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On the predictability of string theory

Mass spectra of the squashed seven-sphere

Master's Thesis in Physics and Astronomy

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MASTER'S THESIS

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Cover: An artists interpretation of Albert Einstein sitting on Anti-de Sitter space squashing spheres, under the picture of two spheres playing squash. By Roger Skogholm, rogerarts.se.

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Abstract

Eleven-dimensional supergravity can be compactified on $AdS_4 \times M_7$ where M_7 can be chosen as the so called squashed seven-sphere. This allows two solutions, depending on the orientation of the sphere, one with $\mathcal{N} = 1$ supersymmetry and one non-supersymmetric. Both solutions correspond to spontaneous symmetry breakings of the round sphere. However, recent conjectures concerning the string landscape have led to the statement that non-supersymmetric AdS vacua should be unstable. By studying the standard model of particle physics one can make certain predictions about the masses of neutrinos from this conjecture, leading to measureable predictions from string theory.

This thesis aims to study the mass spectra of the squashed sphere in order to gain some insight into whether or not any instabilities occur. This could then either disprove or strengthen the proposed conjectures. The full spectrum is not found but some helpful steps along the way have been provided. In particular, only the scalar, vector and spinor spectra are studied.

The thesis also discusses some of the conjectures regarding the string landscape in more detail and shows how these can be used to make predictions in particle physics. The concepts of Kaluza-Klein compactification and supersymmetry are also introduced, as well as some group theoretical methods for studying differential operators on manifolds. The full mass spectra of the round seven-sphere are derived using two different methods.

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Johannes Aspman, Gothenburg, June 2018

Contents

1 Introduction

In the beginning of the twentieth century two theories were formulated that forever changed how we look upon our Universe. One described the physics of the very small, the other of the very large. These two theories were quantum mechanics and general relativity. Physicists have since been working on a way of combining the two into one, and thus creating one sole theory of everything. Today, more than a century after Einstein introduced relativity, and almost a century after Heisenberg and Schrödinger formulated the foundations of quantum mechanics, we are still not sure exactly what this theory is. Our best candidate to date is something called M-theory [1]. It combines all the previous candidates, the five string theories and eleven-dimensional supergravity, into one. However, this theory is so complex that we have not even been able to figure out what the M stands for (the leading candidates are mystery, magic or membrane).

M-theory offers a unique framework for combining all four fundamental forces of Nature. The idea is that instead of just looking at point particles as the fundamental entity, it includes one-dimensional strings and higher-dimensional objects called p-branes (the generalisation of membranes to p dimensions). This seemingly simple modification has shown us that the Universe might be a whole lot stranger than we could have ever imagined before. One of these strange and remarkable features of M-theory concerns the dimensionality of spacetime. Most people are probably used to think of Nature as being four-dimensional, three space dimensions and one time. But if M-theory is correct, we should have eleven dimensions. So where could these extra dimensions be hiding?

One possible answer lies at the very heart of this thesis, and it is a concept called *compactification* [2, 3]. The idea is that we consider the extra dimensions as being too small for us to detect. This has some profound consequences for the theory that we will discuss in detail. One being that the geometry of the extra dimensions gives us information about the masses of the particles in the four-dimensional theory.

In this thesis we will mostly study one special type of compactification where we consider our four-dimensional spacetime to be anti-de Sitter (AdS) and the seven extra dimensions to make up a manifold which we call the squashed seven-sphere [4]. This is similar to an ordinary round sphere in seven dimensions but with a particular deformation added to the metric.

Since the full M-theory is still unknown we will instead work in its low-energy limit. There we find the theory of eleven-dimensional supergravity [5]. This is a theory that combines the ideas of supersymmetry, a symmetry between bosons and fermions, with the concepts of general relativity [6].

A problem that arises when working in string/M-theory or supergravity is that it does not seem to give us a unique four-dimensional effective theory corresponding to our everyday world. In fact, it seems to be giving us an enormous landscape of possible theories [7]. This landscape is so huge that almost all effective theories seem possible. So why does our Universe look the way it does? In recent years this question has led to several conjectures being proposed about the consistency of these effective theories [8, 9].

These conjectures show us that there is an even larger area of inconsistent effective theories surrounding the string landscape, an area that has been dubbed the *swampland* [8].

One conjecture is the Weak Gravity Conjecture (WGC) [10, 11, 12]. This roughly states that a consistent theory of quantum gravity must have gravity as its weakest force. A consequence of this statement (or at least from a sharpened version discussed in Chapter 3) is that non-supersymmetric AdS vacua can not be stable [11]. This sharpened WGC has then been used to make predictions regarding the masses of the neutrinos in the standard model of particle physics [13]. It has also been shown to motivate the addition of some *beyond the standard model* (BSM) particles, like for example extra Weyl fermions, which could then lead to measureable predictions from string theory.

The squashed sphere has been shown to give a non-supersymmetric AdS vacuum as one of its solutions [4]. In order to either disprove or strengthen the WGC we therefore need to study the squashed sphere in more detail[a]. The aim of this thesis is to study the mass spectra of the squashed sphere, and in that way try to contribute to the search for potential instabilities.

1.1 Outline of thesis

The thesis is organised as follows. We begin our journey in Chapter 2 by discussing the concept of compactification in detail. This is the foundation of the whole thesis, and the discussion is needed in order to be able to fully understand the motivation of the thesis project. The motivation is then discussed in Chapter 3, where we wade through the swampland of inconsistent effective theories. We there discuss the instability of non-supersymmetric AdS vacua as well as some other conjectures regarding quantum gravity. The aim of the thesis project is then to contribute to the work of either disproving or strengthening one of these conjectures by analysing the instabilities of the squashed sphere. The mathematical toolbox needed to decipher the spectra is the subject of Chapter 4. We show how one can get geometrical information about a manifold by considering it as a *coset space*. The chapter ends by giving a short description of the relevant differential operators that will be studied. Chapter 5 gives a preparatory example of the different methods we want to use in order to solve for the mass spectra on a coset manifold. The main results of this thesis are found in Chapters 6 and 7 where we explore the squashed seven-sphere in detail. Chapter 8 summarises the results and discuss where to go from here.

The thesis aims to be accessible for an arbitrary master's level student of theoretical physics, meaning that some knowledge about general relativity, differential geometry, quantum field theory and string theory is assumed. The main text also assumes some familiarity with group theory and supersymmetry. However, since these are not subjects that the typical master student may be familiar with we include some introductory notes on these topics in the end, along with some other useful knowledge. Appendix A presents the conventions used throughout the thesis. In Appendix B we introduce the concepts of group theory. Supersymmetry is introduced in Appendix C. There we construct the theories of super-Yang-Mills in ten dimensions and supergravity in eleven dimensions. Note also that this introduction to supersymmetry was written together with Adrian Padellaro[b] as a joint effort to gain the background knowledge needed for our separate master's projects. Appendix D gives a brief introduction to the octonions while Appendix E discusses the Dirac gamma matrices in seven dimensions.

[a] This point was first raised by M.Duff in conversations with my supervisor.

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2

Kaluza-Klein compactification

This thesis deals with eleven-dimensional supergravity, which is the low-energy limit of the only available candidate for a theory of everything, M-theory. The idea that we might live in eleven dimensions, instead of the four that we are used to, probably seems strange to most people, and one could certainly ask oneself about the relevance of such a theory. In this chapter we discuss one possibility of how to deal with these extra dimensions in order to get a sensible theory out of eleven dimensions. The solution is to think of these extra dimensions as being very small, so small that we are not equipped to feel them. This is the subject of *compactification*.

A picture that might help when thinking about compactification is to consider a tightrope-dancer. When balancing on a thin rope far up in the air the dancer can only move forwards or backwards on the rope, i.e., he or she is only able to use one space dimension. If we instead imagine a small mite living on the rope. This mite can go around the rope, inside the rope and back and forth on the rope, and is thus able to move around freely in three space dimensions. In a similar fashion we can think of the extra seven dimensions from eleven-dimensional supergravity as being too small for us to notice when we move around in them. This is the so called *Kaluza-Klein* mechanism, named after its first explorers the German mathematician Theodor Kaluza, [2], and the Swedish physicist Oskar Klein, [3].

We will begin by studying the simplest example of compactifying only one dimension. This is relevant when going from M-theory in eleven dimensions to superstrings in ten dimensions, or when compactifying the standard model in the way discussed in Chapter 3. This is also what Kaluza and Klein did when trying to combine gravity and electromagnetism by going from five dimensions to four [2, 3]. From this we move on to discuss more general aspects of compactification and analyse what restrictions we can put on the geometry of the different dimensions. We then make a specific ansatz leading to a discussion of the mass spectra arising due to the compactness of the extra dimensions. In particular, this allows us to derive the mass operators of eleven-dimensional supergravity compactified to $AdS_4 \times M_7$. This will then lay the foundation for the rest of the thesis.

2.1 Toroidal compactification

We will start by discussing the simplest type of compactification, that of compactifying only one dimension on a circle (or one-dimensional torus). This is done as an illuminating example of how the Kaluza-Klein mechanism works, but also because it is a very fundamental concept in string theory, giving us the so called T-duality. We will also use this kind of compactification in the next chapter when we discuss the Weak Gravity Conjecture.

Consider a general $D(= d + 1)$-dimensional metric given by

$$ds^2 = g^D_{MN} dx^M dx^N. \tag{2.1}$$

We single out the dimension given by x^d and compactify it by identifying

$$x^d \cong x^d + 2\pi\mathcal{R}, \tag{2.2}$$

i.e., it now describes a circle with radius $\mathcal{R}$. The D-dimensional indices, M, N, can be split into $M = (\mu, d)$ where $\mu = 0, \ldots, d-1$ (the x^μ coordinates are still considered non-compact). This means that the metric splits into $g^D_{MN} = \{g_{\mu\nu}, g_{\mu d}, g_{dd}\}$, where the different parts corresponds to a metric, a vector and a scalar in the d-dimensional spacetime. The line element can now be parameterised according to

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{dd}(dx^d + A_\mu dx^\mu)^2, \tag{2.3}$$

where $g_{dd}A_\mu = g_{\mu d}$. Here we only consider the case where $g_{\mu\nu}$, A_μ and g_{dd} are independent of the compact coordinate x^d. The line element is invariant under reparameterisations

$$x'^d = x^d + \lambda(x^\mu), \tag{2.4}$$

if we at the same time allow the vector to transform as

$$A'_\mu = A_\mu - \partial_\mu\lambda. \tag{2.5}$$

We thus see that gauge transformations arise as a consequence of the compactification from higher dimensions [14]. This is what Kaluza and Klein first understood and in this way they tried to combine Einstein's theory of gravity with Maxwell's theory of electromagnetism by going from five dimensions down to four [2, 3].

Consider now a massless scalar field, ϕ, in D dimensions, and expand the x^d dependence of this field in a complete set of eigenfunctions

$$\phi(x^M) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu)e^{inx^d/\mathcal{R}}, \tag{2.6}$$

where the momentum in the x^d direction is quantised as $p_d = n/\mathcal{R}$, and n is usually refered to as the Kaluza-Klein excitation number. The Klein-Gordon[a] equation for this massless scalar is

$$0 = \partial_M\partial^M\phi = \partial_\mu\partial^\mu\phi + \partial_d\partial^d\phi \implies \partial_\mu\partial^\mu\phi_n(x^\mu) = \frac{n^2}{\mathcal{R}^2}\phi_n(x^\mu). \tag{2.7}$$

We see here that the massless modes of the full D-dimensional theory become an infinite tower of massive d-dimensional fields, labeled by n. We also see that the mass spectrum depends on the geometry of the compact dimension (here only the radius $\mathcal{R}$). This is the core concept of Kaluza-Klein compactification [4]. It is also the foundation of this thesis, where we study the mass spectra arising when compactifying eleven-dimensional supergravity on a specific kind of geometry, namely the so called squashed seven-sphere. The idea is precisely this, that the geometry of the compact dimensions determine the mass spectra of the four-dimensional theory.

One can also see that the d-dimensional mass-squared operator is now given by

$$M^2 = -p_\mu p^\mu = \frac{n^2}{\mathcal{R}^2}, \tag{2.8}$$

so that for distances much larger than the compact radii $\mathcal{R}$ we will not see the x^d-dependent fields and the theory becomes d-dimensional [14].

[a]Note that the Klein in the Klein-Gordon equation is the same as in Kaluza-Klein theory.

If we now define $g_{dd} \equiv e^{2\sigma}$ we get the Ricci scalar in D dimensions as

$$R = R_d - 2e^{-\sigma}\Box e^{\sigma} - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}, \tag{2.9}$$

where R_d is the d-dimensional Ricci scalar and $F_{\mu\nu}$ the field strength of A_μ [14]. The graviton-dilaton action of bosonic string theory becomes

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int d^D x\sqrt{-g^D}e^{-2\Phi}\left(R + 4\nabla_\mu\Phi\nabla^\mu\Phi\right) \\ &= \frac{\pi\mathcal{R}}{\kappa^2}\int d^d x\sqrt{-g^d}e^{-2\Phi_d}\left(R_d - \partial_\mu\sigma\partial^\mu\sigma + 4\partial_\mu\Phi_d\partial^\mu\Phi_d - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}\right), \end{aligned} \tag{2.10}$$

where g^d is the determinant of $g_{\mu\nu}$ and we have introduced the d-dimensional dilaton $\Phi_d = \Phi - \sigma/2$ [14]. From this action we see that Φ and σ are massless. They will then label degenerate solutions of the field equations. Although these solutions are degenerate the physics will still change when we vary Φ or σ. We then call these parameters for the *moduli* of the theory, and the space that they parameterise for the *moduli space* of toroidal compactification. We will discuss this some more below, but first we consider the effects of introducing strings.

2.1.1 T-duality in string theory

Compactification also allows for some really remarkable string corrections. This is due to the fact that a closed string can wind around the compactified dimension.

We consider a two-dimensional conformal field theory on the world-sheet of a closed string with one periodic scalar field

$$X \cong X + 2\pi\mathcal{R}. \tag{2.11}$$

The world-sheet action, and thus the equations of motion, are unchanged by this identification, and in particular the theory is still conformally invariant [14]. We should also demand that the states are single-valued under this identification. This implies that the center-of-mass momentum is quantised as $p = \frac{n}{R}$ just as in the ordinary field theory case discussed above.

There is, however, another effect present only in string theory (or at least only in theories including extended objects). This is the winding mentioned above. We define the *winding number*, W, by

$$X(\sigma + \pi) = X(\sigma) + 2\pi\mathcal{R}W, \qquad W \in \mathbb{Z}, \tag{2.12}$$

where $0 \leq \sigma \leq \pi$ for the closed string. In other words the winding number is the number of times the closed string winds around the compactified dimension. The sign of W gives us the orientation of the winding [15].

The mass-squared operator for a closed bosonic string with winding number W will become [15]

$$M^2 = \frac{n^2}{\mathcal{R}^2} + \frac{W^2\mathcal{R}^2}{\alpha'^2} + \frac{2}{\alpha'}(N_L + N_R - 2), \tag{2.13}$$

where α' is the Regge slope, with spacetime-dimension length-squared. One can see that M^2 is invariant under the interchange

$$\mathcal{R} \to \mathcal{R}' = \frac{\alpha'}{\mathcal{R}}, \qquad n \leftrightarrow W. \tag{2.14}$$

This means that the theories in the limits $\mathcal{R} \to 0$ and $\mathcal{R} \to \infty$ are identical. We refer to this duality as *T-duality*. A self-dual theory arise at $\mathcal{R} = \sqrt{\alpha'}$ [15]. One should also note that the level-matching condition of ordinary closed bosonic string theory, i.e. $N_L = N_R$ is modified if both W and n are non-zero, so that we now have $N_R - N_L = Wn$ [15].

With T-duality one can relate different string theories, like type IIA and type IIB, to each other. This is a very large area of research in string theory, but since this thesis mostly deal with supergravity we will not go into more detail on T-duality here. The interested reader should look up further details in some of the great textbooks on string theory ([14] or [15] are good places to start), because it is a truly remarkable phenomenon. We simply state that T-duality seems to be telling us something very fundamental about string theory, namely that strings have a different concept of geometry than we are used to. Note also that T-duality seems to be an exact, i.e. non-perturbative, symmetry [14].

T-duality is present in open string theory as well. This introduces some other exciting effects such as interchanging Dirichlet and Neumann boundary conditions, and the emergence of D-branes [15], but we will not go into this here.

2.1.2 Compactification on T^k

Instead of compactifying only one dimension on a circle we can compactify several dimensions on a higher-dimensional torus, T^k. The line element can now be written as

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu + g_{mn} dY^m dY^n, \tag{2.15}$$

where Y^m, $m = 1, \dots, k$, are the coordinates that make up the k-dimensional torus, all having period 2π, and g_{mn} is the metric of the torus. For simplicity we can consider the rectangular torus. This has all internal circles perpendicular and the metric is thus given by

$$g_{mn} = \frac{1}{\alpha'} \mathcal{R}_m^2 \delta_{mn} \quad \text{(no sum)}, \tag{2.16}$$

where $\mathcal{R}_m$ is the radius of circle m [15]. We now do the identification

$$Y^m(\sigma + \pi, \tau) = Y^m(\sigma, \tau) + 2\pi W^m, \quad W^m \in \mathbb{Z}, \tag{2.17}$$

where W^m is the winding number for the mth dimension of the torus. Expanding this in modes one finds that the difference between the momenta of the left- and right-moving sectors is given by

$$p_L^m - p_R^m = 2W^m \tag{2.18}$$

If we imagine that we do not have any anti-symmetric background field the momenta will simply be quantised as

$$p_L^m + p_R^m = K_m, \tag{2.19}$$

where K_m corresponds to the Kaluza-Klein excitation number [15].

If, however, the anti-symmetric background field, B_{MN}, is turned on we find some new interesting features of the theory. We get new scalar fields in the d-dimensional theory from the two-form field strength, B_{MN}, when we split it according to the index structure as $B_{\mu\nu}, B_{\mu n}$ and B_{mn}. Combining this with the scalars from the metric we now have a total of $\frac{k(k+1)}{2} + \frac{k(k-1)}{2} = k^2$ scalars.

The relevant part of the action for closed bosonic strings are [15]

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \left(g_{mn}\eta^{\alpha\beta} - B_{mn}\epsilon^{\alpha\beta} \right) \partial_\alpha Y^m \partial_\beta Y^n. \tag{2.20}$$

By expanding the internal field components in modes in the usual string theory way one can find that

$$K_m = g_{mn}(p_L^n + p_R^n) + B_{mn}(p_L^n - p_R^n), \quad \text{with } K_m \in \mathbb{Z}. \tag{2.21}$$

This can be solved together with Eq.(2.18) to give

$$\begin{aligned} p_L^m &= W^m + g^{mn}\left(\frac{1}{2}K_n - B_{np}W^p\right), \\ p_R^m &= -W^m + g^{mn}\left(\frac{1}{2}K_n - B_{np}W^p\right), \end{aligned} \tag{2.22}$$

which in turn gives us the mass operator [15]

$$M^2 = 2g_{mn}\left(p_L^m p_L^n + p_R^m p_R^n\right) + 4(N_R + N_L - 2). \tag{2.23}$$

So we see that, although the B_{mn} term in the action is a total derivative and doesn't have any local effect on the theory, it does affect the spectrum. These massless scalar fields will therefore also take part in the parameterisation of the moduli space when compactifying string theory on T^k. This is what we will discuss next.

2.1.3 Moduli space of toroidal compactification

We have seen that the massless scalar fields arising from compactification will affect the physics of the theory, but not the action. We call these k^2 massless scalar fields labeling the different vacua for *moduli*, and the space that they span for the *moduli space* of the compactification [15].

The moduli space for the field theory can be expressed as the coset space

$$O(k,k;\mathbb{R})/O(k,\mathbb{R}) \times O(k,\mathbb{R}). \tag{2.24}$$

We can easily see that the dimensions add up, since we have $\frac{2k(2k-1)}{2} - k(k-1) = k^2$ from the above space, in agreement with the number of scalars from B_{mn} and g_{mn}.

However, we saw earlier that when compactifying string theory on a circle we find T-duality which relates theories with inverse radius. In T^k compactification we can write the metric-dependent term in the mass-squared operator as

$$g_{mn}\left(p_L^m p_L^n + p_R^m p_R^n\right) = \begin{pmatrix} W & K \end{pmatrix} G^{-1} \begin{pmatrix} W \\ K \end{pmatrix}, \tag{2.25}$$

where G is a $2k \times 2k$ matrix

$$G = \begin{pmatrix} \frac{1}{2}g^{-1} & -g^{-1}B \\ Bg^{-1} & 2(g - Bg^{-1}B). \end{pmatrix} \tag{2.26}$$

The T-duality symmetry then generalises to the interchange [15]

$$W^m \leftrightarrow K_m, \quad G \leftrightarrow G^{-1}. \tag{2.27}$$

There are also some additional discrete shift symmetries given by

$$B_{mn} \to B_{mn} + \frac{1}{2}N_{mn}, \quad K_m \to K_m + N_{mn}W^n, \tag{2.28}$$

where N_{mn} is an anti-symmetric integer-valued matrix [15]. These symmetries can be represented by the matrix operators

$$
\begin{aligned}
I &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
S &= \begin{pmatrix} 1 & 0 \\ N_{mn} & 1 \end{pmatrix},
\end{aligned}
\tag{2.29}
$$

acting on G by AGA^T, where A is either I or S. Here I is the inversion and S the shift. These operators generate the group $O(k,k;\mathbb{Z})$ [15]. This means that the physical moduli space of T^k compactification of string theory is the quotient space [15]

$$
O(k,k;\mathbb{Z})\Big\backslash O(k,k;\mathbb{R}) \Big/ [O(k,\mathbb{R}) \times O(k,\mathbb{R})]. \tag{2.30}
$$

We will now turn to more general aspects of compactification, and see that many of the ideas from toroidal compactification resurfaces.

2.2 Compactification of supergravity

Now that we have seen some examples of how compactification works we will turn to a more general discussion. We will only discuss the case of compactifying eleven-dimensional supergravity, however, most of the things discussed are equivalent in string theory. After this general discussion we will go on to a specific ansatz, in the next section, which will be used to derive the mass operators of interest for this thesis.

We are generally interested in compactifying some $(4+k)$-dimensional theory to a product manifold $M_{4+k} = M_4 \times M_k$. In order for us to follow through on the idea that the extra dimensions should be small we need to choose M_k as a compact manifold. This could, for example, be a torus T^k, a sphere S^k or something else.

Since all observations of our Universe, to date, seem to indicate that our four-dimensional spacetime is maximally symmetric, we most often choose M_4 to be such a space [15]. This means that we can express the Riemann tensor of M_4 as

$$
R_{\mu\nu\rho\sigma} = \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{2.31}
$$

where $g_{\mu\nu}$ is the four-dimensional metric tensor, and, as usual, R is the Ricci scalar, proportional to the four-dimensional cosmological constant. We have three possible solutions, $R=0$ implies Minkowski, $R<0$ anti-de Sitter (AdS) and $R>0$ de Sitter (dS) spacetime.

We now start from the equations of motions of eleven-dimensional supergravity. These are derived in Appendix C, but we will shift the notation slightly in order to better conform with our main reference [4]. Compared to the appendix we will now denote the field strength of the three-form as $F \equiv 2H$, and the three-form itself will be denoted A_{MNP} instead of B_{MNP}. We will change the sign of the Riemann tensor so that AdS has negative curvature scalar (as stated above). Capital letters will not be superindices but instead ordinary spacetime indices in eleven dimensions, $M, N, \ldots$ for curved and $A, B, \ldots$ for flat indices. In this notation the supergravity equations of motions are

$$
\begin{aligned}
&R_{MN}(\tilde{\omega}) - \frac{1}{2}g_{MN}R(\tilde{\omega}) = \frac{1}{3}\left(\tilde{F}_{MPQR}\tilde{F}_N{}^{PQR} - \frac{1}{8}g_{MN}\tilde{F}_{PQRS}\tilde{F}^{PQRS}\right), \\
&\hat{\Gamma}^{MNP}\tilde{D}_N(\tilde{\omega})\psi_P = 0, \\
&D_M\tilde{F}^{MPQR} = -\frac{1}{576}\epsilon^{PQRM_1\ldots M_8}\tilde{F}_{M_1\ldots M_4}\tilde{F}_{M_5\ldots M_8},
\end{aligned}
\tag{2.32}
$$

plus the Bianchi identity for $\tilde{F}$ [4]. Here $\tilde{\omega}$ denotes the spin connection resulting from making the choice $T_{ab}{}^{c} = 0$ for the torsion, and $\tilde{D}(\tilde{\omega}) = D_M(\tilde{\omega}) + \frac{1}{144}\left(\hat{\Gamma}_M{}^{PQRS} + 8\hat{\Gamma}^{[PQR}g_M^{S]}\right)\tilde{F}_{PQRS}$ [4]. Here $\hat{\Gamma}$ are the eleven-dimensional gamma matrices. The coordinates of eleven dimensions will split into those of the four-dimensional maximally symmetric space, x^μ $(\mu = 0, 1, 2, 3)$, and those of the seven-dimensional compact manifold, y^m $(m = 4, \ldots, 7)$, so that we can write

$$x^M = (x^\mu, y^m). \tag{2.33}$$

As before, this will allow us to split tensors into different components depending on their index structure. For example the three-form can be split as $A_{MNP} = (A_{\mu\nu\rho}, A_{\mu\nu p}, A_{\mu np}, A_{mnp})$. The internal indices (mnp) do not transform under four-dimensional Lorentz transformations so this three-form now gives us one three-form $A_{\mu\nu\rho}$, seven two-forms, $A_{\mu\nu p}$, 21 vectors $A_{\mu np}$ and 35 scalars A_{mnp} in the four-dimensional theory. We can do the same thing to the vielbein and the gravitino, giving us [4]

$$\begin{aligned} e_M{}^A &= \begin{pmatrix} e_\mu{}^\alpha & e_\mu{}^a \\ e_m{}^\alpha & e_m{}^a \end{pmatrix}, \\ \psi_M &= (\psi_\mu, \psi_m), \end{aligned} \tag{2.34}$$

note however that the spinor index of the gravitino will require some extra care. We will deal with this shortly.

When giving vacuum expectation values (VEVs) to the fields we must be careful not to break the Lorentz symmetry or the maximal symmetry of the four-dimensional manifold. This implies that any VEV must be proportional to an invariant tensor. The candidates are the scalar, the metric and the Levi-Civita tensor. By studying the index structures of the VEVs we can then see that we must set [4]

$$\begin{aligned} &\langle e_\mu{}^a \rangle = \langle e_m{}^\alpha \rangle = 0, \\ &\langle F_{\mu\nu\rho q} \rangle = \langle F_{\mu\nu pq} \rangle = \langle F_{\mu npq} \rangle = 0, \\ &\langle \psi_M \rangle = 0. \end{aligned} \tag{2.35}$$

The only non-zero VEVs are therefore $\langle e_\mu{}^\alpha \rangle$, $\langle e_m{}^a \rangle$, $\langle F_{mnpq} \rangle$ and $\langle F_{\mu\nu\rho\sigma} \rangle$.

Due to us wanting a product space, $M_4 \times M_7$, we should choose

$$\begin{aligned} \langle e_\mu{}^\alpha \rangle &= \mathring{e}_\mu{}^\alpha(x), \\ \langle e_m{}^a \rangle &= \mathring{e}_m{}^a(y), \end{aligned} \tag{2.36}$$

where the circle on top indicates that the object under it is a background value.

Note, however, that we could have considered a more general warped product space where $\langle e_\mu{}^\alpha \rangle = f(y)\mathring{e}_\mu{}^\alpha(x)$ for some warp factor $f(y)$. These warped product spaces are interesting in many cases of string theory, but in this thesis we will only concern ourselves with spaces satisfying $f(y) = 1$.

The diagonality of the vielbein implies that the differential operators will split into two parts. The Klein-Gordon equation for the massless scalar fields of the eleven-dimensional theory will for example split into

$$0 = \Box_{11}\phi = \Box_4\phi + \Box_7\phi, \tag{2.37}$$

which means that the differential operator in the seven-dimensional theory corresponds to a mass operator in the four-dimensional theory, equivalent to what we saw in the case of toroidal compactification, Eq.(2.7).

As stated above, maximal symmetry in the external spacetime also implies that the VEV of the spinor fields must vanish. We are, however, still interested in having unbroken supersymmetry in the four-dimensional theory. This means that we must constrain the VEV to stay zero after a supersymmetry transformation. Using the transformation found in Appendix C we get

$$\langle\delta_\varepsilon\psi\rangle = \langle\tilde{D}_M\varepsilon\rangle = \langle\left(D_M + \frac{1}{144}\left(\hat{\Gamma}_M{}^{PQRS} + 8\hat{\Gamma}^{[PQR}g_M^{S]}\right)\tilde{F}_{PQRS}\right)\varepsilon\rangle = 0. \tag{2.38}$$

We can start by considering the case of non-flux compactification, meaning that all the components of F are set to zero. This gives us

$$D_M\varepsilon = 0. \tag{2.39}$$

This is the so called *Killing spinor equation.* It tells us that ε is a covariantly constant spinor. The number of available Killing spinors is the same as the number of available supersymmetries, i.e., if we have only one spinor satisfying Eq.(2.39) we have $\mathcal{N} = 1$ supersymmetry, and so on [4]. Since we assume that the full 11-dimensional manifold is a product manifold we can decompose the Killing spinor into a sum of terms having the form [4]

$$\varepsilon(x, y) = \varepsilon(x) \otimes \eta(y). \tag{2.40}$$

If we start by considering the covariantly constant Killing spinor in the four-dimensional maximally symmetric theory we find

$$D_\mu\varepsilon(x) = 0 \implies [D_\mu, D_\nu]\varepsilon(x) = \frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\varepsilon(x) = 0 \implies R = 0, \tag{2.41}$$

which means that we are in Minkowski spacetime. In the same way we find in the internal space that

$$D_m\eta = 0 \implies [D_m, D_n]\eta = \frac{1}{4}R_{mnpq}\gamma^{pq}\eta = 0, \tag{2.42}$$

which is called the *integrability* condition [4]. Note that, since we do not assume the internal manifold to be maximally symmetric, this does not imply that it must be flat [4]. Most of the known solutions are, however, Ricci-flat, which means that $R_{mn} = 0$. Probably the most important type of Ricci-flat manifolds are the so called Calabi-Yau (CY) manifolds. This is a class of complex manifolds that exist in any dimension and are generalisations of the $K3$-manifold [15]. Since they are complex this means that they only exist in even real dimensions and therefore they are mostly relevant when going from ten dimensions to $M_4 \times M_6$, or perhaps when compactifying M-theory to something other than $M_4 \times M_7$ like for example $M_5 \times CY_3$. This is a very large subject in string theory, but since we are interested in compactifying to $M_4 \times M_7$ we once again simply refer the interested reader to a good book on string theory, for example Becker, Becker & Schwarz, which has over hundred pages on Calabi-Yau compactification [15].

The known solutions of Ricci-flat seven-dimensional Einstein manifolds are T^7, $K3\times T^3$, $CY_3 \times S^1$ and the so called Joyce manifolds, which are compact G_2 manifolds.

The story is modified when we allow non-zero flux, i.e. non-zero F_{mnpq} and/or $F_{\mu\nu\rho\sigma}$. These fluxes must be proportional to the Levi-Civita tensor. From the Bianchi identity $dF = 0$ together with Eq.(2.35) we find

$$0 = 5(dF)_{m\nu\rho\sigma\tau} = 5\partial_{[m}F_{\nu\rho\sigma\tau]} = \partial_m F_{\nu\rho\sigma\tau} - \underbrace{4\partial_{[\nu}F_{\rho\sigma\tau]m}}_{=0}, \tag{2.43}$$

so that $F_{\mu\nu\rho\sigma}$ must be independent of y. In the same way we see that F_{mnpq} must be x independent [4].

We now have three options. Either we can keep both $F_{\mu\nu\rho\sigma}$ and F_{mnpq}, or we can put one of them to zero while keeping the other. In the next section we will set F_{mnpq} to zero, and keep only the flux in the external space, but for now we can briefly discuss what happens in the other two cases.

If we keep the F terms in both spaces we can write

$$\begin{aligned} \mathring{F}_{\mu\nu\rho\sigma} &= 2m\epsilon_{\mu\nu\rho\sigma}, \\ \mathring{F}_{mnpq} &\neq 0, \end{aligned} \tag{2.44}$$

where the $2m$ factor is just for convenience. This means that the equations of motion for F_{mnpq} and R_{MN} from Eq.(2.32) turns into

$$\begin{aligned} \nabla_m F^{mnpq} &= \frac{1}{6} m \epsilon^{npqrstu} F_{rstu}, \\ R_{\mu\nu} &= \frac{1}{3}\left(-16m^2 - \frac{1}{12}F_{mnpq}F^{mnpq}\right) g_{\mu\nu}, \\ R_{mn} &= \frac{1}{3}\left(F_{mpqr}F_n{}^{pqr} - \frac{1}{12}g_{mn}F_{pqrs}F^{pqrs} + 8m^2 g_{mn}\right). \end{aligned} \tag{2.45}$$

In order to have $R_{\mu\nu} = \Lambda g_{\mu\nu}$, i.e., maximal symmetry in the external spacetime, we must demand that $F^2 = F_{mnpq}F^{mnpq}$ is constant. We can, however, see that we are still in AdS, since $F^2 \geq 0 \implies g^{\mu\nu}R_{\mu\nu} < 0$. The splitting of the spinor is now more complex, and there are many different types of solution. One case where it is possible to prove that all supersymmetries are broken is found by setting

$$F_{mnpq} \propto \bar{\eta}\Gamma_{mnpq}\eta, \tag{2.46}$$

where η is the Killing spinor [4]. This also implies that $A_{mnp} \propto \bar{\eta}\Gamma_{mnp}\eta$, which are just the structure constants of the octonions (see Appendix D). One possibility is to put this into the squashed sphere compactification. This will however render a somewhat different spectra than the one we study, especially leading to broken parity [4]. This is discussed at more length by Duff et al. in Chapter 10 of [4].

Next we can investigate what happens when we put $F_{\mu\nu\rho\sigma} = 0$ and $F_{mnpq} \propto \epsilon_{mnpq}$. The important point here is that in contrast to what happens in the next section we will not get spontaneous compactification to four dimensions. Instead this will lead to a topology of the type $M_4^{comp.} \times AdS_7$ [4]. This does not, however, align with what we are trying to accomplish by thinking of the extra dimensions as very small. There is perhaps some possibility that we are the ones living in the small dimensions and because of the compactness of these we do not feel the other seven dimensions. But in this thesis we focus on the $AdS_4 \times M_7$ solution. The interested reader could for example look in [16].

2.2.1 The Freund-Rubin ansatz

We will now discuss the last case mentioned above, namely to put

$$\begin{aligned} \langle F_{mnpq}\rangle &= 0, \\ \langle F_{\mu\nu\rho\sigma}\rangle &= 3m\mathring{\epsilon}_{\mu\nu\rho\sigma}(x), \end{aligned} \tag{2.47}$$

where the $3m$ proportionality constant is just a choice. This is called the Freund-Rubin ansatz [4]. With this ansatz we will be able to solve the equations of motions. Since the VEV of the gravitino must be zero due to maximal symmetry we need not concern ourselves with its equations of motions. Einstein's equations become

$$\begin{aligned}
&\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{g}_{\mu\nu}\mathring{R} = 3m^2\left(\mathring{\epsilon}_{\mu\rho\sigma\tau}\mathring{\epsilon}_{\nu}{}^{\rho\sigma\tau} - \frac{1}{8}\mathring{g}_{\mu\nu}\mathring{\epsilon}^2\right),\\
&\mathring{R}_{mn} - \frac{1}{2}\mathring{g}_{mn}\mathring{R} = -\frac{3}{8}m^2\mathring{g}_{mn}\mathring{\epsilon}^2,\\
&\mathring{R}_{\mu n} = 0.
\end{aligned} \tag{2.48}$$

These are solved by

$$\begin{aligned}
\mathring{R}_{\mu\nu} &= -12m^2\mathring{g}_{\mu\nu},\\
\mathring{R}_{mn} &= 6m^2\mathring{g}_{mn},
\end{aligned} \tag{2.49}$$

where $\mathring{g}_{\mu\nu}$ has Minkowski signature and $\mathring{g}_{mn}$ Euclidean. The maximally symmetric four-dimensional vacuum is thus seen to be AdS, since we have $\Lambda = -12m^2$.

We next decompose the eleven-dimensional gamma matrices of $SO(1,10)$, $\hat{\Gamma}$, into $SO(1,3)\times SO(7)$ generators by writing

$$\hat{\Gamma}_A = (\gamma_\alpha \otimes \mathbb{1}, \gamma_5 \otimes \Gamma_a), \tag{2.50}$$

where

$$\begin{aligned}
&\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta},\\
&\{\Gamma_a, \Gamma_b\} = 2\delta_{ab},\\
&\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3,
\end{aligned} \tag{2.51}$$

which allows us to split the covariant derivative into [4]

$$\begin{aligned}
\mathring{\tilde{D}}_\mu &= \mathring{D}_\mu + m\mathring{e}_\mu{}^\alpha\gamma_\alpha\gamma_5,\\
\mathring{\tilde{D}}_m &= \mathring{D}_m - i\frac{m}{2}\mathring{e}_m{}^a\Gamma_a.
\end{aligned} \tag{2.52}$$

The spinor is split in the same way as before into a sum of terms $\varepsilon(x,y) = \varepsilon(x)\otimes\eta(y)$, which together with the splitting of the covariant derivative implies that the condition for the supersymmetry transformation of the gravitino is

$$\begin{aligned}
\mathring{\tilde{D}}_\mu\varepsilon(x) &= 0,\\
\mathring{\tilde{D}}_m\eta(y) &= 0.
\end{aligned} \tag{2.53}$$

These are now the Killing spinor equations for the external and internal spaces, giving us the maximal number of unbroken supersymmetries in each space [4]. In AdS spacetime this is four. We will later discuss the round seven-sphere which allows for eight unbroken symmetries and the squashed seven-sphere which has two solutions, one with $\mathcal{N}=1$ and one with $\mathcal{N}=0$, depending on which orientation is used [4]. This orientation reversal will be refered to as "skew-whiffing".

The Killing spinor equation for the external space becomes

$$\mathring{\tilde{D}}_m\eta(y) = \left(\partial_m + \frac{1}{4}\omega_m{}^{ab}\Gamma_{ab} - \frac{i}{2}me_m{}^a\Gamma_a\right)\eta = 0 \implies \nabla_m\eta = i\frac{m}{2}e_m{}^a\Gamma_a\eta, \tag{2.54}$$

where ∇_m is the covariant derivative with only the spin connection term. This gives us the integrability condition

$$[\tilde{D}_m, \tilde{D}_n]\eta = \frac{1}{4}W_{mn}{}^{ab}\Gamma_{ab}\eta = 0, \tag{2.55}$$

where ${W_{mn}}^{ab}$ is the Weyl tensor [4]. These linear combinations of the $Spin(7)$ generators Γ_{ab} generate a subgroup of $Spin(7)$ corresponding to the holonomy group, $\mathcal{H}$, of the connection in Eq.(2.54). This means that the maximal number of unbroken supersymmetries is equal to the number of spinors left invariant by $\mathcal{H}$, which in turn is given by the number of singlets in the decomposition of the 8 representation of $Spin(7)$ under $\mathcal{H}$ [4].

2.2.2 The mass spectrum

We will now very roughly outline the derivation of the four-dimensional mass operators found by Freund-Rubin compactification on an arbitrary M_7. This is done just to give you an idea of the steps needed. A more detailed derivation is found in [4]. The idea is to linearise the eleven-dimensional field equations around some arbitrary background, and then find the mass matrix by substituting the linearised fields into the harmonic expansions on M_7. The spectrum is then found by diagonalising the mass matrix for each field [4].

The fluctuations are defined by [4]

$$g_{MN}(x,y) = \mathring{g}_{MN}(x,y) + h_{MN}(x,y), \tag{2.56}$$
$$\Psi_M(x,y) = 0 + \psi_M(x,y), \tag{2.57}$$
$$A_{MNP}(x,y) = \mathring{A}_{MNP}(x,y) + a_{MNP}(x,y), \tag{2.58}$$

with $\mathring{F}_{MNPQ} = 4\mathring{\nabla}_{[M}\mathring{A}_{NPQ]}$. We will also write $f_{MNPQ} = 4\partial_{[M}a_{NPQ]}$. The next step is to put these expansions into the equations of motion and keep only terms up to linear order. For example, Einstein's equations turn into [4]

$$\begin{aligned}\delta R_{MN} =& \frac{1}{2}\mathring{\Delta}_L h_{MN} + \mathring{\nabla}_{(M}\mathring{\nabla}^P h_{N)P} - \frac{1}{2}\mathring{\nabla}_M\mathring{\nabla}_N {h_P}^P\\ =& \frac{2}{3}{\mathring{F}_{(M}}^{PQR} f_{N)PQR} - \frac{1}{18}\mathring{g}_{MN}\mathring{F}_{PQRS}f^{PQRS} - {\mathring{F}_{(M}}^{PQR}{\mathring{F}_{N)}}^{S}{}_{QR}h_{PS}\\ &- \frac{1}{36}h_{MN}\mathring{F}_{PQRS}\mathring{F}^{PQRS} + \frac{1}{9}\mathring{g}_{MN}\mathring{F}_{PQRS}{\mathring{F}_T}^{QRS}h^{PT}.\end{aligned} \tag{2.59}$$

Here we introduced the Lichnerowicz operator, Δ_L, acting on transverse, traceless, symmetric tensors through $\Delta_L h_{mn} \equiv -\Box h_{mn} - 2R_{mpnq}h^{pq} + 2{R_{(m}}^p h_{n)p}$. Next, we split all indices into spacetime and internal indices, and use the Freund-Rubin ansatz. This gives us a lot of equations. The $(\mu\nu)$-part of Einstein's equations becomes (we can now drop the $\circ$-notation)

$$\begin{aligned}&\nabla_L h_{\mu\nu} + 2\nabla_{(\mu}\nabla^\rho h_{\nu)\rho} - \nabla_\mu\nabla_\nu {h_\rho}^\rho + 2\nabla_{(\mu}\nabla^m h_{\nu)m} - \nabla_\mu\nabla_\nu {h_m}^m\\ &= \frac{2}{3}m g_{\mu\nu}\epsilon^{\rho\sigma\tau\epsilon}f_{\rho\sigma\tau\epsilon} + 24m^2 g_{\mu\nu}{h^\rho}_\rho - 24m^2 h_{\mu\nu},\end{aligned} \tag{2.60}$$

and so on [4]. Since pure gauge modes are of no interest to us we will use a particular gauge that simplifies a lot of things. This is

$$\begin{aligned}&\nabla^m h_{m\nu} = 0,\\ &\nabla^m(h_{mn} - \frac{1}{7}g_{mn}{h^p}_p) = 0,\\ &\nabla^m a_{mNP} = 0,\\ &\Gamma^m\psi_m(x,y) = 0.\end{aligned} \tag{2.61}$$

With this choice we can easily split all the fields into x and y dependent parts, e.g., $h_{\mu\nu}(x,y) = h_{\mu\nu}(x)Y(y)$, $h_{\mu n}(x,y) = B_\mu(x)Y_n(y)$, and so on [4]. Here $Y_{m_1\ldots m_p}(y)$ are transverse modes of the Hodge-de Rahm operator, Δ_p, discussed in more detail in Chapter 4.

Now, we simply (again, see [4] for details) expand all equations, using the known qualities of the different modes, and match the results with the relevant mass terms in AdS. The resulting mass operators are collected in Table 2.1 [4]. The different superscripts label different towers of states. The operators are independent of which topology used as the compact manifold, and are therefore relevant for both the round seven-sphere in Chapter 5 and the squashed sphere of Chapter 7. In this thesis we will simply concern ourselves with calculating the eigenvalues of the different differential operators, not the complete mass spectrum, i.e., we skip all the constants appearing in the table. The differential operators appearing in the table are all discussed in more detail in Chapter 4.

In their derivation Duff et al., [4], find that certain modes must be excluded from some of the towers. This is, however, not something we need to care about since we are only interested in finding the eigenvalues of the operators, but it is certainly important when analysing the actual spectrum.

spin	mass operator
2^+	Δ_0
$(3/2)^{(1),(2)}$	$\not{D}_{1/2} + 7m/2$
$1^{-(1),(2)}$	$\Delta_1 + 12m^2 \pm 6m(\Delta_1 + 4m^2)^{1/2}$
1^+	Δ_2
$(1/2)^{(4),(1)}$	$\not{D}_{1/2} - 9m/2$
$(1/2)^{(3),(2)}$	$3m/2 - \not{D}_{3/2}$
$0^{+(1),(3)}$	$\Delta_0 + 44m^2 \pm 12m(\Delta_0 + 9m^2)^{1/2}$
$0^{+(2)}$	$\Delta_L - 4m^2$
$0^{-(1),(2)}$	$Q^2 + 6mQ + 8m^2$

Table 2.1: Mass operators for Freund-Rubin compactification on an arbitrary internal manifold.

2.2.3 Further demands on M_7

In order to have massless Yang-Mills fields in the external space we need to demand that M_7 admits the existence of some Killing vector fields that generates the isometry group of M_7 [4]. The most obvious example would of course be a group manifold. The only group manifold in seven dimensions that is also Einstein is the flat seven-torus, $T^7 = [S^1]^7$, with the isometry group $U(1)^7$ [4]. We will, however, not deal with this here but instead focus on another type of manifolds admitting a group of motions. These are the homogeneuous spaces corresponding to cosets G/H discussed in Chapter 4. We will especially look at the round seven-sphere $SO(8)/SO(7)$ and the squashed seven-sphere $\frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$. Other solutions are also possible [4].

2.3 Compactification on a G_2 manifold

A G_2 manifold is a manifold that has as its holonomy group the exceptional Lie group G_2. Whenever you compactify from eleven dimensions to $M_4 \times M_7$ and want to have $\mathcal{N} = 1$ supersymmetry you will constrain M_7 to have G_2 holonomy. This means that there should be one G_2-covariantly constant Killing spinor, i.e.

$$\nabla_{G_2}\eta = 0. \tag{2.62}$$

In the Freund-Rubin solution this is exactly what we have. The Killing spinor equation is

$$\nabla_a \eta = i\frac{m}{2}\Gamma_a \eta, \tag{2.63}$$

when ∇ is the covariant derivative of the tangent space group, $SO(7)$, but we can make it G_2 covariant by introducing

$$\tilde{\nabla}_a \eta \equiv \left(\nabla_a - i\frac{m}{2}\Gamma_a\right)\eta = 0. \tag{2.64}$$

As stated in Appendix B G_2 is one of the five exceptional Lie groups. It has rank 2 and dimension 14. It is a subgroup of $Spin(7)$ which is the covering group of $SO(7)$. The reason why we get $\mathcal{N} = 1$ supersymmetry in G_2 compactification is that the spinor of $Spin(7)$ decomposes as $8 = 7 + 1$ under G_2, and the singlet exactly corresponds to the covariantly constant Killing spinor [15], as per the discussion in Section 2.2.1.

A G_2 manifold is characterised by a real, covariantly constant (with respect to the G_2 covariant derivative), three-form, ϕ_{abc}. This also has a dual four-form, usually denoted $\psi_{abcd} = \star\phi_{abc}$. The three-form defines a metric on M_7 [17]

$$g_{ab} = \frac{\sqrt{\mathrm{Det}(g)}}{3!4!}\phi_{ac_1c_2}\phi_{bc_3c_4}\phi_{c_5c_6c_7}\epsilon^{c_1\cdots c_7} = \frac{1}{4!}\phi_{ac_1c_2}\phi_{bc_3c_4}\psi^{c_1c_2c_3c_4}. \tag{2.65}$$

If the Killing spinor is Majorana and well defined over the whole of M_7 the three-form can also be expressed as [17]

$$\phi_{abc} = -i\bar{\eta}\Gamma_{abc}\eta. \tag{2.66}$$

The three-form can be chosen as the structure constants of the octonions, a_{abc}, and we will use the above relation in Appendix D to derive all possible contractions between a_{abc} and its dual four-form c_{abcd}. Since this identity holds for any three-form of a G_2 manifold with a well-defined Majorana Killing spinor, the contraction identities found in Appendix D will hold for any three-form characterising such a G_2 manifold.

3

Quantum gravity and the swampland

One of the big problems with string theory, in its present formulation, is that it does not seem to single out our Universe as the only solution [7]. In fact it gives us an enormously vast landscape of different possible theories, depending on, for example, how we compactify the extra dimensions. This landscape is so huge that it almost seems like you could pick any reasonable effective theory and find it in some limit of string theory. However, in recent years several ideas on how to exclude seemingly consistent effective theories from this landscape of possibilities have been proposed. It has been conjectured that there exists an even vaster area of inconsistent theories called the *swampland* [8]. Vafa and others have proposed a collection of specific conjectures concerning how to distinguish a consistent theory from an inconsistent one.

Here we discuss some of the proposed criteria and show examples where they are upheld in string theory and supergravity. We also discuss how one can apply them to the standard model of particle physics in order to put constraints on neutrino masses. Several of these conjectures have been used in other cases as well, resulting in some rather non-trivial predictions [18, 19]. It is therefore of great relevance to either prove or disprove these conjectures, which is the motivation of this thesis, as discussed below.

Before turning to the conjectures of the swampland we briefly discuss gauge/gravity duality, since this is needed for some of the later discussions.

3.1 Gauge/gravity duality

One of the most exciting areas to emerge from string theory in the last twenty years is the gauge/gravity duality (sometimes referred to as the AdS/CFT duality) [20]. This describes a correspondence between operators in a D-dimensional quantum field theory and local fields in a $(D+1)$-dimensional theory of gravity [21]. The statement is that the field theory lives on the boundary of the gravity theory, and this is therefore sometimes called holography [22].

One can study a system of N coincident $D3$-branes in type IIB string theory and take different limits of the 't Hooft coupling $\lambda \equiv 4\pi g_s N$, where g_s is the string coupling and N is held fixed (and large). In the limit $\lambda \gg 1$ the branes will collapse into a black brane (the generalisation of a black hole), and one can find that the near horizon limit is described by $AdS_5 \times S^5$ [22]. In the other limit, $\lambda \ll 1$, one instead finds that the low-energy excitations corresponds to $SU(N)$ gauge fields that realise a four-dimensional conformal field theory called $\mathcal{N} = 4$ super-Yang-Mills [22]. We then have two different theories in the opposite limits, and the conjecture is that these are equivalent [20].

The duality has not yet been proven, but it has been succesfully tested in a lot of different cases, and no counterexamples have been found. The gauge/gravity duality is therefore strongly believed to be true [22].

One very useful thing about the correspondence is that it relates strongly coupled systems in one limit to weakly coupled systems in the other limit [22]. So instead of doing difficult calculations in a strongly coupled field theory we can go over to the weakly coupled gravity theory, and vice versa. This allows us to perform calculations that previously were thought to be nearly impossible.

The idea that a gravity theory has a dual field theory picture will be of great use to us when we discuss the conjectures regarding the string landscape and the swampland.

3.2 The Weak Gravity Conjecture

The most important conjecture for this thesis is the Weak Gravity Conjecture (WGC). The simplest version of the WGC can be stated roughly as follows: Given a quantum field theory coupled to gravity there must exist some object with an elementary gauge charge, where the corresponding repulsive force is greater than the attractive force of gravity [10].

In a $U(1)$ gauge theory coupled to gravity we can consider two objects with mass m and minimal equal charge q placed a distance r from each other. The repulsive electric force on the two objects will be $F_e \sim \frac{q^2}{r^2}$. The attractive force due to gravity is given by $F_g \sim \frac{m^2 G}{r^2} \sim \frac{m^2}{M_{Pl}^2 r^2}$, where M_{Pl} is the Planck mass, and G Newton's constant. Now, the WGC claims that $F_g \leq F_e$ which implies that the lightest state in the spectrum satisfies

$$\frac{m}{M_{Pl}} \leq q. \tag{3.1}$$

There are several motivations for the WGC. One is that it certainly is true in our Universe, the gravitational force being by far the weakest of the four fundamental forces of Nature. Another argument is that it seems to always be true in string theory, our only known UV consistent theory able to couple a quantum field theory to gravity [10].

As an example we can study the spectrum of heterotic string theory. The heterotic string is constructed by combining bosonic degrees of freedom in one direction with supersymmetric degrees of freedom in the other direction. We will use the convention that the left-movers are bosonic and the right-movers supersymmetric. The bosonic string lives in 26 dimensions and the superstring in 10, so in order to combine these one must compactify the extra 16 dimensions in some way. This should be done on some even and self-dual lattice [23]. These conditions on the lattice arise at the one-loop level. There are only two even and self-dual lattices of dimension 16, namely the weight lattices of $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$. This gives us two possible heterotic theories, and we call them the $E_8 \times E_8$ and the $SO(32)$ heterotic strings. However, this distinction is only available in ten dimensions; once we compactify a number of dimensions the two theories will share the same moduli space [15].

When compactifying the heterotic string on the d-dimensional torus T^d we get a $U(1)^{16+2d}$ gauge symmetry, whose charges make up an even self-dual lattice, i.e. $(p_L, p_R) \in \Gamma^{16+d,d}$. This is so because we need to compactify $16 + d$ of the left-moving dimensions and d of the right-moving ones [15].

The mass spectra of the heterotic string is given by

$$\frac{1}{2}M^2 = \frac{1}{2}p_L^2 + N_L - 1 = \frac{1}{2}p_R^2 + N_R, \tag{3.2}$$

where N_L and N_R are the oscillator contribution for the left- and right-movers, respectively [15]. We are interested in the lowest mass states for a given set of charges. By writing $\frac{1}{2}(p_L^2 - p_R^2) = N_R - N_L + 1$, we see that if we put $N_L = N_R = 0$ we must have $p_L^2 - p_R^2 = 2$. On the other hand, if $p_L^2 - p_R^2 < 2$ we can have $N_R = 0$ but N_L must be non-zero. This state has $\frac{1}{2}M^2 = \frac{1}{2}p_R^2$, and is a so called BPS state. These are massive short representations of extended supersymmetry algebras, where the mass equals the central charge (for more information on extended supersymmetry, and its representations, see Appendix C). The BPS state saturates the WGC bound. If instead $p_L^2 - p_R^2 > 2$ we can put $N_L = 0$ which gives us $\frac{1}{2}M^2 = \frac{1}{2}p_L^2 - 1$. This is not a BPS state. It obeys the strict inequality $M^2 < p_L^2$. So we see that the heterotic string gives us some examples of the WGC being valid.

