Iterative Regularization and Adaptivity for an Electromagnetic Coefficient Inverse Problem

John Bondestam Malmberg[1,a)] and Larisa Beilina[1,b)]

[1]*Department of Mathematical Sciences, Chalmers University of Technology and Gothenburg University, 41296 Gothenburg, Sweden*

[a)]Corresponding author: john.bondestam.malmberg@chalmers.se
[b)]larisa@chalmers.se

Abstract. We study how the choice of the regularization parameter affects the quality of the reconstruction of the dielectric permittivity for an inhomogeneous medium, with data consisting of boundary observations of the electric field. Our method is based on the minimization of a Tikhonov functional and uses a finite element method for computations of the electric field. We conclude that the choice of the regularization parameter does not affect the quality of the reconstruction significantly in the studied cases, and can even be removed with results not significantly different from those with regularization.

Introduction

We consider the problem of determination of a variable dielectric permittivity, or wave propagation speed, of an isotropic, non-magnetic, and non-conductive medium, from boundary observations of the electric field generated by an incident plane wave. A numerical method, based on the minimization of a Tikhonov functional and utilizing adaptive finite elements, for the solution of this problem has previously been studied by our research group [1, 2, 3]. In those studies, the regularization parameter for the Tikhonov functional was taken as a constant, selected by trial and error, to give the best reconstruction. In this note we compare that method to an iterative updating of the regularization parameter as a part of the minimization procedure, by a method presented in [4].

The direct and inverse problems

The dielectric medium in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ is characterized by a dielectric permittivity function $\varepsilon = \varepsilon(\mathbf{x})$. We assume that this permittivity belongs to the set of admissible permittivities U^ε, consisting of those ε such that $\varepsilon \in C(\overline{\Omega})$, $\nabla\varepsilon \in [L_\infty(\Omega)]^3$, $\varepsilon|_T \in P^1(T)\ \forall T \in \mathcal{T}_{\text{fine}}$, $1 \leq \varepsilon(\mathbf{x}) \leq M\ \forall \mathbf{x} \in \Omega$, and $\nabla\varepsilon|_{\partial\Omega} \equiv 0$. Here, $P^1(T)$ denotes the set of polynomials of degree not greater than 1 over T, $\mathcal{T}_{\text{fine}}$ denotes a very fine "ideal" triangulation of Ω into tetrahedra $\{T\}$ (see [5]), and $M > 1$ is some known upper bound.

Let $T > 0$ be a given stopping time. We write $\Omega_T := \Omega \times (0, T)$ and $\partial\Omega_T := \partial\Omega \times (0, T)$. Given a permittivity $\varepsilon \in U^\varepsilon$, we describe the electric field $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ by the following system of equations, defining

International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2016)
AIP Conf. Proc. 1863, 370002-1–370002-4; doi: 10.1063/1.4992549
Published by AIP Publishing. 978-0-7354-1538-6/$30.00

the direct problem:

$$\begin{aligned} &\varepsilon\partial_t^2\mathbf{E} + \nabla\times(\nabla\times\mathbf{E}) - s\nabla(\nabla\cdot(\varepsilon\mathbf{E})) = 0 && \text{in } \Omega_T, \\ &\partial_{\mathbf{n}}\mathbf{E} = \mathbf{P} && \text{on } \partial\Omega_T, \\ &\mathbf{E}(\cdot, 0) = \partial_t\mathbf{E}(\cdot, 0) = 0 && \text{in } \Omega, \end{aligned} \tag{1}$$

where $\mathbf{P} \in [L_2(\partial\Omega_T)]^3$ is a known function describing the plane wave (see [1]). The system (1) then has a weak solution $\mathbf{E}$ such that $\mathbf{E} \in V^{\mathrm{dir}} := \{\mathbf{E} \in [H^1(\Omega_T)]^3 : \mathbf{E}(\cdot, 0) \equiv 0\}$.

The reconstruction problem described in the introduction can now be stated as a coefficient inverse problem for the system (1): *Given observed data* $\mathbf{G} = \mathbf{G}(\mathbf{x}, t)$ *for the electric field on* $\Gamma_T := \Gamma \times (0, T)$*,* $\Gamma \subset \Omega$*, determine* $\varepsilon \in U^\varepsilon$ *such that the corresponding solution* $\mathbf{E}$ *to* (1) *satisfies* $\mathbf{E} = \mathbf{G}$ *on* Γ_T.

