beneficial, both filter equations were implemented, and evaluated on a ground target tracking scenario. The preliminary results indicate that there is no substantial difference between the two weight equations, when it comes to performance of the filter.

For the cardinality update, the aproach presented in this report provides the same equations as is presented in (Ulmke *et al.* 2007). Since the weights sum to the expected number of targets for both setups, it is only in the distribution of the weights among the components that the two descriptions differ.

Future work is needed in order to determine if there is a motivation for using the one or the other approach. It is also necessary to compare the two considered filter equations with the original GM-CPHD description in (Vo *et al.* 2006).

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