We can also motivate the conjecture by considering an electrically charged black hole with mass M and charge Q. The action of such an object in four dimensions is

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right). \tag{3.3}$$

The most general static and spherically symmetric solution is the Reissner-Nordström metric

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega^2, \tag{3.4}$$

where $\Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, and we use units where $G = 1$ [15]. The event horizon of this black hole is at $r_H = M \pm \sqrt{M^2 - Q^2}$, and we see that it can only be present for $M \geq |Q|$. If $M < |Q|$ we get a so called naked singularity. These are, however, forbidden by the cosmic censorship hypothesis [15].

In particular, an extremal black hole is a black hole saturating the Reissner-Nordström bound, i.e., with $M = |Q|$. We know that all (non-BPS) black holes should be allowed to send out Hawking radiation. But, if the WGC is incorrect we have $m > q$ for all states in the spectrum. This means that when the black hole radiates a particle it will afterwards have $M' < |Q'|$, violating the extremality bound. The only solution is for the WGC to hold with at least one type of particle in the spectrum satisfying Eq.(3.1).

The WGC was first introduced by Arkani-Hamed et al. in [10], and there they also give a generalisation of the conjecture to include more general types of charged objects. As an example, consider a p-form Abelian gauge field in D dimensions. Then there must exist electrically and magnetically charged objects of $D-1$ and $D-p-1$ dimensions respectively. According to the generalised conjecture these then have tensions satisfying

$$T_e \leq \left(\frac{g^2}{G}\right)^{1/2}, \qquad T_m \leq \left(\frac{1}{g^2 G}\right)^{1/2}, \tag{3.5}$$

where g is the coupling constant having dimension $m^{p+1-D/2}$ [10]. The bounds are saturated by BPS states but are strict for other types of objects. This fact, together with the case of the heterotic string, motivates a small sharpening of the WGC that we will discuss next.

3.3 A sharpened WGC, and AdS instability

Ooguri and Vafa, [11], have proposed a sharpening of the WGC allowing the inequality of Eq.(3.1) to be saturated *if and only if* the underlying theory is supersymmetric and the states saturating the bound are BPS states.

One motivation for the sharpened conjecture comes from considering extremal black holes, again. If we allow these black holes to emit particles at the bound small perturbations could easily tip the scale in the wrong direction, and the black hole would thus violate the WGC. In this case we must have a good reason for why this does not happen. In the supersymmetric case the BPS states do not allow for these fluctuations and instead provide us with a robustness of the conjecture [11].

The sharpened conjecture is further motivated by the cases we have already seen, e.g. the heterotic string, and the charged branes discussed in the previous section. In connection to those two cases it has been shown that string corrections to the mass/charge ratio of extremal black holes also aligns with the sharpened conjecture [24]. The idea is that if the mass/charge ratio of the extremal black hole is exact it would yield an infinite number of exactly stable particles, which seems highly improbable.

Kats et al., [24], study the higher derivative corrections, such as, e.g., R^2 or F^4 terms, to the action of the non-supersymmetric charged black holes arising when compactifying the heterotic string on a d-dimensional torus. When discarding the contribution from the dilaton field this gives a $(10-d)$-dimensional Reissner-Nordström black hole charged under the $U(1)$ gauge field of the heterotic string (either $SO(32)$ or $E_8 \times E_8$). They find that the leading order corrections are of the form $M = |Q|(1 - \varepsilon(Q))$, with $\varepsilon(Q) \geq 0$ [24].

When the dilaton field is included it does not give a Reissner-Nordström black hole but something called a GHS black hole (for Garfinkle, Horowitz and Strominger) [25]. This is defined by

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r\left(r - \frac{Q^2 e^{-2\phi}}{M}\right) d\Omega^2, \tag{3.6}$$

where

$$e^{-2\phi} = e^{-2\phi_0}\left(1 - \frac{Q^2 e^{-2\phi_0}}{Mr}\right), \tag{3.7}$$

and ϕ_0 is the limit of ϕ at infinity [24]. This has a dilatonic type charge $D = -Q^2/M$ which for an extremal black hole is $D = -M$. Kats et al. find that the corrections to this charge is given by $D \sim -|Q|(1 - \alpha'/Q^2)$, and to the mass/charge ratio by $M \sim |Q|(1 - \alpha'/Q^2)$. This means that the net force between two particles of mass M, with dilatonic charge D and $U(1)$ gauge charge Q, $F \sim \frac{Q^2 - D^2 - M^2}{r^2}$, will be repulsive. This is in agreement with the sharpened WGC [24].

The proposed sharpening of the WGC may seem like an innocent modification, but, as we will now see, it comes with an important consequence for AdS vacua supported by fluxes.

Let us consider an AdS vacuum supported by a flux, i.e. with a $(p+1)$-form gauge potential with field strength along AdS given by $F_{i_1 \ldots i_{p+2}} \sim \epsilon_{i_1 \ldots i_{p+2}}$. A good example of such an AdS vacuum is the one appearing in the Freund-Rubin compactification of 11D supergravity discussed in the previous chapter. It has been shown that a non-supersymmetric AdS space of this type is unstable if there is a p-brane charged with respect to the flux with charge smaller than tension [26]. This is due to the fact that a spherical brane can be created that expands to the AdS boundary in finite time, since the repulsive force of the charge will win over the tension. This reduces the flux, implying that the vacuum is unstable. It could however be a slow expansion and the system could possibly be quasi-stable [26]. The sharpened WGC states that these branes must be present, and a corollary of the sharpened version is therefore that non-supersymmetric AdS vacua supported by fluxes must be unstable [11]. This is a rather remarkable result from such a simple conjecture, giving us strict constraints on what we can expect from a consistent theory of quantum gravity. It is therefore imperative that the sharpened WGC is tested and either proven or falsified, which is the motivation for this thesis.

In the holographic dual picture the situation gets even worse. Due to gravitational time delay the lifetime will get shorter and shorter as we move closer to the horizon, eventually becoming instantaneous in the near horizon limit. This implies that the dual field theory could not even be meta-stable [11].

3.3.1 Compactifying the SM and constraining neutrino masses

As an example of the large footprint the sharpened conjecture leaves we can study the Standard Model (SM). Arkani-Hamed et al., [27], has shown that compactifying the SM plus gravity to two or three dimensions results in a rich landscape of possible vacua.

We will compactify the SM on a circle (compare with Section 2.1). This produces a reduced action of the gravitational sector

$$S_{gr} \to \int d^3x \sqrt{-g_{(3)}} 2\pi r \left[\frac{1}{2} M_{(4)}^2 R_{(3)} - \frac{1}{4} \frac{\mathcal{R}^4}{r^4} V_{\mu\nu} V^{\mu\nu} - M_{(4)}^2 \left(\frac{\partial \mathcal{R}}{\mathcal{R}} \right)^2 - \frac{r^2 \Lambda_{(4)}}{\mathcal{R}^2} \right], \quad (3.8)$$

where R is the Ricci scalar, $V_{\mu\nu}$ the field strength of the graviphoton, Λ the cosmological constant, M is the reduced Planck mass, $\mathcal{R}$ the radion field and r is a scale parameter proportional to the expectation value of $\mathcal{R}$ [27]. Parenthesised indices indicate which dimension the quantity belongs to, for instance, $\Lambda_{(4)}$ is the four-dimensional cosmological constant.

Because of the 4D cosmological constant term in Eq.(3.8) the classical potential will grow quickly when we make the circle smaller, but since the cosmological constant is so small the quantum effects become important. At one-loop level the corrections comes from the Casimir energy [27]. For a particle of mass m the contribution is of the form $e^{-2\pi m \mathcal{R}}$ when $\mathcal{R} \gg 1/m$ [27]. Thus, for any $\mathcal{R}$, the only relevant particles are those with $m < 1/\mathcal{R}$, since the others are exponentially suppressed. They contribute $\sim \mp \frac{n_0}{720\pi} \frac{r^3}{\mathcal{R}^6}$ to the effective potential of a massless state, where $\mp$ are for bosons/fermions (with periodic boundary conditions), and n_0 is the number of degrees of freedom [27]. In the standard model we only have two massless particles, namely the graviton and the photon, each having two bosonic degrees of freedom. The effective potential is then

$$V(R) = \frac{2\pi r^3 \Lambda_{(4)}}{\mathcal{R}^2} - \frac{4r^3}{720\pi \mathcal{R}^6}. \quad (3.9)$$

Comparing the two terms we see that for large radii the classical part wins and the vacua will expand, while for small radii the quantum effects take over and the compact dimension shrinks. This gives us a maximum at

$$\mathcal{R}_{max} = \left(\frac{1}{120\pi^2 \Lambda_{(4)}} \right)^{1/4} \approx 20\mu\text{m}, \quad (3.10)$$

using the current experimental value of $\Lambda_{(4)}$ [27]. The associated mass scale is then $m = \frac{1}{2\pi \mathcal{R}_{max}} \approx 10^{-3}$eV, close to the neutrino mass scale around $10^{-2} - 10^{-1}$eV [13].

When we start from a radius smaller than $\mathcal{R}_{max}$ the dimension will shrink. However, when we reach a level where the inverse radius is comparable to the mass of the lightest particles in the spectra their contribution must also be accounted for. In the SM the lightest particles are the neutrinos, which according to the above will contribute to the potential with a different sign, since they are fermions. The potential will now look something like

$$V(R) = \frac{2\pi r^3 \Lambda_{(4)}}{\mathcal{R}^2} - \frac{4r^3}{720\pi \mathcal{R}^6} + \sum_{i=\nu_e,\nu_\mu,\nu_\tau} \frac{n_i r^3}{720\pi \mathcal{R}^6} \Theta(\mathcal{R}_i - \mathcal{R}), \quad (3.11)$$

with Θ being a step function and $\mathcal{R}_i = 1/m_{\nu_i}$ [13]. Every neutrino has two degrees of freedom if they are Majorana and four if they are Dirac. We now see that, depending on the masses of the neutrinos, we could also get a minimum for the potential.

At the moment we do not know the masses of the neutrinos, but they are constrained by measurements on solar and atmospheric oscillations [13]. From these measurements we only know the difference in mass (squared) between the neutrinos. If we denote this difference between neutrino i and j as Δm^2_{ij} the experimental values are

$$\begin{aligned}\Delta m^2_{21} &= (7.53 \pm 0.18) \times 10^{-5}\mathrm{eV}^2, \\ \Delta m^2_{32} &= (2.44 \pm 0.06) \times 10^{-3}\mathrm{eV}^2, \ \text{ for normal hierachy (NH)}, \\ \Delta m^2_{32} &= (2.51 \pm 0.06) \times 10^{-3}\mathrm{eV}^2, \ \text{ for inverted hierachy (IH)},\end{aligned} \tag{3.12}$$

where normal hierachy means that $m_{\nu_1} < m_{\nu_2} < m_{\nu_3}$ and inverted means $m_{\nu_3} < m_{\nu_1} < m_{\nu_2}$ [13]. We do not know which ordering is the correct one. From NH we now find

$$m_{\nu_2} \approx 8.6 \times 10^{-3}\mathrm{eV}, \qquad m_{\nu_3} \approx 4.9 \times 10^{-2}\mathrm{eV}, \tag{3.13}$$

where ν_1 can be arbitrarily light, and IH implies

$$m_{\nu_1} \approx m_{\nu_2} \approx 4.9 \times 10^{-2}\mathrm{eV}, \tag{3.14}$$

with ν_3 being arbitrarily light [13]. Using these experimental values one finds that for Majorana neutrinos the minima will always be present, independent of the mass of the lightest neutrino. This means that we have a new SM vacuum with negative potential, i.e. we have $AdS_3 \times S^1$ [27]. If the neutrinos instead are Dirac one finds that the vacua can be either dS, AdS or Minkowski [27]. Ibañez et al., [13], found that the bounds of the lightest neutrino are the ones given in Table 3.1.

	NH	IH
Minkowski	$m_{\nu_1} < 6.7$meV	$m_{\nu_3} < 2.1$meV
de Sitter	6.7meV $< m_{\nu_1} <$ 7.7meV	2.1meV $< m_{\nu_3} <$ 2.56meV
anti-de Sitter	7.7meV $< m_{\nu_1}$	2.56meV $< m_{\nu_3}$

Table 3.1: Mass ranges of lightest neutrinos for different vacua configurations according to [13].

Since the SM is non-supersymmetric, this will also be true of these vacua. The sharpened WGC then tells us that if the *AdS* vacua are stable the underlying effective theory, i.e. the four-dimensional SM, will belong to the swampland, and hence lack a consistent UV completion. We thus see that Majorana neutrinos in the minimal SM is contradicted by the WGC, and that the mass of the lightest Dirac neutrino is heavily constrained.

It could also be interesting to study the effect of the uncertainties in the value of the cosmological constant. From Eq.(3.11) we see that the minimum is highly dependent on the values of $\Lambda_{(4)}$. Ibañez et al., [13], found that in order to avoid AdS vacua, when using Majorana neutrinos, the cosmological constant would need to be much larger than the present experimental values. For the Dirac case, however, they found a lower bound on $\Lambda_{(4)}$. This is very interesting since it is probably the only known example of a theory not involving dark energy yielding a non-vanishing value of the cosmological constant [13]. By plugging in the present experimental value they also found a lowest bound on the lightest Dirac neutrino in order to avoid the AdS vacuum. This bound is $m_{\nu_1} > 7.7 \times 10^{-3}$eV for NH and $m_{\nu_3} > 2.56 \times 10^{-3}$eV for IH [13]. This would then mean that the WGC predicts the neutrinos of the standard model to be massive Dirac fermions.

One could also wonder what would happen if we went beyond the standard model, and perhaps included some extra fermions. Ibañez et al. show that adding one or two light Weyl fermions to the SM makes it possible to avoid the AdS vacuum even if one considers Majorana neutrinos [13]. If we add two Weyl fermions to the SM the bounds on the Majorana neutrinos even reaches a level measureable through neutrinoless double beta decay. The same result is found by adding an extra Dirac fermion or gravitino [13]. It is also possible to add other types of particles, like axions, to the SM. This leads to a large number of different bounds on the mass ranges of the neutrinos in order to avoid the AdS vacua. As an example of the results we collect the bounds connected to normal hierarchy in Table 3.2, for inverse hierarchy a lower bound on the axion mass is also found. An equivalent analysis is done for the compactification of the SM on a two-torus, with similar results. The complete results are collected in Table 5 of [13].

Model	Majorana (NH)	Dirac (NH)
SM	no	$m_{\nu_1} \leq 7.7 \times 10^{-3}$
SM + Weyl	$m_{\nu_1} \leq 0.9 \times 10^{-2}$ $m_W \leq 1.2 \times 10^{-2}$	$m_{\nu_1} \leq 1.5 \times 10^{-2}$
SM + Dirac	$m_D \leq 2 \times 10^{-2}$	yes
SM + 1 axion	no	$m_{\nu_1} \leq 7.7 \times 10^{-3}$
≥ 2 axions	yes	yes

Table 3.2: Bounds for lightest neutrino masses, in eV, to avoid an AdS vacuum for different models [13]. A yes (no) implicates that we always (never) avoid the vacuum.

The above discussion has shown us that the WGC can be used to make predictions regarding the masses of the neutrinos of the SM as well as the possible addition of some BSM physics. Some additions even lead to measurable values of the masses. Such a prediction has long been awaited by the string theory community, and it is therefore of great importance to either prove or disprove this conjecture.

3.4 Instability of the squashed sphere

It has been shown that the squashed seven-sphere allows for two different AdS vacua, one with $\mathcal{N} = 1$ supersymmetry and one with $\mathcal{N} = 0$ [4]. The two solutions are related by the so called “skew-whiffing” which means that the orientation of the sphere is reversed [4].

Duff et al., [4], also show that both these solutions satisfy the Breitenlohner-Freedman stability criteria, which is the AdS analogue of the positive mass criteria in Minkowski space.

However, if the sharpened WGC is correct we should have no stable non-supersymmetric AdS vacua. So how does this align with the above statement? Well, fortunately there are other ways that a non-supersymmetric solution can be rendered unstable.

One possibility is that the answer lies in perturbative effects, in coupling constant space, due to massless global singlet marginal operators (GSMOs). Marginal operators are operators of the same dimension as spacetime. Global singlet marginal operators are then marginal operators that are invariant (singlets) under the available gauge symmetries [28].

The argument uses that a supergravity solution, $AdS_4 \times M_7$, is dual to a conformal field theory in the large N limit, through gauge/gravity duality [20]. There could then be some $1/N$ corrections to the beta functions of the dual field theory which could lead to tadpoles on the gravity side, effectively shifting the true vacuum of the supergravity theory [29]. The kind of operators that could be vulnerable to these corrections are just the massless GSMOs [29].

The beta function for a coupling constant, g, connected to a certain operator can be expanded in the $1/N$ perturbation as

$$\beta = \kappa(g - g_*) + \frac{a}{N} + \dots, \tag{3.15}$$

for some constant a, and $\kappa = \delta - D$, where δ is the dimension of the operator and D is the dimension of spacetime [28]. The fixed point at $N \to \infty$ is given by $g = g_*$. For non-marginal operators we have $\kappa \neq 0$ and the beta function has a zero at $g - g_* = -\frac{a}{N\kappa}$, so the effect is just to shift the fixed point. But, when we have marginal operators there is no zero and the conformal fixed point is removed. This would then cause the theory to flow towards some far away point in coupling constant space. The dual theory will therefore not exist for large but finite N, but only in the $N \to \infty$ limit [28].

Murugan, [28], argues that, for $\mathcal{N} = 2$ supersymmetry, all non-supersymmetric solutions due to skew-whiffing should contain at least one such GSMO. This is due to the fact that all supergravity compactifications of the form $AdS_4 \times M_7$ has an AdS_4 massless gauge multiplet transforming in the adjoint representation of the symmetry group of M_7. The dual field theory must then also contain such a multiplet, corresponding to the supersymmetric completion of the conserved global vector currents of the symmetry group. This multiplet is conserved under skew-whiffing since it only depends on the global symmetry group of the field theory. The scalars, π and S, of this multiplet for $\mathcal{N} = 2$, could then be used to form a condensate

$$\mathcal{O} \sim \mathrm{Tr}(\pi S), \tag{3.16}$$

which is thus a GSMO of the theory. This could then lead to the vacuum being unstable [28]. The problem is that for $\mathcal{N} = 1$, which we are interested in, the gauge multiplet does not contain the scalars. It should still be possible to create similar condensates but it is unclear exactly where they should appear. With this thesis we therefore aim to study the mass spectra of the squashed sphere. If we could find the complete spectra we could use it to look for the corresponding condensates of Murugan, and thus either strengthen the WGC or perhaps disprove it.

Another possibility is that the instability is due to non-perturbative effects related to Witten's bubble of nothing [30]. Witten showed that an $\mathbb{R}_9 \times S^1$ geometry admits a solution where a bubble of nothing can nucleate and expand until it takes over the whole of spacetime [30]. Further investigations have also been able to show that the same thing can happen in AdS geometries [28]. It is in general, however, very complicated to see if these bubbles can be constructed. One needs to know a whole lot more about the theory. Many geometries have been shown to allow these bubble solutions and at the moment there are no indications that the squashed seven-sphere should be an exception.

3.5 Moduli spaces and the swampland

In Chapter 2 we briefly discussed the moduli space of compactifying string theory on the k-dimensional torus. The moduli space is of course a more general concept than this. It describes the set of parameters that defines a certain solution of a theory. This could be something as abstract as the eccentricity of an ellipse, or it could be something more concrete as the free parameters of the standard model. The idea is that a transformation in the moduli space corresponds to changing the specific solution, but not the theory, you are looking at. For example you can move around in the moduli space of the ellipse and get ellipses of different eccentricity but all of them will still be ellipses. The same thing happens in compactification of string theory, you can change the vacuum by changing the value of the scalar fields, but we still consider it the same theory (please note that some authors have another interpretation of the word theory, interchanging it with what we call a specific solution of a theory).

In an article by Ooguri & Vafa, [9], they propose five conjectures regarding the moduli space of a consistent quantum field theory coupled to gravity. We will briefly discuss them here and then look at an example where they are satisfied, namely type IIB supergravity.

As we saw in Chapter 2 toroidal compactification of string theory led to the moduli space being parameterised by massless scalar fields. In string theory this is always the case, i.e. there are no free parameters in string theory [15]. But we could very well imagine that it is possible to construct a consistent theory of quantum gravity where there are fixed coupling constants that does not depend on scalar fields. The values of these constants would then also be part of the parameterisation of the moduli space. However, Ooguri & Vafa's first conjecture forbids these imagined theories. The moduli space, $\mathcal{M}$, of a consistent theory of quantum gravity is conjectured to be parameterised solely by the inequivalent expectation values of massless scalar fields [9]. This is something that of course already is at the core of string theory, and since no other consistent theories of quantum gravity are available at the moment we leave the discussion at that.

A natural metric for the moduli space is now given by the kinetic term of the available scalar fields [14]. We thus write

$$ds^2 = g_{ij}(\phi)d\phi^i d\phi^j, \tag{3.17}$$

for some massless scalars ϕ_i, we will see an explicit example of this below. The second conjecture now concerns distances in the moduli space. The shortest geodesic between two points in the moduli space is conjectured to be able to take arbitrarily large values. I.e., for a fixed point $p_0 \in \mathcal{M}$ one can always find another point $p \in \mathcal{M}$ such that $d(p, p_0) > T$ for any positive T. In other words, the moduli space must be non-compact [9]. This conjecture is true in all known examples of string theory [9]. Connected to this is the third conjecture: The theory in the limit $T = d(p, p_0) \to \infty$, for some fixed point p_0, will have an infinite tower of light states, compared to the theory at p_0, with mass decreasing as $m \sim e^{-\alpha T}$, for some $\alpha > 0$ [9].

The third conjecture implies that low energy effective theories defined in a specific point of the moduli space can only be related to other effective theories defined by neighbouring points, or, put in another way, the low energy effective description will break down when we move too far in the moduli space [12].

If we take some effective field theory and compactify it on a circle the theory will have the radius of the circle, $\mathcal{R}$, as its moduli. The metric of the moduli space will therefore be

$$ds^2 = \left(\frac{d\mathcal{R}}{\mathcal{R}}\right)^2. \tag{3.18}$$

The distance between a fixed point $\mathcal{R}_0$ and some other point $\mathcal{R}$ is then given by

$$\int_{\mathcal{R}_0}^{\mathcal{R}} \frac{d\mathcal{R}}{\mathcal{R}} = \log \mathcal{R} - \log \mathcal{R}_0. \tag{3.19}$$

We directly see that the second conjecture is satisfied, since we are able to always find an $\mathcal{R}$ that gives an arbitrary size to this distance. In particular, if we take the limit of $\mathcal{R} \to \infty$ we see that we get light Kaluza-Klein states, since $m \sim \frac{1}{\mathcal{R}}$. But we also have $T \sim \log \mathcal{R} \implies m \sim e^{-T}$, which aligns with the third conjecture.

If we instead take the limit $\mathcal{R} \to 0$ we should again expect some light states, since $\lim_{\mathcal{R}\to 0} \int_{\mathcal{R}}^{\mathcal{R}_0} \frac{d\mathcal{R}}{\mathcal{R}}$ diverges. The states can, however, not be particle states, since these are given by Kaluza-Klein modes which are highly massive in this limit. So the only option is that these states are some extended objects (e.g. strings or branes) that wrap around the circle. Conjecture three thus has the remarkable corollary that a consistent theory of quantum gravity must include extended objects capable of wrapping a compact dimension [12].

The fourth conjecture states that the curvature of the moduli space is strictly negative near the points at infinity (if the dimension is larger than one) [9]. This is given by the fact that there are points infinitely far away while the volume of the space is finite.

The final conjecture is the statement that the moduli space should be simply-connected [11]. In all known examples from string theory the moduli space is found by the quotient

$$\mathcal{M} = \mathcal{T}/\Gamma, \tag{3.20}$$

where $\mathcal{T}$ is some simply-connected space (often a so called *Teichmüller* space), and Γ is some group action on $\mathcal{T}$ [9]. For example, the space $O(d,d;\mathbb{R})/[O(d,\mathbb{R}) \times O(d,\mathbb{R})]$ of toroidal compactification is a Teichmüller space, while the T-duality symmetry $O(d,d;\mathbb{Z})$ is a group action on it, implying that the moduli space of toroidal compactification is of the above type [14]. Further, all known Γ's in string theory are generated by group elements, g_i, having fixed points. These fixed points are then gauge symmetries of $\mathcal{M}$. We could equally well consider the whole of Γ as a broken gauge symmetry of $\mathcal{M}$ [9].

If every element of Γ can be decomposed to elements with fixed points this implies that $\mathcal{M}$ is simply-connected, since all loops $\gamma \subset \mathcal{M}$ can be identified with an element in Γ of the form $\prod g_i$, implying that every segment of the loop can be contracted to a point [9].

We now turn to an example that includes all of these conjectures, type IIB supergravity in ten dimensions. This theory has two scalar fields, namely the dilaton, ϕ, and the axion, χ. The corresponding kinetic term in the Lagrangian is given by [31]

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + e^{2\phi}\partial_\mu\chi\partial^\mu\chi\right). \tag{3.21}$$

By introducing the variable $\lambda = \chi + ie^{-\phi}$, and thus combining the two real scalar fields into one complex, we can rewrite the Lagrangian as

$$\mathcal{L}_{\text{kin}} = 2\frac{\partial_\mu\lambda\partial^\mu\lambda^*}{(\lambda-\lambda^*)^2}. \tag{3.22}$$

This means that the metric of the moduli space can be expressed as

$$ds^2 = \frac{d\lambda d\lambda^*}{(\text{Im}\lambda)^2}. \tag{3.23}$$

The geodesic distance from a fixed point $\lambda_0 \in \mathcal{M}$ to another point $\lambda \in \mathcal{M}$ is given by

$$T \sim \log\left(\text{Im}\lambda\right) - \log\left(\text{Im}\lambda_0\right). \tag{3.24}$$

This is of course divergent, and when we take the $\lambda \to i\infty$ limit we see that there are light states with mass $\sim e^{-T}$. So that conjectures two and three are upheld [9].

It is easy to show that the line element is invariant under the transformations

$$\lambda \to \lambda + 1, \qquad \lambda \to -1/\lambda, \tag{3.25}$$

generating the group $SL(2,\mathbb{Z})$. In fact, it can be shown that the full moduli space can be described as the coset space [31]

$$\mathcal{M} = SL(2,\mathbb{Z})\Big\backslash SL(2,\mathbb{R})\Big/SO(2). \tag{3.26}$$

By integrating the line element over the fundamental region of this coset space one can see that the volume of the moduli space is finite [15]. But, at the same time it allows for infinite distances. This must mean that the metric has a constant negative curvature, which in turn means that conjecture four is satisfied.

Because both $\lambda \to \lambda + 1$ and $\lambda \to -1/\lambda$ have fix points in the fundamental region of $SL(2, \mathbb{Z})$ the moduli space must be simply connected. This means that the last conjecture is also true for this case. And thus, all the conjecture are true for type IIB supergravity [9].

Ooguri & Vafa, [9], give a few more examples, and also discuss some counter examples when a quantum field theory is decoupled from gravity. This means that these conjectures need only be true in a theory of quantum gravity.

3.6 Symmetries in quantum gravity

Another criteria for the swampland is that we can not have any global symmetries in a consistent theory of quantum gravity. This can be motivated from a number of different directions, for instance, by studying C, P and T violation in particle physics or inflation and baryon number non-conservation. We will only give one argument using our beloved black holes. The interested reader is referred to Witten's discussion on the matter in [32].

Consider a particle charged under some global symmetry of the theory. This could for example be lepton or baryon number conservation. If this particle gets eaten by a black hole any information of the symmetry is lost to us, due to the no-hair theorem. Even when the black hole evaporate due to Hawking radiation we still do not get the information back. This implies that black hole formation and evaporation does not conserve global symmetries. On the other hand, gauge symmetries such as electric charge are conserved by black holes. Since, for instance, an electrically charged black hole will radiate away electrically charged particles. This implies that a theory combining quantum theory and gravity should only have gauge symmetries as its fundamental symmetries, no global symmetries [32].

String theory once again gives us further motivation for this criteria, since all symmetries appearing in string theory are either local or approximate [32].

The criteria of only having local symmetries is connected to the WGC by the following argument. When we take the limit $g \to 0$, for the coupling of some gauge symmetry, the local symmetry will effectively become global. There must therefore be something that stops this limit from being taken smoothly. The answer is given by the WGC.

The conjecture states that we have some light charged particle with

$$m_e \leq g_e M_{Pl}, \tag{3.27}$$

and the same should hold for a magnetic monopole, with the inverse coupling. The mass of these monopoles should also be proportional to the energy stored in their magnetic fields, which is linearly divergent [10]. The effective theory must then break down at a scale given by

$$\Lambda \leq g_e M_{Pl}. \tag{3.28}$$

So that when the gauge coupling goes to zero the cut off also goes to zero, stopping the limit from being taken smoothly [10].

We now leave the discussion of the string landscape and the swampland. The next chapter will provide us with some of the mathematical tools needed to decypher the mass spectra of the squashed seven-sphere.

4

Geometry of coset spaces

We now discuss some connections between group theory and geometry. In particular, two important equations are derived. The first describes how to find the eigenvalues of a covariant derivative acting on some harmonic function on a coset space, this is our so called *master equation* (4.17). The second formula connects the Riemann tensor of a coset space with the corresponding structure constants, Eq.(4.23). The chapter ends by a discussion about the differential operators appearing in the mass operators of Table 2.1.

4.1 Lie groups acting on manifolds

For a physicist it is natural to think of a Lie group as the set of transformations acting on a certain manifold. For instance, in quantum mechanics we have the rotations in $\mathbb{R}^3$ represented by the group $SO(3)$, the Poincaré group acting on Minkowski spacetime, and so on. In this chapter we will expand this notion to more general situations. We will study a Lie group, G, a manifold, M, and the differentiable mapping $\sigma : G \times M \to M$ satisfying

$$\begin{aligned} &1)\ \ \sigma(e,m) = m \qquad\qquad \text{for any } m \in M, \text{ and } e \text{ being the identity element} \\ &2)\ \ \sigma(g_1, \sigma(g_2, m)) = \sigma(g_1 g_2, m). \end{aligned} \tag{4.1}$$

Moreover, the action is said to be *transitive* if, for any $m_1, m_2 \in M$, there exists an element $g \in G$ such that $\sigma(g, m_1) = m_2$, i.e. that every point on the manifold can be reached from every other point on the manifold [33].

We also define the *isotropy group* of $m \in M$ as the subgroup of G satisfying

$$H(m) = \{g \in G | \sigma(g, m) = m\}. \tag{4.2}$$

Sometimes H is also called the *stabiliser* or *little group* of m, and it is a Lie subgroup of G [33]. The isotropy group can be thought of as the group that fixes a point on the manifold. For example, if we stand at the north pole of a two-sphere we can rotate around the polar axis without changing our position. The two-sphere therefore has $U(1)$ as its isotropy group.

The *tangent space group* of an n-dimensional Riemannian manifold is always $SO(n)$. This is the group of frame rotations that leaves the metric invariant. The isotropy group is always a subgroup of the tangent space group [4].

Now, for any subgroup, H, of G we can construct the coset space G/H which will be a manifold called a *homogeneous space*, with $\dim(G/H) = \dim G - \dim H$. Especially, if G acts transitively on the manifold M and $H(m)$ is the isotropy group of m, then $G/H(m)$ is a homogeneous space homeomorphic to M [33]. In this context G is usually referred to as the *isometry group* of M. Note that some extra care may be needed if the groups are non-compact.

A good example of a coset space is the two-sphere, S^2. The group that acts transitively on S^2 is $SO(3)$ and the isotropy group is $SO(2)$. This means that one can express the two-sphere as the coset space

$$SO(3)/SO(2) = S^2. \tag{4.3}$$

This is generalised to higher dimensions by [33]

$$SO(n+1)/SO(n) = S^n. \tag{4.4}$$

That this is true can be proven by Hopf fibration, but we will not go through this here.

It is important to note that the coset G/H is only a group if H is a normal subgroup of G. For instance, this is not the case for $SO(2)$ in $SO(3)$ so that S^2 is not a group manifold. The same holds for S^n in general. However, in the special case of $S^3 = SO(4)/SO(3)$ we have a group structure, which is also expected since S^3 is isomorphic to $SU(2)$ [33].

A certain manifold can often be described by several different cosets. For example $SO(8)$, $Spin(7)$, $SU(4)$ and $Sp(2)$ all act transitively on S^7 and they have the corresponding isotropy groups $SO(7)$, G_2, $SU(3)$ and $Sp(1)$. This means that we can express S^7 by [4]

$$\frac{SO(8)}{SO(7)}, \quad \frac{Spin(7)}{G_2}, \quad \frac{SU(4)}{SU(3)}, \quad \text{or} \quad \frac{Sp(2)}{Sp(1)}. \tag{4.5}$$

There may be several different ways in which to embed the subgroup, H, in G, this could then lead to different topologies [4].

4.2 A master equation

We will now study the eigenvalues of differential operators acting on the harmonic functions living on a coset manifold. These transform as irreducible representations of the isometry group (G). This will lead us to a master equation, Eq.(4.17), that is used many times throughout the thesis. The derivation closely follows the ones given in [4] and [34]. In order to find a general expression independent of which representation we are in at the moment we will suppress the indices of the tensors or spinors as well as the tangent space indices which characterise the irreducible representation.

Consider a Lie group, G, and its subgroup H. The complement will be denoted K, so that $G = H \times K$. We only study reductive groups where $[H, K] \subset K$, since all known coset space compactifications of eleven-dimensional supergravity are of this type [4]. One can note that demanding reductivity is the same thing as saying that H is the isotropy group of the coset space. If G is symmetric, as is the case for ordinary spheres $S^n = SO(n+1)/SO(n)$, we will also have $[K, K] \subset H$ [34].

The indices $M, N, \dots$ will run over all generators of G, $a, b, c, \dots$ will be flat indices in G/H, $m, n, p, \dots$ curved indices in G/H and $i, j, \dots$ are indices in H. The complete set of generators on G will hence be denoted T_M and they split into two types, namely the generators of H, T_i, and the generators of K, T_a.

We denote coordinates on G/H by y^m and we can associate a unique element in G, L_y, with every point y^m through

$$L_y \equiv e^{y^a T_a}, \tag{4.6}$$

where $y^a = e_m{}^a y^m$, and $e_m{}^a$ are the components of the vielbein on G/H [34]. In other words, L_y is the element in G that represents the coset G/H. This implies that, with $g \in G$, we have left translation, defined by

$$gL_y = L_{y'} h, \tag{4.7}$$

where $L_{y'}$ is a new point and $h \in H$, both uniquely defined by the action of g. We want to find an expression for the one-form $L_y^{-1}dL_y = e^M T_M$ in the algebra of G, and we do this by defining the connection on H, $\Omega^i = \Omega_m{}^i dy^m$, through

$$L_y^{-1}dL_y = e^a T_a + \Omega^i T_i \iff \partial_m L_y^{-1} = -(e_m{}^a T_a + \Omega_m{}^i T_i)L_y^{-1}, \tag{4.8}$$

were we have used that $0 = d(L_y^{-1}L_y) = (dL_y^{-1})L_y + L_y^{-1}dL_y$, and $e^a = e_m{}^a dy^m$.

Using the relevant representation of L_y^{-1} we can obtain any given harmonic Y, which means that we equally well can write

$$\partial_m Y = -(e_m{}^a T_a + \Omega_m{}^i T_i)Y. \tag{4.9}$$

The fact that $d^2 = 0$ gives us

$$d(L_y^{-1}dL_y) = (dL_y^{-1})(dL_y) = (dL_y^{-1})L_y L_y^{-1}(dL_y) = ((dL_y^{-1})L_y)(L_y^{-1}dL_y). \tag{4.10}$$

We thus see that $\omega \equiv L_y^{-1}dL_y$ satisfies the Maurer-Cartan equation $d\omega + \omega^2 = 0$, since, as we saw before, $(dL_y^{-1})L_y = -L_y^{-1}dL_y$. This implies that

$$d\omega = d(e^a T_a) + d(\Omega^i T_i) = -\omega^2. \tag{4.11}$$

Considering only reductive algebras we have

$$\begin{aligned}(e^a T_a + \Omega^i T_i)^2 &= e^a \wedge e^b T_a T_b + \Omega^i \wedge \Omega^j T_i T_j + 2\Omega^i T_i \wedge e^a T_a \\ &= \frac{1}{2} e^a \wedge e^b f_{ab}{}^c T_c + \underbrace{\frac{1}{2}\Omega^i \wedge \Omega^j f_{ij}{}^k T_k}_{=0} + \Omega^i \wedge e^a f_{ia}{}^b T_b \\ &= \frac{1}{2} e^b \wedge e^c f_{bc}{}^a T_a + \Omega^i \wedge e^b f_{ib}{}^a T_a.\end{aligned} \tag{4.12}$$

If we now project out the coefficent of T_a in Eq.(4.11) and use the above result we get

$$de^a = -\frac{1}{2} e^b \wedge e^c f_{bc}{}^a - \Omega^i \wedge e^b f_{ib}{}^a. \tag{4.13}$$

The spin connection is defined to be torsionless, $de^a = -\omega^{ab} \wedge e^b$. Comparing this to the above expression we find

$$\omega_m{}^{ab} = -\Omega^i_m f_i{}^{ab} - \frac{1}{2} e^c{}_m f_c{}^{ab}. \tag{4.14}$$

Since G is a reductive algebra and T_i are the generators of the Lie algebra of H we can fix the embedding of H in the tangent space group by writing

$$T_i = -f_i{}^{bc}\Sigma_{bc}, \tag{4.15}$$

where Σ are the generators of the tangent space group, $SO(n)$, [4].

Plugging all of the above results into the definition of a covariant derivative acting on a harmonic Y we have

$$\begin{aligned}\nabla_m Y = \partial_m Y + \omega_m{}^{ab}\Sigma_{ab} Y &= \Big[- \Big(e^a_m T_a + \underbrace{\Omega_m{}^i T_i}_{= -\Omega^i_m f_i{}^{bc}\Sigma_{bc}} \Big) - \Big(\Omega_m{}^j f_j{}^{ab} + \frac{1}{2} e^c_m f_c{}^{ab} \Big) \Sigma_{ab} \Big] Y \\ &= -e_m{}^a T_a Y - \frac{1}{2} e_m{}^c f_c{}^{ab}\Sigma_{ab} Y.\end{aligned} \tag{4.16}$$

This is our so called master equation. To underline the importance of this in the work to come we state the final result one more time

$$\nabla_m Y + \frac{1}{2} e_m{}^c f^{ab}{}_c \Sigma_{ab} Y = -e_m{}^a T_a Y. \tag{4.17}$$

We will use this equation when deriving the mass spectra of both the round and the squashed seven-spheres.

4.3 The curvature

In this section we build on the above results to find a formula for the Riemann tensor expressed solely in the structure constants of G and some vielbeins, Eq.(4.23). The derivation follows that of [34] and [35].

Without loss of generality we can assume the Killing form to be block diagonal, so that $\kappa_{MN} = f_{MR}{}^S f_{NS}{}^R = 0$ for $M \in H$ and $N \in K$.

As we saw in Eqs.(4.13) and (4.14) the vielbein and H connection satisfy

$$\partial_{[m} e_{n]}{}^a + f_{bc}{}^a e_m{}^b e_n{}^c + f_{ib}{}^a \Omega_{[m}{}^i e_{n]}{}^b = 0, \tag{4.18}$$

$$\partial_{[m} \Omega_{n]}{}^i + f_{jk}{}^i \Omega_m{}^j \Omega_n{}^k + f_{ab}{}^i e_m{}^a e_n{}^b = 0. \tag{4.19}$$

A connection, $\omega_m{}^b{}_c(y)$, defining parallel transport on the manifold is called *invariant* if it also commutes with all group elements. One can construct the most general form of an invariant connection by writing

$$\omega_m{}^b{}_c(y) = \Omega_m{}^i f_{ic}{}^b + e_m{}^a \omega_a{}^b{}_c(0). \tag{4.20}$$

Here $\omega_a{}^b{}_c(0)$ is an arbitrary invariant tensor in the adjoint representation of H, restricted to the coset [35]. One such set of tensors are the structure constants, which means that we can choose $\omega_a{}^b{}_c(0) = \frac{1}{2} f_{ac}{}^b$. This will give us the same expression as in Eq.(4.14). With this we can rewrite Equation (4.18) as

$$\partial_{[m} e_{n]}{}^a + \omega_{[m}{}^a{}_{|b|} e_{n]}{}^b = 0. \tag{4.21}$$

The Riemann tensor is defined in the usual way

$$R_{mn}{}^a{}_b \equiv 2\partial_{[m} \omega_{n]}{}^a{}_b + 2\omega_{[m}{}^a{}_{|c|} \omega_{n]}{}^c{}_b. \tag{4.22}$$

Plugging in the results of Equations (4.19) and (4.21), rewriting it a bit and using the Jacobi identity for the structure constants we find our formula for the Riemann tensor expressed only in the structure constants of G and vielbeins [35]

$$R_{mn}{}^a{}_b = \left(\frac{1}{2} f_{bc}{}^a f_{de}{}^c + f_{bi}{}^a f_{de}{}^i + \frac{1}{2} f_{dc}{}^a f_{eb}{}^c \right) e_{[m}{}^d e_{n]}{}^e. \tag{4.23}$$

Note that $f_{ab}{}^c = 0$ for symmetric coset spaces. This formula will be used in Chapter 6 when calculating the Riemann tensor of the squashed seven-sphere.

4.4 Differential operators on forms, spinors and tensors

Since our master equation concerns differential operators acting on some arbitrary harmonic functions, we must know what these operators are. In seven dimensions we are interested in differential forms of rank zero up to three (higher ranks are related to these through the Hodge star operator [4]). The relevant operator for these are the corresponding Hodge-de Rahm operators. For supergravity we are also interested in the Dirac operator acting on spin-1/2 and the corresponding operator acting on the gravitino, with spin-3/2. The graviton is a symmetric transverse and traceless mode and the corresponding operator is the Lichnerowicz operator. We start by discussing the Hodge-de Rahm operator acting on zero-forms, one-forms, two-forms and three-forms.

4.4.1 Hodge-de Rahm

For a general p-form

$$\omega = \frac{1}{p!}\omega_{m_1 \dots m_p} dx^{m_1} \wedge \dots \wedge dx^{m_p}, \tag{4.24}$$

the Hodge star operator is defined by

$$\star\,\omega \equiv \frac{\sqrt{|g|}}{p!(D-p)!}\omega_{m_1\dots m_p}\epsilon^{m_1\dots m_p}{}_{m_{p+1}\dots m_D} dx^{m_{p+1}} \wedge \dots \wedge dx^{m_D}, \tag{4.25}$$

where D is the number of dimensions. We define the exterior derivative, d, as mapping p-forms to $(p+1)$-forms

$$d\omega = (p+1)\partial_{[m_1}\omega_{m_2\dots m_{p+1}]}. \tag{4.26}$$

The adjoint of d, $\delta \equiv (-1)^p \star d \star$, then maps a p-form into a $(p-1)$-form

$$\delta\omega = -\nabla^n \omega_{n m_1 \dots m_{p-1}}. \tag{4.27}$$

If $d\omega = 0$ we call ω *closed*, if $\delta\omega = 0$ we call it *coclosed*. If ω can be expressed as $d\alpha$ for some form α we call ω *exact*, and *coexact* if $\omega = \delta\alpha$ [4].

We can now define the Hodge-de Rahm operator, Δ, as [33]

$$\Delta \equiv d\delta + \delta d. \tag{4.28}$$

A form satisfying $\Delta\omega = 0$ is called *harmonic*. The norm of ω can be defined in the usual manner [4]

$$(\omega, \omega) \equiv \int d^7x \sqrt{g}\omega_{m_1\dots m_p}\omega^{m_1 \dots m_p}. \tag{4.29}$$

Due to the adjoint relation between d and δ one can show that they satisfy [33]

$$(\omega, d\omega) = (\delta\omega, \omega), \tag{4.30}$$

which means that we have

$$(\omega, \Delta\omega) = (\omega, d\delta\omega + \delta d\omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega), \tag{4.31}$$

and hence $\Delta \geq 0$, with equality if and only if ω is closed and coclosed [33].

We will need the explicit expressions for the Hodge-de Rahm operator acting on zero-, one-, two- and three-forms, since the others are related to these by the Hodge star operator in seven dimensions. Hodge-de Rahm acting on a p-form will henceforth be denoted Δ_p. Using the expression for ω in Eq.(4.24), and acting with Δ on a zero-form, ϕ, in D dimensions, we find [33]

$$\begin{aligned}
\Delta_0\phi &= d\delta\phi + \delta d\phi = \delta d\phi = -\star d \star (\partial_m\phi dx^m) \\
&= -\star d \left(\frac{\sqrt{g}}{(D-1)!}\partial_m\phi g^{mn}\epsilon_{np_2\dots p_D} dx^{p_2} \wedge \dots \wedge dx^{p_D} \right) \\
&= -\star \frac{1}{(D-1)!}\partial_{p_1}[\sqrt{g}g^{nm}\partial_m\phi]\epsilon_{np_2\dots p_D} dx^{p_1} \wedge \dots \wedge dx^{p_D} \\
&= -\frac{1}{\sqrt{g}}\partial_n[\sqrt{g}g^{nm}\partial_m\phi] = -\Box\phi.
\end{aligned} \tag{4.32}$$

In a similar way we can find explicit expressions for all Hodge-de Rahm operators. The results being [4]

$$\begin{aligned}
\Delta_0\phi &= -\Box\phi, \\
\Delta_1\omega_m &= -\Box\omega_m + R_m{}^n\omega_n, \\
\Delta_2\omega_{mn} &= -\Box\omega_{mn} - 2R_{mnpq}\omega^{pq} - 2R^p{}_{[m}\omega_{n]p}, \\
\Delta_3\omega_{mnp} &= -\Box\omega_{mnp} + 6R^q{}_{[mn}{}^r\omega_{p]qr} + 3R_{[m}{}^q\omega_{np]q}.
\end{aligned} \tag{4.33}$$

Another operator that we will need is the linear operator $Q = \star d$. The eigenfunctions of Q can be shown to be in a one-to-one correspondence with the eigenfunctions of the Δ_3 operator, or more precisely one can show that $Q^2 \sim \Delta_3$ [4].

4.4.2 Dirac and Lichnerowicz

The Dirac operator acting on a regular spin-1/2 fermion is defined in the usual way

$$(\not{D}_{1/2}\psi)_\alpha \equiv i(\gamma^a)_\alpha{}^\beta\nabla_a\psi_\beta, \tag{4.34}$$

where ∇_a is a covariant derivative without an affine connection. When acting on the gravitino we need to include the affine connection acting on the vector index of the spin-3/2 particle. We write

$$\not{D}_{3/2}\psi_{n,\alpha} \equiv i(\gamma_m)_\alpha{}^\beta(D_m\psi)_{n,\beta}, \tag{4.35}$$

where D_m now has an affine connection term.

For spaces with $R_{mn} = 6m^2 g_{mn}$ one can prove that the eigenvalues, λ, of the Dirac operator are bounded by [4]

$$|\lambda| \geq \frac{7}{2}m. \tag{4.36}$$

This is saturated by the Killing spinor discussed in Chapter 2.

The final operator of interest to us is the so called Lichnerowicz operator. This is an operator that acts on transverse trace-free symmetric tensors, and it is defined by [4]

$$\Delta_L h_{mn} \equiv -\Box h_{mn} - 2R_{mpnq}h^{pq} + 2R_{(m}{}^p h_{n)p}. \tag{4.37}$$

This is not positive semi-definite in general, and in fact one needs some knowledge about the Riemann tensor in order to put any bounds on the eigenvalue spectrum [4].

5

The round seven-sphere

We will now study the spectrum of the round seven-sphere. This is done in two ways in preparation for the more complicated case of the squashed sphere. Both methods use our master equation of Chapter 4, Eq.(4.17). The first method uses the square of the master equation in order to be able to express the eigenvalues in terms of the quadratic Casimir operator. In the second method we expand the spherical tensors, or spinors, in modes and act on these with the master equation, for the different differential operators.

We will first need to discuss the geometry of the sphere, as well as the representations appearing in the spectrum. This is not done in much detail, since a more thorough presentation is available in [4]. Any uncertainties that may appear can certainly be resolved by studying that paper.

Note that in this chapter we denote the $SO(7)$ gamma matrices by γ_a and the $SO(8)$ matrices by Γ_a.

5.1 The sphere as a coset space

As previously explained the n-dimensional sphere S^n can be expressed as the coset space $SO(n+1)/SO(n)$, i.e., the isometry group divided by the isotropy group. Moreover, $SO(n+1)$ acts transitively on S^n so that we also have a homogeneous space, as a sphere should be. As discussed in Chapter 4, the seven-sphere can be described by several different coset spaces, but we will choose to only work with $SO(8)/SO(7)$. The tangent space group is of course also $SO(7)$, which simplifies things a bit.

We embed the sphere in $\mathbb{R}^8$ as the surface of radius $1/m$

$$\delta_{MN} dy^M dy^N = m^{-2}, \tag{5.1}$$

where y^M are Cartesian coordinates, $M = 1, \dots, 8$ and the metric δ_{MN} is the ordinary eight-dimensional Kronecker delta, invariant under $SO(8)$.

Since $S^7 \cong SO(8)/SO(7)$ the induced metric on the sphere will be [4]

$$ds^2 = \left(\delta_{mn} + \frac{y^m y^n}{m^{-2} - y^p y^p}\right) dy^m dy^n, \tag{5.2}$$

where $m, n, p = 1, 2, \dots, 7$.

This is consistent with the sphere having the Riemann tensor

$$R_{mnpq} = m^2(g_{mp}g_{nq} - g_{mq}g_{np}), \tag{5.3}$$

as expected from a maximally symmetric space [36]. This can also be found using Eq.(4.23) with the generators of $SO(8)/SO(7)$. The Ricci tensor is then given by $R_{mn} = 6m^2 g_{mn}$ and the Ricci scalar by $R = 42m^2$.

5.2 Representations

The massless spectrum of eleven-dimensional supergravity is given by the multiplet consisting of one graviton (transforming in $1 = (0,0,0,0)$ under $SO(8)$), eight gravitinos ($8_s = (0,0,0,1)$), 28 spin 1 fields ($28 = (0,1,0,0)$), 56 spin 1/2 fields ($56_s = (1,0,1,0)$), 35 scalars ($35_v = (2,0,0,0)$) and 35 pseudoscalars ($35_c = (0,0,2,0)$). It should then be possible to derive all massive multiplets from this massless one by multiplication with some massive representation of $SO(8)$ [37],

$$R \otimes \{1, 8_s, 28, 56_s, 35_v, 35_c\}\,. \tag{5.4}$$

Since it is known that the massive gravitons are in one-to-one correspondence with the spherical $SO(8)$ tensors sitting in the $(n,0,0,0)$ representation, we use this as the multiplying representation above. This multiplication is done in [37]. However we must also add lower helicity states to the spin-2, 3/2 and 1 fields to make them massive, in a form of Higgs mechanism [37]. The result is infinite towers of representations given by

$$\begin{aligned}
&\text{spin-2}: \ (n,0,0,0),\\
&\text{spin-3/2}: \ (n,0,0,1)\oplus(n-1,0,1,0),\\
&\text{spin-1}: \ (n-1,0,1,1)\oplus(n,1,0,0)\oplus(n-2,1,0,0),\\
&\text{spin-1/2}: \ (n+1,0,1,0)\oplus(n-1,1,1,0)\oplus(n-2,1,0,1)\oplus(n-2,0,0,1),\\
&\text{spin-0}: \ (n+2,0,0,0)\oplus(n-2,2,0,0)\oplus(n-2,0,0,0)\oplus(n,0,2,0)\oplus(n-2,0,0,2).
\end{aligned} \tag{5.5}$$

Since the Dynkin labels must be non-negative integers the representations above with negative labels, e.g. $(n-2,0,0,0)$ for $n=0,1$, does not exist for those n. For every n Eq.(5.5) gives an irreducible representation of $OSp(8,4)$ ($n=0$ is the massless multiplet) [4].

The massless $SO(8)$ multiplet decompose under $SO(7)$ as [38]

$$\begin{aligned}
&(0,0,0,0)\to(0,0,0),\\
&(0,0,0,1)\to(0,0,1),\\
&(0,1,0,0)\to(1,0,0)\oplus(0,1,0),\\
&(1,0,1,0)\to(1,0,1)\oplus(0,0,1),\\
&(2,0,0,0)\to(2,0,0)\oplus(1,0,0)\oplus(0,0,0),\\
&(0,0,2,0)\to(0,0,2).
\end{aligned} \tag{5.6}$$

5.3 Quadratic Casimir approach

We now turn to the calculation of the eigenvalues of all the differential operators found in Table 2.1 for the above representations.

The first method discussed is one where we use the results of Chapter 4 and express the differential operator as a generator of G/H. Since the round seven-sphere is a symmetric space the structure constants in Eq. (4.17) are zero and we have

$$\nabla_a\phi = cT_a\phi, \tag{5.7}$$

the constant c is added for dimensional reasons. Squaring this gives

$$\Box\phi = c^2T_aT_a\phi = -c^2(C_G - C_H)\phi, \tag{5.8}$$

where C_G and C_H are the quadratic Casimir operators belonging to the respective groups. In the last equality we simply rewrote $T_aT_a = T_MT_M - T_iT_i = -(C_G - C_H)$, and we have chosen the Killing form to be diagonal, as can always be done for compact semi-simple Lie algebras. The indices $i, j, \ldots$ are here taken to be in H.

The quadratic Casimirs for the vector representation ((0,1,0,0) and (1,0,0) respectively) of $SO(8)$ and $SO(7)$ can be calculated using the formulas of Appendix B, along with the normalisation $(\lambda_i, \lambda_j) = \delta_{ij}$. This gives us $C_G = 12$ and $C_H = 6$, so that $-\Box\phi_a = 6c^2\phi_a$.

To find the constant we use our knowledge of the Killing vectors. There are 28 Killing vectors on S^7 satisfying the Killing equations

$$\nabla_{(a}K_{b)} = 0. \tag{5.9}$$

Acting with another derivative on this, and using that the Killing vectors are transverse, one finds

$$\begin{aligned} 2\nabla^a\nabla_{(a}K_{b)} &= \Box K_b + \nabla^a\nabla_b K_a = \Box K_b + [\nabla^a, \nabla_b]K_a \\ &= \Box K_b + R^a{}_b K_a = \Box K_b + 6m^2 K_b = 0. \end{aligned} \tag{5.10}$$

Comparing this with the above expression for $-\Box\phi_a$ one sees that $c^2 = m^2$. Now we are ready to turn to the differential operators in Table 2.1.