We seek a solution to this problem through minimization of the Tikhonov functional

$$F_\alpha(\varepsilon) := \frac{1}{2}\,\|(\mathbf{E}_\varepsilon - \mathbf{G})z\|^2_{L_2(\Gamma_T)} + \frac{\alpha}{2}\,\|\varepsilon - \varepsilon_0\|^2_{L_2(\Omega)}, \tag{2}$$

where $\mathbf{E}_\varepsilon$ is the solution to (1) for the given permittivity ε, $z = z(t)$ is a smooth cut-off function, which ensures data compatibility in the adjoint problem which appears in the minimization procedure, $\alpha > 0$ is a regularization parameter, and ε_0 is an initial approximation to the true permittivity. In our computations, we have used the homogeneous background initial approximation $\varepsilon_0 \equiv 1$. We describe how to choose α below.

The gradient of F_α with respect to ε, which will be required for the minimization, can be computed as $F'_\alpha(\varepsilon) = \alpha(\varepsilon-\varepsilon_0) - \int_0^T \left(\partial_t\mathbf{E}_\varepsilon\cdot\partial_t\boldsymbol{\lambda}_\varepsilon - s(\nabla\cdot\mathbf{E}_\varepsilon)(\nabla\cdot\boldsymbol{\lambda}_\varepsilon) + s\nabla\cdot((\nabla\cdot\boldsymbol{\lambda}_\varepsilon)\mathbf{E}_\varepsilon)\right)\mathrm{d}t$. Here $\boldsymbol{\lambda}_\varepsilon \in V^{\mathrm{adj}} := \{\boldsymbol{\lambda} \in [H^1(\Omega_T)]^3 : \boldsymbol{\lambda}(\cdot, T) \equiv 0\}$ denotes the weak solution to an adjoint problem to (1), with $-(\mathbf{E}_\varepsilon - \mathbf{G})z$ as Neumann boundary data on Γ_T.

The adaptive algorithm and iterative regularization

Since we cannot in general compute exact solutions $\mathbf{E}_\varepsilon$ and $\boldsymbol{\lambda}_\varepsilon$, we use approximations $\mathbf{E}_h$ and $\boldsymbol{\lambda}_h$, and obtain a corresponding approximate permittivity ε_h. These approximations are computed by a finite element method with subspaces $V_h^{\mathrm{dir}} \subset V^{\mathrm{dir}}$, $V_h^{\mathrm{adj}} \subset V^{\mathrm{adj}}$, and subset $U_h^\varepsilon \subset U^\varepsilon$, consisting of piecewise polynomials of degree no greater than 1 over a triangulation $\mathcal{T}_h$ of Ω into tetrahedra of size h, and a partition $\mathcal{I}_\tau$ of $(0, T)$ into subintervals of length $\tau \propto h$. Details of the approximation procedure can be found in [1, 2, 3].

The principle of the adaptive algorithm can be outlined as follows. Given initial coarse $\mathcal{T}_h$ and $\mathcal{I}_\tau$, we:

1. Minimize the Tikhonov functional of (2), using the conjugate gradient method, with functions approximated by elements of U_h^ε, V_h^{dir}, and V_h^{adj}.
2. Refine the triangulation $\mathcal{T}_h$ by subdividing those tetrahedra where the approximation error $|F_\alpha(\varepsilon) - F_\alpha(\varepsilon_h)|$ is large. If necessary for stability, refine $\mathcal{I}_\tau$ as well.
3. Repeat steps 2 and 3 until a stopping criterion is satisfied.

In order to perform the second step, it is essential to be able to estimate in which regions the approximation error is large. To this end, we have developed the following recommendation for the refinement of the triangulation (see Mesh Refinement Recommendation 1 of [3]): *Refine the triangulation in those regions where the absolute value of the residual* $R_\varepsilon := \alpha(\varepsilon_h - \varepsilon_0) - \int_0^T \partial_t\mathbf{E}_h \cdot \partial_t\boldsymbol{\lambda}_h \,\mathrm{d}t + \frac{s}{2h}\int_0^T [(\nabla\cdot\boldsymbol{\lambda}_h)(\mathbf{n}\cdot\mathbf{E})]\,\mathrm{d}t$ *is within a predefined relative tolerance of its maximal value.* In R_ε, $[(\nabla\cdot\boldsymbol{\lambda}_h)(\mathbf{n}\cdot\mathbf{E})]$ denotes the spatial jumps of the discontinuous function $(\nabla\cdot\boldsymbol{\lambda}_h)(\mathbf{n}\cdot\mathbf{E})$ across the faces of the tetrahedra in $\mathcal{T}_h$.