5.3.1 The integer ranked forms

The Hodge-de Rahm operator acting on a scalar, Δ_0, is simply equal to $-\Box$, so that the spectrum of the scalar is given by

$$-\Box\phi = m^2(C_G - C_H)\phi. \tag{5.11}$$

For the $SO(8)$ representation $(p,0,0,0)$ we have $C_G = p(p+6)$ and for $(0,0,0)$, in $SO(7)$, $C_H = 0$. This implies that

$$\Delta_0\phi = m^2 p(p+6)\phi. \tag{5.12}$$

Hodge-de Rahm acting on a vector, or one-form, is equal to $\Delta_1\phi_m = -\Box\phi_m + R_m{}^n\phi_n$. For S^7 $R_m{}^n = 6m^2\delta^n_m$. The spectrum of the vector representation, $(p-1,1,0,0)$ in $SO(8)$, is thus

$$\Delta_1\phi_m = m^2[p(p+6) - 1]\phi_m + 6m^2\phi_m = m^2[p(p+6) + 5]\phi_m, \tag{5.13}$$

since, according to the formulas in Appendix B, $C_G = p(p+6)+5$ and for $(1,0,0)$ in $SO(7)$ we have, as mentioned above, $C_H = 6$.

For the two-form

$$\Delta_2\phi_{mn} = -\Box\phi_{mn} - 2R_{mpnq}\phi^{pq} - 2R^p{}_{[m}\phi_{n]p}. \tag{5.14}$$

In this case the curvature terms are

$$\begin{aligned} -2R_{mpnq}\phi^{pq} - 2R^p{}_{[m}\phi_{n]p} &= -2m^2(g_{mn}g_{pq} - g_{mq}g_{np})\phi^{pq} - 12m^2\delta^p_{[m}\phi_{n]p} \\ &= 2m^2 g_{mq}g_{np}\phi^{pq} + 12m^2\phi_{mn} = -2m^2\phi_{mn} + 12m^2\phi_{mn} \\ &= 10m^2\phi_{mn}. \end{aligned} \tag{5.15}$$

The representation $(p-1,0,1,1)$ has $C_G = p(p+6)+8$ and $(0,1,0)$ gives $C_H = 10$. This means that

$$\Delta_2\phi_{mn} = m^2[p(p+6) - 2]\phi_{mn} + 10m^2\phi_{mn} = m^2[p(p+6) + 8]\phi_{mn}. \tag{5.16}$$

For the three-forms we are actually interested in the linear operator $Q = *d$, but since Q^2 and Δ_3 have the same eigenvalues, as mentioned in Chapter 4, we will here study the Hodge-de Rahm operator on a three-form. When doing the mode expansion method below, the linear operator will be used instead. We have

$$\Delta_3\phi_{mnp} = -\Box\phi_{mnp} + 6R^q{}_{[mn}{}^r\phi_{p]qr} + 3R_{[m}{}^r\phi_{np]r}, \tag{5.17}$$

and

$$\begin{aligned} 6R^q{}_{[mn}{}^r\phi_{p]qr} + 3R_{[m}{}^r\phi_{np]r} &= 6m^2(\delta^q_{[n}\delta^r_m - g^{qr}g_{[mn})\phi_{p]qr} + 18m^2\delta^r_{[m}\phi_{np]r} \\ &= -6m^2\phi_{mnp} + 18m^2\phi_{mnp} = 12m^2\phi_{mnp}, \end{aligned} \tag{5.18}$$

so that for $(p-1,0,2,0) \to (0,0,2)$ or $(p-1,0,0,2) \to (0,0,2)$ we get

$$\Delta_3\phi_{mnp} = m^2[p(p+6)+9-12+12] = m^2(p+3)^2 \implies Q\phi_{mnp} = \pm m(p+3), \tag{5.19}$$

since $C_G = p(p+6)+9$ and $C_H = 12$ in both cases. Using this method it is not possible for us to decide which sign goes with which representation.

5.3.2 The Lichnerowicz operator

The Lichnerowicz operator, Δ_L, acting on a symmetric, traceless and transverse tensor h_{mn} is defined by

$$\Delta_L h_{mn} = -\Box h_{mn} - 2R_{mpnq}h^{pq} + 2R_{(m}{}^p h_{n)p}. \tag{5.20}$$

For the seven-sphere the curvature terms are

$$\begin{aligned} -2R_{mpnq}h^{pq} + 2R_{(m}{}^p h_{n)p} &= -2m^2(g_{mn}g_{pq} - g_{mq}g_{np})h^{pq} + 12m^2\delta^p_{(m}h_{n)p} \\ &= \underbrace{-2m^2 g_{mn}h^p{}_p}_{=0,\text{ traceless}} + 2m^2h_{mn} + 12m^2h_{mn} = 14m^2h_{mn}. \end{aligned} \tag{5.21}$$

The representation $(p-2,2,0,0)$ goes to $(2,0,0)$ which means that $C_G = p(p+6)+12$ and $C_H = 14$. The Lichnerowicz operator thus has the eigenvalues

$$\Delta_L h_{mn} = m^2[p(p+6)+12-14+14]h_{mn} = m^2[p(p+6)+12]h_{mn}. \tag{5.22}$$

5.3.3 The half-integer spins

To find the eigenvalues of the Dirac operator we use a similar approach. We start with the ordinary Dirac operator

$$\not{D}_{1/2}\psi = i\gamma^n\nabla_n\psi = \lambda\psi, \tag{5.23}$$

and act on this with another Dirac operator, i.e.,

$$-\gamma^m\nabla_m\gamma^n\nabla_n\psi = -\underbrace{\gamma^m\gamma^n}_{g^{mn}+\gamma^{mn}}\nabla_m\nabla_n\psi = -\Box\psi - \gamma^{mn}\nabla_{[m}\nabla_{n]}\psi = \lambda^2\psi. \tag{5.24}$$

We then use the identities

$$\nabla_{[m}\nabla_{n]}\psi_\alpha = \frac{1}{2}R_{mn\alpha}{}^\beta\psi_\beta = \frac{1}{8}R_{mn}{}^{pq}(\gamma_{pq})_\alpha{}^\beta\psi_\beta, \tag{5.25}$$

and

$$\gamma^{mn}\gamma_{pq} = \gamma^{mn}{}_{pq} - 4\delta^{[m}{}_{[p}\gamma^{n]}{}_{q]} - 2\delta^{mn}_{[pq]}. \tag{5.26}$$

Plugging this into Eq.(5.24) gives us

$$\begin{aligned}\lambda^2\psi &= -\Box\psi - \frac{m^2}{8}\left(\gamma^{mn}{}_{pq} - 4\delta^{[m}{}_{[p}\gamma^{n]}{}_{q]} - 2\delta^{mn}_{[pq]}\right)(g_{mp}g_{nq} - g_{mq}g_{np})\psi \\ &= -\Box\psi + \frac{m^2}{4}\delta^{mn}_{[pq]}(g_{mp}g_{nq} - g_{mq}g_{np})\psi = -\Box\psi + \frac{1}{4}R\psi = -\Box\psi + \frac{42m^2}{4}\psi.\end{aligned} \tag{5.27}$$

The $SO(8)$ representations for the spin-1/2 are $(p,0,1,0)$ and $(p,0,0,1)$. These gives us $C_G = p(p+7)+7$ and they both go to the $(0,0,1)$ representation of $SO(7)$, with $C_H = \frac{21}{4}$. This means that $-\Box = [p(p+7) + \frac{7}{4}]m^2$, and

$$\lambda^2\psi = \left[p(p+7) + \frac{7}{4} + \frac{42}{4}\right]m^2\psi = \left(p + \frac{7}{2}\right)^2 m^2\psi \implies \lambda = \pm\left(p + \frac{7}{2}\right)m. \tag{5.28}$$

However, we again see that this method is unable to tell us which sign goes with which representation. This problem will be resolved when we use the method of expanding in modes below.

The eigenvalues of the Dirac operator when acting on a spin-3/2 field are found in the same way, however the covariant derivative must now also include a connection for the vector index of the gravitino, we denote this derivative with D_n. The Dirac operator is then

$$\not{D}_{3/2}\psi_p = i\gamma^n(D_n\psi)_p = \lambda\psi_p, \tag{5.29}$$

and we act with one more Dirac operator to get

$$\begin{aligned}\lambda^2\psi^\alpha &= -\gamma^m\gamma^n D_m D_n\psi^\alpha_p = -\gamma^{mn}D_{[m}D_{n]}\psi^\alpha_p - \Box\psi^\alpha_p \\ &= -\gamma^{mn}\frac{1}{2}\Big(R_{mnp}{}^q\psi^\alpha_q + \underbrace{R_{mn}{}^\alpha{}_\beta}_{=\frac{1}{4}R_{mn}{}^{qr}(\gamma_{qr})^\alpha{}_\beta}\psi^\beta_p\Big) - \Box\psi^\alpha_p \\ &= \not{D}^2_{1/2}\psi_p - \frac{1}{2}\gamma^{mn}R_{mnp}{}^q\psi_q.\end{aligned} \tag{5.30}$$

We also have

$$\frac{1}{2}\gamma^{mn}(g_{mp}\delta^q_n - \delta^q_m g_{np}) = \frac{1}{2}\left(\gamma_p{}^q - \gamma^q{}_p\right) = \gamma_p{}^q = \gamma_p\gamma^q - \delta^q_p, \tag{5.31}$$

and because of the gauge choice we made earlier, $\gamma^m\psi_m = 0$ (Eq.(2.61)), we can now write

$$\left(\gamma_p\gamma^q - \delta^q_p\right)\psi_q = -\psi_p. \tag{5.32}$$

This finally gives us the eigenvalues of the spin-3/2 operator as

$$\lambda^2\psi_p = (\not{D}^2_{1/2} + 1)\psi_p = \left(-\Box + \frac{42}{4}m^2 + 1\right)\psi_p. \tag{5.33}$$

We have the representations $(p-1,1,1,0)$ and $(p-1,1,0,1)$ in $SO(8)$ and these gives us $C_G = p(p+7)+13$, while the $SO(7)$ representation is $(1,0,1)$, giving us $C_H = \frac{49}{4}$. This means that $-\Box\psi_m = m^2[p(p+7) + \frac{3}{4}]\psi_m$ and

$$\lambda^2 = \left[p(p+7) + \frac{49}{4}\right]m^2 = \left(p + \frac{7}{2}\right)^2 m^2 \implies \lambda = \pm\left(p + \frac{7}{2}\right)m. \tag{5.34}$$

We can not say anything about the sign this time either. We will now turn to the other method, where we expand the harmonics in modes.

5.4 Mode expansion method

As mentioned earlier there is another method one can use to find the eigenvalues of the different differential operators. This is inspired by an article from Gibbons & Perry [39]. Since the curvature contributions will be unchanged we will only concern ourselves with the derivatives. The spherical tensor is expressed as

$$T = T_{M_1 \ldots M_n} y^{M_1} \ldots y^{M_n}, \tag{5.35}$$

where $T_{M_1 \ldots M_n}$ has the symmetry properties of the irreducible representation we are in. For example, if we take the vector representation we write the tensor of rank $n+1$ as

$$T_{PQM_1 \ldots M_{n-1}}, \tag{5.36}$$

which is anti-symmetric in P and Q and symmetric in the M_i's. It is also traceless on any pair of indices. We will not write out the T in the following but simply impose its symmetries on the y's, so that the vector is given by

$$y^{[M} y^{M_1]} y^{M_2} \ldots y^{M_n}. \tag{5.37}$$

In particular, we will look at how these tensors split into the irreps of $SO(7)$, so that if we have an $SO(7)$ index a we write the vector as

$$\phi_a = y_a^{[M} y^{M_1]} y^{M_2} \ldots y^{M_n}. \tag{5.38}$$

It is also natural to here introduce a notation using the Young tableaux of Appendix B. The Young tableau of the vector, $(p-1,1,0,0)$, is for example $\begin{array}{l} \overbrace{\boxed{\bullet}\boxed{\bullet}\boxed{\bullet}\boxed{\bullet}\cdot\boxed{\bullet}}^{p} \\ \boxed{a} \end{array}$, and we thus write the vector as $\phi_a^{\yng(4,1)}$. We now see directly from the tableau how to expand the tensor in modes.

5.4.1 The scalar

Using this method we expand the scalar representation, $(p,0,0,0)$, in modes as

$$\phi^{\yng(5)} = y^{(M_1} \ldots y^{M_p)}. \tag{5.39}$$

The y's are $SO(8)$ vectors and the $SO(8)$ indices M_i split into $(a, 8 = \cdot)$, where a are seven dimensional $SO(7)$ indices (we will often omit the 8 index, since it is just a scalar index).

By acting with a derivative on y^M and using our master equation again, we find

$$\nabla_a y^M = -(T_a y^M)_8. \tag{5.40}$$

Since the generators of $SO(8)$ can be written as $(T_{MN})^{PQ} = 2\delta_{MN}^{PQ}$ we directly see that the index structure must be

$$\nabla_a y^M = (-T_{a8})^{8b} y_b^M = -y_a^M. \tag{5.41}$$

When taking the next derivative, in order to end up with the box operator, it can either hit the same y^M or a new one. So we need to know two things: First

$$\nabla_a y^{(M} \nabla_a y^{N)} = y_a^{(M} y_a^{N)}. \tag{5.42}$$

To evaluate this we need to use the full $SO(8)$ vectors. In $SO(8)$ the Kronecker delta is left invariant, i.e.,

$$\delta^{MN} = y_P^M y_Q^N \delta^{PQ} = y_a^M y_a^N + y_8^M y_8^N \implies y_a^M y_a^N = \delta^{MN} - y^M y^N. \tag{5.43}$$

But the mode expansion must be traceless. This means that the delta term will be zero, and we have

$$y_a^{(M} y_a^{N)} = -y^{(M} y^{N)}. \tag{5.44}$$

We also need to know what happens if the derivative acts two times on the same y. This can be checked in steps. One sees that

$$\nabla^b y_a^M = (T_a y^M)^b = (T_{a8})^{b8} y_8^M = \delta_a^b y^M, \tag{5.45}$$

which tells us that

$$\nabla^a \nabla_a y^M = -\nabla^a y_a^M = -7y^M. \tag{5.46}$$

Now, when the first derivative hits we get one term from every y, adding up to a total of p terms. We then get $p(p-1)$ terms when the next derivative act on another y and p terms, with an extra seven in them, from when the second derivative act on the same y. This means that we have

$$\Box\phi^{\square\square\square\square\square} = -[p(p-1) + 7p]\phi = -[p(p+6)]\phi, \tag{5.47}$$

which is the same result we found when using the Casimir method.

5.4.2 The one-form

As previously stated the vector is expanded as

$$\phi_a^{\boxplus\square\square\square} = y_a^{[M} y^{M_1]} y^{M_2} \dots y^{M_p}. \tag{5.48}$$

We are again interested in the box operator acting on this expansion. The only new thing compared to the scalar case will be the added anti-symmetry. From the first derivative we get

$$\begin{aligned}\nabla_b \phi_a^{\boxplus\square\square\square} &= (\nabla_b y_a^{[M}) y^{M_1]} y^{M_2} \dots y^{M_p} + y_a^{[M} (\nabla_b y^{M_1]}) y^{M_2} \dots y^{M_p} + \underbrace{y_a^{[M} y^{M_1]} \nabla_b y^{M_2} \dots y^{M_p}}_{p-1 \text{ such terms.}} \\ &= \delta_{ab} y^{[M} y^{M_1]} y^{M_2} \dots y^{M_p} - y_a^{[M} y_b^{M_1]} y^{M_2} \dots y^{M_p} - \underbrace{y_a^{[M} y^{M_1]} y_b^{M_2} \dots y^{M_p}}_{p-1 \text{ such terms.}}.\end{aligned} \tag{5.49}$$

Acting with the second derivative we get lots of terms, we can evaluate this in steps. When acting on the first term in the equation above we get

$$\begin{aligned}&\nabla^b \delta_{ab} y^{[M} y^{M_1]} y^{M_2} \dots + \delta_{ab} y^{[M} \nabla^b y^{M_1]} y^{M_2} \dots + \underbrace{y^{[M} y^{M_1]} \nabla^b y^{M_2}}_{p-1 \text{ terms}} \dots \\ &= -y_a^{[M} y^{M_1]} y^{M_2} \dots - y^{[M} y_a^{M_1]} y^{M_2} \dots ,\end{aligned} \tag{5.50}$$

where we used that $y^{[M} y^{M_1]} = 0$ because the scalars commute. But $y^{[M} y_a^{N]} = -y_a^{[M} y^{N]}$ so the two remaining terms cancel. Using the same arguments, and the results from the scalar case, when acting on the next two terms in Eq.(5.49) we end up with

$$\Box\phi_a^{\boxplus\square\square\square} = -[p(p+6) - 1]\phi_a^{\boxplus\square\square\square}, \tag{5.51}$$

precisely as in the case of the Casimir method. To get the eigenvalues of Δ_1 we simply add the contribution from the Ricci tensor.

5.4.3 The two-form

For the representation $(p-1,0,1,1)$ we could expand using two spinors, but since an even number of spinor indices gives a tensor we can also use Young tableaux to understand how to split the two-form in modes. The only way to construct a two-form Young tableau in this representation is by writing $\overbrace{\begin{ytableau} \bullet & \bullet & \bullet & \bullet & \cdot & \bullet \\ a \\ b \end{ytableau}}^{p}$ with $p-1$ symmetric and three anti-symmetric boxes. This gives us the mode expansion

$$\phi_{ab}^{\tiny\yng(5,1,1)} = y_a^{[M} y_b^N y^{M_1]} y^{M_2} \dots y^{M_p}. \tag{5.52}$$

Acting with ∇_c on this we get four different kinds of terms, depending on which y it acts on. Acting a second time we get four or five terms for every one of the first four. We skip writing them out here. After some cancellations and using the previous results one ends up with

$$\Box\phi_{ab}^{\tiny\yng(5,1,1)} = -[p(p+6)-2]\phi_{ab}^{\tiny\yng(5,1,1)}, \tag{5.53}$$

as before.

5.4.4 The Lichnerowicz operator

The Lichnerowicz operator act on h_{ab} in the representation $(p-2,2,0,0)$ which is the Young tableau $\begin{ytableau} \bullet & \bullet & \bullet & \bullet & \cdot & \bullet \\ a & b \end{ytableau}$. We express this as

$$h_{ab}^{\tiny\yng(5,2)} = y_a^{[M} y^{M_1]} y_b^{[N} y^{M_2]} y^{M_3} \dots y^{M_p}. \tag{5.54}$$

Acting with the first derivative gives us

$$\begin{aligned}\nabla_c h_{ab}^{\tiny\yng(5,2)} =& \delta_{ac} y^{[M} y^{M_1]} y_b^{[N} y^{M_2]} \dots - y_a^{[M} y_c^{M_1]} \dots + \delta_{bc} y_a^{[M} y^{M_1]} y^{[N} y^{M_2]} \dots \\ & - y_a^{[M} y^{M_1]} y_b^{[N} y_c^{M_2]} \dots - \underbrace{y_a^{[M} y^{M_1]} y_b^{[N} y^{M_2]} y_c^{M_3} \dots}_{p-1 \text{ such terms.}}\end{aligned} \tag{5.55}$$

Acting with the second derivative and rewriting some terms in the same way as before, using the symmetries of the modes, we again find the same results as above, namely

$$-\Box h_{ab}^{\tiny\yng(5,2)} = [p(p+6)-2] h_{ab}^{\tiny\yng(5,2)}. \tag{5.56}$$

5.4.5 The Q-operator

Q is the linear operator $\star d$ acting on a three-form, $\phi = \phi_{abc} dx^a dx^b dx^c$. We express the three-form as

$$\phi_{abc}^{\tiny\yng(6,1,1,1)} = y_a^{[M} y_b^N y_c^P y^{M_1]} y^{M_2} \dots y^{M_p}. \tag{5.57}$$

The derivative in Q gives us

$$\nabla_a \begin{ytableau} \bullet & \bullet & \bullet & \cdot & \bullet \\ a \\ b \\ c \end{ytableau} = -\begin{ytableau} d & \bullet & \bullet & \cdot & \bullet \\ a \\ b \\ c \end{ytableau} - (p-1)\begin{ytableau} \bullet & d & \bullet & \cdot & \bullet \\ a \\ b \\ c \end{ytableau}. \tag{5.58}$$

The second tableau can be turned into the form of the first by writing

$$\begin{aligned} 0 &= 5 y_a^{[M} y_b^N y_c^P y^{M_1} y_d^{M_2]} = 4 y_a^{[M} y_b^N y_c^P y^{M_1]} y_d^{M_2} - y_a^{[M} y_b^N y_c^P y_d^{M_2]} y^{M_1} \\ \implies & y_a^{[M} y_b^N y_c^P y^{M_1]} y_d^{M_2} = \frac{1}{4} y_a^{[M} y_b^N y_c^P y_d^{M_2]} y^{M_1}. \end{aligned} \tag{5.59}$$

We thus have

$$d\;\begin{array}{|c|c|c|c|c|}\hline \bullet & \bullet & \bullet & \cdot & \bullet \\ \hline a \\ \cline{1-1} b \\ \cline{1-1} c \\ \cline{1-1}\end{array} = -\frac{1}{4}(p+3)\begin{array}{|c|c|c|c|c|}\hline d & \bullet & \bullet & \cdot & \bullet \\ \hline a \\ \cline{1-1} b \\ \cline{1-1} c \\ \cline{1-1}\end{array}\,. \tag{5.60}$$

The Hodge-star operator will then give us

$$(\star d\phi)_{efg} = -\frac{p+3}{4}\frac{1}{3!}\epsilon_{efg}{}^{abcd}y_a^{[M}y_b^N y_c^P y_d^{M_1]}y^{M_2}\dots y^{M_p}. \tag{5.61}$$

Now, in eight dimensions we can form the projection operators

$$P_\pm \equiv \frac{1}{2}\left(\delta^{N_1\dots N_4}_{M_1\dots M_4} \pm \frac{1}{4!}\epsilon_{M_1\dots M_4}{}^{N_1\dots N_4}\right). \tag{5.62}$$

This means that we can write

$$\frac{1}{4!}\epsilon_{M_1\dots M_8}T^{M_5\dots M_8} = \pm T_{M_1\dots M_4}, \tag{5.63}$$

which, in our case, implies

$$\epsilon_{efg8}{}^{abcd}y_a y_b y_c y_d = \pm 4! y_e y_f y_g y, \tag{5.64}$$

and we get

$$(\star d\phi)_{abc} = \mp(p+3)y_a^{[M}y_b^N y_c^P y^{M_1]}y^{M_2}\dots y^{M_p} = \mp(p+3)\phi_{abc}. \tag{5.65}$$

5.4.6 Spin-1/2

For the spinor representation the generators are the gamma matrices. As mentioned earlier we will denote the eight-dimensional matrices with Γ^M and the seven-dimensional by γ^m. Using our master equation on a spinor Σ^A_α we have

$$\nabla_a\Sigma^A_\alpha = \frac{1}{2}(\Gamma_{a8})_\alpha{}^\beta\Sigma^A_\beta, \tag{5.66}$$

where $A = (\alpha, \dot\alpha)$ are the spinor indices of $SO(8)$ belonging to either $(0,0,1,0)$ or $(0,0,0,1)$.

We can construct the $SO(8)$ matrices explicitly as

$$\Gamma^M = \left\{\begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0\end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0\end{pmatrix}\right\}, \tag{5.67}$$

so that the chiral gamma matrix is

$$\Gamma^1\dots\Gamma^8 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \tag{5.68}$$

given that $\sigma^1\dots\sigma^7 = 1$.

The σ-matrices are anti-symmetric and real, and we want the $SO(7)$ matrices to be hermitian. This means that we can define them as $\gamma^a = \pm i\sigma^a$. We also want them to satisfy $\gamma^1\dots\gamma^7 = +i$, so we choose the minus sign, i.e., $\gamma^a = -i\sigma^a$. This gives us

$$\Gamma^M = \left\{\underbrace{\begin{pmatrix} 0 & i\gamma^a \\ -i\gamma^a & 0\end{pmatrix}}_{\Gamma^a}, \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0\end{pmatrix}}_{\Gamma^8}\right\}. \tag{5.69}$$

The $SO(8)$ matrices satisfy

$$y_P^M \Sigma_C^A \Sigma_D^B (\Gamma^P)^{CD} = (\Gamma^M)^{AB}, \tag{5.70}$$

which means that we can write

$$\begin{aligned} &y_a^M \Sigma_C^A \Sigma_D^B (\Gamma^a)^{CD} + y^M \Sigma_C^A \Sigma_D^B (\Gamma^8)^{CD} = (\Gamma^M)^{AB} \\ \implies &y_a^M \Sigma_D^B (\Gamma^a)^{AD} + y^M \Sigma_D^B (\Gamma^8)^{AD} = (\Gamma^M)^{CB} \Sigma_C^A. \end{aligned} \tag{5.71}$$

If we now choose $A = \alpha$ we get

$$y_a^M \Sigma_D^B (\Gamma^a)^{\alpha D} = -y^M \Sigma_D^B (\Gamma^8)^{\alpha D} + \underbrace{(\Gamma^M)^{CB} \Sigma_C^\alpha}_{=0} \implies i y_a^M (\gamma^a)_\alpha{}^\beta \Sigma_\beta^A = y^M \Sigma_\alpha^A, \tag{5.72}$$

if we instead had chosen $A = \dot{\alpha}$ we would have found $i y_a^M (\gamma^a)_\alpha{}^\beta \Sigma_\beta^A = -y^M \Sigma_\alpha^A$. Note that we do not make any difference between α and $\dot{\alpha}$ in $SO(7)$, since there is only one spinor representation, $(0,0,1)$.

We can now find the eigenvalues of the Dirac operator. The representation $(p,0,0,1)$ corresponds to α and $(p,0,1,0)$ to $\dot{\alpha}$. The spinor is expanded as

$$\psi_\alpha = y^{(M_1} \dots y^{M_p)} \Sigma_\alpha^A. \tag{5.73}$$

By first acting with the covariant derivative we get

$$\nabla_a \psi_\alpha = -p y_a^{(M_1} y^{M_2} \dots y^{M_p)} + \frac{1}{2} y^{(M_1} \dots y^{M_p)} (\Gamma_{a8})_\alpha{}^\beta \Sigma_\beta^A. \tag{5.74}$$

One easily finds that

$$\Gamma_{a8} = \begin{pmatrix} -i\gamma^a & 0 \\ 0 & i\gamma^a \end{pmatrix}, \tag{5.75}$$

giving us $(\Gamma_{a8})_\alpha{}^\beta = -i\gamma^a$ and $(\Gamma_{a8})_{\dot{\alpha}}{}^{\dot{\beta}} = i\gamma^a$, so that

$$\nabla_a \psi_\alpha = -p y_a^{(M_1} y^{M_2} \dots y^{M_p)} - i\frac{1}{2} y^{(M_1} \dots y^{M_p)} (\gamma_a)_\alpha{}^\beta \Sigma_\beta^A. \tag{5.76}$$

Acting with $i\gamma^a$ on this we get

$$i(\gamma^a)_\alpha{}^\beta \nabla_a \psi_\beta = -ip(\gamma^a)_\alpha{}^\beta y_a^{(M_1} \dots y^{M_p)} \Sigma_\beta^A + \frac{1}{2} y^{(M_1} \dots y^{M_p)} (\gamma^a)_\alpha{}^\beta (\gamma_a)_\beta{}^\gamma \Sigma_\gamma^A. \tag{5.77}$$

Using the results of Eq. (5.72) this becomes

$$-p y^{(M_1} \dots y^{M_p)} \Sigma_\alpha^A - \frac{7}{2} y^{(M_1} \dots y^{M_p)} \Sigma_\alpha^A = -(p + \frac{7}{2}) y^{(M_1} \dots y^{M_p)} \Sigma_\alpha^A \tag{5.78}$$

Exactly as when using the quadratic Casimir approach, but we now see that we get the minus sign for the $(p,0,0,1)$ representation. Had we chosen the $\dot{\alpha}$ instead we would have found the plus sign belonging to the $(p,0,1,0)$ representation.

5.4.7 Spin-$3/2$

The last operator we will deal with is the spin 3/2 operator, $\not{D}_{3/2}$. This is done in the same way as the spin 1/2 operator, the only difference is that we start in the $SO(8)$ representation $(p-1,1,0,1)$. We expand as

$$\psi_{a,\alpha} = y_a^{[M} y^{M_1]} y^{M_2} \dots y^{M_p} \Sigma_\alpha^A. \tag{5.79}$$

We act with the covariant derivative first, and get

$$D_b \psi_{a,\alpha} = \delta_{ab} y^{[M} y^{M_1]} \dots - y_a^{[M} y_b^{M_1]} \dots - (p-1) y_a^{[M} y^{M_1]} y_b^{M_2} \dots + \frac{i}{2} y_a^{[M} y^{M_1]} \dots y^{M_p} (\Gamma_{b8})_\alpha{}^\beta \Sigma_\beta^A. \tag{5.80}$$

We now act with the $SO(7)$ gamma matrices, and multiply with i, on each block of this (either on the α, β part or the $\dot{\alpha}, \dot{\beta}$ part of the gamma matrices)

$$\begin{aligned} i(\gamma^b)_\alpha{}^\beta D_b \psi_{a,\beta} =& \underbrace{i\gamma_a y^{[M} y^{M_1]} \dots}_{=0} - i y_a^{[M} \gamma^b y_b^{M_1]} \dots - i(p-1) y_a^{[M} y^{M_1]} \gamma^b y_b^{M_2} \dots \\ &+ \frac{1}{2} y_a^{[M} y^{M_1]} y^{M_2} \dots y^{M_p} \gamma^b \gamma_b \Sigma_\alpha^A \\ =& -\psi_{a,\alpha} - (p-1)\psi_{a,\alpha} - \frac{7}{2}\psi_{a,\alpha} = -(p+\frac{7}{2})\psi_{a,\alpha}, \end{aligned} \tag{5.81}$$

where we once again used the results of Eq.(5.72).

Now that we have shown several examples of how our methods work we will move on to the squashed sphere. This is not a symmetric space so the structure constants of Eq.(4.17) will not vanish automatically. This will lead to more complex situations. First we need to discuss the geometry of the squashed sphere in order to find the structure constants and the Riemann tensor, which we will need in order to derive the eigenvalues.

6

Geometry of the squashed sphere

In this chapter we study the geometry of the squashed seven-sphere. This is mainly done by regarding the sphere as the coset space $\frac{Sp(2)\times Sp(1)^C}{Sp(1)^A\times Sp(1)^{B+C}}$ and using the results of Chapter 4 to derive the curvature and Killing metric. Here $Sp(1)^A$ and $Sp(1)^B$ make up the $SO(4) \cong Sp(1) \times Sp(1)$ subgroup of $Sp(2)$, and $Sp(1)^{B+C}$ denotes the diagonal $Sp(1)$ subgroup of $Sp(1)^B \times Sp(1)^C$. We will also briefly discuss other ways of looking at the geometry. But first, a few words on the meaning of squashing.

6.1 How to squash a sphere

There are several, equivalent, ways of describing the squashed seven-sphere. As mentioned above we will mostly consider it as a coset space and work from a group theoretical perspective. But to see the meaning of squashing it is easier if we use another approach.

The squashed seven-sphere corresponds to a spontaneous symmetry breaking of the round sphere. The seven-sphere can be recognised as an S^3 bundle over S^4 [4]. This means that we can express the metric as a combination of the S^4 metric and a gauge part coming from the $S^3 \sim SU(2)$, in a sort of inverse Kaluza-Klein mechanism [4]. This metric can be written as

$$ds^2 = d\mu^2 + \frac{1}{4}\sin^2\mu\Sigma_i^2 + \lambda^2\left(\sigma_i - \cos^2\frac{\mu}{2}\Sigma_i\right), \tag{6.1}$$

where $d\mu^2 + \frac{1}{4}\sin^2\mu\Sigma_i^2$ is the S^4 metric, $0 \leq \mu < \pi$, and both Σ_i and σ_i are one-forms satisfying the $SU(2)$ algebra [4]

$$d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k, \qquad i, j, k = 1, 2, 3, \tag{6.2}$$

and correspondingly for Σ_i. The $\cos^2\frac{\mu}{2}\Sigma_i$ term is the gauge potential of an $SU(2)$ instanton. Note that if this was zero we would simply have $S^4 \times S^3$ [4]. The parameter λ is a constant that deforms the $SU(2)$ fibres relative the S^4 space. There are two values of λ that gives us an Einstein metric, these are $\lambda^2 = 1$ and $\lambda^2 = \frac{1}{5}$, where $\lambda^2 = 1$ corresponds to the round sphere and $\lambda^2 = \frac{1}{5}$ to the squashed sphere [4]. Note that all values of λ could be considered as some squashing of the sphere but only these two gives us an Einstein metric, therefore when we talk about the squashed sphere we always refer to the $\lambda^2 = \frac{1}{5}$ case.

6.2 The algebra of the coset space

We now turn to considering the squashed sphere as the coset space $\frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$, beginning by deriving the algebra of this space.

Since $Sp(2)$ is isomorphic to $SO(5)$, which has $SO(4)$ as a subgroup we will start there. We can realise $SO(4)$ with the help of the gamma matrices along with the conventions

$$\gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ \bar{\sigma}^\alpha & 0 \end{pmatrix}, \qquad \{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}, \tag{6.3}$$

where

$$\begin{aligned} \sigma^\alpha &= (-i\mathbb{1}, \sigma^i), \\ \bar{\sigma}^\alpha &= (+i\mathbb{1}, \sigma^i), \end{aligned} \tag{6.4}$$

and σ_i are the ordinary Pauli matrices, satisfying $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$.

The generators of $SO(4)$ are the usual anti-symmetric combinations of two matrices, $\gamma^{\alpha\beta}$. Using the conventions above these are

$$\gamma^{\alpha\beta} = \begin{pmatrix} \sigma^{\alpha\beta} & 0 \\ 0 & \bar{\sigma}^{\alpha\beta} \end{pmatrix} \equiv \begin{pmatrix} \sigma^{[\alpha}\bar{\sigma}^{\beta]} & 0 \\ 0 & \bar{\sigma}^{[\alpha}\sigma^{\beta]} \end{pmatrix}. \tag{6.5}$$

To go from $SO(4)$ to $SO(5)$ we use the chiral gamma matrix, γ^5, and add the generators $\gamma^{\alpha 5} = \gamma^{[\alpha}\gamma^{5]}$ to the ones given above. With our conventions these have the form

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Longrightarrow \gamma^{\alpha 5} = \begin{pmatrix} 0 & -\sigma^\alpha \\ \bar{\sigma}^\alpha & 0 \end{pmatrix}. \tag{6.6}$$

It is easy to check that the set $\{\gamma^{\alpha\beta}, \gamma^{\alpha 5}\}$ satisfies the $SO(5)$ algebra.

To keep track of the different $Sp(1)$ algebras appearing in the coset space we denote them with different superscripts A, B or C. As mentioned in the beginning of the chapter $Sp(1)^A \times Sp(1)^B$ is the $SO(4)$ subgroup of $Sp(2)$, and $Sp(1)^C$ is the algebra appearing in G together with $Sp(2)$. We then embed $Sp(1)^A \times Sp(1)^{B+C}$ in $Sp(2)$ by introducing the generators [40]

$$\begin{aligned} Sp_1^A :\ & i_i \equiv -\frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \\ Sp_1^B :\ & j_i \equiv -\frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_i \end{pmatrix}, \\ Sp_1^C :\ & l_i \equiv -\frac{i}{2}\sigma_i. \end{aligned} \tag{6.7}$$

Next, we define

$$\begin{aligned} H_i &\equiv I_i \equiv -\frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}, \\ H_{\hat{i}} &\equiv J_i + L_i, \end{aligned} \tag{6.8}$$

where $J_i \equiv -\frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_i \end{pmatrix} \otimes \mathbb{1}_{2\times 2}$ and $L_i \equiv -\frac{i}{2}\mathbb{1}_{4\times 4} \otimes \sigma_i$. These six generators, H_i and $H_{\hat{i}}$, comprise the H in the coset G/H [40].

For the complement, in the decomposition $G = H \times K$, we define the generators [40]

$$\begin{aligned} Q_\alpha &\equiv \frac{1}{2}\begin{pmatrix} 0 & -\sigma^\alpha \\ \bar{\sigma}^\alpha & 0 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}, \\ T_{\hat{a}} &\equiv \frac{2}{\sqrt{5}}(J_i - \frac{3}{2}L_i). \end{aligned} \tag{6.9}$$

Note that Q_α are just the $\gamma^{\alpha 5}$ generators mentioned earlier. The normalisation has here been chosen for later convenience, and the relative factor between J_i and L_i in $T_{\hat{a}}$ comes from demanding the Killing form to be diagonal.

We will also use the 't Hooft symbol defined by

$$\begin{aligned}
\sigma^{\alpha\beta} &\equiv i\eta_i^{\alpha\beta}\sigma^i \implies \begin{cases} \eta_i^{oj} = -\delta_i^j, \\ \eta_i^{jk} = \epsilon_{ijk}, \end{cases} \\
\bar{\sigma}^{\alpha\beta} &\equiv i\bar{\eta}_i^{\alpha\beta}\sigma^i \implies \begin{cases} \bar{\eta}_i^{oj} = +\delta_i^j, \\ \bar{\eta}_i^{jk} = \epsilon_{ijk}. \end{cases}
\end{aligned} \tag{6.10}$$

By using the above generators we can now find all the structure constants of the coset. This can then be used to find the Riemann tensor, as in Chapter 4. Since we will not always know where the calculations are going to end up, we use the indices i, j, k for the intermediate steps and then place the correct indices where they should be in the end.

The easiest structure constants to find are the ones from $[H, H]$. We can directly see that, since $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}$,

$$[H_i, H_j] = \epsilon_{ijk}H_k. \tag{6.11}$$

The different $Sp(1)$ algebras all commute with each other, which means that

$$[H_{\hat{i}}, H_{\hat{j}}] = [J_{\hat{i}} + L_{\hat{i}}, J_{\hat{j}} + L_{\hat{j}}] = [J_{\hat{i}}, J_{\hat{j}}] + [L_{\hat{i}}, L_{\hat{j}}] = \epsilon_{\hat{i}\hat{j}\hat{k}}(J_{\hat{k}} + L_{\hat{k}}) = \epsilon_{\hat{i}\hat{j}\hat{k}}H_{\hat{k}}. \tag{6.12}$$

The structure constants involving the $T_{\hat{a}}$ generators gives us

$$\begin{aligned}
\left[H_{\hat{i}}, T_{\hat{a}}\right] &= \left[J_{\hat{i}} + L_{\hat{i}}, \frac{2}{\sqrt{5}}(J_{\hat{a}} - \frac{3}{2}L_{\hat{a}})\right] = \frac{2}{\sqrt{5}}\underbrace{[J_{\hat{i}}, J_{\hat{a}}]}_{=\epsilon_{ijk}J_k} - \frac{3}{\sqrt{5}}\underbrace{[L_{\hat{i}}, L_{\hat{a}}]}_{=\epsilon_{ijk}L_k} \\
&= \frac{2}{\sqrt{5}}\epsilon_{ijk}\Big(J_k - \frac{3}{2}L_k\Big) = \epsilon_{\hat{i}\hat{a}\hat{b}}T_{\hat{b}},
\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}
\left[T_{\hat{a}}, T_{\hat{b}}\right] &= \frac{4}{5}\left[(J_i - \frac{3}{2}L_i), (J_j - \frac{3}{2}L_j)\right] = \frac{4}{5}\left([J_i, J_j] + \frac{9}{4}[L_i, L_j]\right) \\
&= \frac{4}{5}\epsilon_{ijk}\left(J_k + \frac{9}{4}L_k\right).
\end{aligned} \tag{6.14}$$

The squashed sphere is not a symmetric space, and therefore has $[K, K] \subset K + H$. This means that the above expression should be able to give us both something in $H_{\hat{i}}$ and in $T_{\hat{a}}$. With this in mind we write

$$\begin{aligned}
\left[T_{\hat{a}}, T_{\hat{b}}\right] &= \epsilon_{ijk}\left[c_1(J_k + L_k) + c_2\frac{2}{\sqrt{5}}(J_k - \frac{3}{2}L_k)\right] \\
&= \epsilon_{ijk}\left[J_k\left(c_1 + c_2\frac{2}{\sqrt{5}}\right) + L_k\left(c_1 - c_2\frac{3}{\sqrt{5}}\right)\right],
\end{aligned} \tag{6.15}$$

for some constants c_1 and c_2. Comparing this with the above result we get a system of equations for c_1 and c_2 which is solved by $c_1 = \frac{6}{5}$ and $c_2 = -\frac{1}{\sqrt{5}}$, which means that

$$\left[T_{\hat{a}}, T_{\hat{b}}\right] = \frac{6}{5}\epsilon_{\hat{a}\hat{b}\hat{k}}H_{\hat{k}} - \frac{1}{\sqrt{5}}\epsilon_{\hat{a}\hat{b}\hat{c}}T_{\hat{c}}. \tag{6.16}$$

Turning to the structure constants involving Q_α

$$\begin{aligned}
[H_i, Q_\alpha] &= \left[-\frac{i}{2}\begin{pmatrix}\sigma_i & 0 \\ 0 & 0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0 & -\sigma^\alpha \\ \bar{\sigma}^\alpha & 0\end{pmatrix}\right] = \frac{i}{4}\begin{pmatrix}0 & \sigma^i\sigma^\alpha \\ \bar{\sigma}^\alpha\sigma^i & 0\end{pmatrix} \\
&= \frac{i}{4}\left[\begin{pmatrix}0 & i\eta_k^{i\alpha}\sigma^k \\ i\bar{\eta}_k^{\alpha i}\sigma^k & 0\end{pmatrix} + \begin{pmatrix}0 & \delta^{i\alpha} \\ \delta^{i\alpha} & 0\end{pmatrix}\right].
\end{aligned} \tag{6.17}$$

In particular, if we set $\alpha = a = 1,2,3$, and write $\delta^{ia}\mathbb{1} = i\sigma^0\delta^{ia}$ or $-i\bar{\sigma}^0\delta^{ia}$, we find

$$[H_i, Q_a] = \frac{1}{2}\epsilon_{iab}Q^b + \frac{1}{2}\delta^{ia}Q_0. \tag{6.18}$$

Setting $\alpha = 0$ instead gives us

$$[H_i, Q_0] = -\frac{1}{2}\delta^i_b Q^b. \tag{6.19}$$

The bracket $[Q_\alpha, Q_\beta]$ is evaluated as

$$\begin{aligned}[Q_\alpha, Q_\beta] &= \frac{1}{4}\left[\begin{pmatrix}0 & -\sigma^\alpha\\ \bar{\sigma}^\alpha & 0\end{pmatrix}, \begin{pmatrix}0 & -\sigma^\beta\\ \bar{\sigma}^\beta & 0\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}\sigma^\beta\bar{\sigma}^\alpha - \sigma^\alpha\bar{\sigma}^\beta & 0\\ 0 & \bar{\sigma}^\beta\sigma^\alpha - \bar{\sigma}^\alpha\sigma^\beta\end{pmatrix}\\ &= \frac{i}{2}\begin{pmatrix}\eta_k^{\beta\alpha}\sigma^k & 0\\ 0 & \bar{\eta}_k^{\beta\alpha}\sigma^k\end{pmatrix}.\end{aligned} \tag{6.20}$$

If $\alpha = a, \beta = 0$ we get

$$[Q_a, Q_0] = \frac{i}{2}\delta^a_k\begin{pmatrix}-\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} = \delta^a_k\Bigg[\underbrace{-\frac{i}{2}\begin{pmatrix}\sigma^k & 0\\ 0 & 0\end{pmatrix}}_{=H_k} + \frac{i}{2}\begin{pmatrix}0 & 0\\ 0 & \sigma^k\end{pmatrix}\Bigg], \tag{6.21}$$

and one can write

$$\frac{i}{2}\begin{pmatrix}0 & 0\\ 0 & \sigma^k\end{pmatrix} = -J_i = c_1(J_i + L_i) + c_2\frac{2}{\sqrt{5}}(J_i - \frac{3}{2}L_i), \tag{6.22}$$

which is solved by $c_1 = -\frac{3}{5}$ and $c_2 = -\frac{1}{\sqrt{5}}$ implying that

$$[Q_a, Q_0] = \delta^k_a H_k - \frac{3}{5}\delta^{\hat{k}}_a H_{\hat{k}} - \frac{1}{\sqrt{5}}\delta^{\hat{c}}_a T_{\hat{c}}. \tag{6.23}$$

Setting $\alpha = a, \beta = b$ instead leads to

$$[Q_a, Q_b] = \frac{i}{2}\epsilon_{bak}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} = \epsilon_{abk}H_k + \frac{3}{5}\epsilon_{ab\hat{k}}H_{\hat{k}} + \frac{1}{\sqrt{5}}\epsilon_{ab\hat{c}}T_{\hat{c}}. \tag{6.24}$$

From a quick calculation one can see that L_i commutes with Q_α. This helps us a great deal in evaluating the last two commutators. Here we have

$$\begin{aligned}\left[H_{\hat{i}}, Q_\alpha\right] &= [J_i + L_i, Q_\alpha] = [J_i, Q_\alpha] = -\frac{i}{4}\left[\begin{pmatrix}0 & 0\\ 0 & \sigma_i\end{pmatrix}, \begin{pmatrix}0 & -\sigma_\alpha\\ \bar{\sigma}_\alpha & 0\end{pmatrix}\right]\\ &= -\frac{i}{4}\begin{pmatrix}0 & \sigma_\alpha\sigma_i\\ \sigma_i\bar{\sigma}_\alpha & 0\end{pmatrix} = \frac{1}{4}\begin{pmatrix}0 & \eta_k^{\alpha i}\sigma^k\\ \bar{\eta}_k^{i\alpha}\sigma^k & 0\end{pmatrix} - \frac{i}{4}\delta^{\alpha i}\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}.\end{aligned} \tag{6.25}$$

Now, taking $\alpha = a$ gives us

$$[H_{\hat{i}}, Q_a] = \frac{1}{2}\epsilon_{\hat{i}ab}Q_b - \frac{1}{2}\delta_{\hat{i}a}Q_0, \tag{6.26}$$

and $\alpha = 0$ has

$$[H_{\hat{i}}, Q_0] = \frac{1}{2}\delta_{\hat{i}b}Q_b, \tag{6.27}$$

since $\delta_i^{\alpha=0} = 0$.

Finally, the last commutator to check is

$$[T_{\hat{a}}, Q_\beta] = \frac{2}{\sqrt{5}}[J_{\hat{a}} - \frac{3}{2}L_{\hat{a}}, Q_\beta] = \frac{2}{\sqrt{5}}[J_{\hat{a}}, Q_\beta], \tag{6.28}$$

and we just evaluated this commutator, so multiplying the above results with the new prefactor we get, for $\beta = b$

$$[T_{\hat{a}}, Q_b] = \frac{1}{\sqrt{5}}\epsilon_{\hat{a}bc}Q_c - \frac{1}{\sqrt{5}}\delta_{\hat{a}b}Q_0, \tag{6.29}$$

and for $\beta = 0$

$$[T_{\hat{a}}, Q_0] = \frac{1}{\sqrt{5}}\delta^c_{\hat{a}}Q_c. \tag{6.30}$$

Collecting all these results the non-zero structure constants are

$$\begin{aligned}
&{f_{ij}}^k = \epsilon_{ijk}, && {f_{\hat{i}\hat{j}}}^{\hat{k}} = \epsilon_{\hat{i}\hat{j}\hat{k}}, \\
&{f_{ia}}^0 = {f_{0i}}^a = \frac{1}{2}\delta_{ai}, && {f_{a0}}^i = \delta^i_a, \\
&{f_{ia}}^b = \frac{1}{2}\epsilon_{iab}, && {f_{ab}}^i = \epsilon_{abi}, \\
&{f_{\hat{i}a}}^0 = {f_{0\hat{i}}}^a = -\frac{1}{2}\delta_{a\hat{i}}, && {f_{0a}}^{\hat{i}} = \frac{3}{5}\delta^{\hat{i}}_a, \\
&{f_{\hat{i}a}}^b = \frac{1}{2}\epsilon_{iab}, && {f_{ab}}^{\hat{i}} = \frac{3}{5}\epsilon_{ab\hat{i}}, \\
&{f_{\hat{i}\hat{a}}}^{\hat{b}} = \epsilon_{\hat{i}\hat{a}\hat{b}}, && {f_{\hat{a}\hat{b}}}^{\hat{i}} = \frac{6}{5}\epsilon_{\hat{a}\hat{b}\hat{i}}, \\
&{f_{\hat{a}\hat{b}}}^{\hat{c}} = -\frac{1}{\sqrt{5}}\epsilon_{\hat{a}\hat{b}\hat{c}}, && {f_{\hat{a}b}}^c = {f_{bc}}^{\hat{a}} = \frac{1}{\sqrt{5}}\epsilon_{\hat{a}bc}, \\
&{f_{\hat{a}b}}^0 = {f_{0\hat{a}}}^b = {f_{b0}}^{\hat{a}} = -\frac{1}{\sqrt{5}}\delta_{\hat{a}b}.
\end{aligned} \tag{6.31}$$

Note that we get exactly the results of Bais et al., [35], if we instead take $Q_\alpha = \frac{\sqrt{5}}{9\sqrt{2}}\begin{pmatrix} 0 & -\sigma_\alpha \\ \bar{\sigma}_\alpha & 0 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}$ and $T_{\hat{a}} = \frac{2\sqrt{2}}{9}(J_i - \frac{3}{2}L_i)$. However, as we will see, our normalisations better aligns with Duff et al., [4].

By using the expressions for the octonionic structure constants found in Appendix D

$$a_{0i\hat{j}} = -\delta_{ij}, \qquad a_{ij\hat{k}} = -\epsilon_{ijk} \quad \text{and} \quad a_{\hat{i}\hat{j}\hat{k}}, \tag{6.32}$$

we can see that the structure constants of the complement, i.e, ${f_{\hat{a}\hat{b}}}^{\hat{c}}$, ${f_{\hat{a}b}}^c$ and ${f_{\hat{a}b}}^0$, can be combined into one expression, ${f_{\bar{a}\bar{b}}}^{\bar{c}} = -\frac{1}{\sqrt{5}}a_{\bar{a}\bar{b}\bar{c}}$, where $\bar{a} = (0, \hat{i}, i)$. This is very useful since these are exactly the structure constants appearing in our master equation, Eq.(4.17), and we have derived several identities for these structure constants in Appendix D.

6.3 Killing metric and curvature

With the structure constants given by Eq.(6.31) we can now see that the Killing metric is diagonal. Since

$$\kappa_{MN} \equiv {f_{MR}}^S {f_{NS}}^R \tag{6.33}$$

gives us

$$\kappa_{ij} = -2\left(\epsilon_{ikl}\epsilon_{jkl} + \frac{1}{4}\delta_{ai}\delta_{ja} + \epsilon_{iab}\epsilon_{jab}\right) = -2\left(\delta_{ij} + \frac{1}{4}\delta_{ij} + \frac{1}{4}\delta_{ij}\right) = -3\delta_{ij}, \tag{6.34}$$

$$\kappa_{\hat{i}\hat{j}} = -2\left(\epsilon_{\hat{i}\hat{k}\hat{l}}\epsilon_{\hat{j}\hat{k}\hat{l}} + \frac{1}{4}\delta_{\hat{i}\hat{j}} + \frac{1}{4}\epsilon_{\hat{i}ab}\epsilon_{\hat{j}ab} + \epsilon_{\hat{i}\hat{a}\hat{b}}\epsilon_{\hat{j}\hat{a}\hat{b}}\right) = -2\delta_{\hat{i}\hat{j}}\left(1 - \frac{1}{4} - \frac{1}{4} + 1\right) = -5\delta_{\hat{i}\hat{j}}, \tag{6.35}$$

$$\kappa_{00} = -2\left(\frac{1}{4}\delta_{ai}\delta^{ai} + \frac{3}{10}\delta_{a\hat{i}}\delta^{\hat{i}a} + \frac{1}{5}\delta_{\hat{a}b}\delta_{\hat{a}b}\right) = -\frac{9}{2}, \tag{6.36}$$

$$\begin{aligned}\kappa_{ab} &= -2\left(\frac{1}{2}\delta_{ai}\delta^{i}_{b} + \frac{1}{2}\epsilon_{aci}\epsilon_{bci} + \frac{3}{10}\delta_{a\hat{i}}\delta^{\hat{i}}_{b} + \frac{3}{10}\epsilon_{ac\hat{i}}\epsilon_{bc\hat{i}} + \frac{1}{5}\epsilon_{a\hat{c}d}\epsilon_{b\hat{c}d} + \frac{1}{5}\delta_{a\hat{c}}\delta_{b\hat{c}}\right) \\ &= -2\delta_{ab}\left(\frac{1}{2} + 1 + \frac{3}{10} + \frac{3}{5} + \frac{2}{5} + \frac{1}{5}\right) = -6\delta_{ab},\end{aligned} \tag{6.37}$$

and we get the same result for $\kappa_{\hat{a}\hat{b}}$.

The curvature two-form can be evaluated using Eq.(4.23). For example, we have

$$R^{0}{}_{a} = \left(\frac{1}{4}f_{a\hat{c}}{}^{0}f_{DE}{}^{\hat{c}} + \frac{1}{2}f_{a\hat{i}}{}^{0}f_{DE}{}^{\hat{i}} + \frac{1}{2}f_{ai}{}^{0}f_{DE}{}^{i} + \frac{1}{4}f_{Dc}{}^{0}f_{Ea}{}^{c} + \frac{1}{4}f_{D\hat{c}}{}^{0}f_{Ea}{}^{\hat{c}}\right)e^{D}e^{E}, \tag{6.38}$$

where (DE) can be either $(de), (\hat{d}\hat{e}), (d0)$ or $(0e)$ according to Eq.(6.31). Evaluating this further we find

$$\begin{aligned}R^{0}{}_{a} =& \left(-\frac{1}{20}\epsilon_{\hat{d}\hat{e}a} + \frac{6}{20}\epsilon_{\hat{d}\hat{e}a} - \frac{1}{20}\epsilon_{\hat{e}a\hat{d}}\right)e^{\hat{d}}e^{\hat{e}} + \Bigg(\underbrace{\frac{1}{20}\epsilon_{dea} + \frac{3}{20}\epsilon_{dea} - \frac{1}{4}\epsilon_{dea} + \frac{1}{20}\epsilon_{dea}}_{=0}\Bigg)e^{d}e^{e} \\ &+ \left(-\frac{1}{20}\delta_{ad} - \frac{3}{20}\delta_{ad} - \frac{1}{4}\delta_{ad} + \frac{1}{20}\delta_{ad}\right)e^{d}e^{0} + \left(\frac{1}{20}\delta_{ae} + \frac{3}{20}\delta_{ae} + \frac{1}{4}\delta_{ae}\right)e^{0}e^{e} \\ =& \frac{1}{5}\epsilon_{\hat{d}\hat{e}a}e^{\hat{d}}e^{\hat{e}} - \frac{8}{20}\delta_{ad}e^{d}e^{0} + \frac{9}{20}\delta_{ae}e^{0}e^{e} = \frac{17}{20}e^{0}e^{a} + \frac{1}{5}\epsilon_{a\hat{d}\hat{e}}e^{\hat{d}}e^{\hat{e}}.\end{aligned} \tag{6.39}$$

In the same way we find the other non-zero curvature forms to be

$$\begin{aligned}R^{0}{}_{\hat{a}} &= \frac{1}{20}e^{0}e^{\hat{a}} - \frac{1}{5}\epsilon_{\hat{a}b\hat{c}}e^{b}e^{\hat{c}}, \\ R^{a}{}_{b} &= \frac{17}{20}e^{a}e^{b} + \frac{2}{5}\delta^{a}_{\hat{d}}\delta_{b\hat{e}}e^{\hat{d}}e^{\hat{e}}, \\ R^{a}{}_{\hat{b}} &= \frac{1}{20}e^{a}e^{\hat{b}} - \frac{1}{5}\epsilon_{a\hat{b}\hat{c}}e^{0}e^{\hat{c}} + \frac{1}{5}\delta_{a\hat{d}}\delta_{\hat{b}e}e^{\hat{d}}e^{e} + \frac{1}{5}\delta_{a\hat{b}}e^{c}e^{\hat{c}}, \\ R^{\hat{a}}{}_{\hat{b}} &= \frac{2}{5}\epsilon_{\hat{a}\hat{b}c}e^{0}e^{c} + \frac{2}{5}\delta_{\hat{a}d}\delta_{\hat{b}e}e^{d}e^{e} + \frac{5}{4}e^{\hat{a}}e^{\hat{b}}.\end{aligned} \tag{6.40}$$

We now see the benefit of using our normalisation, for these are now the same forms as found by Awada et al., [41], and later used by Duff et al., [4]. Note that Awada et al. use a metric equivalent to Eq.(6.1) in their derivation. From these we can calculate the curvature scalar, using $R^{A}{}_{B} = \frac{1}{2}R^{A}{}_{BCD}e^{C} \wedge e^{D}$, and we find

$$R = \frac{89}{10}. \tag{6.41}$$

To relate this to the ordinary round sphere we will define $m^2 \equiv \frac{9}{20}$ so that $R = 42m^2$ and $R_{ab} = 6m^2 g_{ab}$ exactly as in the round case.

It will also be useful to calculate explicit expressions for the spin connection. We use Eq.(4.14) to find

$$
\begin{aligned}
\omega_m{}^0{}_a &= \frac{1}{2}\Omega^i_m \delta_{ai} - \frac{1}{2}\Omega^{\hat{i}}_m \delta_{a\hat{i}} - \frac{1}{2\sqrt{5}} e^{\hat{c}}_m \delta_{a\hat{c}}, \\
\omega_m{}^0{}_{\hat{a}} &= \frac{1}{2\sqrt{5}} e^c_m \delta_{c\hat{a}}, \\
\omega_m{}^a{}_b &= \frac{1}{2}\Omega^i_m \epsilon_{iba} + \frac{1}{2}\Omega^{\hat{i}}_m \epsilon_{\hat{i}ba} + \frac{1}{2\sqrt{5}} e^{\hat{c}}_m \epsilon_{\hat{c}ba}, \\
\omega_m{}^a{}_{\hat{b}} &= -\frac{1}{2\sqrt{5}} e^0_m \delta_{a\hat{b}} + \frac{1}{2\sqrt{5}} e^c_m \epsilon_{c\hat{b}a}, \\
\omega_m{}^{\hat{a}}{}_{\hat{b}} &= \Omega^{\hat{i}}_m \epsilon_{\hat{i}\hat{b}\hat{a}} - \frac{1}{2\sqrt{5}} e^{\hat{c}}_m \epsilon_{\hat{c}\hat{b}\hat{a}}.
\end{aligned}
\tag{6.42}
$$

To be able to use these we must also find an explicit formula for Ω^i_m. This is done by using Eq.(8.1.17) in [4], and solving for Ω^i_m. This gives us

$$
\begin{aligned}
\Omega^i_m &= -2e_m{}^a \delta^i_a \cot\mu + \sqrt{5} e_m{}^{\hat{a}} \delta^i_{\hat{a}}, \\
\Omega^{\hat{i}}_m &= \frac{6}{2\sqrt{5}} e_m{}^{\hat{a}} \delta^{\hat{i}}_{\hat{a}},
\end{aligned}
\tag{6.43}
$$

where μ is the parameter appearing in Eq.(6.1).

6.4 The squashed sphere as a G_2 manifold

We saw in Chapter 2 that the integrability condition for the Killing spinor only includes the Weyl tensor. So in order to discuss the holonomy of the squashed sphere we need to find an expression for this.

The Weyl tensor is defined by the relation

$$
W_{abcd} \equiv R_{abcd} - m^2(g_{ac}g_{bc} - g_{ad}g_{bc}). \tag{6.44}
$$

The integrability condition, Eq.(2.55), only involves a term of the form $W_{ab}{}^{cd}\gamma_{cd}$ which we will define as W_{ab}. It is then a straightforward, but somewhat lengthy, task to use Eq.(6.40) to find that the only non-zero terms of W_{ab} are [4]

$$
\begin{aligned}
W_{oi} &= \frac{4}{5}\left[\gamma_{0i} + \frac{1}{2}\epsilon_{ijk}\gamma_{\hat{j}\hat{k}}\right], \\
W_{ij} &= \frac{4}{5}\left[\gamma_{ij} + \gamma_{\hat{i}\hat{j}}\right], \\
W_{i\hat{j}} &= \frac{4}{5}\left[-\gamma_{i\hat{j}} - \frac{1}{2}\gamma_{j\hat{i}} + \frac{1}{2}\delta_{ij}\gamma_{k\hat{k}} - \frac{1}{2}\epsilon_{ijk}\gamma_{0\hat{k}}\right], \\
W_{o\hat{i}} &= -\frac{4}{5}\left[\gamma_{0\hat{i}} + \frac{1}{2}\epsilon_{ijk}\gamma_{j\hat{k}}\right], \\
W_{\hat{i}\hat{j}} &= \frac{4}{5}\left[2\gamma_{\hat{i}\hat{j}} + \gamma_{ij} + \epsilon_{ijk}\gamma_{0k}\right].
\end{aligned}
\tag{6.45}
$$

We can also see that [4]

$$
W_{ii} = 0, \qquad W_{0\hat{i}} = \epsilon_{ijk} W_{j\hat{k}}, \qquad \text{and} \qquad W_{\hat{i}\hat{j}} = W_{ij} + \epsilon_{ijk} W_{0k}. \tag{6.46}
$$

From Eq.(6.45) together with the identifications of Eq.(6.46) we get 14 independent W_{ab}. This means that the holonomy group of the squashed sphere, with the connection of Eq.(2.55), is generated by these 14 generators. So what is this group?

Well, the fact that the dimensionality is 14, together with our previous discussions on certain manifolds, hints at the possibility of the squashed sphere being a G_2 manifold. And, this is in fact the case [4]. The 14 generators generate the spinor representation of G_2 [35].

This implies that there is only one Killing spinor, as per the discussion in Section 2.3. The Killing spinor is then covariantly constant under G_2-covariance, i.e.,

$$\tilde{\nabla}_a\eta = \left(\nabla_a - i\frac{m}{2}\gamma_a\right)\eta = 0, \tag{6.47}$$

and we can use it to write the three-form as

$$\phi_{abc} = -i\bar{\eta}\gamma_{abc}\eta. \tag{6.48}$$

These things will be used in the next chapter when we discuss the Killing spinor equation in more detail. There we will use the octonionic structure constants, a_{abc}, as the three-form and normalise the Killing spinor according to $\bar{\eta}\eta = 1$.

7

Mass spectra of the squashed seven-sphere

This chapter presents the main results of the thesis. We calculate the eigenvalues of some mass operators on the squashed seven-sphere. The operators dealt with are the ones acting on the scalar, the vector and the spinor. This analysis turn out to be immensely more difficult than for the round sphere, and we will not complete the task. The mass operators are still the same as in the round case, but the differential operators are not. This is due to the squashed sphere not being a symmetric space, and the ${f^{ab}}_c$ part of Eq.(4.17) will therefore generally not vanish.

We start by discussing how the different representations split into representations of G and H. The tangent space group is of course still $SO(7)$, as in the round case, and as for all seven-dimensional manifolds.

7.1 Representations

From the discussion of the round sphere we know that the scalar, spinor and vector correspond to the $SO(8)$ representations

$$(n,0,0,0), \qquad (n,0,0,1)_s, (n,0,1,0)_c, \qquad \text{and} \qquad (n-1,1,0,0). \tag{7.1}$$

So we want to decompose these under $Sp(2) \times Sp(1)$ to find the representations under G. These decompositions are listed in Chapter 8 of [4], so we skip replicating this here. What happens is that they get decomposed into sums of several terms. For example, the scalar decomposes as

$$(n,0,0,0) \to \sum_{r=0}^{[n/2]} (n-2r, r; n-2r), \tag{7.2}$$

where $[n/2]$ denotes the integer part of $n/2$, and $(p,q;r)$ is a representation of $Sp(2) \times Sp(1)$ [42].

The other interesting thing to know is how our G representations decompose under H, in G/H. There are several books of tables giving us these decompositions, we have used McKay & Patera [38].

We will often use the $(10,1)$ representation of the previous chapter. From McKay & Patera, [38], we find that this decomposes under $SU(2)^A \times SU(2)^{B+C}$ as

$$(10,1) \to (1,1) \oplus (0,2) \oplus (0,0). \tag{7.3}$$

One also finds that the spinor $(0,0,1)$ of $SO(7)$ decomposes as [38]

$$(0,0,1) \to (1,1) \oplus (0,2) \oplus (0,0), \tag{7.4}$$

so it seems like the $(10,1)$ representation could be very useful when dissecting the Dirac spectra of the spinor. The vector decomposes as

$$(1,0,0) \to (1,1) \oplus (0,2). \tag{7.5}$$

Note that, in order to find these decompositions, one has to go via the G_2 decomposition of $SO(7)$.

7.2 The scalar

We begin with the easiest spectra, the scalar. Since the $SO(7)$ generators of the scalar is zero, the f-term in the master equation will vanish once again. Using the quadratic Casimir approach we thus simply find

$$\Delta_0\phi = -\Box\phi = -T_aT_a\phi. \tag{7.6}$$

In the same way as before we can write $T_aT_a = -\frac{20}{9}m^2(C_G - C_H)$, where the $\frac{20}{9}m^2$ is added for dimensional reasons, note that it is numerically equal to 1 (we saw in the previous chapter that $m^2 = \frac{9}{20}$). For the scalar we have $C_H = 0$ which means that

$$\Delta_0\phi = \frac{20}{9}m^2C_G, \tag{7.7}$$

in agreement with the results of Nilsson & Pope, [42].

7.3 Killing spinor equation

Next we analyse the Killing spinor equation on the squashed sphere. This gives us some first indications of the problems arising when squashing the sphere.

As seen in Chapter 2 the Killing spinor equation is

$$(\nabla_a\eta)_\beta = i\frac{m}{2}(\gamma_a)_{\beta\gamma}\eta_\gamma, \tag{7.8}$$

and the Killing spinor must obey the integrability condition

$$W_{mn}{}^{ab}\gamma_{ab}\eta = 0, \tag{7.9}$$

where W is the Weyl tensor.

We can construct a useful representation of the seven-dimensional gamma matrices by writing [43]

$$(\gamma_a)_{\beta\gamma} = \begin{cases} (\gamma_a)_{bc} = -ia_{abc}, & a,b,c = 1,\dots,7, \\ (\gamma_a)_{b8} = -i\delta_{ab}, & \end{cases} \tag{7.10}$$

where a_{abc} are the octonionic structure constants introduced in Appendix D. This representation is discussed in more detail in Appendix E. There we also construct explicit expressions for $\gamma^{(2)}$, $\gamma^{(3)}$ and $\gamma^{(4)}$ which will be needed below. One important thing to remember is the symmetry properties. In seven dimensions we have that $\gamma^{(1)}$ and $\gamma^{(2)}$ are anti-symmetric while $\gamma^{(0)}$ and $\gamma^{(3)}$ are symmetric. These properties are then reflected around $\gamma^{(3)}$, so that $\gamma_{(4)}$ and $\gamma_{(7)}$ are symmetric while $\gamma_{(5)}$ and $\gamma_{(6)}$ are anti-symmetric.

We can now solve the integrability condition and find an explicit expression for the Killing spinor. This is done by first writing

$$(W_{ab})_{\gamma\delta}\eta_\delta = 0 \iff (W_{ab})_{\gamma d}\eta_d = -(W_{ab})_{\gamma 8}\eta_8. \tag{7.11}$$

The non-zero parts of the Weyl tensor are listed in Eq.(6.45). These gives us, for example,

$$\begin{aligned}(W_{0i})_{\gamma d}\eta_d &= \frac{4}{5}\left[(\gamma_{0i})_{\gamma d}\eta_d + \frac{1}{2}\epsilon_{ijk}(\gamma_{\hat{j}\hat{k}})_{\gamma d}\eta_d\right] \\ &= -(W_{0i})_{\gamma 8}\eta_8 = -\frac{4}{5}\left[(\gamma_{0i})_{\gamma 8}\eta_8 + \frac{1}{2}\epsilon_{ijk}(\gamma_{\hat{j}\hat{k}})_{\gamma 8}\eta_8,\right]\end{aligned} \tag{7.12}$$

which, using the expressions in Appendices D and E for the octonions and the gamma matrices, and setting $\gamma = 0$, implies

$$\delta_{id}\eta_d + \frac{1}{2}\epsilon_{ijk}c_{\hat{j}\hat{k}0d}\eta_d = 0 \implies \delta_{id}\eta_d + \frac{1}{2}\epsilon_{ijk}\epsilon_{jkl}\delta_{ld}\eta_d = 2\delta_{id}\eta_d = 0 \implies \eta_i = 0. \tag{7.13}$$

Continuing in the same manner we find that the Killing spinor has the simple form

$$\eta = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}, \tag{7.14}$$

with perhaps some function in front.

Turning to the Killing spinor equation we now find

$$(\nabla_a\eta)_\beta = i\frac{m}{2}(\gamma_a)_{\beta\gamma}\eta_\gamma = i\frac{m}{2}\left((\gamma_a)_{\beta c}\eta_c + (\gamma_a)_{\beta 8}\eta_8\right). \tag{7.15}$$

If we now take $\beta = 8$ this becomes

$$0 = -\frac{m}{2}\delta_{ac}\eta_c = \partial_a\eta_8 + \frac{1}{4}\omega_a{}^{de}(\gamma_{de})_{\beta\gamma}\eta_\gamma = \partial_a\eta_8. \tag{7.16}$$

This means that there should not be any function in front of η in Eq.(7.14). If we instead put $\beta = b$ we find

$$\frac{m}{2}\delta_{ab}\eta_8 = -\frac{1}{8}f_a{}^{de}a_{deb}\eta_8 = \frac{1}{8\sqrt{5}}a_{ade}a_{deb}\eta_8 = \frac{3}{4\sqrt{5}}\delta_{ab}\eta_8 = \frac{m}{2}\delta_{ab}\eta_8. \tag{7.17}$$

This proves that the Killing spinor of Eq.(7.14) solves the Killing spinor equation.

Using the Killing spinor equation and the expression $a_{abc} = -i\bar{\eta}\gamma_{abc}\eta$ from Appendix D we can also find an identity that is needed later on

$$\nabla_a a_{bcd} = -i\nabla_a(\bar{\eta}\gamma_{bcd}\eta) = -2i\bar{\eta}\gamma_{bcd}\nabla_a\eta = m\bar{\eta}\gamma_{bcd}\gamma_a\eta = -m\bar{\eta}\gamma_{abcd}\eta = -mc_{abcd}. \tag{7.18}$$

7.4 Spin 1

The eigenvalues of the vector harmonics have already been found by Yamagishi, [44], but we will employ the methods of Chapter 5, in particular the quadratic Casimir approach, to see that this still work for the squashed sphere. We find that they need to be somewhat modified, or rather that we need to square the master equation one extra time. This analysis is also done by Duff et al., [4], but we will do it in more detail.

The relevant differential operator for the one-forms are of course the Hodge-de Rahm operator. Acting on vectors this is

$$\kappa^2 Y_a = \Delta_1 Y_a = -\Box Y_a + R_{ab} Y_b = -\Box Y_a + 6m^2 Y_a, \tag{7.19}$$

where, once again, $m^2 = \frac{9}{20}$ for the squashed sphere.

Taking our master equation (Eq.(4.17)) acting on a vector and squaring it we have

$$\left(\nabla^b + \frac{1}{2} f^{ef}{}_b \Sigma_{ef}\right)\left(\nabla_b + \frac{1}{2} f^{cd}{}_b \Sigma_{cd}\right) Y_a = (T_b T_b Y)_a. \tag{7.20}$$

Evaluating this, and using that the tangent space generators in the vector representation are just δ^{ab}_{cd}, we get

$$\begin{aligned}(T_b T_b Y)_a &= \Box Y_a + \frac{1}{2}(\nabla^b f^{cd}{}_b)\delta^{ae}_{cd} Y_e + \frac{1}{2} f^{cd}{}_b \delta^{ae}_{cd} \nabla^b Y_e + \frac{1}{2} f^{ef}{}_b \Sigma_{ef} \nabla_b Y_a + \frac{1}{4} f^{ef}{}_b f^{cd}{}_b \Sigma_{ef} \Sigma_{cd} Y_a \\ &= \Box Y_a + \frac{1}{2}(\nabla^b f^{ac}{}_b) Y_c + \frac{1}{2} f^{ac}{}_b \nabla^b Y_c + \frac{1}{2} f^{ac}{}_b \nabla_b Y_c + \frac{1}{4} f^{cd}{}_b f^{ac}{}_b Y_d \\ &= \Box Y_a - \frac{1}{2}(\nabla^b f_{abc}) Y_c - f_{abc} \nabla_b Y_c - \frac{1}{4} f_{abc} f_{dbc} Y_d.\end{aligned} \tag{7.21}$$

We again press on the fact that since we are working in the orthogonal groups, there is no difference between upper or lower indices, and we raise and lower them as we please.

Now, the structure constants of the coset space can be expressed as $f_{abc} = -\frac{2}{3} m a_{abc}$, which we saw in the previous chapter, and the octonionic structure constants are divergenceless, i.e., $\nabla^a a_{abc} = 0$. This means that

$$(T_b T_b Y)_a = \Box Y_a + \frac{2}{3} m a_{abc} \nabla_b Y_c - \frac{m^2}{9} a_{abc} a_{dbc} Y_d = \Box Y_a + \frac{2}{3} m a_{abc} \nabla_b Y_c - \frac{2}{3} m^2 Y_a, \tag{7.22}$$

where we used the identity $a_{abc} a_{dbc} = 6\delta_{ad}$ from Appendix D. We can once again write $(T_b T_b Y)_a = -\frac{20}{9} m^2 (C_G - C_H) Y_a$. Together with Eq.(7.19), which gives us

$$\Box Y_a = 6m^2 Y_a - \kappa^2 Y_a, \tag{7.23}$$

this means that

$$\begin{aligned}&-\frac{20}{9} m^2 (C_G - C_H) Y_a = 6m^2 Y_a - \kappa^2 Y_a + \frac{2}{3} m a_{abc} \nabla_b Y_c - \frac{2}{3} m^2 Y_a \\ &\iff a_{abc} \nabla_b Y_c = -8m Y_a + \frac{3}{2}\frac{\kappa^2}{m} Y_a - \frac{10}{3} m (C_G - C_H).\end{aligned} \tag{7.24}$$

Using the Killing forms found in Chapter 6 we can see that $C_H = \frac{6}{5} C_{SU(2)_A} + 2 C_{SU(2)_{B+C}}$, and applying the formulas for the Casimirs found in Appendix B we get $C_H = \frac{12}{5}$. Using this above gives us

$$a_{abc} \nabla_b Y_c = -8m Y_a + \frac{3}{2}\frac{\kappa^2}{m} Y_a + \frac{10}{3}\frac{12}{5} m - \frac{10}{3} m C_G = \frac{3}{2}\frac{\kappa^2}{m} Y_a - \frac{10}{3} m C_G. \tag{7.25}$$

If we now define the operator $DY_a \equiv a_{abc} \nabla_b Y_c$ and square this we find

$$D^2 Y_a = a_{ade} \nabla_d a_{ebc} \nabla_b Y_c = a_{ade} (\nabla_d a_{ebc}) \nabla_b Y_c + a_{ade} a_{ebc} \nabla_d \nabla_b Y_c. \tag{7.26}$$

From Appendix D we know that $a_{ade} a_{ebc} = 2\delta^{ad}_{bc} - c_{adbc}$, and we can also see that $2\delta^{ad}_{bc} \nabla_d \nabla_b = \nabla_c \nabla_a - \Box \delta_{ac}$. Therefore

$$D^2 Y_a = a_{ade} (\nabla_d a_{ebc}) \nabla_b Y_c + \nabla_c \nabla_a Y_c - \Box Y_a - c_{adbc} \nabla_d \nabla_b Y_c. \tag{7.27}$$

Now, the anti-symmetric properties of c_{abcd} implies that $c_{abcd}\nabla_b\nabla_c Y_d$ is simply a Bianchi identity for the Riemann tensor, so that term is zero. Since the one-forms are divergenceless we can write $\nabla_c\nabla_a Y_c = [\nabla_c, \nabla_a]Y_c = R_{ab}Y_b$. So we only need to analyse the term $a_{ade}(\nabla_d a_{ebc})\nabla_b Y_c$. When evaluating the Killing spinor equation we saw that $\nabla_a a_{bcd} = -mc_{abcd}$ and in Appendix D we derive the identity $a_{ade}c_{debc} = -4a_{abc}$. Plugging all of this in we find

$$D^2 Y_a = 4ma_{abc}\nabla_b Y_c + \kappa^2 Y_a = 4mDY_a + \kappa^2 Y_a. \tag{7.28}$$

This means that, from Eq.(7.25), we must have

$$\begin{aligned}
&\left(\frac{3}{2}\frac{\kappa^2}{m} - \frac{10}{3}mC_G\right)^2 = 4m\left(\frac{3}{2}\frac{\kappa^2}{m} - \frac{10}{3}mC_G\right) + \kappa^2 = 7\kappa^2 - \frac{40}{3}m^2 C_G \\
\Longleftrightarrow\ &\kappa^4 - \frac{m^2\kappa^2}{9}(28 + 40C_G) = -\frac{400}{81}m^4 C_G^2 - \frac{160}{27}m^4 C_G \\
\Longleftrightarrow\ &\left(\kappa^2 - \frac{m^2}{9}(14 + 20C_G)\right)^2 = \frac{m^4}{81}(14 + 20C_G)^2 - \frac{400}{81}m^4 C_G^2 - \frac{160}{27}m^4 C_G \\
&= \frac{196}{81}m^4 + \frac{80}{81}m^4 C_G \\
\Longleftrightarrow\ &\kappa^2 = \frac{20}{9}m^2\left(C_G + \frac{7}{10} \pm \frac{1}{\sqrt{5}}\sqrt{C_G + \frac{49}{20}}\right).
\end{aligned} \tag{7.29}$$

We can see that these eigenvalues perfectly align with the ones found by Yamagishi, [44]. So this does not give us anything new. But it shows us that this method should work. However, when we next turn to the Dirac equation we will encounter some peculiarities.

7.5 Dirac operator

The eigenvalues of the Dirac operator has been found by Nilsson & Pope, [42], in a completely different way than the methods we work with in this thesis. However, when we go to higher spins this method becomes extremely complicated. The hope is that our methods could more easily be generalised to the higher spin cases. It is then very important that we understand these methods completely. We therefore start with the Dirac case, where we know the eigenvalues, and try to resolve as many question marks as possible.

7.5.1 Solving Dirac directly

We begin by trying to solve the Dirac equation directly using the generators of the $(10,1)$ representation discussed in the previous chapter. Our master equation can be used to express the Dirac operator as

$$i\gamma^a\nabla_a\psi = -i\gamma^a\left(\frac{1}{2}f^{bc}{}_a\Sigma_{bc} + T_a\right)\psi = -i\left(\frac{1}{8}f^{bc}{}_a\gamma^a\gamma_{bc} + \gamma^a T_a\right)\psi. \tag{7.30}$$

As stated above, the $(10,1)$ representation split into $(1,1)\oplus(0,2)\oplus(0,0)$ under H. These representations have the dimensions 4, 3 and 1 respectively. So, in order to see what is happening, we want to use our gamma matrices, with the octonions, and try to get our $(10,1)$ basis into this splitted, $4+3+1$, form.

The indices of the different $Sp(1)$ algebras will be denoted as $I_i : A$, $J_i : \dot{A}$ and $L_i : \tilde{A}$, so that, for example, the $(1,1)$ representation has indices $(A, \dot{A})$. Since $H_i = I_i$, we see that these only have indices without any dots or tildes. This indicates that we should change the basis so that H_i lives only in the first 4×4 block. Following the approach of Aspman & Nilsson, [45], we start by interchanging columns two and three, where we think in 2×2 blocks, and then interchange the corresponding rows. For example, this means that we do the transformation

$$2iH_{\hat{1}} = \begin{pmatrix} 0 & 0 & \mathbb{1} & 0 \\ 0 & \sigma_1 & 0 & \mathbb{1} \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & \sigma_1 \end{pmatrix} \to \begin{pmatrix} 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & \sigma_1 & \mathbb{1} \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1} & \sigma_1 \end{pmatrix} \to \begin{pmatrix} 0 & \mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & \sigma_1 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & \sigma_1 \end{pmatrix}, \tag{7.31}$$

and, especially, this means that we get

$$H_i = -\frac{i}{2} \begin{pmatrix} \sigma_i & 0 & 0 & 0 \\ 0 & \sigma_i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.32}$$

as we wanted. Note that we define the tensor product to mean that we put the left term into the right term, i.e.,

$$\sigma_1 \otimes \mathbb{1} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \tag{7.33}$$

instead of the, perhaps, more usual reversed meaning. This is only a definition, and this particular choice simplifies a lot of things for us.

We apply the corresponding transformations to all the generators, and the result is

$$\begin{aligned} H_i &= -\frac{i}{2} \begin{pmatrix} \sigma_i \otimes \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \\ H_{\hat{i}} &= -\frac{i}{2} \begin{pmatrix} \mathbb{1} \otimes \sigma_i & 0 \\ 0 & \sigma_i \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_i \end{pmatrix}, \\ T_{\hat{a}} &= -\frac{i}{\sqrt{5}} \begin{pmatrix} -\frac{3}{2}\mathbb{1} \otimes \sigma_a & 0 \\ 0 & \sigma_a \otimes \mathbb{1} - \frac{3}{2}\mathbb{1} \otimes \sigma_a \end{pmatrix}, \\ Q_\alpha &= \frac{1}{2} \begin{pmatrix} 0 & -\sigma_\alpha \otimes \mathbb{1} \\ \bar{\sigma}_\alpha \otimes \mathbb{1} & 0 \end{pmatrix}. \end{aligned} \tag{7.34}$$

Next, we want to do a change of basis that resembles what we do in quantum mechanics when we go from $\frac{1}{2} \otimes \frac{1}{2}$ to $1 \oplus 0$. This is done to every 4×4 block, in the generators, to bring them into the same basis as the gamma matrices. Starting with, for example, the first block in H_i

$$H_i^{(1)} = -\frac{i}{2}\sigma_i \otimes \mathbb{1} = -\frac{i}{2}(\sigma_i)_A{}^B \delta_{\dot{A}}{}^{\dot{B}}, \tag{7.35}$$

we find the $(\cdot,\cdot)$ entry by multiplying with $\frac{1}{\sqrt{2}}\varepsilon$ from both sides, i.e. $\frac{1}{2}\varepsilon H_i^{(1)} \varepsilon^T$, where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ [40]. Doing this we get

$$-\frac{i}{2}\frac{1}{\sqrt{2}}\varepsilon^{A\dot{A}}(\sigma_i)_{\dot{A}}{}^{\dot{B}} \delta_A{}^B \frac{1}{\sqrt{2}}\varepsilon_{\dot{B}B} = -\frac{i}{4}\varepsilon^{A\dot{A}}(\sigma_i)_{\dot{A}}{}^{\dot{B}}\varepsilon_{\dot{B}A} = -\frac{i}{4}\mathrm{Tr}(\varepsilon\sigma_i\varepsilon^T) = 0. \tag{7.36}$$

The $(j,\cdot)$ entry is instead found by multiplying with $\frac{i}{\sqrt{2}}(\sigma_j)^{A\dot{A}} = \frac{i}{\sqrt{2}}\varepsilon^{A\dot{A}}(\sigma_j)_{\dot{A}}{}^{\dot{B}}$ from the left and with $\frac{1}{\sqrt{2}}\varepsilon_{\dot{B}B}$ from the right (the $(\cdot,j)$ entry is found by the opposite multiplication but with a $-i$ in front of the σ_j). For $(j,\cdot)$ we thus have

$$-\frac{i}{2}\frac{i}{\sqrt{2}}\varepsilon^{A\dot{A}}(\sigma_j)_{\dot{A}}{}^{\dot{B}}(\sigma_i)_{\dot{B}}{}^{\dot{C}}\delta_A{}^B\frac{1}{\sqrt{2}}\varepsilon_{\dot{C}B}=\frac{1}{4}\varepsilon^{A\dot{A}}(\sigma_j\sigma_i)_{\dot{A}}{}^{\dot{B}}\varepsilon_{\dot{B}A}=\frac{1}{4}\mathrm{Tr}(\varepsilon\sigma_j\sigma_i\varepsilon^T)=-\frac{1}{2}\delta_{ij}. \tag{7.37}$$

In the same way we find that the $(\cdot,j)$ entry is equal to $\frac{1}{2}\delta_{ij}$, which we already knew since the matrix still had to be anti-hermitian after the change of basis. Lastly we find the (j,k) entry by multiplying with $i\sigma_j$ from the left and $-i\sigma_k$ from the right, as

$$\begin{aligned}-\frac{i}{2}\frac{i}{\sqrt{2}}\varepsilon^{A\dot{A}}(\sigma_j)_{\dot{A}}{}^{\dot{B}}(\sigma_i)_{\dot{B}}{}^{\dot{C}}\delta_A{}^B(-i)(\sigma_k)_{\dot{C}}{}^{\dot{D}}\frac{1}{\sqrt{2}}\varepsilon_{\dot{D}B}&=-\frac{i}{4}\varepsilon^{A\dot{A}}(\sigma_j\sigma_i\sigma_k)_{\dot{A}}{}^{\dot{B}}\varepsilon_{\dot{B}B}\\&=-\frac{i}{4}\mathrm{Tr}(\varepsilon\sigma_j\sigma_i\sigma_k\varepsilon^T)=-\frac{i}{4}2i\epsilon_{jik}=-\frac{1}{2}\epsilon_{ijk}.\end{aligned} \tag{7.38}$$

We recognise that we can combine these entries and write the full block as a t' Hooft symbol, $-\frac{1}{2}\eta_i^{\alpha\beta}$, so that we now have

$$H_i=-\frac{1}{2}\begin{pmatrix}(\eta_i)^{\alpha\beta} & 0\\ 0 & 0\end{pmatrix}. \tag{7.39}$$

We do the same thing for $H_{\hat{i}}$ and $T_{\hat{a}}$ and end up with

$$\begin{aligned}H_{\hat{i}}&=-\frac{1}{2}\left(\begin{array}{c|cc}(\bar{\eta}_{\hat{i}})^{\alpha\beta} & & \\ \hline & (\epsilon_i)_{jk} & 0\\ \cline{2-3} & 0 & 0\end{array}\right),\\ T_{\hat{a}}&=\frac{1}{2\sqrt{5}}\left(\begin{array}{c|cc}3(\bar{\eta}_{\hat{a}})^{\beta\gamma} & & \\ \hline & (\epsilon_a)_{jk} & -5\delta_{aj}\\ \cline{2-3} & 5\delta_{ak} & 0\end{array}\right),\end{aligned} \tag{7.40}$$

where for example $(\epsilon_a)_{jk}$ are 3×3 matrices, M_{jk}, enumerated by a, and all the empty blocks are filled with zeros. We see that these generators are now almost in the $4+3+1$ form.

We also want to perform this change of basis on the second and third block of the Q_α generators, but this time we skip the i when finding the $(j,\cdot)$ and the (j,k) entries. The result is

$$Q_\alpha=\frac{1}{2}\left(\begin{array}{c|cc}0 & (\bar{\eta}_\alpha)_\beta{}^{\hat{j}} & \delta_{\alpha\beta}\\ \hline -(\bar{\eta}_\alpha)^{\hat{k}}{}_\gamma & 0 & \\ -\delta_{\alpha\gamma} & & \end{array}\right), \tag{7.41}$$

where now $(\bar{\eta}_\alpha)_\beta{}^{\hat{j}}$ is a 4×3 matrix and $(\bar{\eta}_\alpha)^{\hat{k}}{}_\gamma$ a 3×4 matrix, both enumerated by α.

It is a good idea to check whether or not we have done anything illegal. This is done by verifying that our refurnished generators still satisfy the algebra. We will not go through this here, but by using the identities for the 't Hooft symbol, found in the end of [46], one soon finds that the whole algebra is still satisfied (this can also be done rather quickly by using a computer). For example, since $\eta_i^{\alpha\beta}\eta_j^{\beta\gamma} = -\delta_{ij}\delta^{\alpha\gamma} - \epsilon_{ijk}\eta_k^{\alpha\gamma}$, we directly see that $[H_i, H_j] = \epsilon_{ijk}H_k$ as expected.

Before turning to the Dirac equation it is useful to first express the gamma matrices in a similar form as the generators. By using that $a_{\alpha\beta\hat{i}} = -\bar{\eta}^{\hat{i}}_{\alpha\beta}$ and $a_{\hat{i}\hat{j}\hat{k}} = \epsilon_{\hat{i}\hat{j}\hat{k}}$ we find

$$
\begin{aligned}
\gamma_{\hat{i}} &= i \left(\begin{array}{c|c} (\bar{\eta}_i)^{\alpha\beta} & \\ \hline & \begin{array}{c|c} -(\epsilon_i)_{jk} & \delta_{ij} \\ \hline -\delta_{ik} & 0 \end{array} \end{array} \right), \\
\gamma_\alpha &= i \left(\begin{array}{c|c} 0 & \begin{array}{c|c} -(\bar{\eta}_\alpha)_\beta{}^{\hat{j}} & -\delta_{\alpha\beta} \end{array} \\ \hline \begin{array}{c} (\bar{\eta}_\alpha)^{\hat{k}}{}_\gamma \\ \hline \delta_{\alpha\gamma} \end{array} & 0 \end{array} \right) = -2iQ_\alpha.
\end{aligned}
\tag{7.42}
$$

To solve Eq.(7.30) we also need to evaluate the f-term, i.e. $\frac{1}{8}\gamma^a f_a{}^{bc}\gamma_{bc} = -\frac{1}{8\sqrt{5}}a_{abc}\gamma_{abc}$. We use the expression given in Appendix E for γ_{abc} in our basis. This gives us

$$
\begin{aligned}
(a_{abc}\gamma_{abc})_m{}^n &= -6i\delta_{mn}, \\
(a_{abc}\gamma_{abc})_m{}^8 &= 0, \\
(a_{abc}\gamma_{abc})_8{}^8 &= 42i,
\end{aligned}
\tag{7.43}
$$

which means that

$$
\frac{1}{8}\gamma^a f_a{}^{bc}\gamma_{bc} = \begin{cases} \frac{3i}{4\sqrt{5}}\delta_{mn}, & \text{for } (\mu,\nu) = (m,n), \\ -\frac{21i}{4\sqrt{5}}, & \text{for } (\mu,\nu) = (8,8). \end{cases}
\tag{7.44}
$$

The Dirac equation now reads

$$
i\gamma^a \nabla_a \psi = -\left(i\gamma^{\hat{i}} T_{\hat{i}} + i\gamma^\alpha Q_\alpha + \frac{i}{8}\gamma^a f_a{}^{bc}\gamma_{bc} \right)\psi = \lambda\psi.
\tag{7.45}
$$

Remarkably, this gives us a completely diagonal matrix

$$\lambda\psi = 2\delta_{AB}\psi + \frac{1}{4\sqrt{5}}\begin{pmatrix} -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}\psi. \tag{7.46}$$

This should then be compared to the results found by Nilsson & Pope [42]. This is done in more detail in the next chapter, we can just mention here that the results do not agree. For some reason this method does not seem to give us the whole picture. We therefore move on to the method corresponding to what we did for the vector.

7.5.2 Quadratic Casimir approach

The next method we will use is similar to the vector case, i.e. a variation of the quadratic Casimir approach of the round sphere. We start by noting that the result found for the round sphere, namely Eq. (5.27),

$$\lambda^2\psi = -\Box\psi + \frac{R}{4}\psi = -\Box\psi + \frac{21}{2}m^2\psi, \tag{7.47}$$

is still valid for the squashed sphere, since only the Ricci tensor and Ricci scalar enters in the calculation, and these are equal for the round and squashed sphere [35]. So with this in mind we square our master equation acting on a spinor,

$$(T_bT_b\psi)_\alpha = \Box\psi_\alpha + \frac{1}{4}f^{de}{}_c(\gamma_{de})_\alpha{}^\beta(\nabla^c\psi)_\beta + \frac{1}{64}f^{fg}{}_c f^{de}{}_c(\gamma_{gf}\gamma_{de})_\alpha{}^\beta\psi_\beta + \frac{1}{8}(\nabla^c f^{de}{}_c)(\gamma_{de}\psi)_\alpha, \tag{7.48}$$

where we used that the tangent space generators $\Sigma_{ab} = \frac{1}{4}(\gamma_{ab})$. Using now again that $f_{abc} = -\frac{2m}{3}a_{abc}$ and that $\nabla^a a_{abc} = 0$ we get

$$\begin{aligned}(T_bT_b\psi)_\alpha &= \Box\psi_\alpha - \frac{m}{6}a_{abc}(\gamma_{ab})_\alpha{}^\beta(\nabla^c\psi)_\beta + \frac{1}{144}m^2 a_{ade}a_{abc}(\gamma_{de}\gamma_{bc})_\alpha{}^\beta\psi_\beta \\ &= \Box\psi_\alpha - \frac{m}{6}a_{abc}(\gamma_{ab})_\alpha{}^\beta(\nabla^c\psi)_\beta + \frac{1}{144}m^2\left(2\delta^{de}_{bc} - c_{debc}\right)\left(\gamma_{debc} - 4\gamma^{[d}{}_{[b}\delta^{e]}_{c]} - 2\delta^{de}_{bc}\right)\psi \\ &= \Box\psi_\alpha - \frac{m}{6}a_{abc}(\gamma_{ab})_\alpha{}^\beta(\nabla^c\psi)_\beta - \frac{1}{144}m^2 c_{abcd}(\gamma_{abcd})_\alpha{}^\beta\psi_\beta - \frac{7}{12}m^2\psi_\alpha.\end{aligned} \tag{7.49}$$

Our expressions for the gamma matrices, found in Appendix E, implies that

$$c_{abcd}\gamma_{abcd} = 24\left(\begin{array}{c|c} -\delta_{ab} & 0 \\ \hline 0 & 7 \end{array}\right) \equiv 24A. \tag{7.50}$$

As before, we write $T_bT_b = -\frac{20}{9}m^2(C_G - C_H)$ and then move around some things to get (omitting spinor indices)

$$\begin{aligned}a_{abc}\gamma_{ab}\nabla^c\psi &= \frac{6}{m}\Box\psi - \frac{7}{2}m\psi - mA\psi + \frac{40}{3}m(C_G - C_H)\psi \\ &= \left(63m - \frac{6}{m}\lambda^2 - \frac{7}{2}m - mA + \frac{40}{3}m(C_G - C_H)\right)\psi.\end{aligned} \tag{7.51}$$

If we now want to follow the same recipe as in the spin-1 case we should square the operator on the left hand side of this equation and try and find some equation for this square that can be solved. If we define $\tilde{D}\psi \equiv a_{abc}\gamma_{ab}\nabla^c\psi$ and square this we find

$$\begin{aligned}\tilde{D}^2\psi =& -16m\tilde{D}\psi - 2ma_{cde}\gamma_{cdef}\nabla^f\psi - 4\gamma_{cd}\nabla^c\nabla^d\psi - 4c_{cdef}\gamma_{ef}\nabla^c\nabla^d\psi \\ &- 12\Box\psi + a_{cef}a_{dgh}\gamma_{efgh}\nabla^c\nabla^d\psi,\end{aligned} \tag{7.52}$$

where we have skipped all the intermediate steps, it is just a lot of manipulation of the octonionic structure constants and of the gamma matrices. We also used that $\nabla_a a_{bcd} = -mc_{abcd}$ as before.

Eq.(7.52) is, however, troublesome. The $\gamma^{(4)}$ terms will give us some symmetric combination of two covariant derivatives, and this is not something we know how to evaluate. So instead we use a little trick. We can add the Dirac operator to the operator we want to square without violating any rules. This should give us some terms with $\gamma^{(3)}$ in them. In seven dimensions $\gamma^{(3)}$ and $\gamma^{(4)}$ are dual, so with some luck and a good choice of parameters we could perhaps get rid of the bad parts.

So we define the operator $D \equiv a_{cde}(\gamma_{cd})_\alpha{}^\beta(\nabla^e\psi)_\beta + \xi(\gamma^c)_\alpha{}^\beta(\nabla_c\psi)_\beta$, where ξ is a parameter we want to decide, and then square this (once more omitting the spinor indices). We skip the intermediate steps also this time and just give the result

$$\begin{aligned}D^2\psi =& -16ma_{abc}\gamma_{ab}\nabla^c\psi - 4c_{abcd}\gamma_{ab}\nabla^c\nabla^d\psi + (\xi^2-4)\gamma_{ab}\nabla^a\nabla^b\psi + (\xi^2-12)\Box\psi \\ &- 4\xi a_{abc}\gamma_a\nabla^b\nabla^c\psi - 2ma_{abc}\gamma_{abcd}\nabla^d\psi - m\xi c_{abcd}\gamma_{abc}\nabla^d\psi + a_{abc}a_{def}\gamma_{bcde}\nabla^a\nabla^f\psi \\ &+ 2\xi a_{abc}\gamma_{dab}\nabla^{(c}\nabla^{d)}\psi.\end{aligned} \tag{7.53}$$

The parts we want to get rid of are the ones with $\gamma^{(3)}$ or $\gamma^{(4)}$. So we dualise

$$-m\xi c_{abcd}\gamma_{abc} = -m\xi\frac{1}{6}\epsilon_{abcdefg}a_{efg}\gamma_{abc} = -im\xi a_{efg}\gamma_{defg}. \tag{7.54}$$

Comparing this to the term with $a_{abc}\gamma_{abcd}$ above we see that these two will cancel if $\xi = -2i$.

The two last terms with $\gamma^{(3)}$ and $\gamma^{(4)}$ will actually also cancel when we put $\xi = -2i$, this can be checked in the same way as above, but we can also evaluate the terms using our expressions for the gamma matrices. If we for example take the $(8,8)$ component of the two terms we have

$$\begin{aligned}(a_{abc}a_{def}\gamma_{abef} + 2\xi a_{ab(c}\gamma_{d)ab})_8{}^8 &= a_{abc}a_{def}c_{abef} + 2i\xi a_{ab(c}a_{d)ab} \\ &= -4a_{abc}a_{dab} + 12i\xi\delta_{cd} = -24\delta_{cd} + 12i\xi\delta_{cd},\end{aligned} \tag{7.55}$$

which is zero for $\xi = -2i$. By doing the same thing for the other component we find that they all cancel each other for the same value of ξ.

So, if we now put $\xi = -2i$ we have

$$D^2\psi = -16ma_{abc}\gamma_{ab}\nabla^c\psi - 4c_{abcd}\gamma_{ab}\nabla^c\nabla^d\psi - 8\gamma_{ab}\nabla^a\nabla^b\psi - 16\Box\psi + 8ia_{abc}\gamma_a\nabla^b\nabla^c\psi. \tag{7.56}$$

We can next use that $\nabla^{[c}\nabla^{d]}\psi_\alpha = \frac{1}{8}R^{cd}{}_{ef}(\gamma^{ef})_\alpha{}^\beta\psi_\beta$ as we saw in Chapter 5, and that $\Box = \frac{R}{4} - \lambda^2$. This gives us

$$\begin{aligned}D^2\psi &= -16ma_{abc}\gamma_{ab}\nabla^c\psi - 4R\psi + 16\lambda^2 + 2R + ia_{abc}\gamma_a R_{bc}{}^{de}\gamma_{de} - \frac{1}{2}c_{abcd}\gamma_{ab}\gamma_{ef}R_{cd}{}^{ef} \\ &= -16ma_{abc}\gamma_{ab}\nabla_c - 2R + 16\lambda^2 + ia_{abc}\gamma_{ade}R_{bc}{}^{de} - \frac{1}{2}c_{abcd}\gamma_{cdef}R_{abef}.\end{aligned} \tag{7.57}$$

Dualising between $\gamma^{(3)}$ and $\gamma^{(4)}$, as well as between a and c, turns the last term into

$$\begin{aligned}
-\frac{1}{2}c_{abcd}\gamma_{cdef}R_{abef} &= \frac{i}{12\cdot 6}\epsilon_{abcd}{}^{ijk}\epsilon_{cdef}{}^{mnp}a_{ijk}\gamma_{mnp}R_{abef} = \frac{i}{12\cdot 6}2\cdot 5!\delta^{abijk}_{efmnp}a_{ijk}\gamma_{mnp}R_{abef} \\
&= \frac{i}{3}\left(\delta^{ab}_{ef}a_{ijk}\gamma_{ijk}R_{abef} + 6\delta^{i[a}_{ef}\delta^{b]jk}_{mnp}a_{ijk}\gamma_{mnp}R_{abef} + 3\delta^{ij}_{ef}\delta^{abk}_{mnp}a_{ijk}\gamma_{mnp}R_{abef}\right) \\
&= \frac{i}{3}Ra_{abc}\gamma_{abc} - 2ia_{abc}\gamma_{dbc}R_{ad} + ia_{abc}\gamma_{ade}R_{bcde} \\
&= 14im^2a_{abc}\gamma_{abc} - 12im^2a_{abc}\gamma_{abc} + ia_{abc}\gamma_{ade}R_{bcde} \\
&= 2im^2a_{abc}\gamma_{abc} + ia_{abc}\gamma_{ade}R_{bcde}.
\end{aligned} \tag{7.58}$$

The second term in this expression will add with the corresponding term of Eq.(7.57). The Riemann tensor can be written as

$$R_{abcd} = W_{abcd} + 2m^2\delta^{ab}_{cd}, \tag{7.59}$$

where W_{abcd} is the Weyl tensor, which means that

$$2ia_{abc}\gamma_{ade}R_{bcde} = 2ia_{abc}\gamma_{ade}\left(W_{bcde} + 2m^2\delta^{bc}_{de}\right) = 2ia_{abc}\gamma_a W_{bc} + 4im^2a_{abc}\gamma_{abc}, \tag{7.60}$$

where $W_{ab} \equiv W_{abcd}\gamma_{cd}$. We can then use the expressions in Eq.(6.45) to evaluate the part involving the Weyl tensor. After a bit of calculation, and splitting of the indices into $(0, i, \hat{i})$, we find that this part is actually zero [40].

The gamma matrices of Appendix E tells us that

$$a_{abc}\gamma_{abc} = 6i\left(\begin{array}{c|c} -\delta_{ab} & 0 \\ \hline 0 & 7 \end{array}\right) \equiv 6iA. \tag{7.61}$$

Putting all of this together

$$\begin{aligned}
D^2\psi &= -16ma_{abc}\gamma_{ab}\nabla_c\psi + \left(-2R + 16\lambda^2 - 36A\right)\psi \\
&= -16mD\psi - 32im\gamma^a\nabla_a\psi + \left(-84m^2 + 16\lambda^2 - 36m^2A\right)\psi,
\end{aligned} \tag{7.62}$$

where we added and subtracted $16m\xi\gamma^a\nabla_a\psi$ on the left side in order to get the full D operator back. This can be re-expressed as

$$(D + 8m)^2\psi = 16(\lambda - m)^2\psi - 36(1 + A)\psi. \tag{7.63}$$

Using Eq.(7.51) we can see that

$$D\psi = \frac{179}{3}m\psi - \frac{6}{m}(\lambda + \frac{m}{6})^2\psi - mA\psi + \frac{40}{3}m(C_G - C_H)\psi. \tag{7.64}$$

If we now assume that D and A commute, so that we can diagonalise them simultaneously, we can plug Eq.(7.64) into Eq.(7.63) to find the eigenvalues. Setting $\alpha = a$ gives us four roots, these should correspond to the $(1, 1)$ and $(0, 2)$ representations so we put $C_H = \frac{12}{5}$, as before. The result is

$$\begin{aligned}
\lambda_{1,2} &= -\frac{m}{2} \pm \frac{2\sqrt{5}}{3}m\sqrt{C_G + \frac{63}{20}}, \\
\lambda_{3,4} &= \frac{m}{6} \pm \frac{2\sqrt{5}}{3}m\sqrt{C_G + \frac{49}{20}}.
\end{aligned} \tag{7.65}$$

These should then also be compared to the results of Nilsson & Pope, [42]. This is also done in the next chapter. We can of course directly see that these do not give the same results as in the previous example, and they also do not align perfectly with Nilsson & Pope.

For the $\alpha = 8$ component, corresponding to the singlet representation, $(0,0)$, we put $C_G = C_H = 0$ which gives us one root corresponding to the known singlet

$$\lambda = -\frac{7}{2}m, \tag{7.66}$$

and three roots with other, nonsensical (complex-valued and strange), eigenvalues which we throw away.

7.5.3 G_2 covariance

A third option is to define a G_2 covariant derivative and use this to evaluate the Dirac equation. As discussed in Chapter 2 this derivative is

$$\tilde{\nabla}_a \equiv \nabla_a - i\frac{m}{2}\gamma_a, \tag{7.67}$$

where ∇_a is the ordinary covariant derivative used in the previous sections. Using this derivative will change things a whole lot. We rewrite our master equation as

$$\tilde{\nabla}_a\psi - \frac{m}{12}a_{abc}\gamma_{bc}\psi = -(T_a + \frac{im}{2}\gamma_a)\psi, \tag{7.68}$$

where we simply added the term to make the derivative G_2 covariant. Squaring this we find

$$\left(\tilde{\Box} - \frac{m}{6}a_{abc}\gamma_{bc}\tilde{\nabla}_a - \frac{m^2}{144}c_{abcd}\gamma_{abcd} - \frac{7}{12}m^2\right)\psi = \left(-\frac{20}{9}m^2(C_G - C_H) - \frac{7m^2}{4} + im\gamma_a T_a\right)\psi. \tag{7.69}$$

We can use our master equation to rewrite the last term as

$$im\gamma_a T_a\psi = -im\gamma_a\left(\nabla_a - \frac{m}{12}a_{abc}\gamma_{bc}\right)\psi = -m\lambda\psi + i\frac{m^2}{12}a_{abc}\gamma_{abc}\psi \equiv -m\lambda\psi - \frac{m^2}{2}A\psi, \tag{7.70}$$

where A is the same matrix as appearing in Eq.(7.50). This means that

$$\begin{aligned} a_{abc}\gamma_{ab}\tilde{\nabla}_c\psi &= \frac{6}{m}\tilde{\Box}\psi + 7m\psi + 6\lambda\psi + \frac{40}{3}m(C_G - C_H)\psi + 2mA\psi \\ &= \frac{6}{m}\Box\psi - \frac{7}{2}m\psi + \frac{40}{3}m(C_G - C_H)\psi + 2mA\psi, \end{aligned} \tag{7.71}$$

where we used that $\tilde{\Box} = \Box - im\gamma^a\nabla_a - \frac{7}{4}m^2$.

We follow the recipe of the previous section and define the operator

$$D \equiv a_{abc}\gamma_{ab}\tilde{\nabla}_c + \xi\gamma_a\tilde{\nabla}_a. \tag{7.72}$$

When squaring this we note that this new derivative satisfies $\tilde{\nabla}_a a_{bcd} = 0$ and $[\tilde{\nabla}_a, \tilde{\nabla}_b]\psi = \frac{1}{4}W_{ab}\psi$. Then we can use dualisation to find that $\xi = -2i$ again cancels some bad terms, and, since $\gamma_{ab}W_{ab} = 0$, almost all terms actually cancel. In the end, the only thing left is

$$D^2\psi = -16\tilde{\Box}\psi = -16\left(\Box - im\gamma^a\nabla_a - \frac{7}{4}m^2\right)\psi = 16\left(\lambda^2 + \frac{m}{2}\right)^2 - 144m^2. \tag{7.73}$$

From Eq.(7.71) we find

$$
\begin{aligned}
D\psi &= \left(63m - \frac{7}{2}m - \frac{6}{m}\lambda^2 + \frac{40}{3}m(C_G - C_H) + 2mA - 2\lambda - 7m\right)\psi \\
&= \left(\frac{105}{2}m - \frac{6}{m}\lambda^2 - 2\lambda + \frac{40}{3}(C_G - C_H) + 2mA\right)\psi.
\end{aligned} \tag{7.74}
$$

Combining Eqs.(7.73) and (7.74) we now have a quartic equation for λ, again assuming that D and A commute. We still hope that the 8th component correspond to the singlet so we set $C_G = C_H = 0$. This does in fact give us the same eigenvalue as before, namely

$$
\lambda = -\frac{7}{2}m, \tag{7.75}
$$

which is the known eigenvalue of the singlet [4]. However, the roots we throw away are different, i.e., the full equation is not the same as before. For the other block, $\alpha = 1, \ldots, 7$, the quartic equation does not factorise in a nice way, and we can not get any sensible eigenvalues out of it. So something seems to be missing in this method.

8

Conclusions and outlook

In this thesis we have studied the mass spectra of eleven-dimensional supergravity compactified on the squashed seven-sphere. We now give a short summary of the motivation for this study and then move on to discuss the results and where to go from here.

In Chapter 3 we discussed some proposed conjectures about features of a consistent theory of quantum gravity. One of these was the sharpened version of the Weak Gravity Conjecture (WGC) [10]. This led to the corollary that non-supersymmetric AdS vacua supported by fluxes must be unstable [11]. We saw that one could use this to make predictions regarding the masses of neutrinos and the size of the cosmological constant. In particular we saw that, in order to have Majorana neutrinos, we need to include some beyond the standard model physics [13]. This could then lead to measurable predictions from string theory, something that has long been looked for.

The squashed seven-sphere allows for one non-supersymmetric AdS vacua supported by a flux. It therefore provides us with a great opportunity to test the WGC. In order to look for possible instabilities it is important that we know more about the mass spectra of the theory.

In this thesis we studied a few different methods that could be used to find the mass spectra. These were applied to the ordinary round seven-sphere with great success, however when moving on to the squashed sphere some problems were encountered. This was most evident in the Dirac operator spectra. We first tried to solve for the Dirac operator directly by using a specific representation of the generators, namely the $(10, 1)$ representation dicussed in Chapter 6. This gave us the eigenvalues

$$\begin{aligned}
\lambda_1 &= 2 - \frac{15}{4\sqrt{5}}, \\
\lambda_2 &= 2 + \frac{17}{4\sqrt{5}}, \\
\lambda_3 &= 2 + \frac{9}{4\sqrt{5}}.
\end{aligned} \tag{8.1}$$

When instead using a method that involved squaring our master equation, Eq.(4.17), we found the eigenvalues

$$\begin{aligned}
\lambda_{1,2} &= -\frac{m}{2} \pm \frac{2\sqrt{5}}{3} m \sqrt{C_G + \frac{63}{20}}, \\
\lambda_{3,4} &= \frac{m}{6} \pm \frac{2\sqrt{5}}{3} m \sqrt{C_G + \frac{49}{20}}, \\
\lambda_0 &= -\frac{7}{2} m,
\end{aligned} \tag{8.2}$$

plus three non-sensical roots that we threw away. None of these perfectly align with the previous results of Nilsson & Pope, [42]. They found that

$$\begin{aligned} \lambda &= -\frac{m}{2} \pm \frac{2\sqrt{5}}{3} m \sqrt{C_G + \frac{81}{20}}, \\ \lambda &= \frac{m}{6} \pm \frac{2\sqrt{5}}{3} m \sqrt{C_G + \frac{49}{20}}. \end{aligned} \tag{8.3}$$

We thus see that the second method came really close. The only difference is the 63 under the square root of $\lambda_{1,2}$, instead of the 81 from [42]. Another possible problem is that λ_0, which is the correct eigenvalue of the singlet representation, only comes out for exactly that representation, i.e. when setting $C_G = C_H = 0$, otherwise the resulting quartic equation does not factorise in a good way. This is also in contrast to Nilsson & Pope who found that the singlet is contained in the whole tower having the 81 eigenvalue above, meaning that it is the special case of that eigenvalue when $C_G = 0$.

The problem here is, probably, that we assumed our operator $D = a_{abc}\gamma_{ab}\nabla_c - 2i\gamma^a\nabla^a$ to commute with the traceless matrix

$$A \equiv \left(\begin{array}{c|c} -\delta_{ab} & 0 \\ \hline 0 & 7 \end{array} \right). \tag{8.4}$$

If this is not the case we should not be able to diagonalise the two operators simultaneously [40]. This has actually been shown to be the problem in the last few days by Nilsson & Pope and the solution will be discussed in a future paper [40].

By using that our $(10,1)$ representation has $C_G = \frac{19}{4}$, since $C_G = C_{Sp(2)} + 3C_{Sp(1)}$ [4], we find that the eigenvalues of Nilsson & Pope are

$$\begin{aligned} \lambda &= \frac{-3 \pm 8\sqrt{11}}{4\sqrt{5}}, \\ \lambda &= \frac{1 \pm 24}{4\sqrt{5}}. \end{aligned} \tag{8.5}$$

These obviously do not align with Eq.(8.1), but we can see that if we instead had

$$\begin{aligned} \lambda_1 &= -\frac{2}{\sqrt{5}} - \frac{15}{4\sqrt{5}}, \\ \lambda_2 &= \frac{2}{\sqrt{5}} + \frac{17}{4\sqrt{5}}, \end{aligned} \tag{8.6}$$

we would have had the results of the second row in Eq.(8.5).

We also used a method that incorporated the G_2 holonomy of the squashed sphere. This gave us the correct eigenvalue for the singlet, i.e., $\lambda = -\frac{7}{2}m$, but again only when we set $C_G = C_H = 0$, otherwise the equation did not factorise in a manageable way. For the other components it did not factorise in a good way at all, so something seems to be missing from this analysis. Here we again assumed that our differential operator commuted with the traceless matrix, so this could at least be part of the problem also in this case.

As mentioned above the problem with the second method has been resolved, but the other two methods are still not understood completely.

The same type of problems that arose for the Dirac equation also seems to be present for the higher spins. It would therefore be very interesting if these could be resolved completely in the case where we know the answer, so that we can fully trust our methods when moving on to the unknown.

In addition to our representation of the squashed seven-sphere as the coset space $\frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$, it has been shown that one can express it as the coset space $\frac{Sp(2)}{Sp(1)}$, where $Sp(1)$ is one of the algebras in the $SO(4) \cong Sp(1) \times Sp(1)$ subgroup of $Sp(2)$ [47]. This coset space has however not been studied in as much detail as the one we consider. A corresponding treatment of this coset space would therefore be very interesting, and could perhaps lead to some simplifications.

A

Conventions

Dear Reader, if the thought of living in a world with eleven dimensions worries you, you will probably tremble in fear when facing the immense vastness of the convention-space of theoretical physics. This appendix will serve as a guide through this wilderness.

A.1 Weights

Symmetrisation and anti-symmetrisation, written $y^{(a}y^{b)}$ and $y^{[a}y^{b]}$ respectively, is defined to have weight one, meaning that $T_{[ab]} = T_{ab}$ if the tensor is anti-symmetric in a and b. We thus have, for example,

$$y^{[a}y^{b]} = \frac{1}{2}(y^a y^b - y^b y^a). \tag{A.1}$$

With this convention, an arbitrary tensor T_{ab} splits into its symmetric and anti-symmetric parts as

$$T_{ab} = T_{(ab)} + T_{[ab]}. \tag{A.2}$$

A.2 A matter of signs

The Minkowski metric will always have the signature $\eta^{\alpha\beta} = \mathrm{diag}(-1, +1, +1, \ldots, +1)$.

The two-dimensional Pauli sigma matrices are defined in the usual way as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.3}$$

and we define

$$\begin{aligned} \sigma^\alpha &\equiv (-i\mathbb{1}, \sigma^i), \qquad \alpha = 0, i, \quad i = 1,2,3, \\ \bar{\sigma}^\alpha &\equiv (+i\mathbb{1}, \sigma^i). \end{aligned} \tag{A.4}$$

The sigma matrices are generalised to higher dimensions through the introduction of the gamma-matrices, γ^α, which in four dimensions are constructed as

$$\gamma^\alpha \equiv \begin{pmatrix} 0 & \sigma^\alpha \\ \bar{\sigma}^\alpha & 0 \end{pmatrix}. \tag{A.5}$$

These satisfy the Clifford algebra $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$. It may be worth noting that this is in contrast to our main reference [4], who uses $\{\gamma^\alpha, \gamma^\beta\} = -2\eta^{\alpha\beta}$. In even dimensions we can create the chiral gamma-matrix, and we will always choose it to have the form

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{A.6}$$

The general definition is, in D dimensions,

$$\gamma^{D+1} \equiv (i)\gamma^1\gamma^2\cdots\gamma^D, \tag{A.7}$$

where the inclusion of the i depends on whether or not we can have both Majorana and Weyl fermions in the given dimension. In odd dimensions we instead find that

$$\gamma^1\cdots\gamma^D = \pm(i)\mathbb{1}, \tag{A.8}$$

where we choose the plus sign. We will do most of our work in seven dimensions, where we thus use the convention

$$\gamma^1\cdots\gamma^7 = i\mathbb{1}. \tag{A.9}$$

The anti-symmetrised combination of gamma-matrices, which we write for example $\gamma^{\alpha\beta\gamma} \equiv \gamma^{[\alpha}\gamma^{\beta}\gamma^{\gamma]}$, and similarily for other numbers of matrices, will appear frequently. Sometimes we will denote it as $\gamma^{(n)}$, so that we have $\gamma^{(2)} = \gamma^{ab}$ and so on. The set of anti-symmetrised combinations of different numbers of matrices form a complete basis of the Clifford algebra.

We also introduce the 't Hooft symbol $\eta_i^{\alpha\beta}$, and its dual $\bar{\eta}_i^{\alpha\beta}$, defined by

$$\begin{aligned} \sigma^{\alpha\beta} &\equiv i\eta_i^{\alpha\beta}\sigma^i, \\ \bar{\sigma}^{\alpha\beta} &\equiv i\bar{\eta}_i^{\alpha\beta}\sigma^i, \end{aligned} \tag{A.10}$$

which in our conventions satisfy

$$\begin{aligned} \eta_k^{\alpha\beta} &: \begin{cases} \eta_j^{0i} = -\delta_j^i, \\ \eta_k^{ij} = \epsilon_{ijk} \end{cases} \\ \bar{\eta}_k^{\alpha\beta} &: \begin{cases} \eta_j^{0i} = +\delta_j^i, \\ \eta_k^{ij} = \epsilon_{ijk}. \end{cases} \end{aligned} \tag{A.11}$$

There are several identities involving the 't Hooft symbols that we will need. Most of them can be found in the end of [46].

A.3 Indices

The matter of indices is a very tricky one in this type of work, since we move around in so many different dimensions, use a lot of different sets of operators and so on. We will try to be as clear as possible every time we change the meaning of the indices. The general rule is that capital letters belong to higher dimensions than lower case letter. When discussing compactification we thus use capital letters for the full dimension and split it into $\mu, \nu, \ldots$ for spacetime and $m, n, \ldots$ for the compact dimensions. i, j, k will most often be reserved for three-dimensional indices, as for example when we discuss the algebra of the squashed seven-sphere, or the Pauli matrices σ^i.

The beginning of the alphabet, i.e., a, b, c, A, B, C or α, β, γ etc., will usually indicate flat indices while $m, n, p, M, N, P, \mu, \nu, \rho$ etc. usually indicate curved indices. Greek letters will many times also be reserved for spinor type indices, so that for example we usually write $(\gamma^a)_\alpha{}^\beta\nabla_a\psi_\beta$, for the Dirac operator.

When we work in the orthogonal groups we raise and lower indices with δ_{ab}, so we don't really need to bother with the right placing. We will therefore be a bit sloppy with the indices at those occasions.

B

Group theory and Lie algebras

This appendix discusses the theories of groups and Lie algebras. These are both extremely huge subjects on their own and we will simply give a brief introduction and state the necessary results. If however the subjects presented in this chapter feels unfamiliar or uncomfortable to you, we highly recommend looking them up in some of the available text books. Our main references are Fuchs & Schweigert [48], Wybourne [49], Hamermesh [50] and Ramond [51]. Group theory is probably one of the most fascinating subjects out there so you will not regret it.

The sections discussing the eigenvalues of the quadratic Casimir and the dimension of irreducible representations will be a bit more technical. This is partly because the results have not been found anywhere else and also because they are crucial for the work in this thesis.

B.1 Groups

A group, G, is a set of elements, g, with an associated composition law (multiplication, addition, etc.) that satisfies the following four axioms,

$$\begin{aligned} &1)\ \text{If } g_1, g_2 \in G \implies g_1 g_2 = g_3 \in G, && \text{(closure)} \\ &2)\ \text{If } g_1, g_2, g_3 \in G \implies (g_1 g_2) g_3 = g_1 (g_2 g_3), && \text{(associativity)} \\ &3)\ \text{There is a unique element, } e, \text{ called the identity element,} \\ &\quad\ \text{such that } eg = ge = g, \forall g \in G. \\ &4)\ \text{There is a unique element, } g^{-1} \in G, \text{ called the inverse of } g, \\ &\quad\ \text{satisfying } gg^{-1} = g^{-1} g = e. \end{aligned} \tag{B.1}$$

Note that instead of writing out any symbol (+, -, $\times$, etc.) for the composition law we simply put two elements next to each other. The set could be finite or infinite and it could be discrete or continuous.

If $g_1 g_2 = g_2 g_1$ the group is called *Abelian*. One example of an Abelian group is the integers, which is a discrete group under addition. The identity element is zero, and the inverse of any element is minus the element. It is easily seen that this satisfies all the axioms of group theory.

The class of groups that we are most interested in here are the so called Lie groups. They are characterised by the fact that they are differentiable manifolds. Lie groups can be either finite- or infinite-dimensional. The classical examples that we will study are $SO(n), SU(n)$ and $Sp(n)$.

A *subgroup* of G is a subset of the elements in G satisfying the group axioms on their own. Note that a group G will always contain at least two subgroups, namely the identity element, and the group itself. If H is a subgroup of G we can construct a right or left *coset* by taking an element in G not in H and compose this element from the right or left with the whole of H. Two cosets are then either disjoint or identical as sets. A *normal* subgroup is defined as satisfying $xHx^{-1} = H$ for all $x \in G$. This is the same as saying that the right and left cosets constructed from H are the same. If G does not contain any normal subgroups (except the trivial ones $\{e\}$ and $\{G\}$) it is said to be *simple*.

A *representation* of a group is defined as a set of matrices satisfying the multiplication laws of the group, i.e. $ab = c \implies \Gamma(a)\Gamma(b) = \Gamma(c)$, for $a, b, c \in G$ and $\Gamma(a)$ being the matrix representing the element a. The representation is said to be *isomorphic* to the group if it is one-to-one, or *homomorphic* if many-to-one. Note that a representation is only defined up to a similarity transformation since, if $\Gamma_i \to \Gamma_i' = S^{-1}\Gamma_i S$, for all Γ_i, and $\mathrm{Det}(S) \neq 0$, we have

$$\Gamma_i'\Gamma_j' = S^{-1}\Gamma_i S S^{-1}\Gamma_j S = S^{-1}\Gamma_i\Gamma_j S = (\Gamma_i\Gamma_j)', \tag{B.2}$$

so Γ_i and Γ_i' are equivalent representations. Note also that if $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are two representations of a group, then so is $\tilde{\Gamma}_i = \begin{pmatrix} \Gamma_i^{(1)} & 0 \\ 0 & \Gamma_i^{(2)} \end{pmatrix}$. However, since $\tilde{\Gamma}$ is block diagonal, it is said to be a *reducible* representation. On the contrary, if there are no similarity transformations taking Γ into block diagonal form, then Γ is said to be an *irreducible* representation, or irrep for short.

B.2 The symmetric group and Young tableaux

There is a very important family of groups called the symmetric groups, S_n. These are the groups of all permutations of n objects. This group has several important applications, for example when we discuss wavefunctions for several particles in quantum mechanics. Consider a system of n identical particles, specified by their coordinates x_n. The wavefunction must then have certain symmetry properties under permutations of these coordinates, i.e., be in an irreducible representation of S_n. We denote an object in S_n permuting particles x_i and x_j by (ij). This acts on the wavefunction by

$$(ij)\psi(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_n) = \psi(x_1, x_2, \ldots, x_j, \ldots, x_i, \ldots, x_n). \tag{B.3}$$

We can take the example of S_2 to further illustrate how this works. The group S_2 has two elements, the identity e and the permutation (12). It also has two irreps, corresponding to a symmetric or an anti-symmetric function. These are $e = (12) = 1$ and $e = 1$, $(12) = -1$. The unit element can therefore be decomposed as

$$e = \frac{1}{2}(e + (12)) + \frac{1}{2}(e - (12)), \tag{B.4}$$

and this decomposes the wavefunction into two parts, corresponding to the symmetric and the anti-symmetric part (this is the same as we do with any tensor when we split it into symmetric and anti-symmetric parts). If we consider the wavefunction to have the form $\psi(x_1, x_2) = \phi(x_1)\phi(x_2)$, we get

$$\psi \to \frac{1}{2}\underbrace{(\phi_1(x_1)\phi_2(x_2) + \phi_1(x_2)\phi_2(x_1))}_{\text{symmetric}} + \frac{1}{2}\underbrace{(\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1))}_{\text{anti-symmetric}}. \tag{B.5}$$

This can be represented by the notion of Young tableaux. We let each function ϕ_i be represented by a box, $\Box$, and each particle location, x_i, by the number i. When we place several boxes together the rule is that horisontal boxes are symmetric and vertical boxes are anti-symmetric. The resulting diagrams must not grow vertically in the left-to-right direction, or horisontally in the top-to-bottom direction. This means, for example, that $\ydiagram{2,3}$ is not allowed but $\ydiagram{3,2,1}$ is. Dimensions of the irreps are then found by counting the number of ways one can place the numbers $1, \ldots, n$ into the n boxes so that they increase both horisontally and vertically. In S_2 we only have $\ytableaushort{12}$, and $\ytableaushort{1,2}$, so that we have two one-dimensional irreps.

When instead going to S_3 we can construct so called *hooked* diagrams of the form $\ytableaushort{12,3}$. However, here we have the possibility of placing the numbers in another order, namely, $\ytableaushort{13,2}$. This indicates that there is a two-dimensional irrep under S_3. Note that S_3 also has the irreps $\ytableaushort{123}$ and $\ytableaushort{1,2,3}$, so that in total we have three different irreps, two of dimension one and one of dimension two.

The counting of all possible ways to place the numbers in boxes gets rather complicated when going to higher S_n's. But, as always in group theory, there is a nice trick. This is the so called *hook* formula. It works like this: In each box you place a number corresponding to how many boxes are to the right of your box plus how many boxes are directly below your box, and add the number one for your box. For the hooked diagram of S_3 we get $\ytableaushort{31,1}$. The hook formula is then to take $n!$, for S_n, divided by the product of all the numbers you have filled in. For S_3 we thus get $\frac{3!}{3} = 2$, which we already saw before.

Now, to find the corresponding wavefunction of a given diagram we need to introduce Young operators. We once again use S_3 as an example. The diagram $\ydiagram{3}$ gives us the operator corresponding to the symmetric case

$$R_S \equiv e + (12) + (13) + \ldots, \tag{B.6}$$

where all terms have a plus sign. If we instead take the diagram $\ydiagram{1,1,1}$ we find the anti-symmetric operator

$$R_A \equiv e - (12) + (13) - \ldots, \tag{B.7}$$

where even permutations have a plus sign and odd ones have a minus sign. For the hooked diagrams we need to define two operators corresponding to the symmetries of the first row and the first column. For $\ytableaushort{12,3}$ we get

$$\begin{aligned} P &\equiv e + (12), && \text{top row,} \\ Q &\equiv e - (13), && \text{first column.} \end{aligned} \tag{B.8}$$

We then construct the Young operator as

$$Y \equiv QP = e + (12) - (13) - (123). \tag{B.9}$$

The diagram $\ytableaushort{13,2}$ instead gives us

$$Y' = e - (12) + (13) - (132). \tag{B.10}$$

We can now decompose the unit element as

$$e = \frac{1}{6}R_S + \frac{1}{6}R_A + \frac{1}{3}Y + \frac{1}{3}Y' \implies \psi \to \psi_{\ydiagram{3}} + \psi_{\ydiagram{1,1,1}} + (\psi_{\ydiagram{2,1}} + \psi'_{\ydiagram{2,1}}). \tag{B.11}$$

This is very useful when searching for decompositions of wavefunctions or more general tensors.

There is another useful formula connected to the Young tableaux. This counts the number of ways we can place numbers that are strictly increasing in the vertical direction but only weakly increasing in the horisontal direction. For example, this could count the degrees of freedom of a tensor having the symmetry properties implied by the tableaux-structure in its indices. For a tensor in n dimensions we place an imagined n in the first box, and then $n+1$ in the next horisontal box and so on. When going down one line we place an $n-1$ in the first box, and then step up as we go to the right as before. The multiplication of all these numbers is then the numerator of a quotient that will give us the degrees of freedom. The denominator is the same as in the hook formula above, i.e. the product of all the numbers corresponding to how many boxes are to the left and below each box.

An example is certainly in order. Take a rank six tensor in six dimensions, with symmetry properties implied by $\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}$, i.e., it could perhaps be written $T_{[ab][cd](ef)}$, where $a, b, \ldots, f = 1, 2, \ldots, 6$. We start with the numerator, we place a six in the first box, then a seven, an eight and finally a nine. In the next line we have five and six. This means that the numerator is $6 \cdot 7 \cdot 8 \cdot 9 \cdot 5 \cdot 6$. To find the denominator we fill in the numbers as before, this gives us $\begin{array}{|c|c|c|c|}\hline 5 & 4 & 2 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}$. The tensor thus has

$$N = \frac{6 \cdot 7 \cdot 8 \cdot 9 \cdot 5 \cdot 6}{5 \cdot 4 \cdot 2 \cdot 2} = 3 \cdot 7 \cdot 9 \cdot 6 = 1134 \text{ degrees of freedom.} \tag{B.12}$$

This is a very neat trick to count the degrees of freedom. One can also connect this to the dimension of a specific irrep of the Lie algebras, especially to A_r.

B.3 Lie groups

As previously mentioned, the Lie groups are a class of groups that are also differentiable manifolds. They are usually realised as matrix groups, and there are three classical infinite families, and five exceptional groups. The Lie groups appear frequently in physics applications, the most canonical example perhaps being the generators of angular momentum in quantum mechanics.

The most general matrix group is the general linear group, $GL(n, \mathbb{F})$, which is the group of all matrices with non-zero determinant, defined over some number field $\mathbb{F}$, e.g. the real numbers. We can put some restraints on this group. For instance, if we demand that the matrices have unit determinant we get the special linear groups, $SL(n, \mathbb{F})$.

The classical Lie groups are subgroups of GL with the constraints that they leave some matrix, G, invariant. We get three cases

$$\begin{aligned} &R \in O(n), \ G = \mathbb{1} \implies RGR^T = G, \\ &U \in U(n), \ G = \mathbb{1} \implies UGU^\dagger = G, \\ &M \in Sp(n), \ G = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \implies MGM^T = G. \end{aligned} \tag{B.13}$$

We usually combine these with the constraint of unit determinant, and denote the corresponding groups as $SO(n), SU(n)$ and $Sp(n)$. We can also have different signatures in these groups, this is denoted by for example $SO(p, q)$ which means that it leaves a Minkowski-like metric, with signature (p, q), invariant, as for example the Lorentz group $SO(1, 3)$ which leaves the ordinary four-dimensional Minkowski metric invariant.

It is often a troublesome task to prove whether or not two Lie groups are isomorphic. We first must check that they are topologically the same and then see if they obey the same multiplication table. One simpler way of analysing the groups is to first linearise them. For example, if we have $A \in SL(2,\mathbb{R})$ we can write this as

$$A(a,b,c) = \begin{pmatrix} 1+a & b \\ c & \frac{1+bc}{1+a} \end{pmatrix}. \tag{B.14}$$

The multiplication of two such elements will give us complicated non-linear relations. If we instead linearise A near the identity element (here the unit matrix) we get

$$\begin{aligned} A(\delta a,\delta b,\delta c) &= \begin{pmatrix} 1+\delta a & \delta b \\ \delta c & \frac{1+\delta b\delta c}{1+\delta a} \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \delta a & \delta b \\ \delta c & -\delta a \end{pmatrix} + \mathcal{O}(\delta^2) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \delta b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \delta c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &\equiv \mathbb{1} + \delta a T_1 + \delta b T_2 + \delta c T_3, \end{aligned} \tag{B.15}$$

where T_i are called the (infinitesimal) generators of $SL(2,\mathbb{R})$. The analysis of the linearised group elements introduces us to the subject of Lie algebras.

B.4 Lie algebra

If a system has a certain symmetry generated by some object T_a, according to the above definition of the generators, the corresponding Lie algebra is defined by the relation

$$[T^a, T^b] = f^{ab}{}_c T^c, \tag{B.16}$$

where $f^{ab}{}_c$ are called the *structure constants* of the algebra, and $[\cdot,\cdot]$ is called the *Lie bracket*. The bracket satisfies $[A,B] = -[B,A]$, and is usually realised as a commutator. The algebra must also satisfy the Jacobi identity

$$[T^a,[T^b,T^c]] + [T^b,[T^c,T^a]] + [T^c,[T^a,T^b]] = 0, \tag{B.17}$$

or expressed in the structure constants

$$f^{ad}{}_e f^{bc}{}_d + f^{bd}{}_e f^{ca}{}_d + f^{cd}{}_e f^{ab}{}_d = 0. \tag{B.18}$$

Due to the anti-symmetry of the Lie bracket an Abelian algebra has $[T^a, T^b] = 0$.

The Lie algebra is by definition a vector space, with the Lie bracket taking the rôle of the multiplication. The generators can therefore be seen as basis elements in the tangent space of the group around the identity element. We can then write an arbitrary group element, g, close to the identity element as $g = \mathbb{1} + \theta_a T^a$. Performing this transformation several consecutive times results in an exponential mapping of the algebra to the group, i.e. we get an arbitrary element of the group as $g = e^{\theta_a T^a}$. This exponential mapping does not always work perfectly (but in the cases we are interested in it does). It is, however, very useful. For instance, it tells us that for Abelian groups we have $e^A e^B = e^{A+B}$ so that $(e^A)^{-1} = e^{-A}$. For $SO(n)$ we have that an element satisfies $R^T = R^{-1}$, and since $R = e^A$ this means that $A^T = -A$. We also see that the constraints of unit determinant implies that $\mathrm{Tr}A = 0$, since $1 = \mathrm{Det}(e^A) = e^{\mathrm{Tr}A} \implies \mathrm{Tr}A = 0$.

There is a special representation of any Lie algebra called the *adjoint* representation. This is the representation of the algebra on itself. We can thus write the generators of the adjoint representation as

$$(T^a)^b{}_c = -f^{ab}{}_c. \tag{B.19}$$

The matrices of the adjoint representation have the same dimension as the Lie algebra.

To every Lie group there exist an associated Lie algebra, and we will denote this by lower case letters, so that for example the Lie algebra of the group $SO(n)$ is denoted $so(n)$ (in the main text we will, however, not always make this extinction, but instead let the context tell us whether we mean the group or the algebra). As we saw above $so(n)$ is the set of all anti-symmetric, real $n \times n$ matrices. Note that two Lie groups can have the same algebra but still be different as groups, we then say that they are *locally isomorphic*. We are now ready to classify all finite-dimensional simple Lie algebras.

B.5 Cartan classification

In this section we study the complex versions of $sl(2, \mathbb{R})$, i.e. the algebra spanned by all complex linear combinations of the basis elements. Doing this will lead us to a classification of all the finite-dimensional Lie algebras. This was first done by Cartan and is therefore called the Cartan classification. We will do it rather quickly and skip many details, so the unfamiliar but interested reader should really look up a more detailed derivation, we recommend looking in Fuchs & Schweigert [48].

The idea is to start from the $sl(2)$ algebra, given by

$$\begin{aligned} &[H^1, E^1_\pm] = \pm 2 E^1_\pm, \\ &[E^1_+, E^1_-] = H^1. \end{aligned} \tag{B.20}$$

All quantum physicists out there can compare this with the algebra of angular momentum in QM (there we usually denote $E_\pm$ as $J_\pm$ and H is most often taken as the z-component of the spin operator, J_z). To get a new quantum number that can be measured simultaneously with the one from H^1, we introduce another set of $sl(2)$ generators, such that

$$\begin{aligned} &[H^1, H^2] = 0, \\ &[H^2, E^2_\pm] = \pm 2 E^2_\pm, \\ &[E^2_+, E^2_-] = H^2. \end{aligned} \tag{B.21}$$

The commuting generators, H^i, are called the Cartan generators, and the $E^i_\pm$ are called step-operators. The Cartan generators span a subalgebra, called the Cartan subalgebra, and all states are labeled by two integers, the eigenvalues of H^1 and H^2. This means that the weights of the states are two-dimensional quantities.

We now have some options. One possibility is that all the mixed brackets could be zero, this would however indicate that we only have a direct sum of two $sl(2)$ algebras, and that is not a simple algebra. So to get something truly new we need to allow for non-zero commutators. We thus define

$$[H^i, E^j_\pm] \equiv \pm A^{ji} E^j_\pm, \tag{B.22}$$

where A^{ij} is called the *Cartan matrix*.

We also need to study the brackets $[E^1_\pm, E^2_\pm]$ and $[E^1_\pm, E^2_\mp]$. By acting with H^i on these, i.e. taking $[H^i, [E^1_\pm, E^2_\pm]]$ and $[H^i, [E^1_\pm, E^2_\mp]]$, we see that both of these brackets will give possible new elements (step-operators with fixed weights) in the algebra (this is a good exercise for the interested reader). It is, however, possible to define one of them to be zero in order to find the smallest possible new algebra, so we set $[E^1_\pm, E^2_\mp] = 0$. This will now allow us to decide the complete Cartan matrix by evaluating the different commutators of all the step-operators. We soon find

$$A^{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{B.23}$$

which is the Cartan matrix for $sl(3,\mathbb{R})$.

The rows of the Cartan matrix, corresponding to the eigenvalues of each Cartan generator acting on the step-operators, are called *roots* of the algebra, denoted α_i. For every root α the corresponding step-operators, $E^{\alpha}_{\pm}$, generates an $sl(2)$ subalgebra. The real vector space spanned by the roots is called the *root space* of the algebra, and the dual vector space is called the *weight space*.

We define the positive roots to be the ones where the first non-zero entry in a certain basis is positive, and the simple roots are those positive roots that can not be written as a sum of some other positive roots. All roots can be expressed as linear combinations of the set of simple roots, with all coefficients having the same sign.

Now, in order to do this more generally we start by choosing a maximal set of commuting linearly independent elements among the semisimple elements of g. We denote these as H^i, and we should then have

$$[H^i, H^j] = 0, \quad \text{for } i,j = 1,2,\ldots,r. \tag{B.24}$$

The subalgebra spanned by the H^i's is called the Cartan subalgebra of g, and we will denote it by g_0. Note that a semisimple Lie algebra can have many different Cartan subalgebras, but they are all related. They all also have the same dimension r, called the *rank* of the Lie algebra. The rank gives us the maximal number of quantum numbers available to label states of a system with symmetry algebra g.

The Lie algebra g is spanned by all elements, y, satisfying

$$[h, y] = \alpha_y(h)y, \qquad \text{for } h \in g_0, \tag{B.25}$$

where α_y is the root of the Lie algebra corresponding to the element y [48]. This also means that we can decompose g in a direct sum of vector spaces $g_\alpha = \{y \in g | [h,y] = \alpha(h)x, \forall h \in g_0\}$ as

$$g = \bigoplus_{\alpha} g_\alpha = g_0 \oplus \bigoplus_{\alpha\neq 0} g_\alpha. \tag{B.26}$$

This is called the *root space decomposition* of g relative to g_0. We can thus have a basis constructed entirely from H^i and elements E^α satisfying

$$[H^i, E^\alpha] = \alpha^i E^\alpha. \tag{B.27}$$

This is the *Cartan-Weyl basis*. As we said earlier we will call the vector α^i for a root of g. The set of non-zero root vectors is denoted Δ and the set of positive roots is denoted Δ_+. Lastly, we denote the set of simple roots by Π. The number of simple roots is equal to the rank of the algebra. We can now write the Cartan-Weyl basis as

$$\mathcal{B}_{CW} = \{H^i | i = 1,\ldots,r\} \cup \{E^\alpha | \alpha \in \Delta\}. \tag{B.28}$$

Earlier we saw that also the brackets $[E^\alpha, E^\beta]$ could be non-zero. Using the action of the Cartan generators on this bracket one finds [48]

$$\begin{aligned}
&[E^\alpha, E^\beta] = N_{\alpha,\beta} E^{\alpha+\beta}, \quad \text{if } \alpha+\beta \in \Delta, \\
&[E^\alpha, E^\beta] = g_{ij}\alpha^j H^i, \quad \text{for } \alpha+\beta = 0, \\
&[E^\alpha, E^\beta] = 0, \quad \text{otherwise.}
\end{aligned} \tag{B.29}$$

Here we introduced some metric denoted g_{ij} that is dual to the so called *Killing form*, defined by

$$\kappa^{ij} \equiv f^{ik}{}_{l} f^{jl}{}_{k}. \tag{B.30}$$

If we denote the simple roots as $\alpha^{(i)}$ the Cartan matrix can be defined as

$$A^{ij} \equiv 2\frac{(\alpha^{(i)},\alpha^{(j)})}{(\alpha^{(j)},\alpha^{(j)})}, \tag{B.31}$$

where $(\alpha^{(i)},\alpha^{(j)})$ is the inner product of the two roots, defined by the metric g_{ij}. We can also define the *coroot* of α as

$$\check{\alpha} \equiv \frac{2\alpha}{(\alpha,\alpha)}, \tag{B.32}$$

which turns the equation for the Cartan matrix into

$$A^{ij} = (\alpha^{(i)},\check{\alpha}^{(j)}). \tag{B.33}$$

We will once again refer to the duals of the roots as the *weights* and denote them λ, so that

$$\lambda_i \alpha^j = \delta_i^j, \tag{B.34}$$

in particular, we call the weights dual to the simple coroots for the *fundamental weights*, $\lambda_{(i)}$. One can then use the fundamental weights as the basis of the weight space, this basis is usually refered to as the Dynkin basis. We call the *Dynkin basis* components of a weight for the *Dynkin labels*, these can be used to label an irreducible representation as we will see in a while.

There is a unique root, called the *highest root*, θ, for any simple algebra g such that

$$(\theta,\theta) \geq (\alpha,\alpha) \qquad \forall\alpha\in\Delta. \tag{B.35}$$

Now, in order to classify all simple finite-dimensional Lie algebras it simplifies things a lot if we also fix the constants in Eq.(B.29). This leads us to the Chevalley-Serre basis, defined by the relations [48]

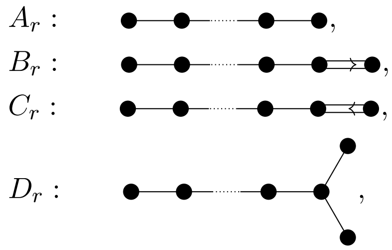
$$\begin{aligned}
&[H^i,H^j] = 0,\\
&[H^i,E^j_\pm] = \pm A^{ji}E^j_\pm,\\
&[E^i_+,E^j_-] = \delta_{ij}H^j,\\
&(\mathrm{ad}_{E^i_\pm})^{1-A^{ji}}E^j_\pm = 0,
\end{aligned} \tag{B.36}$$

where $(\mathrm{ad}_{E^i_\pm})^{1-A^{ji}}E^j_\pm$ means that we act on $E^j_\pm$ $1-A^{ji}$ times with the Lie bracket, e.g. $(\mathrm{ad}_{E^i_\pm})^2E^j_\pm = [E^i_\pm,[E^i_\pm,E^j_\pm]]$ and so on. The relations of Eq.(B.36) completely, and uniquely, characterises the Lie algebra g. From the use of this basis and the definition, we can find that the Cartan matrix satisfy

$$\begin{aligned}
&A^{ii} = 2,\ \forall i,\\
&A^{ij} = 0 \implies A^{ji} = 0,\ \text{for } i\neq j,\\
&A^{ij} \in \{0,-1,-2,-3\} \text{ for } i\neq j,\\
&\mathrm{Det}(A) > 0,
\end{aligned} \tag{B.37}$$

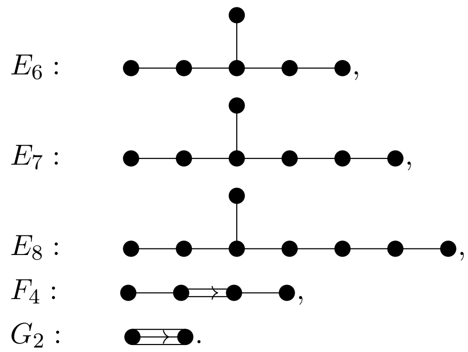
and through these relations we find all simple finite-dimensional Lie algebras [48]. There are four infinite families, A_r, B_r, C_r and D_r (where r denotes the rank of the algebra), called the classical Lie algebras, and five exceptional algebras, E_6, E_7, E_8, F_4 and G_2. One can find that A_r is isomorphic to $su(r+1)$, B_r to $so(2r+1)$, C_r to $sp(r)$ and D_r to $so(2r)$. Table VI of Fuchs & Schweigert lists the complete Cartan matrices of all simple finite-dimensional Lie algebras [48].

The simplest way of representing all Lie algebras is through their Dynkin diagrams. The idea is to associate every simple root with a node in the diagram. Every pair of simple roots are then connected by $\max\{|A^{ij}|, |A^{ji}|\}$ lines. If two roots are of different length we draw an arrow pointing towards the shorter root. The Dynkin diagrams of the finite dimensional classical Lie algebras are



(B.38)

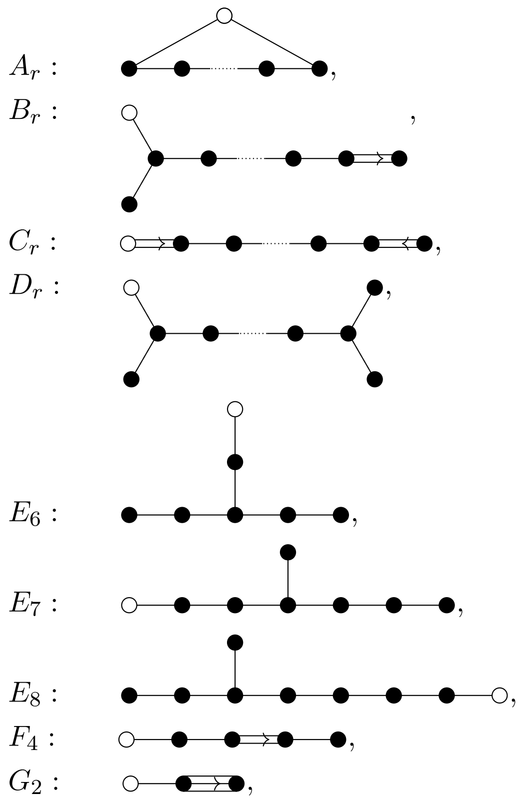
and of the five exceptional algebras



(B.39)

Algebras where all roots are of equal length is called *simply laced*. These are easy to spot when analysing the diagrams, since they are the ones with only single lines between every pair of nodes, i.e., A_r, D_r, E_6, E_7 and E_8.

It is also possible to add a node corresponding to minus the highest root, i.e. $-\theta$, of the algebra. This will generate the so called *extended* Dynkin diagrams. These are



(B.40)

The extended diagrams are also the diagrams of the affine Lie algebras, which has $\mathrm{Det}(A) = 0$. We will, however, not discuss these algebras here.

B.6 Representations of Lie algebras

A specific representation is constructed by first choosing the basis vectors of the representation space to be simultaneous eigenvectors of the Cartan generators, H^i. These basis vectors are denoted as $|\lambda^i\rangle$, so that

$$H^i|\lambda^i\rangle = \lambda^i|\lambda^i\rangle, \tag{B.41}$$

where λ^i are the components of the weight vector $\lambda = (\lambda^1, \ldots, \lambda^r)$. We will henceforth, somewhat misleading, refer also to these components as weights. Note that in the adjoint representation, these are the roots. There is an important theorem in representation theory which states that for a given irreducible representation, the highest weight, denoted $|\Lambda\rangle$, is non-degenerate and fixes the representation uniquely [48]. The highest weight state satisfies

$$E^\alpha|\Lambda\rangle = 0, \qquad \forall\alpha \in \Delta_+. \tag{B.42}$$

The highest weight can be expanded in the weights, λ^i, as

$$\Lambda = \sum_i l^i\lambda^i, \tag{B.43}$$

where the l^i are some expansion coefficients. Another way to specify an irreducible representation is by their Dynkin labels, as mentioned before. These are an ordered set of numbers, $(n^1, n^2, \ldots, n^r)$, defined by

$$n^i \equiv \frac{2(\Lambda, \alpha^i)}{(\alpha^i, \alpha^i)} = (\Lambda, \check{\alpha}^i). \tag{B.44}$$

The roots may also be expressed in the weights. In this thesis we will only need to concern ourselves with the families B_r, C_r and D_r. The results are found in Wybourne [49]

$$\begin{aligned} B_r: &\quad \{\lambda^p, \pm\lambda^p \pm \lambda^q\}_1^r, && \text{(B.45)} \\ C_r: &\quad \{\pm 2\lambda^p, \pm\lambda^p \pm \lambda^q\}_1^r, && \text{(B.46)} \\ D_r: &\quad \{\pm\lambda^p \pm \lambda^q\}_1^r, && \text{(B.47)} \end{aligned}$$

where the ranges of p and q are given at the end of the curly braces. The corresponding positive roots are found by setting the first entry in every root to be positive. Using this together with the expansion of the highest weight we can find formulas for the expansion coefficents, l^i, in terms of the Dynkin labels for all algebras. For these we find [48]

$$\begin{aligned} B_r: &\quad l^k = \sum_{i=k}^{r-1} n^i + \frac{n^r}{2}, \\ C_r: &\quad l^k = \sum_{i=k}^{r} n^i, \\ D_r: &\quad l^k = \begin{cases} \sum_{i=k}^{r-2} n^i + \frac{n^{r-1}+n^r}{2} & \text{for } k = 1, 2, \ldots, r-1, \\ \frac{n^{r-1}-n^r}{2} & \text{for } k = r \end{cases}. \end{aligned} \tag{B.48}$$

If we express the weights in terms of the roots we can use the Cartan matrix to find the inner product of the weights. For B_r, C_r and D_r we have

$$(\lambda^i, \lambda^j) = K\delta^{ij}, \tag{B.49}$$

where K are constants differing between each family of algebras. These are [49]

$$\begin{aligned} B_r: &\quad K = \frac{1}{2(2r-1)}, \\ C_r: &\quad K = \frac{1}{4(r+1)}, \\ D_r: &\quad K = \frac{1}{4(r-1)}. \end{aligned} \tag{B.50}$$

We will later normalise these to one in all cases so that the weights form an orthonormal basis,

$$(\lambda^i, \lambda^j) = \delta^{ij}. \tag{B.51}$$

B.7 Eigenvalues of the quadratic Casimir

The quadratic Casimir operator is defined by

$$C_R \equiv \kappa_{ab} T^a T^b, \tag{B.52}$$

where κ_{ab} is the inverse of the Killing form.

One important property of the Casimir operators is that they commute with all the generators of the algebra, i.e., $[C_R, T^a] = 0$, $\forall\ T^a$. Schur's lemma then tells us that, if we are in an irreducible representation of the algebra, the quadratic Casimir must be a multiple of the identity operator [48]. We thus write

$$C_R = \kappa_{ab} R^a R^b = c_R \mathbb{1}, \tag{B.53}$$

where c_R is a constant that only depends on the representation R.

In the Cartan-Weyl basis we can write the quadratic Casimir in terms of H^i and E^α as [48]

$$C_R = \sum_{j=1}^{r} \kappa_{ij} H^i H^j + \sum_{\alpha\in\Delta_+} (E^\alpha E^{-\alpha} + E^{-\alpha} E^\alpha). \tag{B.54}$$

This can then be rewritten using the definition of the highest weight state

$$\begin{aligned} C_R|\Lambda\rangle &= (\Lambda,\Lambda)|\Lambda\rangle + \sum_{\alpha\in\Delta_+} (E^\alpha E^{-\alpha} + E^{-\alpha} E^\alpha)|\Lambda\rangle \\ &= (\Lambda,\Lambda)|\Lambda\rangle + \sum_{\alpha\in\Delta_+} [E^\alpha, E^{-\alpha}]|\Lambda\rangle \\ &= (\Lambda,\Lambda)|\Lambda\rangle + \sum_{\alpha\in\Delta_+} (\alpha,\Lambda)|\Lambda\rangle = c_R|\Lambda\rangle. \end{aligned} \tag{B.55}$$

Our goal is to express these eigenvalues in the Dynkin labels. It is helpful to define the Weyl vector, δ, as

$$\delta \equiv \frac{1}{2} \sum_{\alpha\in\Delta_+} \alpha, \tag{B.56}$$

i.e. as half the sum of all the positive roots. Using this in the expression for the eigenvalues above we get

$$c_R|\Lambda\rangle = (\Lambda, \Lambda + 2\delta)|\Lambda\rangle. \tag{B.57}$$

The Weyl vector can also be expanded in the weights as

$$\delta = \sum_i \delta^i \lambda^i. \tag{B.58}$$

Solving for δ^i in the different algebras we find [49]

$$\begin{aligned} B_r:&\quad \delta^i = r - i + \frac{1}{2}, \\ C_r:&\quad \delta^i = r - i + 1, \\ D_r:&\quad \delta^i = r - i. \end{aligned} \tag{B.59}$$

By using the expansions of both Λ and δ in the weights we can now find an expression for the eigenvalues of the quadratic Casimir expressed solely in the expansion coefficients l^i, which in turn can be expressed in the Dynkin labels according to Eq.(B.48). First we write

$$c_R = (\Lambda, \Lambda + 2\delta) = \left(\sum_i l^i \lambda^i, \sum_j (l^j + 2\delta^j)\lambda^j \right) = \sum_{i,j} l^i (l^j + 2\delta^j)(\lambda^i, \lambda^j). \tag{B.60}$$

Now, using the results of Eq.(B.45) and (B.59) we find

$$\begin{aligned}
B_r: \quad c_R &= \frac{1}{2(2r-1)} \sum_{i=1}^{r} l^i(l^i + 2r - 2i + 1), \\
C_r: \quad c_R &= \frac{1}{4(r+1)} \sum_{i=1}^{r} l^i(l^i + 2r - 2i + 2), \\
D_r: \quad c_R &= \frac{1}{4(r-1)} \sum_{i=1}^{r} l^i(l^i + 2r - 2i).
\end{aligned} \tag{B.61}$$

Finally, we use Eq.(B.48) to find our sought formulas [40]

$$\begin{aligned}
B_r: \; c_R &= \frac{1}{2(2r-1)} \sum_{i=1}^{r} \left[\left(\sum_{j=i}^{r-1} n^j + \frac{n^r}{2} \right) \left(\sum_{k=i}^{r-1} n^k + \frac{n^r}{2} + 2r - 2i + 1 \right) \right], \\
C_r: \; c_R &= \frac{1}{4(r+1)} \sum_{i=1}^{r} \left(\sum_{j=i}^{r} n^j \left(\sum_{k=i}^{r} n^k + 2r - 2i + 2 \right) \right), \\
D_r: \; c_R &= \frac{1}{4(r-1)} \left\{ \sum_{i=1}^{r-1} \left[\left(\sum_{j=i}^{r-2} n^j + \frac{n^{r-1} + n^r}{2} \right) \left(\sum_{k=i}^{r-2} n^k + \frac{n^{r-1} + n^r}{2} + 2r - 2i \right) \right] \right. \\
&\qquad \left. + \frac{(n^{r-1} - n^r)^2}{4} \right\}.
\end{aligned} \tag{B.62}$$

These are the main results of this section. In particular, we can evaluate these in the algebras we are interested in, namely $sp(2)$, $so(3)$, $so(5)$, $so(7)$ and $so(8)$. This gives us the following results

$$\begin{aligned}
sp(2): \quad c_R =& \frac{1}{12} \left[n_1(n_1+4) + 2n_2(n_2+3) + 2n_1n_2 \right], \\
so(3): \quad c_R =& \frac{1}{2} \frac{[n_1(n_1+2)]}{4}, \\
so(5): \quad c_R =& \frac{1}{6} \left[n_1(n_1+3) + \frac{n_2}{2}(n_2+4) + n_1n_2 \right], \\
so(7): \quad c_R =& \frac{1}{10} \left[n_1(n_1+5) + 2n_2(n_2+4) + \frac{3n_3}{4}(n_3+6) + 2n_1n_2 + 2n_2n_3 + n_1n_3 \right], \\
so(8): \quad c_R =& \frac{1}{12} \Big[n_1(n_1+6) + 2n_2(n_2+5) + n_3(n_3+6) + n_4(n_4+6) \\
& + 2n_1n_2 + n_1n_3 + n_1n_4 + 2n_2n_3 + 2n_2n_4 + n_3n_4 \Big].
\end{aligned} \tag{B.63}$$

From this we can also see that $sp(2) \sim so(5)$ if we only exchange $n_1 \leftrightarrow n_2$. As mentioned above we will later normalise according to $K = 1$ for all algebras, so that one simply drops the prefactors in the above formulas. Next we will derive a formula for the dimension of an irreducible representation as a function of its Dynkin labels.

B.8 Dimension of irreducible representations

If φ is a unitary irreducible representation of a compact semi-simple Lie group, characterised by its highest weight state, Λ, then the dimension of φ is given by the famous Weyl formula [49]

$$\mathrm{Dim}(\varphi) = \prod_{\alpha\in\Delta_+} \frac{(\Lambda+\delta,\alpha)}{(\delta,\alpha)}. \tag{B.64}$$

We use the results from the previous sections, namely the different expansions in the weight vectors, to rewrite this. For B_r we find

$$\mathrm{Dim}(\varphi) = \prod_{\alpha\in\Delta_+} \frac{(\Lambda+\delta,\alpha)}{(\delta,\alpha)} = \prod_{\alpha\in\Delta_+} \frac{\sum_i (l^i+\delta^i)(\lambda^i,\alpha)}{\sum_j \delta^j(\lambda^j,\alpha)}. \tag{B.65}$$

The denominator can be evaluated as

$$\prod_{\alpha\in\Delta_+}\sum_j \delta^j(\lambda^j,\alpha) = \left(\prod_p\sum_i \delta^i(\lambda^i,\lambda^p)\right)\left(\prod_{q,r}\sum_j \delta^j(\lambda^j,\lambda^q+\lambda^r)\right)\left(\prod_{s,t}\sum_k \delta^k(\lambda^k,\lambda^s-\lambda^t)\right), \tag{B.66}$$

where the expressions for the positive roots in terms of the weights in Eq.(B.45) has been used. Using now that $(\lambda^i,\lambda^j)=K\delta^{ij}$ we get

$$K^3\left(\prod_p \delta^p\right)\left(\prod_{q,r}(\delta^q+\delta^r)\right)\left(\prod_{s,t}(\delta^s-\delta^t)\right). \tag{B.67}$$

Defining $m^i \equiv l^i+\delta^i$ we evaluate the numerator in the same way and find

$$\prod_{\alpha\in\Delta_+}\sum_i m^i(\lambda^i,\alpha) = K^3\left(\prod_p m^p\right)\left(\prod_{q,r}(m^q+m^r)\right)\left(\prod_{s,t}(m^s-m^t)\right). \tag{B.68}$$

Combining the two gives us our result

$$\mathrm{Dim}(\varphi) = \left(\prod_p \frac{m^p}{\delta^p}\right)\left(\prod_{q,r}\frac{m^q+m^r}{\delta^q+\delta^r}\right)\left(\prod_{s,t}\frac{m^s-m^t}{\delta^s-\delta^t}\right). \tag{B.69}$$

Doing the same thing for C_r and D_r we find that C_r gives us the same expression while D_r has

$$\mathrm{Dim}(\varphi) = \prod_{p,q}\left(\frac{m^p-m^q}{\delta^p-\delta^q}\right)\left(\frac{m^p+m^q}{\delta^p+\delta^q}\right). \tag{B.70}$$

We can now use the results of Eq.(B.48) and (B.59) to write this in terms of the Dynkin labels. The results are [40]

$$\begin{aligned} B_r:\ \mathrm{Dim}(\varphi) &= \left(\prod_p \frac{\sum_{i=p}^{r-1} n^i + \frac{n^r}{2} + r - p + \frac{1}{2}}{r-p+\frac{1}{2}}\right)\left(\prod_{p,q}\frac{\sum_{i=p}^{r-1} n^i + \sum_{j=q}^{r-1} n^j + n^r + 2r + 1 - p - q}{2r+1-p-q}\right) \\ &\times \left(\prod_{p,q}\frac{\sum_{i=p}^{r-1} n^i - \sum_{j=q}^{r-1} n^j - p + q}{-p+q}\right) \end{aligned} \tag{B.71}$$

$$\begin{aligned} C_r:\ \mathrm{Dim}(\varphi) &= \left(\prod_p \frac{\sum_{i=p}^{r} n^i + r - p + 1}{r-p+1}\right)\left(\prod_{p,q}\frac{\sum_{i=p}^{r} n^i + \sum_{j=q}^{r} n^j + 2r + 2 - p - q}{2r+2-p-q}\right) \\ &\times \left(\prod_{p,q}\frac{\sum_{i=p}^{r} n^i - \sum_{j=q}^{r} n^j - p + q}{-p+q}\right) \end{aligned} \tag{B.72}$$

$$D_r: \ \mathrm{Dim}(\varphi) = \left(\prod_{p,q} \frac{\sum_{i=p}^{r-2} n^i + \frac{n^{r-1} \pm n^r}{2} + \sum_{j=q}^{r-2} n^j + \frac{n^{r-1} \pm n^r}{2} + 2r - p - q}{2r - p - q} \right) \times \left(\prod_{p,q} \frac{\sum_{i=p}^{r-2} n^i + \frac{n^{r-1} \pm n^r}{2} - \sum_{j=q}^{r-2} n^j - \frac{n^{r-1} \pm n^r}{2} - p + q}{-p + q} \right), \tag{B.73}$$

where the $\pm$ is $+$ if $p, q = 1, \ldots, r-1$ and $-$ if $p, q = r$. For the algebras we are interested in we especially find

$$\begin{aligned} sp(2): \ \mathrm{Dim}(\varphi) &= \frac{1}{6}(n_1+1)(n_2+1)(n_1+n_2+2)(n_1+2n_2+3), \\ so(5): \ \mathrm{Dim}(\varphi) &= \frac{1}{6}(n_1+1)(n_2+1)(n_1+n_2+2)(2n_1+1n_2+3), \end{aligned} \tag{B.74}$$

note that these, once again, change into each other under the interchange of $n_1 \leftrightarrow n_2$. For $so(7)$ we get

$$\begin{aligned} \mathrm{Dim}(\varphi) = \frac{1}{720} &(n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(n_2+n_3+2)(2n_2+n_3+3) \\ &\times (n_1+n_2+n_3+3)(n_1+2n_2+n_3+4)(2n_1+2n_2+n_3+5), \end{aligned} \tag{B.75}$$

and for $so(8)$

$$\begin{aligned} \mathrm{Dim}(\varphi) = \frac{1}{4320} &(n_1+1)(n_2+1)(n_3+1)(n_4+1)(n_1+n_2+2)(n_2+n_3+2)(n_2+n_4+2) \\ &\times (n_1+n_2+n_3+3)(n_1+n_2+n_4+3)(n_2+n_3+n_4+3) \\ &\times (n_1+n_2+n_3+n_4+4)(n_1+2n_2+n_3+n_4+5). \end{aligned} \tag{B.76}$$

These formulas will be helpful when considering the particle representations appearing in the supergravity theory compactified on different manifolds.

B.9 Decomposition of groups

Given an algebra g with a subalgebra h, $h \subset g$, (we will refer to g as the *ambient* algebra in this context) we can usually embed h in g, so that irreducible representations of g break up into several irreducible representations of h. These decompositions are called *branching rules*. These rules tell us how a state in the original system, with g symmetry, gets organised into states labeled by some h symmetry. This can be written as

$$\Lambda \mapsto \bigoplus_{M \in P_+} b_{\Lambda M} M, \tag{B.77}$$

where Λ is an irrep of g, M an irrep of h, P_+ is the set of highest weights for irreducible representations, and $b_{\Lambda M}$ are called the branching coefficients. These coefficients gives the multiplicity of M under the decomposition of Λ. A specific branching can also be characterised by a *projection matrix*. If we for example have an irrep of $su(3)$ given by $\begin{pmatrix} n_1 & n_2 \end{pmatrix}$, and a corresponding projection matrix, $\mathcal{P}$, for how an irrep of $su(2)$ is embedded in $su(3)$, then we get the branching rules by writing

$$\mathcal{P} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} \mathcal{P}_1 & \mathcal{P}_2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = (\mathcal{P}_1 n_1 + \mathcal{P}_2 n_2) = (n_{su(2)}). \tag{B.78}$$

There is also a useful rule stating that, if

$$\Lambda \mapsto \bigoplus_{M \in P_+} b_{\Lambda M} M, \quad \text{and} \quad \Xi \mapsto \bigoplus_{\Pi \in P_+} b_{\Xi \Pi} \Pi, \tag{B.79}$$

then

$$\Lambda \otimes \Xi \mapsto \bigoplus_{M,\Pi} b_{\Lambda M} b_{\Xi \Pi} M \otimes \Pi. \tag{B.80}$$

One can classify the different embeddings according to the properties of the smaller algebra under the decomposition. We will most often concern ourselves with the case where h is a maximal subalgebra of g. This means that there are no intermediate steps in the embedding, i.e., we can not write $h \hookrightarrow l \hookrightarrow g$, for some other algebra $l \subset g$. The rules for non-maximal subalgebras follow directly by applying the embedding of the maximal subalgebras in steps.

If all step-operators of h are also step-operators of g we call h for a *regular* subalgebra of g, otherwise we call it a *special* subalgebra.

There is a very simple prescription for finding all regular maximal subalgebras of an algebra. You start from the Dynkin diagram of the ambient algebra and add a node corresponding to minus the highest root. This will yield the so called extended Dynkin diagrams of g, see Eq.(B.40). One then simply removes any node corresponding to some simple root, and receives the Dynkin diagram of a regular maximal subalgebra. Note that if $g = A_r$ one needs to discard two roots in order to find a proper maximal regular subalgebra, since the removal of one node only gives back the original diagram. There are also a few exceptions (five to be exact) when working with the exceptional algebras (sounds almost obvious when you write it out like this), resulting in non-maximal subalgebras [48]. This trick does not work for the special subalgebras. There we instead use that if g is simple, then any n-dimensional representation of g gives us an embedding of g into $sl(n)$ [48]. If the corresponding representation space is selfconjugate and symplectic we can embed it in $sp(n)$, and if it is selfconjugate and orthogonal we can embed it in $so(n)$. Both of these embeddings are special and (except for a few cases) maximal. If g is not simple one can use the matrix representation to look for the embeddings. This is done in [48]. The exceptional algebras must be done on a case-by-case basis while the classical Lie algebras gives some more general results.

C

An introduction to supersymmetry

Johannes Aspman and Adrian Padellaro

This appendix was written together with Adrian Padellaro[a] in a joint effort to gain the necessary background knowledge needed for our separate master's projects. The text serves as an introduction to the subject of supersymmetry. This is intended for the average master's level student of theoretical physics unfamiliar with the subject, and therefore includes a lot of explicit calculations, often more than has been found anywhere else. Since this appendix is written with the intent of being somewhat self-contained there will perhaps be some overlap with the rest of the thesis, this may also lead to some contrasts in conventions, but we will always try and note the differences.

The first section gives a short motivation to why supersymmetry is interesting. We then introduce the simplest possible supersymmetric theory, the Wess-Zumino model. In Sections C.3 and C.4 we discuss representation theory for supersymmetry and $SO(N)$ respectively. After this we introduce two other formalisms more suitable when interactions are included, namely the notions of superfields and superspace, Section C.6 then shows how we can express the coordinate transformations in superspace in a manifestly covariant way. In the last two sections we construct two supersymmetric theories of great importance, super-Yang-Mills in ten dimensions and supergravity in eleven dimensions.

C.1 Introduction and motivation – Why SUSY?

Supersymmetry, or SUSY for short, is a symmetry between fermions and bosons. It relates every boson to a fermion superpartner, and vice versa. The partner of the known bosons are named after the boson with the added suffix *-ino*, for example the photon has as its partner the photino, and so on. The partner of the fermions get a prefix *s* (for scalar, or super, depending on who you are asking), so we have the selectron, the sneutrino, the squark and so on.

In this appendix we will introduce some of the most important aspects of supersymmetry. We will also show how one can incorporate these ideas into supersymmetric theories, such as super-Yang-Mills and supergravity. But first we will try to motivate why it is important, or at least interesting, to study supersymmetry. There are of course a lot of reasons, and we will only mention the ones we find most significant. Although supersymmetry is interesting for both mathematicians and physicists we will focus mostly on the physical applications.

There are three fundamental constants of physics. The speed of light, c, Planck's constant, $\hbar$, and Newton's constant, G. From these one can create the Planck units, which

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should then set the natural scale for fundamental interactions. We get [15]

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35}\text{m}, \tag{C.1}$$

$$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.39 \times 10^{-44}\text{s}, \tag{C.2}$$

$$m_P = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19}\text{GeV/c}^2. \tag{C.3}$$

In the standard model particles acquire their mass by interacting with the Higgs field [52]. The renormalisation of the Higgs interactions gives a large contribution to the mass of the Higgs boson, and this leads to the conclusion that the mass of the Higgs boson should be the greatest mass scale available. However, in 2012 the Higgs boson was experimentally found to have a mass of only around 125GeV/c^2, [53], much lighter than the Planck mass. This discrepancy between the Higgs mass and the Planck mass is called the hierarchy problem of the electroweak scale.

In supersymmetry the hierarchy problem is solved since the superpartners cancel the Planck scale quantum corrections. This is because there is a relative sign present when doing calculations involving fermion loops as opposed to boson loops in quantum field theory, and imposing supersymmetry forces the two loop corrections to exactly cancel each other out [52].

One may note in this context that there is another vastly different scale in Nature, namely the neutrino mass scale, which happens, for unknown reasons, to be very close to the dark energy scale, deduced from the observed value of the cosmological constant, Λ. This is briefly discussed in Chapter 3.

The way in which supersymmetry cancels UV divergences in loop correction calculations has made it possible to construct more finite quantum fields theories, or at least it has been shown that they are finite up to higher order correction than before. It also led to the formulation of certain non-renormalisation theorems giving some insight into whether a theory is renormalisable or not. For instance, one theorem states that as long as supersymmetry is broken a particle who is massless at tree level will still be massless at any finite order of perturbation theory [54].

One theorem also states that if the classical potential vanishes at some point in field space, the effective potential will vanish at that point to all finite orders of perturbation [54].

Another aspect of supersymmetry that involves the standard model is the question of the unification of the three gauge coupling constants which set the scales of three of the fundamental forces, the electromagnetic, the strong and the weak force. These coupling constants are energy dependent and some decades ago, before more precise measurements had been done, they seemed to be able to meet at a certain energy scale. For many years people tried to find a way to unify these forces into one at this scale. Later on when the measurements became more precise it was found that they would not actually meet at one point but instead cross each other at different energies [52].

But, if one includes supersymmetry and the extra superparticles the energy dependence of the coupling constants gets modified. It has been found that this leads to the existence of a unique point where all three coupling constants meet, making it possible to unify the three forces into one single gauge group [52].

The two superpartners of supersymmerty should be of equal mass, but we have not found any superpartners to the known particles at these mass scales. This means that, if supersymmetry is a true symmetry of nature, it must be spontaneously broken in our present Universe.

We consider a single conserved supercharge, which is a spinor, Q_α. As we will see later, the superalgebra is given by [54]

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu. \tag{C.4}$$

This means that the Hamiltonian

$$H = |Q|^2. \tag{C.5}$$

It follows that if supersymmetry is not broken, i.e., that the Q_α annihilate the vacuum, the energy of the vacuum is zero. Since of course

$$H|0\rangle = |Q|^2|0\rangle = 0. \tag{C.6}$$

However, if supersymmetry is spontaneously broken, i.e. $Q_\alpha|0\rangle = \alpha|0\rangle \neq 0$, we find

$$\langle 0|H|0\rangle = \langle 0||Q|^2|0\rangle = \alpha^2 > 0. \tag{C.7}$$

So the energy of the vacuum is positive [54].

As discussed in this thesis the Weak Gravity Conjecture of Ooguri & Vafa, [11], has been shown to imply that non-supersymmetric AdS vacua are unstable.

This could mean that our present Universe possibly is in an unstable state, and should then be moving towards a state where it will have supersymmetry. It could of course be in a local minimum of the potential, but due to quantum tunneling it should still be able to escape and move towards a supersymmetric global ground state.

Lastly, one should also mention that our leading candidate for a theory unifying all the fundamental forces of Nature, string/M-theory, in its present formulation, needs supersymmetry on the world sheet to work [15]. So supersymmetry seems to be a deep fundamental symmetry of Nature.

These are just a few of the reasons why one should study supersymmetry, and hopefully you have been motivated to keep reading this text.

C.2 The Wess-Zumino model

To introduce supersymmetry we start by studying one of the simplest models, called the Wess-Zumino model. This was the first example of a four-dimensional interacting supersymmetric model, introduced by Julius Wess and Bruno Zumino in 1974 [55]. This will involve a lot of explicit calculations and manipulations, but since this is something that you encounter all the time in supersymmetry, we will go through it in detail in this section to familiarise ourselves with the framework.

Consider a field theory consisting of one complex scalar field and one Dirac spinor field. For simplicity we can split the complex scalar field into

$$\phi = \frac{1}{\sqrt{2}}(A + iB), \tag{C.8}$$

where now both A and B are real scalar fields. We thus have the kinetic Lagrangian

$$L_0 = -\frac{1}{2}\partial_a A \partial^a A - \frac{1}{2}\partial_a B \partial^a B + \frac{i}{2}\bar{\psi}\gamma^a\partial_a\psi, \tag{C.9}$$

where we employ the convention that latin indices belong to ordinary spacetime, while greek indices will be used for spinors. This Lagrangian is invariant under the supersymmetry transformations defined by

$$\begin{aligned}
\delta A &= -i\bar{\varepsilon}\psi, \\
\delta B &= -\bar{\varepsilon}\gamma^5\psi, \\
\delta\psi &= \gamma^a\partial_a(A - i\gamma^5 B)\varepsilon, \\
\delta\bar{\psi} &= -\bar{\varepsilon}\partial_a(A - i\gamma^5 B)\gamma^a,
\end{aligned} \tag{C.10}$$

where ε is an anti-commuting (Grassmann) parameter called the *supersymmetry parameter* [54]. These transformations will be derived later when we talk about superspace, but for now we simply state them and check that the given Lagrangian really is invariant under them. We also see that these transforms fermions into bosons and bosons into fermions, which is exactly what supersymmetry represents.

If we transform the Lagrangian accordingly we get

$$\begin{aligned}
\delta L_0 &= -\partial_a\delta A\partial^a A - \partial_a\delta B\partial^a B + \frac{i}{2}\delta\bar{\psi}\gamma^a\partial_a\psi + \frac{i}{2}\bar{\psi}\gamma^a\partial_a\delta\psi \\
&= \delta A\Box A + \delta B\Box B + \frac{i}{2}\delta\bar{\psi}\gamma^a\partial_a\psi + \frac{i}{2}\bar{\psi}\gamma^a\partial_a\delta\psi \\
&= -i\bar{\varepsilon}\psi\Box A - \bar{\varepsilon}\gamma^5\psi\Box B - \frac{i}{2}\bar{\varepsilon}\partial_a(A - i\gamma^5 B)\gamma^a\gamma^b\partial_b\psi + \frac{i}{2}\bar{\psi}\gamma^a\partial_a\gamma^b\partial_b(A - i\gamma^5 B)\varepsilon,
\end{aligned} \tag{C.11}$$

where we used partial integration in the second step and plugged in the given transformations in the last step. We now use that $\gamma^a\gamma^b = \eta^{ab} + \gamma^{ab}$ and that γ^{ab} is anti-symmetric in a and b while $\partial_a\partial_b$ is symmetric, which means that we can drop terms where these two multiply each other. This gives us

$$\delta L_0 = -i\bar{\varepsilon}\psi\Box A - \bar{\varepsilon}\gamma^5\psi\Box B - \frac{i}{2}\bar{\varepsilon}\partial^a(A - i\gamma^5 B)\partial_a\psi - \frac{i}{2}\bar{\varepsilon}\partial_a(A - i\gamma^5 B)\gamma^{ab}\partial_b\psi + \frac{i}{2}\bar{\psi}\Box(A - i\gamma^5 B)\varepsilon. \tag{C.12}$$

Then we perform partial integration, again dropping the γ^{ab} term because of symmetry considerations, leading to

$$\delta L = i\Box A(-\bar{\varepsilon}\psi + \frac{1}{2}\bar{\varepsilon}\psi + \frac{1}{2}\bar{\psi}\varepsilon) + \Box B(-\bar{\varepsilon}\gamma^5\psi + \frac{1}{2}\bar{\varepsilon}\gamma^5\psi + \frac{1}{2}\bar{\psi}\gamma^5\varepsilon). \tag{C.13}$$

Now, for the Lagrangian to be invariant under supersymmetry we want this to vanish, and we see that this requires

$$\begin{aligned}
\bar{\psi}\varepsilon &= \bar{\varepsilon}\psi, \\
\bar{\psi}\gamma^5\varepsilon &= \bar{\varepsilon}\gamma^5\psi,
\end{aligned} \tag{C.14}$$

but this is exactly the demands on a Majorana spinor since we then have [54]

$$\bar{\psi} = \psi^T C, \qquad C^T = -C, \tag{C.15}$$

which implies that we can do a Majorana flip

$$\bar{\psi}\varepsilon = \psi^T C\varepsilon = -\varepsilon^T C^T\psi = \varepsilon^T C\psi = \bar{\varepsilon}\psi, \tag{C.16}$$

and similarly for $\bar{\psi}\gamma^5\varepsilon$ since $C_{\alpha\beta}$ and $(\gamma^5 C)_{\alpha\beta}$ are both anti-symmetric (Section C.4 will discuss this in more detail). This means that if we take the spinor field to be Majorana the Lagrangian will be invariant under the supersymmetry given by (C.10).

Now we want to find the algebra of the supersymmetry transformations. We start by introducing two different transformations δ_1 and δ_2 and then check how the commutator of these act on the different fields. Starting with the scalars, e.g. A (the case for B is equivalent), we have

$$\begin{aligned}\big[\delta_1,\delta_2\big]A &= \delta_1(-i\bar{\varepsilon}_2\psi)-\delta_2(-i\bar{\varepsilon}_1\psi) = -i\bar{\varepsilon}_2(\delta_1\psi)+i\bar{\varepsilon}_1(\delta_2\psi)\\ &= -i\bar{\varepsilon}_2(\gamma^a\partial_a(A-i\gamma^5 B)\varepsilon_1)+i\bar{\varepsilon}_1(\gamma^a\partial_a(A-i\gamma^5 B)\varepsilon_2).\end{aligned} \tag{C.17}$$

As above, we can do a Majorana flip which tells us that

$$\begin{aligned}\bar{\varepsilon}_1\gamma^a\varepsilon_2 &= -\bar{\varepsilon}_2\gamma^a\varepsilon_1,\\ \bar{\varepsilon}_1\gamma^a\gamma^5\varepsilon_2 &= \bar{\varepsilon}_2\gamma^a\gamma^5\varepsilon_1.\end{aligned} \tag{C.18}$$

Plugging this into the expression above we can see that the B-terms will cancel and the A-terms will add, so in the end we have

$$[\delta_1,\delta_2]A = -2i\bar{\varepsilon}_2\gamma^a\varepsilon_1\partial_a A. \tag{C.19}$$

We now define

$$\delta \equiv -i\bar{\varepsilon}Q = -i\bar{\varepsilon}^\alpha Q_\alpha = -i\varepsilon_\alpha Q^\alpha = +i\varepsilon_\alpha\bar{Q}^\alpha. \tag{C.20}$$

Using this in the above expression for the commutator we get

$$(\delta_1\delta_2-\delta_2\delta_1)A = (-i)^2(\bar{\varepsilon}_1^\alpha\bar{\varepsilon}_2^\beta)\{Q_\alpha,Q_\beta\}A = -2i\bar{\varepsilon}_2^\alpha\varepsilon_{1\beta}(\gamma^a)_\alpha{}^\beta\partial_a A, \tag{C.21}$$

but

$$-\bar{\varepsilon}_1^\alpha\bar{\varepsilon}_2^\beta Q_\beta = \bar{\varepsilon}_2^\alpha\bar{\varepsilon}_1^\beta Q_\beta = -\bar{\varepsilon}_2^\alpha\varepsilon_{1\beta}\bar{Q}^\beta, \tag{C.22}$$

so we must have

$$\{Q_\alpha,\bar{Q}^\beta\}A = -2i(\gamma^a)_\alpha{}^\beta\partial_a A. \tag{C.23}$$

Using the Weyl notation with dotted indices, which will become crucial in the later discussions on superspace and supergravity, this can be written as [54]

$$\{Q_\alpha,\bar{Q}_{\dot{\beta}}\}A = -2i\sigma^a_{\alpha\dot{\beta}}\partial_a A. \tag{C.24}$$

Now we want to redo this for the spinor field, ψ, but first we will have a short Fierz interlude. Fierzing is a way of expanding a spinor bilinear form in the gamma basis, i.e., for a bilinear of two spinors $\psi\chi$ we write

$$\psi_\alpha\chi_\beta = x_0 C_{\alpha\beta}\bar{\psi}\chi + x_1\gamma^a_{\alpha\beta}\bar{\psi}\gamma_a\chi+\dots, \tag{C.25}$$

for some coefficients x_i [52]. To find these x_i's we simply check term by term

$$C^{\alpha\beta}\psi_\alpha\chi_\beta = \bar{\psi}\chi = x_0 C^{\alpha\beta}C_{\alpha\beta}\bar{\psi}\chi = -4x_0\bar{\psi}\chi \implies x_0 = -\frac{1}{4}, \tag{C.26}$$

$$(\gamma_b)^{\alpha\beta}\psi_\alpha\chi_\beta = \bar{\psi}\gamma_b\chi = x_1\gamma_b^{\alpha\beta}\gamma^a_{\alpha\beta}\bar{\psi}\gamma_a\chi = 4\delta^a_b x_1\bar{\psi}\gamma_a\chi \implies x_1 = \frac{1}{4}. \tag{C.27}$$

$$(\gamma_{bc})^{\alpha\beta}\psi_\alpha\chi_\beta = \bar{\psi}\gamma_{bc}\chi = x_2\mathrm{Tr}(\gamma_{bc}\gamma^{ad})\bar{\psi}\gamma_{ad}\chi = -8x_2\delta^{ad}_{bc}\bar{\psi}\gamma_{ad}\chi \implies x_2 = -\frac{1}{8}, \tag{C.28}$$

$$\begin{aligned}(\gamma_b\gamma^5)^{\alpha\beta}\psi_\alpha\chi_\beta &= \bar{\psi}\gamma_b\gamma^5\chi = x_3(\gamma_b\gamma^5)^{\alpha\beta}(\gamma^a\gamma^5)_{\alpha\beta}\bar{\psi}\gamma_a\gamma_5\chi = -x_3\mathrm{Tr}(\gamma_b\gamma^5\gamma^a\gamma^5)\bar{\psi}\gamma_a\gamma_5\chi\\ &= 4\delta^a_b x_3\bar{\psi}\gamma_a\gamma_5\chi \implies x_3 = \frac{1}{4},\end{aligned} \tag{C.29}$$

$$(\gamma^5)^{\alpha\beta}\psi_\alpha\chi_\beta = \bar{\psi}\gamma^5\chi = x_4(-\mathrm{Tr}\gamma^5\gamma^5)\bar{\psi}\gamma^5\chi = -4x_4\bar{\psi}\gamma^5\chi \implies x_4 = -\frac{1}{4}, \tag{C.30}$$

So this means that we have

$$\psi_\alpha \chi_\beta = -\frac{1}{4}C_{\alpha\beta}\bar{\psi}\chi + \frac{1}{4}(\gamma^a)_{\alpha\beta}\bar{\psi}\gamma_a\chi - \frac{1}{8}(\gamma^{ab})_{\alpha\beta}\bar{\psi}\gamma_{ab}\chi + \frac{1}{4}(\gamma^a\gamma^5)_{\alpha\beta}\bar{\psi}\gamma_a\gamma_5\chi - \frac{1}{4}(\gamma^5)_{\alpha\beta}\bar{\psi}\gamma^5\chi. \tag{C.31}$$

This will very soon be needed when we now go on to study the commutation of the transformations acting on the spinor

$$\begin{aligned}\left[\delta_1, \delta_2\right]\psi_\alpha &= \delta_1(\gamma^a\partial_a(A - i\gamma^5 B)\varepsilon_2)_\alpha - (1 \leftrightarrow 2) = (\gamma^a\partial_a(-i(\bar{\varepsilon}\psi) + i\gamma^5(\bar{\varepsilon}\gamma^5\psi))\varepsilon_2)_\alpha - (1 \leftrightarrow 2)\\ &= -i(\bar{\varepsilon}_1\partial_a\psi(\gamma^a\varepsilon_2)_\alpha - \bar{\varepsilon}_1\gamma^5\partial_a\psi(\gamma^a\gamma^5\varepsilon_2)_\alpha) - (1 \leftrightarrow 2).\end{aligned} \tag{C.32}$$

We see that we can use

$$\varepsilon_2^{(\alpha}\varepsilon_1^{\beta)} = \frac{1}{4}(\gamma^a)^{\alpha\beta}\bar{\varepsilon}_2\gamma_a\varepsilon_1 - \frac{1}{8}(\gamma^{ab})^{\alpha\beta}\bar{\varepsilon}_2\gamma_{ab}\varepsilon_1 \tag{C.33}$$

from the Fierz identity to rewrite this. We begin with the first term

$$\begin{aligned}\bar{\varepsilon}_1\partial_a\psi(\gamma^a\varepsilon_2)_\alpha - \bar{\varepsilon}_2\partial_a\psi(\gamma^a\varepsilon_1)_\alpha &= \varepsilon_{1\beta}\partial_a\psi^\beta(\gamma^a)_\alpha^\gamma\varepsilon_{2\gamma} - \varepsilon_{2\beta}\partial_a\psi^\beta(\gamma^a)_\alpha^\gamma\varepsilon_{1\gamma} = -2\varepsilon_{1(\beta}\varepsilon_{2\gamma)}\partial_a\psi^\beta(\gamma^a)_\alpha^\gamma\\ &= -\frac{1}{2}\bar{\varepsilon}_1\gamma^b\varepsilon_2(\gamma_b)_{\beta\gamma}\partial_a\psi^\beta(\gamma^a)_\alpha^\gamma + \frac{1}{4}\bar{\varepsilon}_1\gamma^{bc}\varepsilon_2(\gamma_{bc})_{\beta\gamma}\partial_a\psi^\beta(\gamma^a)_\alpha^\gamma\\ &= -\frac{1}{2}\bar{\varepsilon}_1\gamma^b\varepsilon_2(\gamma^a\gamma_b\partial_a\psi)_\alpha + \frac{1}{4}\bar{\varepsilon}_1\gamma^{bc}\varepsilon_2(\gamma^a\gamma_{bc}\partial_a\psi)_\alpha,\end{aligned} \tag{C.34}$$

where we have plugged in the result of Eq.(C.33) in the third step.

The second term is

$$\begin{aligned}-\bar{\varepsilon}_1\gamma^5\partial_a\psi(\gamma^a\gamma^5\varepsilon_2)_\alpha - (1 \leftrightarrow 2) &= -\varepsilon_{1\beta}(\gamma^5\partial_a\psi)^\beta(\gamma^a\gamma^5)_\alpha{}^\gamma\varepsilon_{2\gamma} - (1 \leftrightarrow 2)\\ &= 2\varepsilon_{1(\beta}\varepsilon_{2\gamma)}(\gamma^5\partial_a\psi)^\beta(\gamma^a\gamma^5)_\alpha{}^\gamma.\end{aligned} \tag{C.35}$$

The Fierz identity of Eq.(C.33) can be used again, and gives us

$$\begin{aligned}&\frac{1}{2}\bar{\varepsilon}_1\gamma^b\varepsilon_2(\gamma_b)_{\beta\gamma}(\gamma^5\partial_a\psi)^\beta(\gamma^a\gamma^5)_\alpha{}^\gamma - \frac{1}{4}\bar{\varepsilon}_1\gamma^{bc}\varepsilon_2(\gamma_{bc})_{\beta\gamma}(\gamma^5\partial_a\psi)^\beta(\gamma^a\gamma^5)_\alpha{}^\gamma\\ &= \frac{1}{2}\bar{\varepsilon}_1\gamma^b\varepsilon_2(\gamma^a\gamma^5\gamma_b\gamma^5\partial_a\psi)_\alpha - \frac{1}{4}\bar{\varepsilon}_1\gamma^{bc}\varepsilon_2(\gamma^a\gamma^5\gamma_{bc}\gamma^5\partial_a\psi)_\alpha.\end{aligned} \tag{C.36}$$

We can move the γ^5's next to each other, picking up a sign in the first term but not in the second one. The γ^5's then square to one. Adding these results together we have

$$\begin{aligned}\left[\delta_1, \delta_2\right]\psi_\alpha = -i\Big(&-\frac{1}{2}\bar{\varepsilon}_1\gamma^b\varepsilon_2(\gamma^a\gamma_b\partial_a\psi)_\alpha + \frac{1}{4}\bar{\varepsilon}_1\gamma^{bc}\varepsilon_2(\gamma^a\gamma_{bc}\partial_a\psi)_\alpha\\ &-\frac{1}{2}\bar{\varepsilon}_1\gamma^b\varepsilon_2(\gamma^a\gamma_b\partial_a\psi)_\alpha - \frac{1}{4}\bar{\varepsilon}_1\gamma^{bc}\varepsilon_2(\gamma^a\gamma_{bc}\partial_a\psi)_\alpha\Big)\\ = i\bar{\varepsilon}_1&\gamma^b\varepsilon_2(\gamma^a\gamma_b\partial_a\psi)_\alpha = 2i\bar{\varepsilon}_1\gamma^a\varepsilon_2\partial_a\psi_\alpha - i\bar{\varepsilon}_1\gamma^b\varepsilon_2\gamma_b\gamma^a\partial_a\psi_\alpha\\ = -2i&\bar{\varepsilon}_2\gamma^a\varepsilon_1\partial_a\psi_\alpha + i\bar{\varepsilon}_2\gamma^b\varepsilon_1\gamma_b\gamma^a\partial_a\psi_\alpha.\end{aligned} \tag{C.37}$$

The first term is just what we got for the scalar fields in Eq. (C.19), while we recognise the Dirac equation in the end of the second term. This means that we have found our supersymmetry algebra, but only on-shell. Could this problem be solved?

How about interactions? There may be mass and potential terms that we have not included yet. However, we have seen that it is a rather cumbersome calculation to add terms and then check if we still have supersymmetry. Is there an easier way to do it? This together with the problem of only being on-shell for the algebra is solved when we introduce the concept of superspace in Section C.5.

C.3 Representation theory for D=4 supersymmetry

The Wess-Zumino model contains three fields related by supersymmetry transformations. It is one example of a supermultiplet. We are interested in exploring other possible combinations of fields which can form a supersymmetric Lagrangian. In group theoretical language we are looking for (irreducible) representations of the supersymmetry algebra. In general we can have more than one supersymmetry, leading to larger multiplets.

The supersymmetry algebra is [54]

$$\{Q^i_\alpha, \bar{Q}_{\dot{\beta}j}\} = 2\sigma^a_{\alpha\dot{\beta}} P_a \delta^i_j, \tag{C.38}$$

where the indices $i, j, \dots$ run from $1, \dots, \mathcal{N}$, for $\mathcal{N}$ different supersymmetry generators. In this section we will construct irreducible representations of this algebra for one-particle states where, because $P^2 = -m^2$ is a Casimir operator, all particle states have equal mass [54]. The massive as well as the massless case will be studied.

If we consider the massive case we can boost to the rest frame, where $P_a = (-m, 0, 0, 0)$. Recalling that $\sigma^a = (\sigma^0, \sigma^1, \sigma^2, \sigma^3) = (-\mathbb{1}, \sigma^i)$ (note that this convention differs from the rest of the thesis), where σ^i are the Pauli matrices, we have

$$\sigma^a P_a = -m\sigma^0 = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}. \tag{C.39}$$

The algebra in the rest frame becomes

$$\{Q^i_\alpha, \bar{Q}_{\beta j}\} = 2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}_{\alpha\beta} \delta^i_j = 2m\delta_{\alpha\beta}\delta^i_j. \tag{C.40}$$

Note that we have discarded the dotted indices. Boosting to the rest frame corresponds to a partial gauge fixing, for which the remaining transformations that keep the gauge choice invariant is called the Little group [54]. In our case the Little group is $SO(3)$, rotations of the space components of P_a. But both representations transform identically under the Little group and therefore dotted and undotted indices correspond to the same representation.

We can now define a set of operators

$$\begin{aligned} a^i_1 &\equiv \frac{1}{\sqrt{2m}} Q^i_1, \qquad & a^\dagger_{1i} &\equiv \frac{1}{\sqrt{2m}} \bar{Q}_{1i}, \\ a^i_2 &\equiv \frac{1}{\sqrt{2m}} Q^i_2, \qquad & a^\dagger_{2i} &\equiv \frac{1}{\sqrt{2m}} \bar{Q}_{2i}. \end{aligned} \tag{C.41}$$

With this normalisation these operators fulfill the algebra of creation and annihilation operators as we are used to in quantum mechanics, i.e.

$$\begin{aligned} \{a^i_\alpha, a^j_\beta\} &= \{a^\dagger_{\alpha j}, a^\dagger_{\beta j}\} = 0, \\ \{a^i_\alpha, a^\dagger_{\beta j}\} &= \delta^i_j \delta_{\alpha\beta}. \end{aligned} \tag{C.42}$$

The representations (all the one particle states) of such an algebra are usually constructed from a vacuum. The vacuum $|\Omega\rangle$ is defined by the condition

$$a^i_\beta |\Omega\rangle = 0 \qquad \forall i, \beta. \tag{C.43}$$

We can build all the states in the Hilbert space by acting with creation operators $a^\dagger_{\alpha i}$ on the vacuum. The creation operators change the magnetic quantum number (i.e. the L_z

eigenvalue) of the states they act on by $\pm 1/2$. More specifically $a^\dagger_{1i}$ and $a^\dagger_{2i}$ raise, and lower the quantum number, respectively. The $2\mathcal{N}$ different creation operators generate a total of $2^{2\mathcal{N}}$ states.

To analyse the complete spectrum of particles for any given $\mathcal{N}$ and vacuum with spin S we will employ a group theoretical approach. By defining a new set of operators

$$\begin{aligned}
\Gamma^l &= \frac{1}{\sqrt{2}}\left[a_1^l + a^\dagger_{1l}\right], \quad l = 1, \ldots, \mathcal{N} \\
\Gamma^{\mathcal{N}+l} &= \frac{1}{\sqrt{2}}\left[a_2^l + a^\dagger_{2l}\right], \\
\Gamma^{2\mathcal{N}+l} &= \frac{i}{\sqrt{2}}\left[a_1^l - a^\dagger_{1l}\right], \\
\Gamma^{3\mathcal{N}+l} &= \frac{i}{\sqrt{2}}\left[a_2^l - a^\dagger_{2l}\right]
\end{aligned} \tag{C.44}$$

and a new index $r, t = 1, \ldots, 4\mathcal{N}$ these operators satisfy an $SO(4\mathcal{N})$ invariant algebra

$$\left\{\Gamma^r, \Gamma^t\right\} = \delta^{rt}. \tag{C.45}$$

This shows that our original algebra (C.42) spans a subset of $SO(4\mathcal{N})$ representations (remember that we originally had $2^{2\mathcal{N}}$ states but this algebra would generate $2^{4\mathcal{N}}$ states) [56].

However, physical states are characterised by their $SU(2)$ representation (spin). These are not manifest in the new algebra, instead we can define a third set of operators

$$\begin{aligned}
q^l_\alpha &= a^l_\alpha, \quad l = 1, \ldots, \mathcal{N} \\
q^{\mathcal{N}+l}_\alpha &= \sum_\beta \epsilon_{\alpha\beta} a^\dagger_{\beta l}
\end{aligned} \tag{C.46}$$

which satisfy [56]

$$\begin{aligned}
\left\{q^l_\alpha, q^l_\beta\right\} &= \left\{q^{\mathcal{N}+l}_\alpha, q^{\mathcal{N}+l}_\beta\right\} = 0 \\
\left\{q^l_\alpha, q^{\mathcal{N}+m}_\beta\right\} &= -\epsilon_{\alpha\beta}\delta^l_m \\
\left\{q^{\mathcal{N}+m}_\alpha, q^l_\beta\right\} &= \epsilon_{\alpha\beta}\delta^l_m.
\end{aligned} \tag{C.47}$$

These can be put in a more compact form by introducing indices $p, q = 1, \ldots, 2\mathcal{N}$ and the $2\mathcal{N} \times 2\mathcal{N}$ matrix

$$\Lambda^{pq} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}_{pq}. \tag{C.48}$$

With this notation the algebra becomes

$$\left\{q^p_\alpha, q^q_\beta\right\} = -\epsilon_{\alpha\beta}\Lambda^{pq}. \tag{C.49}$$

We have arrived at an algebra that is manifestly invariant under $SU(2) \times USp(2\mathcal{N})$ which is a subgroup of $SO(4\mathcal{N})$.

The fundamental multiplet (the multiplet constructed from an $S = 0$ vacuum) is given by decomposing the $2^{2\mathcal{N}}$ $SO(4\mathcal{N})$ states into $SU(2) \times USp(2\mathcal{N})$ states. The decomposition is given by [56]

$$2^{2\mathcal{N}} \to (\mathcal{N}+1, 1) + (\mathcal{N}, 2\mathcal{N}) + \cdots + (\mathcal{N}+1-k, [2\mathcal{N}]_k) + \cdots + (1, [2\mathcal{N}]_{\mathcal{N}}). \tag{C.50}$$

The first label is the dimension of the $SU(2)$ representation given by $D = 2S + 1$. The second label is the dimension of the totally anti-symmetric traceless representation of $USp(2\mathcal{N})$ with k indices, given by

$$\begin{aligned}
[M]_k &= \frac{M \dots (M+1-k)}{k!} - \frac{M \dots (M+3-k)}{(k-2)!}, \quad k > 2 \\
[M]_2 &= \frac{M \dots (M+1-k)}{k!} - 1.
\end{aligned}$$

We will demonstrate the method for $\mathcal{N} = 3$. Starting from an $S = 0$ vacuum the decomposition given by equation (C.50) is

$$(4,1) + (3,6) + (2,[6]_2) + (1,[6]_3) = (4,1) + (3,6) + (2,14) + (1,14).$$

If we gather the multiplet content in terms of spin in a table we have

S	Multiplicity
0	14
$\frac{1}{2}$	14
1	6
$\frac{3}{2}$	1

The multiplet constructed from an $S = 1/2$ ground state is given by the $S = 0$ multiplet using angular momentum addition. For example the fourteen $S = 1/2$ particles give rise to fourteen $S = 1$ and $S = 0$ particles when combined with the ground state. The complete decomposition is

$$\begin{aligned}
0 \otimes \frac{1}{2} &= \frac{1}{2}, \\
\frac{1}{2} \otimes \frac{1}{2} &= 1 \oplus 0, \\
1 \otimes \frac{1}{2} &= \frac{3}{2} \oplus \frac{1}{2}, \\
\frac{3}{2} \otimes \frac{1}{2} &= 2 \oplus 1.
\end{aligned} \tag{C.51}$$

In total the $S = 1/2$ multiplet is

S	Multiplicity
0	14
$\frac{1}{2}$	20
1	15
$\frac{3}{2}$	6
2	1

For the massless case there is no rest frame, so the best we can do is to pick a frame where $P_\mu = (-E, 0, 0, E)$. In this frame the SUSY algebra is

$$\{Q^i, \bar{Q}_j\} = 4E\delta^i_j.$$

After introducing the normalised operators, as for the massive case, the algebra becomes the algebra of $\mathcal{N}$ creation and annihilation operators. The operators are ladder operators of helicity rather than spin. Moreover we only have one type (for each supersymmetry) of creation operator. This means we only have one type of index to worry about when it comes to anti-symmetrisation. We introduce a vacuum, this time it is a helicity eigenstate,

$$a^i \left|\Omega_\lambda\right\rangle = 0, \qquad \forall i,$$

where λ denotes the helicity. When constructing the Hilbert space we have $\mathcal{N}$ anti-commuting operators that at most appear once, acting on the vacuum. Therefore we have a total of $2^{\mathcal{N}}$ states. A state with m operators has helicity $\lambda + m/2$ and the multiplicity is the combinatorial factor $\binom{\mathcal{N}}{m}$. For example in the case of $\mathcal{N} = 8$ starting from $\lambda = -2$ we have the following state content [54]

Helicity	Multiplicity
2	1
3/2	8
1	28
1/2	56
0	70
−1/2	56
−1	28
−3/2	8
−2	1.

The choice of $\mathcal{N} = 8$ and $\lambda = -2$ was not arbitrary. It is the unique maximal SUSY multiplet containing spin 2 but no higher spins. For larger $\mathcal{N}$ (or smaller λ) we would get particles with spin higher than 2. We can also see that the number of bosonic and fermionic states are equal, as required for supersymmetry.

The previous example is invariant under CPT, however most multiplets are not. To build a quantum field theory with such a multiplet it is required that one adds a CPT copy of the multiplet to the theory. For example, consider the case of $\mathcal{N} = 1$ starting from $\lambda = 3/2$. The multiplet is $(3/2, 2)$ but CPT requires that we also have $(-2, -3/2)$ states.

C.4 Majorana representation in $SO(1,3)$

For the Wess-Zumino model it turned out that we needed our spinors to be Majorana for the Lagrangian to be invariant under supersymmetry transformations. In this section we will review the construction of a Majorana representation of gamma matrices in four dimensions. We hope that this will make the transition to gamma matrices generalised to higher dimensions easier.

Gamma matrices are a set of matrices γ^μ that satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad \text{where we have chosen} \quad \eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{C.52}$$

If a set of matrices γ^μ satisfy the Clifford algebra then so will also the sets $(\gamma^\mu)^T$, $-(\gamma^\mu)^T$ and $-(\gamma^\mu)^\dagger$. It is then known that these matrices are related by similarity transformations. We define matrices A, C_+ and C_- by

$$A^{-1}\gamma^\mu A = -(\gamma^\mu)^\dagger \tag{C.53}$$

$$C_\pm^{-1}\gamma^\mu C_\pm = \pm(\gamma^\mu)^T \quad . \tag{C.54}$$

We can also find a matrix γ^5 with the properties $\left(\gamma^5\right)^2 = 1$, $\left\{\gamma^5, \gamma^\mu\right\} = 0$ and $\left(\gamma^5\right)^\dagger = \gamma^5$. Which is used to construct a projection matrix.

If we pick one of the matrices $C_\pm$ we can form a basis for all 4×4 matrices using γ^μ. The basis is composed of the matrices

$$\begin{aligned}
&C\\
&\gamma^\mu C\\
&\gamma^{[\mu}\gamma^{\nu]}C\\
&\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]}C\\
&\gamma^5 C.
\end{aligned}$$

If for example $(\gamma^\mu C)^T = -(\gamma^\mu C)$, it is sometimes said that γ^μ is anti-symmetric (or symmetric if there is a plus sign). But this is just abuse of notation and what is really considered is the matrix $\gamma^\mu C$. We expect a basis for 4×4 matrices to have 6 anti-symmetric and 10 symmetric elements.

We will now go ahead and show a way to build a Majorana (real) representation of gamma matrices via tensor products of Pauli matrices. We have four real 2x2 matrices at our disposal: σ^1, $\varepsilon = i\sigma^2$, σ^3 and $\mathbb{1}$. Let

$$\begin{aligned}
\gamma^0 &= \varepsilon\otimes\sigma^3 \Rightarrow \left(\gamma^0\right)^2 = -\mathbb{1}\\
\gamma^1 &= \sigma^1\otimes\sigma^3 \Rightarrow \left(\gamma^1\right)^2 = \mathbb{1}\\
\gamma^2 &= \sigma^3\otimes\sigma^3 \Rightarrow \left(\gamma^2\right)^2 = \mathbb{1}\\
\gamma^3 &= \mathbb{1}\otimes\sigma^1 \Rightarrow \left(\gamma^3\right)^2 = \mathbb{1}.
\end{aligned} \tag{C.55}$$

Since $(A\otimes B)(C\otimes D) = AC\otimes BD$ and we want all gamma matrices to anti-commute, the trick is to make sure that the Pauli matrices pairwise (i.e. each slot in the tensor products) anti-commute an odd number of times.

Now, from equations (C.53) and (C.54) we see that $A = C_- = \gamma^0$ in our representation because the matrices γ^μ are real. Moreover $C_-^T = -C_-$ which we will see is important for the construction of our basis. From equation (C.54) we could also find C_+, a matrix that commutes with γ^1, γ^2 and γ^3 but anti-commutes with γ^0. By trial and error one finds $C_+ = \gamma^1\gamma^2\gamma^3$, which just like C_- is anti-symmetric.

In general a spinor is said to be Majorana if $\bar\psi \equiv \psi^\dagger A = \psi^T C_-$. But in our representation $A = C_-$ and thus a spinor is Majorana in the Majorana representation if $\psi = \psi^*$. A Majorana spinor has half the degrees of freedom compared to a Dirac spinor because we have gone from two complex to two real degrees of freedom.[b]

There exists a second restriction one can put on Dirac spinors to reduce the degrees of freedom due to the matrix γ^5. These are called Weyl spinors, they can be constructed from Dirac spinors using a projection

$$\psi_\pm = \frac{1\pm\gamma^5}{2}\psi. \tag{C.56}$$

One may ask if it is possible for a spinor to be Majorana and Weyl at the same time. Since Majorana spinors are real it would then be necessary for γ^5 to be real as well. We could try to form γ^5 as $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, which indeed would be real. But then $\left(\gamma^5\right)^2 = -\mathbb{1} \neq \mathbb{1}$. The correct γ^5 for $SO(1,3)$ is $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ which is not real. We can therefore conclude that in four dimensions it is not possible for a spinor to be Majorana and Weyl at the same time [54].

[b] Note that although a Dirac spinor has four components they are not independent if they are solutions to the Dirac equation.

We previously observed that the choice of C corresponded to two different sets of basis. The fact that they are different and how they differ is important for supersymmetry. In particular we need to know the symmetry properties under transposition. Denote n anti-symmetrised gammas by $\gamma^{(n)} \equiv \gamma^{[\mu_1} \dots \gamma^{\mu_n]}$. If we transpose $\gamma^{(n)} C_\pm$ we get

$$\left(\gamma^{(n)} C_\pm\right)^T = (-1)^{\frac{n(n-1)}{2}} C_\pm^T \left(\gamma^T\right)^{(n)} = (-1)(\pm)^n (-1)^{\frac{n(n-1)}{2}} \gamma^{(n)} C_\pm \quad \text{for n>0.} \tag{C.57}$$

It is useful to make a table for the two choices of C. The result is Table C.1. For supersymmetry we need $\gamma^\mu C$ to be symmetric and $\gamma^5 C$ to be anti-symmetric, which corresponds to the choice $C = C_-$.

	$C = C_+$		$C = C_-$	
	Symmetric	Anti-symmetric	Symmetric	Anti-symmetric
C		1		1
$\gamma^{(1)} C$		4	4	
$\gamma^{(2)} C$	6		6	
$\gamma^{(3)} C$	4			4
$\gamma^5 C$		1		1

Table C.1: Number of symmetric and anti-symmetric matrices in four dimensions. The numbers in each cell show how many symmetric respectively anti-symmetric matrices you get from each class of matrices, for the two choices of basis. Note that in both cases they add up to 10 symmetric and 6 anti-symmetric matrices.

One could naively try to construct a basis using a symmetric C but then all symmetric matrices would become anti-symmetric and vice versa. This would give us 6 symmetric and 10 anti-symmetric matrices which can not constitute a basis. We will later see that the symmetry properties of C are dimension dependent.

C.4.1 Real representation of $SO(8)$

For the spinor representation of $SO(8) = SO(2 \cdot 4) \equiv D_4$ we are looking for $2^4 = 16$ dimensional matrices satisfying the Clifford algebra with Euclidean signature

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}. \tag{C.58}$$

Thus for all matrices γ^μ we have $(\gamma^\mu)^2 = 1$. This will end up making it so that all matrices are forced to be symmetric if we insist on keeping them real. In terms of tensor products we have

$$\begin{aligned}
\gamma^1 &= \sigma^1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \\
\gamma^2 &= \varepsilon \otimes \sigma^3 \otimes \sigma^1 \otimes \varepsilon \\
\gamma^3 &= \varepsilon \otimes \sigma^3 \otimes \sigma^3 \otimes \varepsilon \\
\gamma^4 &= \varepsilon \otimes \sigma^3 \otimes \varepsilon \otimes \mathbb{1} \\
\gamma^5 &= \varepsilon \otimes \sigma^1 \otimes \varepsilon \otimes \sigma^1 \\
\gamma^6 &= \varepsilon \otimes \sigma^1 \otimes \varepsilon \otimes \sigma^3 \\
\gamma^7 &= \varepsilon \otimes \varepsilon \otimes \mathbb{1} \otimes \mathbb{1} \\
\gamma^8 &= \varepsilon \otimes \sigma^1 \otimes \mathbb{1} \otimes \varepsilon,
\end{aligned} \tag{C.59}$$

and $\gamma^9 \equiv \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7\gamma^8 = \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$. If we solve for $C_\pm$ we find $C_+ = \mathbb{1}$ and $C_- = \gamma^9$. This gives us Table C.2. The important observation here is that γ^9 is in fact real. Therefore we could impose reality (Majorana) and Weyl constraints simultaneously.

	$C = C_+$		$C = C_-$	
	Symmetric	Anti-symmetric	Symmetric	Anti-symmetric
C	1		1	
$\gamma^{(1)}C$	8			8
$\gamma^{(2)}C$		28		28
$\gamma^{(3)}C$		56	56	
$\gamma^{(4)}C$	70		70	
$\gamma^{(5)}C$	56			56
$\gamma^{(6)}C$		28		28
$\gamma^{(7)}C$		8	8	
$\gamma^{9}C$	1		1	

Table C.2: Number of symmetric and anti-symmetric matrices in eight dimensions. The numbers in each cell show how many symmetric respectively anti-symmetric matrices you get from each class of matrices, for the two choices of basis. For both choices of C we get a total of 136 symmetric and 120 anti-symmetric matrices.

C.4.2 Real representation of $SO(1, 9)$

The spinor representation constructed from the Clifford algebra for $SO(1, 9)$ has dimension $2^5 = 32$ with one matrix satisfying $\left(\gamma^0\right)^2 = -\mathbb{1}$. The reason we first constructed the real representation of $SO(8)$ is that it is in fact simple to extend the construction to $SO(1, 9)$. Let γ^a be the eight matrices constructed previously for $SO(8)$, then the eleven matrices $\{\Gamma^a, \Gamma^{11}\}$ for $SO(1, 9)$ are

$$\begin{aligned} \Gamma^0 &= \varepsilon \otimes \mathbb{1}_{16} \\ \Gamma^a &= \sigma^1 \otimes \gamma^a \quad \text{for} \quad a = 1, \ldots, 8 \\ \Gamma^{11} &= \sigma^3 \otimes \mathbb{1}_{16} \\ \Gamma^9 &= \Gamma^0 \ldots \Gamma^8 \Gamma^{11} = -\sigma^1 \otimes \sigma^3 \otimes \mathbb{1}_8. \end{aligned} \tag{C.60}$$

This particular choice of Γ^{11} coincides with the usual projection matrix (which is real for $SO(1, 9)$). $C_- = \Gamma^0$ and $C_+ = \Gamma^1 \ldots \Gamma^9 = -\sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \varepsilon$ both of which are anti-symmetric. We see that spinors can be both Majorana and Weyl in $1 + 9$ dimensions. It is again useful to construct a table over the basis elements with their index structure and symmetry that can be used in calculations, the results are found in Table C.3. For example C_- is off-diagonal, composed of two 16-dimensional pieces $(C_-)_{\alpha\dot{\beta}}$ and $(C_-)_{\dot{\alpha}\beta}$.

	$C = C_-$	
	Symmetric	Anti-symmetric
$C_{\alpha\dot{\beta}}$		1
$\left(\Gamma^{(1)}C\right)_{\alpha\beta}$	10	
$\left(\Gamma^{(2)}C\right)_{\alpha\dot{\beta}}$	45	
$\left(\Gamma^{(3)}C\right)_{\alpha\beta}$		120
$\left(\Gamma^{(4)}C\right)_{\alpha\dot{\beta}}$		210
$\left(\Gamma^{(5)}C\right)_{\alpha\beta}$	252	
$\left(\Gamma^{(6)}C\right)_{\alpha\dot{\beta}}$	210	
$\left(\Gamma^{(7)}C\right)_{\alpha\beta}$		120
$\left(\Gamma^{(8)}C\right)_{\alpha\dot{\beta}}$		45
$\left(\Gamma^{(9)}C\right)_{\alpha\beta}$	10	
$\left(\Gamma^{11}C\right)_{\alpha\dot{\beta}}$	1	

Table C.3: Number of symmetric and anti-symmetric matrices in ten dimensions. The numbers in each cell show how many symmetric respectively anti-symmetric matrices you get from each class of matrices, and their index structure. An analogous table exists where dotted and undotted indices are exchanged. The total number of matrices is 528 symmetric and 496 anti-symmetric, as expected for a basis of 32-dimensional matrices.

C.4.3 Real representation of $SO(1,10)$

We can follow a similar recipe to extend the $SO(8)$ representation to $SO(1,10)$ which is odd-dimensional. If we include γ^9 we have nine anti-commuting matrices which we can extend to eleven by following the same recipe as for $SO(1,9)$. The list of gamma matrices in terms of tensor products is then

$$\begin{aligned}
\Gamma^0 &= \varepsilon \otimes \mathbb{1}_{16} \\
\Gamma^a &= \sigma^1 \otimes \gamma^a \quad \text{for} \quad a = 1, \ldots, 9 \\
\Gamma^{10} &= \sigma^3 \otimes \mathbb{1}_{16}.
\end{aligned} \tag{C.61}$$

There is no projection matrix in odd dimensions. If we tried to construct one in the naive way it would just be proportional to the identity matrix. In 1+10 dimensions we still have $C_- = \Gamma^0$. When constructing the table of basis elements in odd dimensions we have to be careful. Since we no longer have a projection matrix the second half of the table does not constitute new basis elements (they are not linearly independent). The result is Table C.4.

	$C = C_-$	
	Symmetric	Anti-symmetric
$C_{\alpha\beta}$		1
$\left(\Gamma^{(1)}C\right)_{\alpha\beta}$	11	
$\left(\Gamma^{(2)}C\right)_{\alpha\beta}$	55	
$\left(\Gamma^{(3)}C\right)_{\alpha\beta}$		165
$\left(\Gamma^{(4)}C\right)_{\alpha\beta}$		330
$\left(\Gamma^{(5)}C\right)_{\alpha\beta}$	462	

Table C.4: Number of symmetric and anti-symmetric matrices in eleven dimensions dimensions. The numbers in each cell show how many symmetric respectively anti-symmetric matrices you get from each class of matrices, and their index structure. The total number of matrices is 528 symmetric and 496 anti-symmetric, as expected for a basis of 32-dimensional matrices.

C.5 Superfields in four dimensions

A superfield is the extension of a regular field with bosonic coordinates to fields with fermionic or Grassmann coordinates. The new coordinates are $Z^A = (x^a, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$ with θ_α and $\bar{\theta}^{\dot{\alpha}}$ being anti-commuting Grassmann numbers. Coordinates are also parameters for a group element [54]

$$G(x,\theta,\bar{\theta}) = \exp\left\{i\left(-x^a P_a + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right)\right\}. \tag{C.62}$$

The action of $G(0,\varepsilon,\bar{\varepsilon})$ on $G(x,\theta,\bar{\theta})$ is a new group element with new parameters. We say that multiplication of group elements induce motion (transformations) of the parameters (coordinates). It is precisely this representation we are interested in, but of the generators.

Infinitesimal transformations of coordinates are associated with differential operators. We will go ahead and implement the generators of supersymmetry as differential operators with respect to super-coordinates. Although this is the initial idea behind what is called superspace we must now turn it into a meaningful formalism.

First we have to introduce the analogue of differentiation for Grassmann coordinates. Two familiar but modified properties are assumed

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta, \tag{C.63}$$

$$\frac{\partial}{\partial\theta^\alpha}\left(\theta^\beta\theta^\gamma\right) = \left(\frac{\partial}{\partial\theta^\alpha}\theta^\beta\right)\theta^\gamma - \theta^\beta\left(\frac{\partial}{\partial\theta^\alpha}\theta^\gamma\right), \tag{C.64}$$

and similarly for dotted indices. The second property is just the Leibniz rule with a minus sign because ∂_α is assumed to anti-commute with θ^α. Because $\left(\theta^\alpha\theta^\beta\right)^* = \bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\alpha}}$ we also have

$$\left(\frac{\partial}{\partial\theta^\alpha}\theta^\beta\right)^* = -\left(\frac{\partial}{\partial\theta^\alpha}\right)^*\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}{}^{\dot{\beta}} \Rightarrow (\partial_\alpha)^* = -\bar{\partial}_{\dot{\alpha}}. \tag{C.65}$$

With these defining properties of Grassmann derivatives understood we can go ahead and construct a coordinate representation of the supersymmetry generators as follows,

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^a_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_a, \tag{C.66}$$

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\beta\sigma^a_{\beta\dot{\alpha}}\partial_a. \tag{C.67}$$

However we still have to confirm that these in fact do satisfy the supersymmetry algebra. We will check the non-trivial anti-commutator $\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\}$.

$$\begin{aligned}\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} &= -\underbrace{\left\{\partial_\alpha, \bar{\partial}_{\dot{\beta}}\right\}}_{=0} + \left\{\partial_\alpha, i\theta^\gamma \sigma^a_{\gamma\dot{\beta}}\partial_a\right\} + \left\{i\sigma^a_{\alpha\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}\partial_a, \bar{\partial}_{\dot{\beta}}\right\} + \left\{\sigma^a_{\alpha\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}\partial_a, \theta^\delta \sigma^b_{\delta\dot{\beta}}\partial_b\right\} \\ &= 0 + i\sigma^a_{\alpha\dot{\beta}}\partial_a + i\sigma^a_{\alpha\dot{\beta}}\partial_a + \sigma^a_{\alpha\dot{\gamma}}\sigma^b_{\delta\dot{\beta}}\partial_a\partial_b \underbrace{\left\{\bar{\theta}^{\dot{\gamma}}, \theta^\delta\right\}}_{=0} = 2i\sigma^a_{\alpha\dot{\beta}}\partial_a.\end{aligned} \tag{C.68}$$

Therefore our representation is indeed a valid choice for the SUSY algebra.

We now turn to the fields these generators act on, the superfields, in particular scalar superfields. They can be written as power expansions in their Grassmann coordinates in the following way

$$\begin{aligned}\Phi(z) = \phi(x) + \theta^\alpha\psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}}\chi^{\dot{\alpha}}(x) + \frac{1}{2}\theta^\alpha\theta^\beta b_{\alpha\beta}(x) + \frac{1}{2}\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}\tilde{b}^{\dot{\alpha}\dot{\beta}}(x) + \theta^\alpha\sigma^a_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}v_a \\ + \frac{1}{2}\theta^\alpha\theta^\beta\psi'_{\alpha\beta\dot{\gamma}}\bar{\theta}^{\dot{\gamma}} + \frac{1}{2}\theta^\alpha\bar{\chi}'_{\alpha\dot{\beta}\dot{\gamma}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\gamma}} + \frac{1}{4}\theta^\gamma\theta^\delta C_{\gamma\delta\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}}.\end{aligned} \tag{C.69}$$

That the expansion terminates is due to the anti-commuting properties of Grassmann numbers. In four dimensions the indices α and $\dot{\alpha}$ take two values, therefore any term with three θ or three $\bar{\theta}$ has the same Grassmann number at least twice. Grassmann numbers square to zero and therefore the expansion terminates. In higher dimensions the expansion will be longer, but will still terminate after a finite number of terms.

As we will see in a moment this superfield turns out to be a reducible representation of SUSY. To see that we introduce a new set of operators

$$\begin{aligned}D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\sigma^a_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_a, \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\beta\sigma^a_{\beta\dot{\alpha}}\partial_a,\end{aligned} \tag{C.70}$$

which satisfy $\left\{D_\alpha, \bar{D}_{\dot{\alpha}}\right\} = -2i\sigma^a_{\alpha\dot{\beta}}\partial_a$. But perhaps more importantly $\{D, Q\} = 0$ for all combinations of indices on D and Q. Thus D_α and $\bar{D}_{\dot{\alpha}}$ are covariant derivatives with respect to supersymmetric transformations. Now we can put constraints on the superfield with the use of covariant derivatives, which are invariant under supersymmetry, reducing the degrees of freedom of the superfield and therefore also the size of the representation. Two possible choices are $D_\alpha\Phi = 0$ or $\bar{D}_{\dot{\alpha}}\Phi = 0$, a field that satisfies one of these constraints is called a chiral superfield. We will pick the latter and solve the constraint to find our new field using a trick. The trick is to find a similarity transformation T such that

$$\bar{D}_{\dot{\alpha}}\Phi \longrightarrow -\bar{\partial}_{\dot{\alpha}}\tilde{\Phi} = -T^{-1}\bar{\partial}_{\dot{\alpha}}T\Phi = 0. \tag{C.71}$$

Then we can solve $\partial_{\dot{\alpha}}\tilde{\Phi} = 0$ and transform back to $\Phi = T\tilde{\Phi}$. We start off by finding our similarity transformation T satisfying

$$-T^{-1}\bar{\partial}_{\dot{\alpha}}T = \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^a_{\beta\dot{\alpha}}\partial_a, \tag{C.72}$$

which has the formal solution $T = \mathrm{e}^{i\theta^\beta\sigma^a_{\beta\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_a}$ but should be thought of as the corresponding power series (which will be finite). We can go ahead and write out the expansion and simplify

$$T = 1 + i\theta^\alpha\sigma^a_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_a - \frac{1}{2}\theta^\alpha\sigma^a_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\theta^\delta\sigma^b_{\delta\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}\partial_a\partial_b = 1 + i\theta^\alpha\sigma^a_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_a + \frac{1}{4}\theta^2\bar{\theta}^2\Box. \tag{C.73}$$

In the last step we used the following properties

$$\begin{aligned} \theta^\alpha\theta_\alpha &\equiv \theta^2 = \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta \Rightarrow -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2 = \theta^\alpha\theta^\beta, \\ \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} &\equiv \bar\theta^2 = -\epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} \Rightarrow \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta^2 = \theta^{\dot\alpha}\theta^{\dot\beta}, \end{aligned} \tag{C.74}$$

and the trace of Pauli matrices

$$\epsilon^{\alpha\delta}\epsilon^{\dot\beta\dot\gamma}\sigma^a_{\alpha\dot\beta}\sigma^b_{\delta\dot\gamma} = -2\eta^{ab}.$$

Now that we have found a compact expression for T we are almost done. The solution to $\partial_{\dot\alpha}\tilde\Phi = 0$ is

$$\tilde\Phi = \phi(y) + \theta^\alpha\psi_\alpha(y) + \frac{1}{2}\theta^\alpha\theta^\beta b_{\alpha\beta}(y) \equiv \phi(y) + \theta^\alpha\psi_\alpha(y) + \frac{1}{2}\theta^2 F(y). \tag{C.75}$$

All that is left to do is to go back to our old basis

$$\begin{aligned} \Phi = T\tilde\Phi &= \left(1 + i\theta^\alpha\sigma^a_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_a + \frac{1}{4}\theta^2\bar\theta^2\Box\right)\left(\phi + \theta^\gamma\psi_\gamma + \frac{1}{2}\theta^2 F'\right) \\ &= \phi + \theta^\gamma\psi_\gamma + \frac{1}{2}\theta^2 F' + i\theta^\alpha\sigma^a_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_a\phi + i\theta^\alpha\sigma^a_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_a\theta^\gamma\psi_\gamma + \frac{1}{4}\theta^2\bar\theta^2\Box\phi \\ &= \phi(x) + \theta^\alpha\psi_\alpha(x) + \theta^2 F(x) + i\theta^\alpha\sigma^a_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_a\phi(x) - \frac{i}{2}\theta^2\partial_a\psi^\alpha(x)\sigma^a_{\alpha\dot\beta}\bar\theta^{\dot\beta} + \frac{1}{4}\theta^2\bar\theta^2\Box\phi(x). \end{aligned} \tag{C.76}$$

Note that a lot of terms dropped out because they had more than two Grassmann coordinates in them.

C.5.1 Transformations in superspace

So far all that we have done is construct an irreducible superfield. We have not yet seen the utility of using the superspace formalism. It would be useful if we could find the SUSY transformations for the fields in our superfield via the superspace formalism. We start by noting that a transformation of the superfield is generated by Q_α and $\bar Q_{\dot\alpha}$ in the following way

$$\delta_\varepsilon\Phi = \big(\varepsilon Q + \bar\varepsilon\bar Q\big)\Phi \equiv \big(\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha}\big)\Phi. \tag{C.77}$$

If we want to examine the transformation of the field $\phi(x)$ we observe that

$$\phi(x) = \Phi(x,\theta,\bar\theta)\Big|_{\theta=\bar\theta=0}. \tag{C.78}$$

Now it is straightforward to see that, from the definition of the generators and the covariant derivatives, we have

$$\delta_\varepsilon\phi = \delta_\varepsilon\Phi\Big|_{\theta=\bar\theta=0} \equiv \big(\varepsilon Q + \bar\varepsilon\bar Q\big)\Phi\Big|_0 = \big(\varepsilon D + \bar\varepsilon\bar D\big)\Phi\Big|_0 = \varepsilon D\Phi\Big|_0. \tag{C.79}$$

In the last step we used the fact that we are working with a chiral superfield. Thus the transformation for ϕ is $\delta_\varepsilon\phi = \varepsilon^\alpha\psi_\alpha$. We can keep doing this for all our fields. For ψ_α we have

$$\begin{aligned} \delta_\varepsilon(\psi_\alpha) &= \delta_\varepsilon\,(D_\alpha\Phi)\Big|_0 = \big(\varepsilon Q + \bar\varepsilon\bar Q\big)D_\alpha\,\Phi\Big|_0 = \big(\varepsilon D + \bar\varepsilon\bar D\big)D_\alpha\,\Phi\Big|_0 \\ &= \Big(\varepsilon^\beta D_\beta D_\alpha - \bar\varepsilon^{\dot\beta}\Big(\Big\{\bar D_{\dot\beta}, D_\alpha\Big\} - D_\alpha\bar D_{\dot\beta}\Big)\Big)\,\Phi\Big|_0 = \varepsilon^\beta D_\beta D_\alpha\,\Phi\Big|_0 + 2i\bar\varepsilon^{\dot\beta}\sigma^a_{\alpha\dot\beta}\partial_a\,\Phi\Big|_0 \\ &= 2\varepsilon^\beta\epsilon_{\alpha\beta}F + 2i\sigma^a_{\alpha\dot\beta}\bar\varepsilon^{\dot\beta}\partial_a\phi = 2\varepsilon_\alpha F + 2i\sigma^a_{\alpha\dot\beta}\bar\varepsilon^{\dot\beta}\partial_a\phi. \end{aligned} \tag{C.80}$$

Note that we used the fact that

$$D_\beta D_\alpha \, \Phi\Big|_0 = 2\epsilon_{\alpha\beta} F, \tag{C.81}$$

which we can invert to get F in terms of Φ.

$$\epsilon^{\alpha\beta} D_\beta D_\alpha \, \Phi\Big|_0 = 2\epsilon^{\alpha\beta}\epsilon_{\alpha\beta} F = -4F \Rightarrow F = -\frac{1}{4} D^2 \, \Phi\Big|_0 . \tag{C.82}$$

With this result at hand we can find the transformation of F as well.

$$\begin{aligned}
\delta_\varepsilon F &= \left(\varepsilon Q + \bar\varepsilon \bar Q\right)\left(\frac{-1}{4} D^2\right) \Phi\Big|_0 = \quad \{\text{No terms with three } \theta\} \quad = -\frac{1}{4}\bar\varepsilon \bar D D^2 \, \Phi\Big|_0 \\
&= \quad \{\text{ Using that } \Phi \text{ is chiral }\} \quad = \frac{1}{4}\bar\varepsilon^{\dot\beta}\left[\bar D_{\dot\beta}, D^\alpha D_\alpha\right] \Phi\Big|_0 \\
&= \frac{1}{4}\bar\varepsilon^{\dot\beta}\left(-\left\{\bar D_{\dot\beta}, D_\alpha\right\} D^\alpha + D_\alpha \bar D_{\dot\beta} D^\alpha - D^\alpha \left\{D_a, \bar D_{\dot\beta}\right\} + D^\alpha \bar D_{\dot\beta} D_\alpha\right) \Phi\Big|_0 \\
&= -\frac{1}{2}\bar\varepsilon^{\dot\beta} D^\alpha \left\{D_\alpha, \bar D_{\dot\beta}\right\} \Phi\Big|_0 = i\bar\varepsilon^{\dot\beta}\sigma^a_{\alpha\dot\beta}\partial_a D^\alpha \, \Phi\Big|_0 = i\bar\varepsilon^{\dot\beta}\sigma^a_{\alpha\dot\beta}\partial_a \psi^\alpha \\
&= i\varepsilon_{\dot\gamma} \underbrace{\epsilon^{\dot\beta\dot\gamma}\sigma^a_{\alpha\dot\beta}\epsilon^{\alpha\delta}}_{\bar\sigma^{a\dot\gamma\delta}} \partial_a \psi_\delta = i\bar\varepsilon \bar\sigma^a \partial_a \psi .
\end{aligned} \tag{C.83}$$

Interestingly $\delta_\varepsilon F = 0$ is the Dirac equation.

C.5.2 Actions in superspace

At the end of the day we want to be able to write down an action for our models. As we will see it is relatively straightforward to write down a manifest supersymmetric action in superspace. However, the final Lagrangian should not depend on the Grassmann coordinates and therefore we need a method for integrating them out.

Integration is defined such that

$$\begin{aligned}
&\int \mathrm{d}\theta^\alpha \, \theta_\beta = \delta^\alpha_\beta \quad , \quad \int \mathrm{d}\bar\theta^{\dot\alpha} \, \bar\theta_{\dot\beta} = -\delta^{\dot\alpha}_{\dot\beta}, \\
&\int \mathrm{d}^2\theta \, \theta^2 = 1, \\
&\int \mathrm{d}^2\bar\theta \, \bar\theta^2 = 1,
\end{aligned} \tag{C.84}$$

which is consistent with having $\mathrm{d}^2\theta = -\frac{1}{4}\,\mathrm{d}\theta^\alpha\,\mathrm{d}\theta^\beta\,\epsilon_{\alpha\beta}$. This we can check, but first we need the following result

$$\int \mathrm{d}\theta^\alpha \, \theta_\beta \theta_\gamma = -2\delta^\alpha_{[\beta}\theta_{\gamma]}. \tag{C.85}$$

Then we can go ahead and check the consistency of our choice

$$\int \mathrm{d}^2\theta \, \theta^2 = -\frac{1}{4}\epsilon_{\alpha\beta}\int \mathrm{d}\theta^\alpha \, \mathrm{d}\theta^\beta \, \theta_\gamma\theta_\delta\epsilon^{\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta}\epsilon^{\gamma\delta}\int \mathrm{d}\theta^\alpha \, \delta^\beta_{[\gamma}\theta_{\delta]} = \frac{1}{2}\epsilon_{\alpha\beta}\epsilon^{\gamma\delta}\delta^\beta_{[\gamma}\delta^\alpha_{\delta]} = \frac{1}{2}\epsilon_{\alpha\beta}\epsilon^{\beta\alpha} = 1. \tag{C.86}$$

Observe that integration acts just like differentiation for Grassmann coordinates. Therefore we can instead think of it as such and explicitly write

$$\begin{aligned}\int \mathrm{d}\theta^\alpha &\equiv \frac{\partial}{\partial\theta_\alpha} = -\, D^\alpha\Big|_0 ,\\ \int \mathrm{d}\bar\theta^{\dot\alpha} &\equiv -\frac{\partial}{\partial\bar\theta_{\dot\alpha}} = +\, \bar D^{\dot\alpha}\Big|_0 ,\\ \int \mathrm{d}^2\theta &= -\frac{1}{4}\, D^2\Big|_0 ,\\ \int \mathrm{d}^2\bar\theta &= -\frac{1}{4}\, \bar D^2\Big|_0 .\end{aligned} \tag{C.87}$$

Now we can write an action as

$$S \sim \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\, \Phi^*\Phi, \tag{C.88}$$

where $D_\alpha\Phi^* = 0$, then we start integrating. In the next calculation everything following the first equality is evaluated at $\theta = \bar\theta = 0$ but it is not written out explicitly to reduce clutter.

$$\begin{aligned}\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\, \Phi^*\Phi &= \frac{1}{16} D^2\bar D^2\Phi^*\Phi = \frac{1}{16}\epsilon_{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} D^\alpha D^\beta \bar D_{\dot\alpha} \underbrace{\bar D_{\dot\beta}\Phi^*\Phi}_{(\bar D_{\dot\beta}\Phi^*)\Phi + 0}\\ &= \frac{1}{16}\epsilon_{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} D^\alpha D^\beta\left(\bar D_{\dot\alpha}\bar D_{\dot\beta}\Phi^*\right)\Phi.\end{aligned} \tag{C.89}$$

We can look at just the last part

$$\begin{aligned}&D^\alpha D^\beta\left(\bar D_{\dot\alpha}\bar D_{\dot\beta}\Phi^*\right)\Phi = D^\alpha \underbrace{\left(D^\beta \bar D_{\dot\alpha}\bar D_{\dot\beta}\Phi^*\right)}_{\left[D^\beta, \bar D_{\dot\alpha}\bar D_{\dot\beta}\right]\Phi^*} \Phi + D^\alpha\left(\bar D_{\dot\alpha}\bar D_{\dot\beta}\Phi^*\right)D^\beta\Phi\\ &= -4iD^\alpha\left(\epsilon^{\beta\gamma}\sigma^a_{\gamma[\dot\alpha}\bar D_{\dot\beta]}\partial_a\Phi^*\Phi\right) + \left(D^\alpha\bar D_{\dot\alpha}\bar D_{\dot\beta}\Phi^*\right)D^\beta\Phi + \left(\bar D_{\dot\alpha}\bar D_{\dot\beta}\Phi^*\right)D^\alpha D^\beta\Phi\\ &= -4i\left(D^\alpha\epsilon^{\beta\gamma}\sigma^a_{\gamma[\dot\alpha}\bar D_{\dot\beta]}\partial_a\Phi^*\right)\Phi - 8i\epsilon^{[\alpha|\gamma|}\sigma^a_{\gamma[\dot\alpha}\bar D_{\dot\beta]}\partial_a\Phi^*\left(D^{\beta]}\Phi\right) + \left(\bar D_{\dot\alpha}\bar D_{\dot\beta}\Phi^*\right)D^\alpha D^\beta\Phi\\ &= -8\epsilon^{\beta\gamma}\epsilon^{\alpha\delta}\sigma^a_{\gamma[\dot\alpha}\sigma^b_{|\delta|\dot\beta]}(\partial_b\partial_a\Phi^*)\Phi - 8i\epsilon^{\alpha\gamma}\sigma^a_{\gamma[\dot\alpha}\partial_{|a|}\bar\psi_{\dot\beta]}\psi^\beta + 4\epsilon_{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}F^*F.\end{aligned} \tag{C.90}$$

Going back to equation (C.89) we find

$$\begin{aligned}\frac{1}{16}\epsilon_{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}D^\alpha D^\beta\left(\bar D_{\dot\alpha}\bar D_{\dot\beta}\Phi^*\right)\Phi &= \frac{1}{16}\left(-8\sigma^a_{\alpha\dot\alpha}\bar\sigma^{b\dot\alpha\alpha}\partial_b\partial_a\Phi^*\Phi - 8i\psi\sigma^a\partial_a\bar\psi + 16F^*F\right)\\ &= \frac{1}{16}\left(16\eta^{ab}\partial_b\partial_a\Phi^*\Phi + 8i\partial_a\bar\psi\bar\sigma^a\psi + 16|F|^2\right)\\ &= (\Box\Phi^*)\Phi + \frac{i}{2}\partial_a\bar\psi\bar\sigma^a\psi + |F|^2 = -\partial_a\phi^*\partial^a\phi + \frac{i}{2}\partial_a\bar\psi\bar\sigma^a\psi + |F|^2.\end{aligned} \tag{C.91}$$

and so we have found our kinetic Lagrangian

$$L_{\mathrm{kin}} = -\partial_a\phi^*\partial^a\phi + \frac{i}{2}\partial_a\bar\psi\bar\sigma^a\psi + |F|^2. \tag{C.92}$$

Now we can start adding interactions. It is useful to employ a dimensional counting scheme. The superfield Φ is a scalar field, as is evident from the first term in the expansion, and therefore it has mass dimension 1. The second term is $\theta\psi$ where $\psi_\alpha(x)$ is a spinor which has mass dimension 3/2. To match the first term θ^α must have mass dimension $-1/2$ and because $\int \mathrm{d}^2\theta\, \theta^2$ is dimensionless the consistent thing to assign $\mathrm{d}^2\theta$ is dimension 1. It turns out that the unique non-zero combination which gives a dimensionless action using $\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}$ is $\Phi^*\Phi$, the kinetic term. However we can still construct actions using $\int \mathrm{d}^4x \left(\mathrm{d}^2\theta + \mathrm{d}^2\bar{\theta}\right)$ with dimension -3. We write the interaction Lagrangian as

$$L = \int \mathrm{d}^2\theta \left(\lambda\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3\right) + \text{c.c.} \tag{C.93}$$

Here λ, m and g are constants (dimension 2, 1 and 0 respectively). Then we start integrating (note that the evaluated at $\theta = \bar{\theta} = 0$ notation is suppressed).

$$\begin{aligned}
L_\lambda &= \int \mathrm{d}^2\theta\, \lambda\Phi + \text{c.c.} = -\frac{1}{4}D^2\Phi + \text{c.c.} = \lambda F + \lambda^* F^*, \\
L_m &= \frac{1}{2}m \int \mathrm{d}^2\theta\, \lambda\Phi^2 + \text{c.c.} = -\frac{1}{4}m(D^\alpha\Phi)(D_\alpha\Phi) - \frac{1}{4}m\Phi D^2\Phi + \text{c.c.} \\
&= -\frac{1}{4}m(\psi^\alpha\psi_\alpha - 4\phi F) + \text{c.c.} = -\frac{m}{4}\psi^2 - \frac{m^*}{4}\bar{\psi}^2 + m\phi F + m^*\phi^* F^*, \\
L_g &= -\frac{g}{12}D^2\Phi^3 + \text{c.c.} = -\frac{g}{12}D^\alpha\big(3\Phi^2 D_\alpha\Phi\big) + \text{c.c.} = -\frac{g}{4}\big((D^\alpha\Phi^2)D_\alpha\Phi + \Phi^2 D^2\Phi\big) + \text{c.c.} \\
&= -\frac{g}{4}\big(2\Phi(D^\alpha\Phi)D_\alpha\Phi + \Phi^2 D^2\Phi\big) + \text{c.c.} = -\frac{g}{2}\phi\psi^2 + g\phi^2 F - \frac{g^*}{2}\phi^*\bar{\psi}^2 + g^*(\phi^*)^2 F^*.
\end{aligned} \tag{C.94}$$

We see that there are no terms in the Lagrangian which has derivatives acting on the field F. The field has no dynamics, its sole purpose is to make sure that the superfield Φ transforms properly under SUSY (in other words that we get a superfield back after a transformation). Thus, we can use the equation of motion for F (and analogously for F^*),

$$F + \lambda^* + m^*\phi^* + g^*(\phi^*)^2 = 0 \tag{C.95}$$

to eliminate it from the Lagrangian. This would give us the on-shell Lagrangian

$$\begin{aligned}
L_{(on-shell)} = &-\lambda\big(m^*\phi^* + g^*(\phi^*)^2\big) \\
&- \frac{m}{4}\psi^2 - m\phi\big(\lambda^* + m^*\phi^* + g^*(\phi^*)^2\big) \\
&- \frac{g}{2}\phi\psi^2 - g\phi^2\big(m^*\phi^* + g^*(\phi^*)^2\big) + \text{c.c.}
\end{aligned} \tag{C.96}$$

The terms linear in F are a sign of instability because a non-zero F will minimise the potential. If F picks up a VEV $\langle F\rangle$ the transformation of ψ_α becomes

$$\delta\psi_\alpha = 2\varepsilon_\alpha \langle F\rangle + 2\varepsilon_\alpha F + 2i\sigma^a_{\alpha\dot{\beta}}\bar{\varepsilon}^{\dot{\beta}}\partial_a\phi. \tag{C.97}$$

We expect the ground state to have $F(x) = \psi_\alpha(x) = \phi(x) = 0$ but the supersymmetry transformation takes $\psi_\alpha \to 2\varepsilon_\alpha \langle F\rangle$ which is non-zero. This means that our ground state solution has spontaneously broken supersymmetry.

C.6 Differential forms in superspace

We have seen that supersymmetry transformations are equivalent to coordinate transformations in superspace. This means that, in order to be able to formulate a supersymmetric theory of gravity we would like to express these transformations in a manifestly covariant way. This is done by introducing a framework for differential forms in superspace.

We can define a new set of coordinates $Z^M \equiv (x^m, \theta^\mu)$ in superspace, where M is a super-index, m ordinary curved bosonic indices and μ fermionic Grassmann indices. We will use $M, N, P, \ldots$ for curved indices and $A, B, C, \ldots$ for flat, where, e.g., $A = (a, \alpha)$.

The geometrical interpretation of superspace comes from studying the covariant derivatives, as is done in both Maxwell and Yang-Mills theory, where the field strength tensors are in both cases defined by $[D_\mu, D_\nu]\phi = iF_{\mu\nu}\phi$, and in ordinary gravity where $[D_\mu, D_\nu]V^\sigma \sim R_{\mu\nu}{}^\sigma{}_\rho V^\rho$ for the Riemann tensor, or if there is torsion (i.e. $\Gamma^\rho_{\mu\nu} \neq \Gamma^\rho_{\nu\mu}$) we have the Ricci identity

$$[D_\mu, D_\nu]V^\rho = -R_{\mu\nu}{}^\rho{}_\sigma V^\sigma - T_{\mu\nu}{}^\sigma D_\sigma V^\rho. \tag{C.98}$$

Earlier we also saw that in the supersymmetric case we have (in four dimensions) $\{\mathcal{D}_a, \bar{\mathcal{D}}_{\dot{a}}\} = -2i\sigma^\mu_{a\dot{a}}\partial_\mu$. All these identities can be derived in superspace.

A simple superspace to use is in ten dimensions. We then have

$$Z^M = (x^m, \theta^\mu), \tag{C.99}$$

where x^m are 10 real bosonic coordinates and θ^μ are 16 real Grassmann coordinates (which will become Majorana spinors when we express them in flat indices). There are two types of derivatives

$$\partial_m \equiv \frac{\partial}{\partial x^m}, \quad \partial_\mu \equiv \frac{\partial}{\partial \theta^\mu}. \tag{C.100}$$

The introduction of supervielbeins gives us a basis in the tangent plane as

$$E_A{}^M(Z)\partial_M \equiv E_A(Z), \tag{C.101}$$

where E_a is a Lorentz vector and E_α a spinor [54]. Note that the ordering is important in this expression.

In cotangent space we instead have the one-forms

$$dZ^M = (dx^m, d\theta^\mu), \tag{C.102}$$

and in flat coordinates [54]

$$E^A(Z) \equiv dZ^M E_M{}^A(Z), \quad E^A = (\underbrace{E^a}_{\text{bos.}}, \underbrace{E^\alpha}_{\text{ferm.}}), \tag{C.103}$$

where

$$E_M{}^A(Z) = \begin{pmatrix} E_m{}^a & E_m{}^\alpha \\ E_\mu{}^a & E_\mu{}^\alpha \end{pmatrix}. \tag{C.104}$$

Here we see that the diagonal blocks are bosonic and the off-diagonal ones fermionic.

Note that we do not have a super-Lorentz group, i.e., the Lorentz transformations in the tangent plane are only bosonic [54]

$$(E^a, E^\alpha) \xrightarrow{L.T.} (E^b L_b{}^a, E^\beta L_\beta{}^\alpha). \tag{C.105}$$

A super-Lorentz group would have mixed the different parts. This means that the super-aspects of superspace only sits in the coordinate transformations.

Together with the frame fields $E^A(Z)$ we also need a spin connection, which is a one-form. We write

$$\omega_A{}^B = (\omega_a{}^b, \omega_\alpha{}^\beta) = (\omega_a{}^b, \omega^{ab}\frac{1}{4}(\Gamma_{ab})_\alpha{}^\beta), \tag{C.106}$$

which means that we only have one superfield, $\omega_{Ma}{}^{b} = (\omega_{ma}{}^{b}, \omega_{\mu a}{}^{b})$.

The exterior product in superspace is defined by

$$dz^M \wedge dz^N = -(-)^{NM} dz^N \wedge dz^M, \tag{C.107}$$

where $(-)^m = (-1)^0$ and $(-)^\mu = (-1)^1$ [54]. The wedge-sign will sometimes be dropped, but all products of forms are done with this exterior product. This means that we can introduce a general p-form as

$$\overset{(p)}{A} \equiv \frac{1}{p!} dz^{M_1} \wedge dz^{M_2} \wedge \cdots \wedge dz^{M_p} A_{M_p \ldots M_1}, \tag{C.108}$$

where the ordering has been chosen so as to minimise the number of signs entering at later stages. For example, we write a one-form as

$$A = dz^M A_M(z) = dx^m A_m(z) + d\theta^\mu A_\mu(z). \tag{C.109}$$

An exterior derivative is introduced by

$$d \equiv dz^M \partial_M = dx^m \partial_m + d\theta^\mu \partial_\mu. \tag{C.110}$$

This is defined to act through *right action*, which means that

$$d(\overset{(p)}{A} \wedge \overset{(q)}{B}) = \overset{(p)}{A} \wedge d\overset{(q)}{B} + (-1)^q d\overset{(p)}{A} \wedge \overset{(q)}{B}. \tag{C.111}$$

It has all the usual properties of an exterior derivative, i.e. $d(A + B) = dA + dB$, $dd = 0$ and it maps a p-form to a $(p + 1)$-form by

$$d\overset{(p)}{A} = \frac{1}{p!} dz^{M_1} \wedge \cdots \wedge dz^{M_p} dA_{M_p \ldots M_1}, \tag{C.112}$$

since $d(dz) = 0$.

If we write all our equations in terms of differential forms and exterior derivatives they will be manifestly covariant under general coordinate transformations. However, gauge theories also transform covariantly under a local structure group. In Yang-Mills theory this is a compact Lie group and in gravity theories this is the Lorentz group [54]. We consider differential forms which span a representation (X) of this structure group

$$A^a \to A'^a = A^b X_b{}^a(z). \tag{C.113}$$

This implies that the exterior derivative does not map a tensor to a tensor (this was why we needed to introduce a connection earlier).

If we instead introduce a covariant derivative as

$$\mathcal{D}A^a = dA^a + A^b \omega_b{}^a, \tag{C.114}$$

or

$$\mathcal{D}\overset{(p)}{A} = dz^{M_1} \ldots dz^{M_p} dz^N \frac{\partial}{\partial z^N} A_{M_p \ldots M_1}(z) + dz^{M_1} \ldots dz^{M_p} dz^N \omega_N{}^r A_{M_p \ldots M_1}(z) i T^r, \tag{C.115}$$

this maps a p-form into a $(p + 1)$-form and a tensor into a tensor.

We can construct one tensor from the exterior derivative and the connection, this is the curvature (or field strength) tensor

$$R = d\omega + \omega^2. \tag{C.116}$$

The general form for this is a Lie algebra valued two-form

$$R = \frac{1}{2}dz^M dz^N R_{MN} = \frac{1}{2}dz^M dz^N R_{MN}{}^r iT^r, \tag{C.117}$$

while in gravity one usually omit the iT^r, and the algebra is then the Lorentz algebra.

For a general p-form super vector field, $V^A(z)$ we have

$$\mathcal{D}V^A = dV^A + V^B\omega_B{}^A, \tag{C.118}$$

and the Ricci identity is found by applying the covariant derivative twice

$$\begin{aligned}\mathcal{D}\mathcal{D}V^A &= \mathcal{D}(dV^A + V^B\omega_B{}^A) = d^2V^A + d(V^B\omega_B{}^A) + (dV^B)\omega_B{}^A + V^B\omega_B{}^C\omega_C{}^A \\ &= V^B d\omega_B{}^A - dV^B\omega_B{}^A + dV^B\omega_B{}^A + V^B\omega_B{}^C\omega_C{}^A \\ &= V^B(d\omega_B{}^A + \omega_B{}^C\omega_C{}^A) = V^B R_B{}^A.\end{aligned} \tag{C.119}$$

Similarly, for a p-form $V_A(z)$

$$\begin{aligned}\mathcal{D}\mathcal{D}V_A =&\mathcal{D}\left(dV_A - \omega_A{}^B V_B(-1)^p\right) \\ =&d^2V_A - d\left(\omega_A{}^B V_B(-1)^p\right) - (-1)^{p+1}\omega_A{}^B dV_B + \omega_A{}^C\omega_C{}^B V_B(-1)^{p+1}(-1)^p \\ =&-(-1)^p\omega_A{}^B dV_B - (-1)^p(-1)^p d\omega_A{}^B V_B - (-1)^{p+1}\omega_A{}^B dV_B - \omega_A{}^C\omega_C{}^B V_B \\ =&-(d\omega_A{}^B + \omega_A{}^C\omega_C{}^B)V_B = -R_A{}^B V_B.\end{aligned} \tag{C.120}$$

If we write this in flat indices we find

$$\begin{aligned}\mathcal{D}\mathcal{D}V^A &= \mathcal{D}(E^B\mathcal{D}_B V^A) = E^B(\mathcal{D}\mathcal{D}_B V^A) + (\underbrace{\mathcal{D}E^B}_{=T^B})\mathcal{D}_B V^A \\ &= E^B E^C\mathcal{D}_C\mathcal{D}_B V^A + T^B\mathcal{D}_B V^A \\ \implies E^B E^C &\underbrace{\mathcal{D}_C\mathcal{D}_B}_{\frac{1}{2}[\mathcal{D}_C,\mathcal{D}_B\}} V^A = V^D R_D{}^A - T^D\mathcal{D}_D V^A = V^D\frac{1}{2}E^B E^C R_{CBD}{}^A - \frac{1}{2}E^B E^C T_{CB}{}^D\mathcal{D}_D V^A \\ \implies [\mathcal{D}_C,\mathcal{D}_B\}V^A &= (-)^{D(C+B)}V^D R_{CBD}{}^A - T_{CB}{}^D\mathcal{D}_D V^A,\end{aligned} \tag{C.121}$$

where we have introduced the torsion one-form as $T^A = dE^A + E^B\omega_B{}^A$, and the graded commutator, $[\cdot,\cdot\}$, which is an anti-commutator if both arguments are fermionic, otherwise a commutator.[c]

In the same way one can show [54]

$$[\mathcal{D}_C,\mathcal{D}_B\}V_A = -R_{CBA}{}^D V_D - T_{CB}{}^D\mathcal{D}_D V_A. \tag{C.122}$$

Now, higher derivatives will not give us new tensors or forms, since $d^2 = 0$, but instead we will get identities, more precisely the Bianchi identities. There are two types of Bianchi identities. The first type is

$$\mathcal{D}R = 0, \tag{C.123}$$

[c]Note that if we had ordinary left action for the derivative we would have $T^A = dE^A + E^B\omega_B{}^A$, which implies $T_{MN}{}^A = 2(\partial_{[M}E_{N]}{}^A + \omega_{[N}{}^A{}_{|B|}E_{M]}{}^B)$. This has an extra relative sign compared to our expression. The sign is eliminated if we let $\omega \to -\omega$ in superspace.

which is easily seen from

$$\begin{aligned}\mathcal{D}R &= dR + R\omega - \omega R = d(d\omega + \omega^2) + (d\omega + \omega^2)\omega - \omega(d\omega + \omega^2)\\ &= d^2\omega + \omega d\omega - (d\omega)\omega + (d\omega)\omega - \omega d\omega = 0.\end{aligned} \tag{C.124}$$

The second type is of the form

$$\mathcal{D}T^A = E^B R_B{}^A, \tag{C.125}$$

which follow from

$$\begin{aligned}\mathcal{D}T^A &= dT^A + T^B\omega_B{}^A = d(dE^A + E^B\omega_B{}^A) + (dE^B + E^C\omega_C{}^B)\omega_B{}^A\\ &= E^B d\omega_B{}^A + E^C\omega_C{}^B\omega_B{}^A = E^B R_B{}^A.\end{aligned} \tag{C.126}$$

These will be used in the subsequent sections when we do super-Yang-Mills and supergravity.

C.7 Supersymmetric Yang-Mills in D=10

In this section we will study the theory of supersymmetric Yang-Mills in ten dimensions with the goal of finding the equations of motions and the supertransformations of the fields present in the theory. This is done as a preparation for studying supergravity in 11 dimensions, which follow roughly the same procedure, however, more complex.

We choose to do it in ten dimensions since this simplifies a great deal of things. Although one may feel that it is strange to work in ten or eleven dimensions it is quickly found to be much more comfortable than the awkward four dimensions. Ten dimensional super-Yang-Mills is also highly connected to superstring theories. Superstrings are in turn connected to the theory of supergravity in eleven dimensions discussed in the next section.

In ten-dimensional super-Yang-Mills theory we have the superspace coordinates

$$z^M = (x^m, \theta^\mu), \tag{C.127}$$

where $m = 0, \ldots, 9$ are the ordinary, curved, spacetime coordinates and $\mu = 1, \ldots, 16$ are the fermionic coordinates. The Yang-Mills field is now a super-1-form

$$A = dx^m A_m(x,\theta) + d\theta^\mu A_\mu(x,\theta), \tag{C.128}$$

but this implies that, using dimensions appropriate for four dimensions, A_μ has dimension $L^{-1/2}$. This is highly unphysical, and therefore we must hope for this field to vanish.

In flat coordinates this turns into

$$A = E^A A_A = E^a A_a + E^\alpha A_\alpha, \tag{C.129}$$

where A_α is a 16-dimensional Majorana-Weyl spinor. We saw earlier that the Bianchi identities are of the form $\mathcal{D}F = 0$, so if we start from this we can write

$$\begin{aligned}\mathcal{D}F = 0 &\implies \frac{1}{2}\mathcal{D}(E^B E^A F_{AB}) = 0\\ &\implies \frac{1}{2}E^B E^A \underbrace{\mathcal{D}}_{E^C D_C} F_{AB} + \frac{1}{2}E^B(\underbrace{\mathcal{D}E^A}_{T^A})F_{AB} - \frac{1}{2}(\underbrace{\mathcal{D}E^B}_{T^B})E^A F_{AB} = 0\\ &\implies E^C \wedge E^B \wedge E^A D_A F_{BC} + E^C \wedge E^B \wedge E^A T_{AB}{}^D F_{DC} + E^B \wedge E^A \wedge E^C T_{AB}{}^D F_{DC}\\ &= E^C \wedge E^B \wedge E^A \left(D_A F_{BC} + T_{AB}{}^D + (-1)^{BC}(-1)^{AC} T_{AB}{}^D F_{DC}\right) = 0\\ &\implies \frac{1}{2}E^C \wedge E^B \wedge E^A \left(\mathcal{D}_{[A}F_{BC\}} + T_{[AB}{}^D F_{|D|C\}}\right) = 0.\end{aligned} \tag{C.130}$$

So the Bianchi identities for super-Yang-Mills read

$$\mathcal{D}_{[A}F_{BC\}} + T_{[AB}{}^{D}F_{|D|C\}} = 0. \tag{C.131}$$

C.7.1 Solving the Bianchi Identities

The title of this section might seem a bit strange, how can we solve an identity? It is after all an identity so no new information should be available. However, we will find that some constraints can be put on the fields in the theory making it so that the Bianchi identities no longer are identities. Instead they become equations that we can solve to find the actual degrees of freedom in the new (constrained) theory.

We start by splitting the Bianchi identities into four different cases (for the different combinations of the types of indices, α, β, γ or a, b, c). We also note that we only have torsion in the fermionic indices, i.e. only $T_{\alpha\beta}{}^{a}$ is non-zero, and actually it can be put equal to $2i(\Gamma^a)_{\alpha\beta}$. This is easily seen from Eq.(C.121) in the last section. This gives us

$$\begin{aligned}
&\mathcal{D}_{[a}F_{bc]} = 0,\\
&\mathcal{D}_{[a}F_{b\gamma\}} = 2\mathcal{D}_{[a}F_{b]\gamma} + \mathcal{D}_{\gamma}F_{ab} = 0,\\
&\mathcal{D}_{[a}F_{\beta\gamma\}} + T_{[a\beta}{}^{D}F_{|D|\gamma\}} = \mathcal{D}_{a}F_{\beta\gamma} + 2\mathcal{D}_{(\beta}F_{\gamma)a} + 2i(\Gamma^b)_{\beta\gamma}F_{ba} = 0,\\
&\mathcal{D}_{(\alpha}F_{\beta\gamma)} + T_{(\alpha\beta}{}^{D}F_{|D|\gamma)} = \mathcal{D}_{(\alpha}F_{\beta\gamma)} + 2i(\Gamma^a)_{(\alpha\beta}F_{|a|\gamma)} = 0.
\end{aligned} \tag{C.132}$$

Now, since the components of F_{AB} do not mix under Lorentz transformations (this is due to the fact that the flat indices are not superindices) $F_{ab}, F_{\alpha\beta}$ and $F_{a\beta}$ are independent Lorentz tensors. This means that we can constrain one of them without losing any of the Lorentz symmetry. So for instance we could put $F_{\alpha\beta} = 0$ without breaking Lorentz invariance. This will however heavily affect the relations above and this will mean that the Bianchi identities no longer are identities, and we can instead solve them to find new information.

Plugging this constraint into the above equations we now have

$$\begin{aligned}
&\mathcal{D}_{[a}F_{bc]} = 0,\\
&2\mathcal{D}_{[a}F_{b]\gamma} + \mathcal{D}_{\gamma}F_{ab} = 0,\\
&2\mathcal{D}_{(\beta}F_{\gamma)a} + 2i(\Gamma^b)_{\beta\gamma}F_{ba} = 0,\\
&(\Gamma^a)_{(\alpha\beta}F_{|a|\gamma)} = 0.
\end{aligned} \tag{C.133}$$

The last equation could imply that $F_{a\beta} = 0$ but plugging this into the second or third equation would then mean that the entire field F_{AB} is either a constant or equal to zero, and this would in turn mean that the whole theory is meaningless (trivial). So we hope that this is not the case.

We start by trying to solve the last equation, and we do this by decomposing $F_{a\beta}$ into irreducible representations of $SO(1,9)$ by writing

$$F_{a\beta} = \tilde{F}_{a\beta} + (\Gamma_a)_{\beta\gamma}\chi^{\gamma}, \tag{C.134}$$

where $\tilde{F}_{a\beta}$ is such that $(\Gamma^a)^{\alpha\gamma}\tilde{F}_{a\gamma} = 0$. Putting this into the last equation one finds

$$(\Gamma^a)_{(\alpha\beta}\tilde{F}_{|a|\gamma)} + (\Gamma^a)_{(\alpha\beta}(\Gamma_a)_{\gamma)\delta}\chi^{\delta} = 0. \tag{C.135}$$

Now we are lucky, because there is a Fierz identity in ten dimensions that tells us

$$(\Gamma^a)_{(\alpha\beta}(\Gamma_a)_{\gamma)\delta} = 0. \tag{C.136}$$

The proof is easily done by checking the expansion coefficents for the symmetric matrices in $SO(1,9)$. This means that the χ^δ term drops out without giving us any information. On the other hand, we can multiply Eq.(C.135) with $\Gamma^a\Gamma^{(5)}$ to find $\tilde{F}_{a\beta}=0$. This implies that we want χ^α to be non-zero, which in turn implies that $F_{a\beta}=(\Gamma_a)_{\beta\gamma}\chi^\gamma$. Using this in the other equations above, i.e., the second and third of Eq.(C.133), we get

$$\begin{aligned}&\mathcal{D}_\gamma F_{ab}+2\mathcal{D}_{[a}(\Gamma_{b]})_{\gamma\delta}\chi^\delta=0,\\&2\mathcal{D}_{(\beta}(\Gamma_a)_{\gamma)\delta}\chi^\delta+2i(\Gamma^b)_{\beta\gamma}F_{ba}=0.\end{aligned}\tag{C.137}$$

To explore this further we expand $\mathcal{D}_\alpha\chi^\beta$ in gamma matrices. We only need to expand in the ones with one dotted and one regular index in Table C.3, i.e.,

$$\mathcal{D}_\alpha\chi^\beta=\delta_\alpha{}^\beta\phi+(\Gamma^{ab})_\alpha{}^\beta\phi_{ab}+(\Gamma^{abcd})_\alpha{}^\beta\phi_{abcd}.\tag{C.138}$$

If we contract the second equation in Eq.(C.137) with $(\Gamma_c)^{\beta\gamma}$ we find, on the left hand side

$$2(\Gamma_c)^{\beta\gamma}\mathcal{D}_{(\beta}(\Gamma_a)_{\gamma)\delta}\chi^\delta=2(\Gamma_c\Gamma_a)^\beta{}_\delta\mathcal{D}_\beta\chi^\delta=2(\Gamma_{ca})^\beta{}_\delta\mathcal{D}_\beta\chi^\delta+2\eta_{ca}\mathcal{D}_\beta\chi^\beta,\tag{C.139}$$

with one symmetric and one anti-symmetric part. On the right hand side we get

$$2i(\Gamma_c)^{\beta\gamma}(\Gamma^b)_{\beta\gamma}F_{ab}=2i\mathrm{Tr}\left[\Gamma_c\Gamma^b\right]F_{ab}=32iF_{ac}.\tag{C.140}$$

Now, since F_{ac} is anti-symmetric, this tells us that

$$\begin{aligned}&i)\ \mathcal{D}_\beta\chi^\beta=0,\\&ii)\ F_{ab}=\frac{i}{16}(\Gamma_{ab})^\alpha{}_\beta\mathcal{D}_\alpha\chi^\beta.\end{aligned}\tag{C.141}$$

We see that $i)$ implies $\phi=0$ in the expansion of $\mathcal{D}_\alpha\chi^\beta$.

It is also necessary to contract with $(\Gamma_{a_1a_2\dots a_5})^{\beta\gamma}$ to see if we can say anything about the other terms in the expansion. The right hand side will be identically zero since we will have a trace over $(\Gamma_{(5)}\Gamma^b)$. This means that we have

$$\begin{aligned}0=2(\Gamma_{a_1\dots a_5}\Gamma_a)^\beta{}_\delta\mathcal{D}_\beta\chi^\delta&=2(\Gamma_{a_1\dots a_5})^{\beta\gamma}(\Gamma_a)_{\gamma\delta}\left[(\Gamma^{b_1b_2})_\beta{}^\delta\phi_{b_1b_2}+(\Gamma^{c_1\dots c_4})_\beta{}^\delta\phi_{c_1\dots c_4}\right]\\&=2\underbrace{\mathrm{Tr}[\Gamma^{b_1b_2}\Gamma_a\Gamma_{(5)}]}_{=0}\phi_{b_1b_2}+2\mathrm{Tr}(\Gamma^{c_1\dots c_4}\Gamma_a\Gamma_{a_1\dots a_5})\phi_{c_1\dots c_4}\\&=2\mathrm{Tr}[\eta_{ab}\Gamma^{c_1\dots c_4}\Gamma^b\Gamma_{a_1\dots a_5}]\phi_{c_1\dots c_4}\\&=2\cdot16\left(5\eta_{ab}\delta^{bc_1c_2c_3c_4}_{[a_1a_2a_3a_4a_5]}\right)\phi_{c_1\dots c_4}+\eta_{ab}\epsilon^{c_1\dots c_4b}{}_{a_1\dots a_5}\phi_{c_1\dots c_4}\mathrm{Tr}\Gamma^{11}\\&=2\cdot16\cdot5\eta_{a[a_1}\phi_{a_2\dots a_5]}+\eta_{ab}\epsilon^{c_1\dots c_4b}{}_{a_1\dots a_5}\phi_{c_1\dots c_4}\mathrm{Tr}\Gamma^{11}.\end{aligned}\tag{C.142}$$

Then setting $a=a_5$ we see that $\phi_{c_1\dots c_4}=0$, giving us $\mathcal{D}_\alpha\chi^\beta=(\Gamma^{b_1b_2})_\alpha{}^\beta\phi_{b_1b_2}$.

Putting this into $ii)$ in Eq.(C.141) we get

$$F_{ab}=\frac{i}{16}(\Gamma_{ab})^\alpha{}_\beta(\Gamma^{cd})_\alpha{}^\beta\phi_{cd}=-\frac{i}{16}\underbrace{\mathrm{Tr}(\Gamma_{ab}\Gamma^{cd})}_{=-2\cdot16\delta^{cd}_{ab}}\phi_{cd}=2i\delta^{cd}_{ab}\phi_{cd}=2i\phi_{ab}.\tag{C.143}$$

This implies that $\phi_{ab}=-\frac{i}{2}F_{ab}$, so that

$$\mathcal{D}_\alpha\chi^\beta=-\frac{i}{2}(\Gamma^{ab})_\alpha{}^\beta F_{ab}.\tag{C.144}$$

Looking back at Eq.(C.137) we can now see that there are only two ordinary spacetime fields present in the theory, namely $F_{ab}(x)$ and $\chi^\alpha(x)$. The first equation tells us that the θ-expansion of F_{ab} will have χ^α at first order in θ and the second equation tells us that it will come back to F_{ab} at the next order.

However, when we did Wess-Zumino earlier we saw that we needed auxiliary fields to make it supersymmetric off-shell, but they are not present here. This means that we should have some equations of motion hidden in Eq.(C.137).

If we write down Dirac's equation for χ^α

$$\begin{aligned}(\not{\mathcal{D}}\chi)_\alpha &= (\Gamma^a)_{\alpha\beta}\mathcal{D}_a\chi^\beta = -\frac{1}{2i}\{\mathcal{D}_\alpha,\mathcal{D}_\beta\}\chi^\beta = \frac{i}{2}(\mathcal{D}_\alpha\mathcal{D}_\beta+\mathcal{D}_\beta\mathcal{D}_\alpha)\chi^\beta \\ &= \frac{i}{2}\mathcal{D}_\beta(-\frac{i}{2}(\Gamma^{ab})_\alpha{}^\beta F_{ab}) = \frac{1}{4}(\Gamma^{ab})_\alpha{}^\beta\mathcal{D}_\beta F_{ab}.\end{aligned} \tag{C.145}$$

Using the second equation in Eq.(C.133) we can write this as

$$\begin{aligned}-\frac{1}{2}(\Gamma^{ab})_\alpha{}^\beta\mathcal{D}_{[a}F_{b]\beta} &= -\frac{1}{2}(\Gamma^{ab})_\alpha{}^\beta(\mathcal{D}_aF_{b\beta}-\mathcal{D}_bF_{a\beta}) = -\frac{1}{2}(\Gamma^{ab})_\alpha{}^\beta\left(\mathcal{D}_a(\Gamma_b)_{\beta\gamma}\chi^\gamma-\mathcal{D}_b(\Gamma_a)_{\beta\gamma}\chi^\gamma\right) \\ &= -\frac{1}{2}\Big(\mathcal{D}_a(\Gamma^{ab}\Gamma_b)_{\alpha\gamma})-\mathcal{D}_b(\Gamma^{ab}\Gamma_a)_{\alpha\gamma}\Big)\chi^\gamma = -\frac{1}{2}\left(-18\mathcal{D}_a(\Gamma^a)_{\alpha\gamma}\chi^\gamma\right) \\ &= 9(\not{\mathcal{D}}\chi)_\alpha.\end{aligned} \tag{C.146}$$

But $(\not{\mathcal{D}}\chi)_\alpha = 9(\not{\mathcal{D}}\chi)_\alpha$ must imply that $(\not{\mathcal{D}}\chi)_\alpha = 0$ which is just Dirac's equation in absence of any sources.

If we instead take the first equation in Eq.(C.137) and contract it with $\mathcal{D}_\beta(\Gamma^a)^{\beta\gamma}$ we get, for the first term

$$\mathcal{D}_\beta(\Gamma^a)^{\beta\gamma}\mathcal{D}_\gamma F_{ab} = \frac{1}{2}(\Gamma^a)^{\beta\gamma}\{\mathcal{D}_\beta,\mathcal{D}_\gamma\}F_{ab} = -i(\Gamma^a)^{\beta\gamma}(\Gamma_c)_{\beta\gamma}\mathcal{D}^cF_{ab} = -16i\mathcal{D}^aF_{ab}, \tag{C.147}$$

where we have used that $(\Gamma^a)_{\alpha\beta}$ is symmetric, and that $\{\mathcal{D}_\beta,\mathcal{D}_\gamma\} = -2i(\Gamma_c)_{\beta\gamma}\mathcal{D}^c$. For the next term we instead find

$$\begin{aligned}2\mathcal{D}_\beta(\Gamma^a)^{\beta\gamma}(\Gamma_{[b})_{|\gamma\delta|}\mathcal{D}_{a]}\chi^\delta &= \mathcal{D}_\beta\Big((\Gamma^a\Gamma_a)^\beta{}_\delta\mathcal{D}_b\chi^\delta-(\Gamma^a\Gamma_b)^\beta{}_\delta\mathcal{D}_a\chi^\delta\Big) \\ &= \mathcal{D}_\beta\left(10\mathcal{D}_b\chi^\beta-2\mathcal{D}_b\chi^\beta+(\Gamma_b)^{\beta\gamma}\underbrace{(\Gamma_a)_{\gamma\delta}\mathcal{D}_a\chi^\delta}_{=(\not{\mathcal{D}}\chi)_\gamma=0}\right) \\ &= 8\mathcal{D}_\beta\mathcal{D}_b\chi^\beta = 8\Big([\mathcal{D}_\beta,\mathcal{D}_b]+\underbrace{\mathcal{D}_b\mathcal{D}_\beta}_{=0}\Big)\chi^\beta \\ &= 8\Big(-(\Gamma_b)_{\alpha\beta}\{\chi^\alpha,\chi^\beta\}\Big).\end{aligned} \tag{C.148}$$

Combining the two terms we get the Yang-Mills equations of motion with a source term

$$\mathcal{D}^aF_{ab} = -\frac{i}{2}(\Gamma_b)_{\alpha\beta}\{\chi^\alpha,\chi^\beta\}. \tag{C.149}$$

Off-shell Yang-Mills with pure spinors

The constraint we put on F_{AB} above gave us a set of equations only valid on-shell, i.e. we did not find any auxiliary fields. To find the off-shell behaviour of the theory we should instead try to find some lesser constraints to put on the component fields. This turns out to be a rather complicated subject that demands the introduction of so called *pure* spinors. We will not get into this here, but it has been proposed that instead of putting $F_{\alpha\beta} = 0$ we can set

$$F_{\alpha\beta} = (\Gamma^a)_{\alpha\beta} F_a + (\Gamma^{a_1 a_2 \ldots a_5})_{\alpha\beta} F_{a_1 a_2 \ldots a_5}, \tag{C.150}$$

which may lead us to an off-shell super-Yang-Mills theory [57].

C.7.2 Supertransformations and the Lagrangian

Now that we have investigated the Bianchi identities we want to find the supertransformations of the theory and see what kind of a supersymmetric Lagrangian we can build using our fields.

The supertransformations are $\delta_\varepsilon\, \chi^\alpha\big|_{\theta=0} \equiv -i\bar{\varepsilon}Q\, \chi^\alpha\big|_{\theta=0}$, and at $\theta = 0$ we can change $Q \to \mathcal{D}$, but we have to be careful, because $\mathcal{D}$ also contains gauge fields. At $\theta = 0$ we have the transformation

$$\delta_\varepsilon\, \chi^\alpha\bigg|_{\theta=0} = -i\bar{\varepsilon}^\beta \mathcal{D}_\beta\, \chi^\alpha\bigg|_{\theta=0} + i\bar{\varepsilon}(\chi^\alpha A + A\chi^\alpha). \tag{C.151}$$

The gauge fields appearing here are something we want to get rid of. This means that we should probably do a gauge transformation. Since we normally have

$$\delta_\Lambda \chi^\alpha = \chi^\alpha \Lambda - \Lambda \chi^\alpha, \tag{C.152}$$

we can see that choosing $\Lambda = i\bar{\varepsilon}^\beta A_\beta$ means that we can define a supersymmetry transformation as

$$\delta_s \equiv \delta_\varepsilon + \delta_\Lambda \implies \delta_s \chi^\alpha = -i\bar{\varepsilon}^\beta \mathcal{D}_\beta \chi^\alpha = \frac{1}{2}\bar{\varepsilon}^\beta (\Gamma^{ab})_\beta{}^\alpha F_{ab}. \tag{C.153}$$

The transformation of F_{ab} is instead given by

$$\begin{aligned} \delta_s F_{ab} &= -i\bar{\varepsilon}Q F_{ab} + F_{ab}\Lambda - \Lambda F_{ab} = -i\bar{\varepsilon}^\alpha \mathcal{D}_\alpha F_{ab} \\ &= -i\bar{\varepsilon}^\alpha \left(2(\Gamma_{[a})_{|\alpha\beta|} \mathcal{D}_{b]} \chi^\beta\right) = -2i\bar{\varepsilon}^\alpha (\Gamma_{[a})_{|\alpha\beta|} \mathcal{D}_{b]} \chi^\beta. \end{aligned} \tag{C.154}$$

Now, what Lagrangian can we build from our two available fields that is invariant under these SUSY-transformations? We have one field that satisfies the Dirac equation and one that satisfies the Yang-Mills equation, so we should have two terms giving these equations. Since both fields are in the adjoint representation we should also have a trace over the generators. This gives us the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[F^{ab}F_{ab}\right] \pm i\mathrm{Tr}\left[\chi^\alpha (\not{\mathcal{D}}\chi)_\alpha\right]. \tag{C.155}$$

The sign of the Dirac term must be checked, because we have created the field from a superfield without knowing what it was going to turn out to be. The sign should be such that the Yang-Mills equations of motion with a source is correct. To check this we vary our Lagrangian with respect to the components A^r_a. We have

$$\frac{\delta\mathcal{L}}{\delta A^r_a} = \mathrm{Tr}\left[-\delta_A F^{ab}F_{ab} \pm i\chi^\alpha (\Gamma^a)_{\alpha\beta} i\{\delta A_a, \chi^\beta\}\right] = \mathrm{Tr}\left[\left(2\mathcal{D}^a F_{ab} \mp (\Gamma_b)\{\chi^\alpha, \chi^\beta\}\right)\delta A^b\right], \tag{C.156}$$

and we see that the minus sign on the Dirac term gives us the same sign as before for the equations of motion. This means that our Lagrangian is

$$\mathcal{L} = -\mathrm{Tr}\left[\frac{1}{2}F^{ab}F_{ab} + i\chi^\alpha (\not{\mathcal{D}}\chi)_\alpha\right]. \tag{C.157}$$

To see if this Lagrangian is invariant under the SUSY-transformation we simply vary it

$$\delta_s\mathcal{L} = -\mathrm{Tr}\left[F^{ab}\delta_s F_{ab} + i\delta_s\chi^\alpha(\not{\mathcal{D}}\chi)_\alpha + i\chi^\alpha(\not{\mathcal{D}}\delta_s\chi)_\alpha\right]. \tag{C.158}$$

The trace makes it possible for us to do partial integration on one of the Dirac terms, and when we move them around the minus sign from the partial integration will be cancelled by the anti-commuting properties of χ^α, so we will get two equal terms that add up. Using this and plugging in the supersymmetry transformations of the fields we get

$$\begin{aligned}\delta_s\mathcal{L} &= -\mathrm{Tr}\left[-2iF^{ab}\bar{\varepsilon}^\alpha(\Gamma_a)_{\alpha\beta}\mathcal{D}_b\chi^\beta + 2\bar{\varepsilon}^\beta F_{ab}(\Gamma^{ab})_\beta{}^\alpha(\Gamma^c)_{\alpha\gamma}\mathcal{D}_c\chi^\gamma\right]\\ &= \mathrm{Tr}\left[2iF^{ab}\bar{\varepsilon}^\alpha(\Gamma_a)_{\alpha\beta}\mathcal{D}_b\chi^\beta - i\bar{\varepsilon}^\alpha F_{ab}(\Gamma^{ab}\Gamma^c)_{\alpha\beta}\mathcal{D}_c\chi^\beta\right]\\ &= \mathrm{Tr}\left[2iF^{ab}\bar{\varepsilon}^\alpha(\Gamma_a)_{\alpha\beta}\mathcal{D}_b\chi^\beta - 2iF^{ab}\bar{\varepsilon}^\alpha(\Gamma_a)_{\alpha\beta}\mathcal{D}_b\chi^\beta\right] = 0.\end{aligned} \tag{C.159}$$

So our Lagrangian is invariant under the SUSY-transformations. We will leave the super-Yang-Mills theory here and move on to supergravitation in 11 dimensions. This will in great deal follow the same procedure as in this section, but with an increased level of complexity.

C.8 Supergravity in D=11

We have now reached the final supersymmetric theory to be discussed in this appendix, eleven-dimensional supergravity. The outline of this section will in large part follow the procedure of super-Yang-Mills. Starting with finding the Bianchi identities, then impose some constraints to be able to solve them and find the equations of motion. We will then use these to construct a Lagrangian and find supersymmetry transformations (from the supertranslations) that leave this Lagrangian invariant.

Since the procedure largely follows the last section we omit many of the calculations, and instead simply mention the important steps.

Note that supergravity is synonymous with local supersymmetry. In fact, one cannot have a local supersymmetry without gravity, since the superalgebra will contain local translation parameters, i.e. diffeomorphisms, giving us general relativity [21].

There are two interesting coincidences when we consider eleven-dimensional theories. Eleven dimensions is the highest dimension allowing supersymmetry, if we do not consider higher spin theories, when compactifying to four dimensions [58]. Seven extra dimensions is also the lowest number that can contain the $SU(3)\times SU(2)\times U(1)$ gauge group of the Standard model [58]. This seems to point at some deep relevance of eleven-dimensional theories. In fact, our best known candidate for a theory of everything lives in exactly eleven dimensions. This is M-theory, and its low energy limit is eleven-dimensional supergravity.

In eleven-dimensional supergravity we only have three spacetime fields present. These are the vielbein (or graviton), $e_m{}^a$, the spinor (gravitino), ψ_m, and a three index field B_{mnp} (similar to a Maxwell field) [6]. The corresponding field strengths are the curvature, R, the spinor field strength, $S_{mn}\sim\partial_{[a}\psi_{b]}$, and $H = dB$.

The curvature is a Lie algebra valued two form which we define in superspace using the connection as [54]

$$R_A{}^B = \frac{1}{2}E^C E^D R_{DCA}{}^B = dz^M dz^N \partial_N\omega_{MA}{}^B + dz^M\omega_{MA}{}^C dz^N\omega_{NC}{}^B. \tag{C.160}$$

C.8.1 Bianchi identities

The three fields of $d = 11$ supergravity obey the Bianchi identities

$$\begin{aligned} \mathcal{D}T^A &= E^B R_B{}^A, \\ \mathcal{D}R^A{}_B &= 0, \\ dH &= 0. \end{aligned} \tag{C.161}$$

The first two were derived in Section C.6, and the last one follow from the fact that $H \equiv dB$, and $d^2B = 0$.

There is a theorem by Dragon which states that, for $d > 3$ ($d = 3$ is different due to the so called Dragon window [59]), the curvature automatically solves the second equation if the first equation is satisfied [60]. So we need not analyse the second equation any further.

From the first equation we find

$$\mathcal{D}T^D = E^C R_C{}^D \implies \mathcal{D}_{[A}T_{BC\}}{}^D + T_{[AB}{}^E T_{|E|C\}}{}^D = R_{[ABC\}}{}^D, \tag{C.162}$$

while the last equation tells us

$$\begin{aligned} 0 = dH &= d\left(\frac{1}{4!}E^D E^C E^B E^A H_{ABCD}\right) \\ &= \frac{1}{4!}E^D \dots E^A \mathcal{D}H_{ABCD} + \frac{1}{4!}E^D E^C E^B \underbrace{\mathcal{D}E^G}_{=T^G} H_{GBCD} + 3 \text{ more terms} \\ &= \frac{1}{4!}E^D E^C E^B E^A E^F \mathcal{D}_{[F}H_{ABCD\}} + \frac{1}{4!}\frac{1}{2}E^D E^C E^B E^A E^F (-1)^{AF} T_{[AF}{}^G H_{|G|BCD\}} + 3 \text{ more terms} \\ &\implies \mathcal{D}_{[A}H_{BCDE\}} + 2T_{[AB}{}^F H_{|F|CDE\}} = 0. \end{aligned} \tag{C.163}$$

So the Bianchi identities are now

$$\mathcal{D}_{[A}T_{BC\}}{}^D + T_{[AB}{}^E T_{|E|C\}}{}^D = R_{[ABC\}}{}^D, \tag{C.164}$$

$$\mathcal{D}_{[A}H_{BCDE\}} + 2T_{[AB}{}^F H_{|F|CDE\}} = 0. \tag{C.165}$$

Next, we will investigate what constraints one can impose on the fields of the theory.

C.8.2 The constraints

We start by studying the dimensions of the different parts of $T_{AB}{}^C$ and H_{ABCD}. The rules for counting the dimensions are that a spinor index sitting downstairs will contribute $+1/2$ (in dimensions of inverse length), while if it instead sits upstairs it will give $-1/2$. The bosonic indices gives ± 1 respectively. $H_{(4)}$ has dimension -3 and T dimension 0. This gives us

	dimension (L^{-1})
$T_{\alpha\beta}{}^\gamma$	$1/2$
$T_{\alpha\beta}{}^c$	0
$T_{\alpha b}{}^\gamma$	1
$T_{\alpha b}{}^c$	$1/2$
$T_{ab}{}^\gamma$	$3/2$
$T_{ab}{}^c$	1

	dimension (L^{-1})
$H_{\alpha\beta\gamma\delta}$	-1
$H_{a\beta\gamma\delta}$	$-1/2$
$H_{ab\gamma\delta}$	0
$H_{abc\delta}$	$1/2$
H_{abcd}	1

The available field strengths in the theory have dimensions

	dim (L^{-1})
R_{abcd}	2
S_{ab}	3/2
H_{abcd}	1

One can also have something with dimension zero, which can be put proportional to the gamma matrices. This means that we put all components of T and H with other dimensions to zero. The only surviving parts of T are $T_{\alpha b}{}^{\gamma}$, $T_{ab}{}^{\gamma}$, $T_{ab}{}^{c}$ and $T_{\alpha\beta}{}^{c}$ which, because it's dimension is zero, we normalise as $2i(\Gamma^c)_{\alpha\beta}$. Since the spin connection is undetermined up to a tensor we can also make a choice to put $T_{ab}{}^{c} = 0$.

For H we only have two surviving parts, namely H_{abcd} and $H_{ab\gamma\delta}$, which as with the corresponding T-part is put equal to $2i(\Gamma_{ab})_{\gamma\delta}$.

Plugging these constraints into the Bianchi identities of Eq.(C.164) we will get eight equations from the first identity and seven from the last.

For example we can start with the $(\beta\gamma\delta, a)$ equation. This gives us

$$\mathcal{D}_{(\beta}T_{\gamma\delta)}{}^{a} + T_{(\beta\gamma}{}^{e}T_{|e|\delta)}{}^{a} + T_{(\beta\gamma}{}^{\epsilon}T_{|\epsilon|\delta)}{}^{a} = R_{(\beta\gamma\delta)}{}^{a}. \tag{C.166}$$

Using that $T_{\alpha\beta}{}^{\gamma} = T_{a\beta}{}^{c} = 0$, and $\mathcal{D}\Gamma^a = 0$, we see that the left hand side is zero. The right hand side has $R_{(\beta\gamma\delta)}{}^{a}$, but this is identically zero since the two last indices on the Riemann tensor are Lorentz indices, which do not mix fermionic and bosonic. So this equation only tells us that $0 = 0$. Doing similar manipulations on the other equations we end up with

$(\beta\gamma\delta, a)$	$0 = 0$
$(\beta\gamma\delta, \alpha)$	$2i(\Gamma^e)_{(\beta\gamma}T_{\vert e\vert\delta)}{}^{\alpha} = R_{(\beta\gamma\delta)}{}^{\alpha}$
$(\beta\gamma d, a)$	$4iT_{d(\beta}{}^{\epsilon}(\Gamma^a)_{\vert\epsilon\vert\gamma)} = R_{\beta\gamma d}{}^{a}$
$(b\gamma\delta, \alpha)$	$\mathcal{D}_{(\gamma}T_{\delta)b}{}^{\alpha} + i(\Gamma^e)_{\gamma\delta}T_{eb}{}^{\alpha} = R_{b(\gamma\delta)}{}^{\alpha}$
$(\beta cd, a)$	$i(\Gamma^a)_{\alpha\beta}T_{cd}{}^{\alpha} = R_{\beta[cd]}{}^{a}$
$(bc\delta, \alpha)$	$\mathcal{D}_{\delta}T_{bc}{}^{\alpha} + 2\mathcal{D}_{[b}T_{c]\delta}{}^{\alpha} + 2T_{\delta[b}{}^{\epsilon}T_{\vert\epsilon\vert c]}{}^{\alpha}$
(bcd, a)	$0 = R_{[bcd]}{}^{\alpha}$
(bcd, α)	$\mathcal{D}_{[b}T_{cd]}{}^{\alpha} + T_{[bc}{}^{\epsilon}T_{\vert\epsilon\vert d]}{}^{\alpha} = 0$

$(\alpha\beta\gamma\delta\epsilon)$	$0 = 0$
$(a\beta\gamma\delta\epsilon)$	$(\Gamma_a)_{(\alpha\beta}(\Gamma^{ab})_{\gamma\delta)} = 0$
$(ab\gamma\delta\epsilon)$	$0 = 0$
$(a\beta c\delta\epsilon)$	$0 = 0$
$(abc\delta\epsilon)$	$(\Gamma^f)_{\delta\epsilon}H_{fabc} = 6T_{(\epsilon[a}{}^{\eta}(\Gamma_{bc]})_{\vert\eta\vert\delta)}$
$(abcd\epsilon)$	$\mathcal{D}_{\epsilon}H_{abcd} + 12iT_{[ab}{}^{\eta}(\Gamma_{bc]})_{\eta\epsilon} = 0$
$(abcde)$	$\mathcal{D}_{[a}H_{bcde]} = 0$

In 11 dimensions $(\Gamma_a)_{(\alpha\beta}(\Gamma^{ab})_{\gamma\delta)}$ is identically zero. This is shown by contracting with the symmetric gamma matrices. In fact, it suffices to contract with $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(5)}$ since the contribution from the other symmetric matrices are given by the mirroring in $\Gamma^{(11)}$.

From dimensional and index structure arguments we can see that we can put

$$T_{a\beta}{}^{\gamma} = xH_{abcd}(\Gamma^{bcd})_{\beta}{}^{\gamma} + yH_{a_1a_2a_3a_4}(\Gamma_{a}{}^{a_1a_2a_3a_4}), \tag{C.167}$$

with x and y being some constants. These can be solved for using the $(\beta\gamma\delta, \alpha)$ and $(\beta\gamma d, a)$ equations of the left table and the $(abc\delta\epsilon)$ equation of the right table (these are the equations of dimension 1). One needs to follow the example of the Yang-Mills theory and contract with all symmetric gamma matrices. After a bit of manipulation one finds $x = -\frac{1}{36}$ and $y = -\frac{1}{288}$.

The last torsion component, $T_{ab}{}^{\gamma}$ can be decomposed into irreducible parts (as we did for one of the components of F in Yang-Mills)

$$T_{ab}{}^{\gamma} = G_{ab}{}^{\gamma} + 2G_{[a}{}^{\beta}(\Gamma_{b]})_{\beta}{}^{\gamma} + G^{\beta}(\Gamma)_{\beta}{}^{\gamma}, \tag{C.168}$$

with

$$G_{ab}{}^{\gamma}(\Gamma^b)_{\gamma}{}^{\alpha} = G_a{}^{\beta}(\Gamma^a)_{\beta}{}^{\alpha} = 0. \tag{C.169}$$

This means that we have

$$\begin{aligned} T_{ab}\Gamma^b &= G_a\Gamma_b\Gamma^b - G_b\Gamma_a\Gamma^b + G\Gamma_{ab}\Gamma^b \\ &= 11G_a - G_b(2\delta_a^b - \Gamma^b\Gamma_a) + 10G\Gamma_a \\ &= 9G_a + 10G\Gamma_a, \end{aligned} \tag{C.170}$$

$$T_{ab}\Gamma^{ab} = -T_{ab}\Gamma^b\Gamma^a = -9G_a\Gamma^a - 10G\Gamma_a\Gamma^a = -110G, \tag{C.171}$$

and

$$T_{ab}\Gamma^{abc} = T_{ab}(\Gamma^{ab}\Gamma^c - 2\Gamma^{[a}\eta^{b]c}) = 18G^c - 90G\Gamma^c. \tag{C.172}$$

Using this in the equations of dimension 3/2, and again contracting with the symmetric gamma matrices, we obtain conditions on G and G_a, namely that they are zero. It is also seen that $G_{ab}{}^{\gamma}$ drops out. This is good, because $G_{ab}{}^{\gamma}$ is the spin 3/2 analogue of the Weyl tensor in ordinary general relativity. If this did not drop out of Einstein's equations the geometry would be fixed.

The constraints on G and G_a means that

$$T_{ab}\Gamma^a = 0, \tag{C.173}$$

$$T_{ab}\Gamma^{ab} = 0, \tag{C.174}$$

$$T_{ab}\Gamma^{abc} = 0. \tag{C.175}$$

The last equation is the equation of motion for the spin 3/2 field (the gravitino), since the Lagrangian is $\sim \psi_a\Gamma^{abc}\partial_b\psi_c$. Varying this we find $\partial_b\psi_c\Gamma^{abc} = 0$, and we can see from the index structure that $T_{ab}{}^{\gamma} \sim \partial_{[a}\psi_{b]}$.

From the equations of dimension 2 we find the equations of motion for R and H. These are Einstein's equations for R,

$$R_{ab} - \frac{1}{2}\eta_{ab}R = \frac{1}{96}\eta_{ab}H^2 - \frac{1}{12}H_{ac_1c_2c_3}H_b{}^{c_1c_2c_3}, \tag{C.176}$$

with $H^2 = H_{abcd}H^{abcd}$, and for H we get

$$\mathcal{D}^d H_{dabc} = -\frac{1}{1152}\epsilon_{abc}{}^{(4)(\tilde{4})}H_{(4)}H_{(\tilde{4})}. \tag{C.177}$$

Lastly, the dimension 5/2 equation does not give us any new information, it's simply the Bianchi identity for the spin 3/2 field.

We have now solved the Bianchi identities, and found three equations of motion. Next we go on to construct supersymmetry transformations of the fields, and then write down a Lagrangian that is invariant under these transformations.

C.8.3 The supersymmetric transformations

In this section we will define supergauge transformations and see how the different fields transform under these. These transformations are then combined with Lorentz transformations in order to construct the sought supersymmetric transformations. This corresponds to what we did in the Yang-Mills case where we combined the coordinate transformation from superspace with the gauge transformation of the structure group of Yang-Mills, in Eq.(C.153).

We use a parameter ξ^A to characterise an infinitesimal change in coordinates. The transformation of a tensor superfield will then be [54]

$$\delta_\xi V^A = -\xi^M \partial_M V^A + V^B {L_B}^A = -\xi^B {E_B}^M \partial_M V^A + V^B {L_B}^A, \tag{C.178}$$

where ${L_B}^A$ is the representation of the Lorentz group that corresponds to the tensor structure of V. To make this covariant under Lorentz transformations it is necessary to change the derivative to a covariant derivative,

$$\mathcal{D}_M V^A = \partial_M V^A + (-)^{MB} V^B {\omega_{MB}}^A, \tag{C.179}$$

and we write $\mathcal{D}_B V^A = {E_B}^M \mathcal{D}_M V^A$. This gives us

$$\delta_\xi V^A = -\xi^B \mathcal{D}_B V^A + V^B \xi^C {\omega_{CB}}^A + V^B {L_B}^A. \tag{C.180}$$

Since the connection is Lie algebra valued $\xi^C {\omega_{CB}}^A$ acts as a field-dependent Lorentz transformation on V^B. This means that one can put ${L_B}^A = -\xi^C {\omega_{CB}}^A$ resulting in the manifestly covariant (under Lorentz transformations)

$$\delta_\xi V^A = -\xi^C \mathcal{D}_C V^A, \tag{C.181}$$

for any tensor field V^A. In particular, since the vielbein generally transforms as

$$\begin{aligned} \delta {E_M}^A &= -\xi^L \partial_L {E_M}^A - \partial_M \xi^L {E_L}^A + {E_M}^B {L_B}^A \\ &= -\xi^L \big(\partial_L {E_M}^A - (-)^{LM} \partial_M {E_L}^A\big) - \partial_M \xi^A + {E_M}^B {L_B}^A \\ &= -\partial_M \xi^A - \xi^L \big({T_{LM}}^A - {\omega_{LM}}^A + (-)^{ML} {\omega_{ML}}^A\big) + {E_M}^B {L_B}^A, \end{aligned} \tag{C.182}$$

and the connection combines with $\partial_M \xi^A$ to form a covariant derivative, the supergauge transformation of the vielbein will be

$$\delta_\xi {E_M}^A = -\mathcal{D}_M \xi^A - \xi^B {T_{BM}}^A. \tag{C.183}$$

The connection generally transforms as

$$\delta {\omega_{MA}}^B = -\xi^L \partial_L {\omega_{MA}}^B - \partial_M \xi^L {\omega_{LA}}^B + {\omega_{MA}}^C {L_C}^B - (-)^{M(A+C)} {L_A}^C {\omega_{MC}}^B - \partial_M {L_A}^B, \tag{C.184}$$

which means that the supergauge transformation is

$$\delta_\xi {\omega_{MA}}^B = -\xi {R_{CMA}}^B, \tag{C.185}$$

using the definition of ${R_{MNA}}^B$ in Eq.(C.160).

When we did Yang-Mills we combined the supertransformation with a transformation from the structure group. We do a similar thing here and write the supersymmetric transformation of the vielbein as a supergauge transformation combined with an additional Lorentz transformation

$$\delta {E_M}^A = -\mathcal{D}_M \xi^A - \xi^B {T_{BM}}^A + {E_M}^B {L_B}^A. \tag{C.186}$$

Using the higher components of ξ^A, which enter in the derivatives, we can transform the $\theta = 0$ components of the vielbein, and it's inverse, into the forms

$$E_M{}^A\Big|_{\theta=0} = \begin{pmatrix} e_m{}^a(x) & \psi_m{}^\alpha(x) \\ 0 & \delta_\mu{}^\alpha \end{pmatrix}, \qquad E_A{}^M\Big|_{\theta=0} = \begin{pmatrix} e_a{}^m(x) & -\psi_a{}^\mu(x) \\ 0 & \delta_\alpha{}^\mu \end{pmatrix}. \tag{C.187}$$

Note that we henceforth drop the $\theta = 0$ and only write $E_M{}^A\Big|$.

The transformation of the connection can be written in the same way as a combination of a supergauge transformation and a Lorentz one

$$\delta\omega_{MA}{}^B = -\xi^C R_{CMA}{}^B + \omega_{MA}{}^C L_C{}^B - (-)^{M(A+C)} L_A{}^C \omega_{MC}{}^B - \partial_M L_A{}^B. \tag{C.188}$$

Since the connection is Lie algebra valued we can gauge away $\omega_\mu{}^{AB}\Big|$ which means that

$$\begin{aligned} \omega_{mA}{}^B\Big|(z) &= \omega_{mA}{}^B(x), \\ \omega_{\mu A}{}^B\Big|(z) &= 0. \end{aligned} \tag{C.189}$$

We can put the $\theta = 0$ components of ξ^α to ε^α and the zero-components of ξ^a and L_{AB} to zero, while choosing higher components to preserve the gauge. This is in line with what we want to do, since it is the ξ^α components that parameterise the local supersymmetry transformations.

This implies that we must have

$$\delta E_\mu{}^A\Big| = 0. \tag{C.190}$$

Using our results from before, regarding which torsion components are non-zero, we can evaluate this as

$$\begin{aligned} \delta E_\mu{}^A\Big| &= \left(-\partial_\mu - \xi^b\omega_{\mu b}{}^a - \xi^\beta T_{\beta\mu}{}^a + E_\mu{}^b L_b{}^a\right)\Big| \\ &= -\partial_\mu \xi^a\Big| - \varepsilon^\beta 2i(\Gamma^a)_{\beta\mu} = 0 \\ &\implies \xi^a = -2i(\Gamma^a)_{\beta\mu}\varepsilon^\beta\theta^\mu. \end{aligned} \tag{C.191}$$

The transformation of the graviton can be found from

$$\delta e_m{}^a = \delta E_m{}^a\Big| = -\mathcal{D}_m\xi^a\Big| - \xi^B T_{Bm}{}^a\Big| + e_m{}^B L_B{}^a\Big| = -\xi^\beta T_{\beta m}{}^a\Big|, \tag{C.192}$$

where

$$T_{\beta m}{}^a = E_m{}^C T_{\beta C}{}^a = E_m{}^\gamma T_{\beta\gamma}{}^a = 2i\psi_m{}^\gamma(\Gamma)_{\beta\gamma}. \tag{C.193}$$

This gives us

$$\delta e_m{}^a = -2i\varepsilon^\beta\psi_m{}^\gamma(\Gamma^a)_{\beta\gamma}. \tag{C.194}$$

Moving on to the gravitino we have

$$\delta\psi_m{}^\alpha = \delta E_m{}^\alpha\Big| = -\mathcal{D}_m\xi^\alpha\Big| - \xi^\beta T_{\beta m}{}^\alpha\Big|. \tag{C.195}$$

We use results from before to write

$$T_{\beta m}{}^\alpha = E_m{}^C T_{\beta C}{}^\alpha = E_m{}^c T_{\beta c}{}^\alpha = e_m{}^c\left(\frac{1}{36}H_{ca_1a_2a_3}(\Gamma^{a_1a_2a_3})_\beta{}^\alpha + \frac{1}{288}H_{a_1\ldots a_4}(\Gamma_c{}^{a_1\ldots a_4})\right). \tag{C.196}$$

So that the transformation of the gravitino is

$$\begin{aligned}\delta\psi_m &= -\mathcal{D}_m\varepsilon(x) - \varepsilon e_m{}^c\left(-\frac{1}{36}\delta_c^{[a_4}\Gamma^{a_1a_2a_3]} + \frac{1}{288}\Gamma_c{}^{a_1a_2a_3a_4}\right)H_{a_1a_2a_3a_4} \\ &= -\mathcal{D}_m\varepsilon(x) - \frac{1}{288}\varepsilon e_m{}^c\left(\Gamma_c{}^{a_1\dots a_4} + 8\Gamma^{[a_1a_2a_3}\delta_c^{a_4]}\right)H_{a_1\dots a_4} \\ &\equiv -\tilde{\mathcal{D}}_m\varepsilon(x).\end{aligned} \tag{C.197}$$

For the three index field B_{MNP} the general transformation is given by [54]

$$\begin{aligned}\delta B_{MNP} &= -\xi^L\partial_L B_{MNP} - 3\partial_{[M}\xi^L B_{|L|NP\}} \\ &= -\xi^L\left(\partial_L B_{MNP} - 3(-)^{LM}\partial_{[M}B_{|L|NP]}\right) - 3(\partial_{[M}\xi^L)B_{|L|NP]}.\end{aligned} \tag{C.198}$$

The last term is pure gauge and the supersymmetric transformation will be

$$\delta_\xi B_{MNP} = -\xi^L\left(\partial_L B_{MNP} - 3(-)^{LM}\partial_{[M}B_{|L|NP]}\right). \tag{C.199}$$

For the physical field, B_{mnp}, this is just

$$\begin{aligned}4\delta B_{mnp} &= -\varepsilon^\alpha H_{\alpha mnp} = -\varepsilon^\alpha E_p{}^D E_n{}^C E_m{}^B H_{\alpha BCD} = -3\varepsilon^\alpha E_p{}^{[d}E_n{}^c E_m{}^{\beta]}H_{\alpha\beta cd} \\ &= -6i\varepsilon^\alpha e_{[p}{}^d e_n{}^c \psi_{m]}{}^\beta(\Gamma_{cd})_{\alpha\beta} = 6i\varepsilon\psi_{[m}\Gamma_{np]} \\ &\implies \delta B_{mnp} = \frac{3}{2}i\varepsilon\Gamma_{[mn}\psi_{p]}.\end{aligned} \tag{C.200}$$

C.8.4 The Lagrangian

Now, we will rewrite the equations of motions in curved indices, since this is where the physical fields are. After doing this we simply write down a Lagrangian for these fields that gives the equations of motion and that is invariant under the supersymmetric transformations found in the previous section.

However, when going over to curved indices we will find that both the equations of motion and the transformations will be modified.

The only non-zero components of H_{ABCD} are H_{abcd} and $H_{ab\gamma\delta}$ (as we saw earlier), and this is of course also true when we go to curved indices. This means that we can write

$$\begin{aligned}H_{abcd} &= E_a{}^M E_b{}^N E_c{}^P E_d{}^Q H_{MNPQ} = e_a{}^m e_b{}^n e_c{}^p e_d{}^q H_{mnpq} + e_a{}^m e_b{}^n \psi_c{}^\rho\psi_d{}^\sigma H_{mn\rho\sigma} \\ &= e_a{}^m e_b{}^n e_c{}^p e_d{}^q\left(H_{mnpq} + \psi_p{}^\rho\psi_q{}^\sigma H_{mn\rho\sigma}\right) = e_a{}^m e_b{}^n e_c{}^p e_d{}^q\left(H_{mnpq} + 2i\psi_p{}^\rho\psi_q{}^\sigma(\Gamma_{mn})_{\rho\sigma}\right).\end{aligned} \tag{C.201}$$

One can use this to rewrite the equations of motion for H, Eq.(C.177). The left hand side gives us

$$\begin{aligned}\mathcal{D}^d H_{dabc} &= E_R{}^d\mathcal{D}^R\left[e_d{}^q e_a{}^m e_b{}^n e_c{}^p(H_{qmnp} + 2i\psi_n{}^\rho\psi_p{}^\sigma(\Gamma_{qm})_{\rho\sigma})\right] \\ &= e_r{}^d e_d{}^q e_a{}^m e_b{}^n e_c{}^p\mathcal{D}^r(H_{qmnp} + 2i\psi_n{}^\rho\psi_p{}^\sigma(\Gamma_{qm})_{\rho\sigma}) \\ &= \mathcal{D}^q\Big(H_{qmnp} + 2i\psi_n{}^\rho\psi_p{}^\sigma(\Gamma_{qm})_{\rho\sigma}\Big).\end{aligned} \tag{C.202}$$

The right hand side instead becomes

$$\begin{aligned}-\frac{1}{1152}\epsilon_{mnp}{}^{m_1\dots m_8}\Big(&H_{m_1\dots m_4}H_{m_5\dots m_8} + 2i\psi_{m_1}\psi_{m_2}\Gamma_{m_3m_4}H_{m_5\dots m_8} + 2iH_{m_1\dots m_4}\psi_{m_5}\psi_{m_6}\Gamma_{m_7m_8} \\ &- 4\psi_{m_1}\psi_{m_2}\Gamma_{m_3m_4}\psi_{m_5}\psi_{m_6}\Gamma_{m_7m_8}\Big).\end{aligned} \tag{C.203}$$

We can put in some anti-symmetrisations and move around the gamma matrices to write

$$-\frac{1}{1152}\epsilon_{mnp}{}^{m_1\dots m_8}\Big(H_{m_1\dots m_4}H_{m_5\dots m_8}+8i\psi_{[m_1}\Gamma_{m_2m_3}\psi_{m_4]}H_{m_5\dots m_8}+8iH_{m_1\dots m_4}\psi_{[m_5}\Gamma_{m_6m_7}\psi_{m_8]} \\ -64\psi_{[m_1}\Gamma_{m_2m_3}\psi_{m_4]}\psi_{[m_5}\Gamma_{m_6m_7}\psi_{m_8]}\Big). \tag{C.204}$$

If we do the same with the left hand side and then define $\tilde{H}_{mnpq}\equiv H_{mnpq}+8i\psi_{[m}\Gamma_{np}\psi_{q]}$ we find the equations of motion looking as we are used to see them

$$\mathcal{D}^q\tilde{H}_{qmnp}=-\frac{1}{1152}\epsilon_{mnp}{}^{m_1\dots m_8}\tilde{H}_{m_1\dots m_4}\tilde{H}_{m_5\dots m_8}. \tag{C.205}$$

We can do the same for the other fields and get the other two equations of motion as

$$\begin{aligned} &R_{mn}-\frac{1}{2}g_{mn}R=\frac{1}{96}g_{mn}\tilde{H}^2-\frac{1}{12}\tilde{H}_{mp_1p_2p_3}\tilde{H}_n{}^{p_1p_2p_3}, \\ &\Gamma^{mnp}\tilde{\mathcal{D}}_n\psi_p=0, \end{aligned} \tag{C.206}$$

where $\tilde{\mathcal{D}}_m=\mathcal{D}_m+\frac{1}{288}e_m{}^c\left(\Gamma_c{}^{a_1\dots a_4}+8\Gamma^{[a_1a_2a_3}\delta_c^{a_4]}\right)\tilde{H}_{a_1a_2a_3a_4}$. It is important to note that our definition of the spin connection has led us to an expression where the stress tensor has a relative sign as opposed to say M. Duff et al. [4], this means, for example, that in our convention AdS space will end up having a positive curvature scalar.

The transformations will also pick up an $\tilde{H}$, and we have

$$\begin{aligned} \delta e_m{}^a &= -2i\varepsilon(x)\Gamma^a\psi_m, \\ \delta\psi_m &= -\tilde{\mathcal{D}}_m\varepsilon(x), \\ \delta B_{mnp} &= \frac{3}{2}i\varepsilon\Gamma_{[mn}\psi_{p]}, \end{aligned} \tag{C.207}$$

From the available fields we can now build a Lagrangian. One can again follow the same procedure as in the Yang-Mills case. There we created a Lagrangian from the usual terms involving the available fields and then used the fact that the variation of the Lagrangian should give us back the equations of motion to set the relative signs and possible constants. The obvious way to start is with

$$\mathcal{L}=\frac{1}{4}ee_b{}^ne_a{}^mR_{mn}{}^{ab}+\frac{1}{2}\psi_m\Gamma^{mnp}\mathcal{D}_n\psi_p-\frac{1}{192}eH_{mnpq}H^{mnpq}, \tag{C.208}$$

But, when one starts to vary this one quickly notices that an extra term is needed to give the equations for H. This term will be something like $H\wedge H\wedge B$.

The Lagrangian will turn out to be [4, 21, 6]

$$\begin{aligned} \mathcal{L}=&\frac{1}{4}ee_b{}^ne_a{}^mR_{mn}{}^{ab}+\frac{1}{2}\psi_m\Gamma^{mnp}\mathcal{D}_n\psi_p-\frac{1}{192}eH_{mnpq}H^{mnpq} \\ &+\frac{1}{4\cdot(144)^2}e\epsilon^{m_1\dots m_{11}}H_{m_1\dots m_4}H_{m_5\dots m_8}B_{m_9\dots m_{11}} \\ &-\frac{3}{4\cdot 144}e\big(\psi_m\Gamma^{mnwxyz}\psi_n+12\psi^w\Gamma^{xy}\psi^z\big)(H_{wxyz}+\tilde{H}_{wxyz}), \end{aligned} \tag{C.209}$$

where the last terms comes from $\tilde{\mathcal{D}}$ and $\tilde{H}$. This Lagrangian is invariant under the supersymmetry transformations found before and gives the equations of motion when varied with respect to the fields.

With this we end our discussion on supersymmetry and supergravity.

D

Octonions

We here define the octonions and discuss some of their properties. The octonions naturally arise when we study the mass spectra of the squashed seven-sphere. It is therefore of interest to understand what they are and how they work.

D.1 Basics

The octonions are the last of the normed division algebras, the others being the real numbers, complex numbers and the quaternions, and are the generalisation of the complex numbers to include seven imaginary units. The seven imaginary units are defined to satisfy the multiplication rule

$$o_a o_b = -\delta_{ab} + a_{abc} o_c, \tag{D.1}$$

where a_{abc} are the totally anti-symmetric octonionic structure constants defined by [61]

$$a_{abc} = 1, \quad \text{for } (abc) = 123, 246, 435, 367, 651, 572, 714. \tag{D.2}$$

This can also be expressed by splitting the indices as $a = (\hat{\imath}, i, 7)$, where $\hat{\imath} = 1, 2, 3$, and $i = 4, 5, 6$. The non-zero parts are then

$$a_{7i\hat{\jmath}} = -\delta_{ij}, \qquad a_{ij\hat{k}} = -\epsilon_{ijk}, \qquad a_{\hat{\imath}\hat{\jmath}\hat{k}} = \epsilon_{ijk}. \tag{D.3}$$

The complete eight dimensional division algebra is given by adding the real unit $o_0 = 1$ to the set, and we denote the set of octonions as $\mathbb{O}$. That there are only four normed division algebras is a theorem due to Hurwitz [61]. Note that it is not true that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the only division algebras, it is actually rather easy to construct other types. They are, however, the only normed division algebras [61].

A division algebra can be defined as an algebra where the operations of left and right multiplication by any non-zero element can be inverted, and a normed division algebra is a division algebra that is also a normed vector space.

As can be seen from the multiplication rule the octonions are not assocative, i.e., $(o_a o_b) o_c \neq o_a (o_b o_c)$, but they are alternative. This means that the associater, defined by $[o_a o_b o_c] \equiv (o_a o_b) o_c - o_a (o_b o_c)$, is totally anti-symmetric. There is a theorem, connected to Hurwitz', which states that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the only alternative division algebras [61].

Using the multiplication laws of the octonions we can easily prove that, for example by evaluating $(o_a o_b) o_c \delta_{ab}$,

$$a_{acd} a_{bcd} = 6\delta_{ab}, \tag{D.4}$$

and we directly see that we also have $a_{abc} a_{abc} = 42$.

We can define the dual of a_{abc} as

$$c_{abcd} \equiv \frac{1}{6} \epsilon_{abcdefg} a_{efg}, \tag{D.5}$$

where we used the totally anti-symmetric levi-civita symbol, defined by $\epsilon_{1234567} = +1$. From this and the above definition of a_{abc} we can see that c_{abcd} satisfies

$$c_{abcd} = 1, \quad \text{for } (abcd) = 4567, 3751, 6172, 5214, 7423, 1346, 2635, \tag{D.6}$$

or

$$c_{ijk7} = \epsilon_{ijk}, \qquad c_{\hat{i}\hat{j}k7} = -\epsilon_{ijk}, \qquad c_{\hat{i}\hat{j}kl} = \delta^{ij}_{kl}. \tag{D.7}$$

Using this we also see that

$$a_{abe}a_{cde} = 2\delta^{cd}_{ab} - c_{abcd}. \tag{D.8}$$

In the next section we will discuss how one can relate the structure constants to the seven-dimensional gamma matrices using the Killing spinor. After this is done we will be able to derive further identities for contractions of a and c.

D.2 Octonions and the Killing spinor

We can also express the octonionic structure constants using the Killing spinor and the gamma matrices in seven dimensions. As usual we use the convention that these seven, anti-symmetric, matrices satisfy

$$\gamma^1 \cdots \gamma^7 = i\mathbb{1} \iff \gamma^{a_1 \ldots a_7} = i\epsilon^{a_1 \ldots a_7}\mathbb{1}. \tag{D.9}$$

We write

$$a_{abc} = -i\bar{\eta}\gamma_{abc}\eta, \tag{D.10}$$

where η is the Killing spinor and $\bar{\eta}\eta = 1$. Now, from this we find that

$$c_{abcd} \equiv \frac{1}{6}\epsilon_{abcdefg}a_{efg} = -\frac{i}{6}\bar{\eta}\epsilon_{abcdefg}\gamma_{efg}\eta = -\frac{1}{6}\bar{\eta}\gamma^{abcdefg}\gamma_{efg}\eta = \bar{\eta}\gamma_{abcd}\eta. \tag{D.11}$$

There is a very useful Fierz identity involving the Killing spinor. In seven dimensions the symmetric gamma matrices are γ^a and γ^{ab}. This means that $\eta\gamma^a\bar{\eta} = \eta\gamma^{ab}\bar{\eta} = 0$. Using this leads to the Fierz identity

$$\eta\bar{\eta} = \frac{1}{8}\bar{\eta}\eta + \frac{1}{48}\gamma^{(3)}\bar{\eta}\gamma_{(3)}\eta. \tag{D.12}$$

Adding this together with the same expression, but with $\eta = \gamma^a\eta$, we find that

$$\gamma^a\eta\bar{\eta}\gamma_a + \eta\bar{\eta} = \frac{1}{8}(\bar{\eta}\gamma^a\gamma_a\eta + \bar{\eta}\eta) + \frac{1}{48}\gamma^{(3)}\bar{\eta}\left(\gamma^a\gamma_{(3)}\gamma_a + \gamma_{(3)}\right)\eta = \bar{\eta}\eta, \tag{D.13}$$

or, re-expressed using the normalisation $\bar{\eta}\eta = 1$,

$$\gamma^a\eta\bar{\eta}\gamma_a = 1 - \eta\bar{\eta}. \tag{D.14}$$

This means that we have

$$\begin{aligned} a_{abe}a_{cde} &= -\bar{\eta}\gamma_{abe}\eta\bar{\eta}\gamma_{cde}\eta = -\bar{\eta}\gamma_{ab}\gamma_e\eta\bar{\eta}\gamma_e\gamma_{cd}\eta = -\bar{\eta}\gamma_{ab}\gamma_{cd}\eta \\ &= -\bar{\eta}\left(\gamma_{abcd} - 2\delta^{ab}_{cd}\right)\eta = 2\delta^{ab}_{cd} - c_{abcd}. \end{aligned} \tag{D.15}$$

So we see that this expression for the structure constants are in line with the one we had before.

There are other identities following from using the Fierz identity. Since we will need them at different places in the thesis, we derive them here. First off we have

$$\begin{aligned} a_{ade}c_{debc} &= -i\bar{\eta}\gamma_{ade}\eta\bar{\eta}\gamma_{debc}\eta = i\bar{\eta}\gamma_{ad}\gamma_e\eta\bar{\eta}\gamma_e\gamma_{dbc}\eta = i\bar{\eta}\gamma_{ad}\gamma_{dbc}\eta \\ &= 6i\bar{\eta}\gamma^{[a}{}_{[db}\delta^{d]}_{c]}\eta = i\bar{\eta}\left(\gamma^a{}_{db}\delta^d_c + \gamma^a{}_{cd}\delta^d_b + \gamma^a{}_{bc}\delta^d_d - \gamma^d{}_{bc}\delta^a_d\right)\eta \\ &= 4i\bar{\eta}\gamma_{abc}\eta = -4a_{abc}. \end{aligned} \tag{D.16}$$

In the same way we find

$$\begin{aligned} c_{abcd}a_{def} &= -i\bar{\eta}\gamma_{abcd}\eta\bar{\eta}\gamma_{def}\eta = -i\bar{\eta}\gamma_{abc}\gamma_{ef}\eta = 6i\bar{\eta}\gamma^{[ab}{}_{[e}\delta^{c]}_{f]}\eta \\ &= -6a^{[ab}{}_{[e}\delta^{c]}_{f]} = -3(a_{[ab|e|}\delta^f_{c]} - a_{[ab|f|}\delta^e_{c]}). \end{aligned} \tag{D.17}$$

The other identities that we will need are all derived in the same way, so we simply list them here.

The identities for c_{abcd} are

$$\begin{aligned} c_{abcd}c_{defg} &= a_{abc}a_{efg} + 9c^{[ab}{}_{[ef}\delta^{c]}_{g]} - 6\delta^{abc}_{efg}, \\ c_{abcd}c_{cdef} &= 8\delta^{ab}_{ef} - 2c_{abef}, \\ c_{abcd}c_{bcde} &= -24\delta_{ae}, \\ c_{abcd}c_{abcd} &= 168. \end{aligned} \tag{D.18}$$

Contracting c with a gives us the two identities of Eqs. (D.16) and (D.17) as well as

$$c_{abcd}a_{abc} = 0. \tag{D.19}$$

D.3 Octonions and G_2

It may also be worth mentioning that the exceptional group G_2, discussed at different places in the thesis, is highly connected to the octonions. It can be defined as the group of automorphisms of the octonions [61]. In fact, all the exceptional Lie groups can be defined using the octonions. The other four can be defined as the isometry groups of different projective planes. We can define F_4 as the isometry group of the octonionic projective plane $\mathbb{OP}^2$, E_6 of the bioctonionic projective plane $(\mathbb{C}\otimes\mathbb{O})\mathbb{P}^2$, E_7 of the quateroctonioc projective plane $(\mathbb{H}\otimes\mathbb{O})\mathbb{P}^2$ and E_8 of the octooctonionic projective plane $(\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2$ [61].

We can also relate the octonions to the so called G_2 manifolds, the squashed sphere being an example, discussed in Chapter 2. These are characterised by a real, covariantly constant three-form, and this three-form can be chosen as the octonionic structure constants. This is implied by the fact that the three-form can be expressed as

$$\phi_{abc} = -i\bar{\eta}\gamma_{abc}\eta, \tag{D.20}$$

where η is the Killing spinor of the G_2 manifold, see Chapter 2. This is of course exactly the form we saw above for the structure constants of the octonions.

E

Gamma matrices and SO(7)

In Appendix C we gave a general discussion on the gamma matrix representation of $SO(n)$. In this appendix we instead focus on $SO(7)$ since this is the tangent space group of seven-dimensional manifolds. We especially give a useful basis for the seven-dimensional gamma matrices in terms of the octonionic structure constants. This is the basis used in Chapter 7. We will also list a couple of identities involving contractions of gamma matrices, these are valid in all dimensions.

E.1 Gamma matrices in seven dimensions

In seven dimensions we only have one possibility of choosing the symmetry properties of the gamma-basis. The properties are also reflected around $\gamma^{(3)}$ so we only need to list the properties up to $\gamma^{(3)}$. The gamma matrices of $SO(7)$ are eight-dimensional and we therefore want 28 anti-symmetric and 36 symmetric matrices. The only way of choosing them is then

	Symm.	Anti-symm.
$C\mathbb{1}$	1	
$C\gamma^{(1)}$		7
$C\gamma^{(2)}$		21
$C\gamma^{(3)}$	35	

Note that this was used already in Appendix D when deriving the Fierz identity for the Killing spinors.

E.2 Our basis

One can use the octonionic structure constants of the previous chapter to construct a very useful basis for the $SO(7)$ gamma matrices. We define [43]

$$(\gamma_a)_\beta{}^\gamma = \begin{cases} -ia_{abc} & \text{for } (\beta,\gamma)=(b,c), \\ -i\delta_{ab} & \text{for } (\beta,\gamma)=(b,8), \end{cases} \tag{E.1}$$

with $a,b,c = 1,2,\ldots,7$. As we saw before, γ_a is anti-symmetric so that $(\beta,\gamma)=(8,c)$ must give us $+i\delta_{ac}$. Using the properties of the octonions one can easily check that these matrices satisfies

$$\gamma_1\cdots\gamma_7 = i\mathbb{1}, \tag{E.2}$$

as we wish for them to do. From this we can also construct all the $\gamma^{(n)}$, we will however only need $\gamma^{(2)}$, $\gamma^{(3)}$ and $\gamma^{(4)}$, so we skip the rest.

Starting with $\gamma^{(2)}$ we write

$$(\gamma_{ab})_{\gamma\delta} = (\gamma_{[a})_{|\gamma\epsilon|}(\gamma_{b]})_{\epsilon\delta} = (\gamma_{[a})_{|\gamma e|}(\gamma_{b]})_{e\delta} + (\gamma_{[a})_{|\gamma 8|}(\gamma_{b]})_{8\delta}, \tag{E.3}$$

which means that [40]

$$(\gamma_{ab})_{c8} = (\gamma_{[a})_{|ce|}(\gamma_{b]})_{e8} = (-ia_{[a|ce|})(-i\delta_{b]e}) = -\frac{1}{2}(a_{acb} - a_{bca}) = a_{abc}. \tag{E.4}$$

As well as

$$\begin{aligned}(\gamma_{ab})_{cd} &= (\gamma_{[a})_{|ce|}(\gamma_{b]})_{ed} + (\gamma_{[a})_{|c8|}(\gamma_{b]})_{8d} = (-ia_{[a|ce|})(-ia_{b]ed}) + (-i\delta_{[a|c|})(i\delta_{b]d}) \\ &= -\frac{1}{2}(a_{ace}a_{bed} - a_{bce}a_{aed}) + \delta^{ab}_{cd} = \frac{1}{2}(2\delta^{ac}_{bd} - c_{acbd} - 2\delta^{bc}_{ad} + c_{bcad}) + \delta^{ab}_{cd} \\ &= 2\delta^{ab}_{cd} + c_{abcd},\end{aligned} \tag{E.5}$$

where c_{abcd} is the dual of a_{abc}. Here we also used some of the contraction identities of the octonions derived in Appendix D.

The other matrices are found in the same way. For $\gamma^{(3)}$ we have [40]

$$\begin{aligned}(\gamma_{abc})_m{}^n &= 6i\delta^{(m}_{[a}a_{bc]}{}^{n)} - i\delta_{mn}a_{abc}, \\ (\gamma_{abc})_m{}^8 &= ic_{abcm}, \\ (\gamma_{abc})_8{}^8 &= ia_{abc}.\end{aligned} \tag{E.6}$$

While $\gamma^{(4)}$ instead takes the form

$$\begin{aligned}(\gamma_{abcd})_m{}^n &= 8\delta^{(m}_{[a}c_{bcd]}{}^{n)} + c_{abcd}\delta_{mn}, \\ (\gamma_{abcd})_m{}^8 &= 4\delta_{m[a}a_{bcd]}, \\ (\gamma_{abcd})_8{}^8 &= c_{abcd}.\end{aligned} \tag{E.7}$$

A good control that one should do is to check whether these matrices are traceless. This is directly seen by inspection of $\gamma^{(1)}$ and $\gamma^{(2)}$. For $\gamma^{(3)}$ the trace is given by

$$(\gamma_{abc})_m{}^m + (\gamma_{abc})_8{}^8 = 6i\delta^m_{[a}a_{bc]}{}^m - i\delta_{mm}a_{abc} + ia_{abc} = 6ia_{bca} - 7ia_{abc} + ia_{abc} = 0. \tag{E.8}$$

So it is in fact also traceless. The same holds for $\gamma^{(4)}$.

E.3 Contraction identities

Here we simply list a few useful identities involving contractions of gamma matrices representing $SO(n)$, for arbitrary dimension n. Many of these are used in various places of the thesis. The identities can also be found in [4], but since they use the convention that $\{\gamma_a, \gamma_b\} = -2\eta_{ab}$, there are some sign differences between our lists.

$$\gamma_a\gamma_b = \gamma_{ab} + \delta_{ab}, \tag{E.9}$$

$$\gamma_{ab}\gamma_c = \gamma_{abc} + 2\gamma_{[a}\delta_{b]c}, \tag{E.10}$$

$$\gamma^{ab}\gamma_{cd} = \gamma^{ab}{}_{cd} - 4\delta^{[a}_{[c}\gamma^{b]}{}_{d]} - 2\delta^{ab}_{[cd]}, \tag{E.11}$$

$$\gamma^{abc}\gamma_d = \gamma^{abc}{}_d + 3\gamma^{[ab}\delta^{c]}_d, \tag{E.12}$$

$$\gamma_{ab}\gamma^{cde} = \gamma_{ab}{}^{cde} - 6\gamma_{[a}{}^{[cd}\delta^{e]}_{b]} - 6\gamma^{[c}\delta^d_{[a}\delta^{e]}_{b]}, \tag{E.13}$$

$$\gamma^{abc}\gamma_{def} = \gamma^{abc}{}_{def} + 9\gamma^{[ab}{}_{[de}\delta^{c]}_{f]} + 18\gamma^{[a}{}_{[d}\delta^{bc]}_{ef]} - 6\delta^{abc}_{def}, \tag{E.14}$$

$$\gamma^{(n)}\gamma^a = \gamma^{(n)a} + n\gamma^{[b_1\ldots b_{n-1}}\delta^{b_n]a}, \tag{E.15}$$

$$\gamma^{a_1\ldots a_r b_1\ldots b_s}\gamma_{b_s\ldots b_1} = \frac{(n-r)!}{(n-r-s)!}\gamma^{a_1\ldots a_r} \tag{E.16}$$

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