Classical methods for the choice of regularization parameter, such as the (generalized) discrepancy principle [6, 7], rely on multiple minimizations of the Tikhonov functional. In our case, this is computationally expensive since the minimization requires the numerical solution of two partial differential equations

TABLE 1. Maximum values $m_\delta = \max_{\mathbf{x}\in\Omega} \varepsilon(\mathbf{x})$ of the reconstructions on the final twice refined mesh, and largest differences $d_\delta = \max_{\mathbf{x}\in\Omega} |\varepsilon(\mathbf{x}) - \varepsilon_{\mathrm{ni}}(\mathbf{x})|$, where $\varepsilon_{\mathrm{ni}}$ is the reconstruction using non-iterative regularization. For iterative regularization, p is the power in (3). Recall that the value of the true permittivity is 3.0 inside the inclusions.

	non-iterative regularization	iterative regularization $p = 0.1$	$p = 0.2$	$p = 0.5$	$p = 1.0$	no regularization
$m_{0.05}$	2.870533	2.870554	2.870584	2.870642	2.870735	2.870929
$d_{0.05}$	–	0.000038	0.000074	0.000151	0.000262	0.000519
$m_{1.00}$	2.843455	2.843476	2.843513	2.843567	2.843648	2.843846
$d_{1.00}$	–	0.000042	0.000075	0.000162	0.000270	0.000532

at each iteration. Therefore, we study an adaptive regularization method, described in detail in [4], which relies on only one minimization. The idea is to select a sequence of regularization parameters $\alpha = \alpha_n$, where $n \in \mathbb{N}$ is the iteration count in the conjugate gradient method, and corresponding step-sizes β_n, such that $\min_\varepsilon F_{\alpha_n}(\varepsilon) \to 0$ as $n \to \infty$. Under certain assumptions on the boundedness and regularity of the Tikhonov functional, this amounts to selecting α_n and β_n such that $\alpha_0 \geq \ldots \geq \alpha_n > 0$, $\lim_{n\to\infty} \alpha_n = 0$, and $0 < |1 - \alpha_n\beta_n| < 1$. In our computations we use the sequence

$$\alpha_n = \frac{\alpha_0}{(n+1)^p}, \tag{3}$$

with $\alpha_0 = 0.01$, and $p \in (0, 1)$. Optimal step-size β_n is computed explicitly.

Numerical results

As a numerical example, we consider the reconstruction of three spherical inclusions of radius 0.2, with centres $\mathbf{x}_1 = (0.3, 0.3, -0.2)$, $\mathbf{x}_2 = (-0.3, 0.3, 0.0)$, and $\mathbf{x}_3 = (0.0, -0.3, 0.2)$, respectively, with permittivity 3.0 inside the inclusions, and 1.0 outside. We use the computational domain $\Omega = [-1.4, 1.4] \times [-1.4, 1.4] \times [-0.7, 0.7]$.

To generate data, we construct an initial mesh of with mesh size $h = 0.1$ over Ω, then refine it three times locally close to the inclusions. The obtained three times refined mesh is used as $\mathcal{T}_{\mathrm{fine}}$ in U^ε. On this mesh, we solve the direct problem for an incident plane wave at $\{\mathbf{x} \in \overline{\Omega} : x_3 = -0.7\}$ of the form $P(t) = \sin(\omega t)$ for $0 < t \leq 2\pi/\omega$, and $P(t) = 0$ for $t \geq 2\pi/\omega$, with $\omega = 10.0$, and collect observations at the transmission side $\{\mathbf{x} \in \overline{\Omega} : x_3 = 0.7\}$. To these observations, we add a pointwise uniform random additive noise of levels $\delta = 0.05$, and $\delta = 1.00$ respectively. To these data sets, we apply the reconstruction algorithm outlined above. The results are shown in Figure 1 and Table 1.

Conclusion

We see from Figure 1 and Table 1 that the adaptive algorithm performs well for the lower level of noise, reconstructing the correct number of inclusions, in the correct locations, with a reasonable maximum value of the permittivity. For the larger level of noise, the inclusion furthest away from the observation boundary is lost in the reconstruction. These results confirm those of [1, 2, 3].

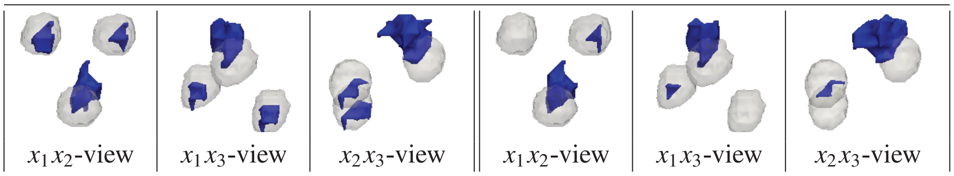

FIGURE 1. Isosurfaces at the level $(\min_{\mathbf{x}} \varepsilon(\mathbf{x}) + \max_{\mathbf{x}} \varepsilon(\mathbf{x}))/2$ for the reconstruction with non-iterative regularization (blue), and reference shapes of the three inclusions (grey, transparent) for noise level 0.05 (left), and 1.00 (right). Reconstructions with iterative regularization and with no regularization are visually indistinguishable from the reconstructions presented here.

The differences presented in Table 1 indicate that the choice of regularization parameter does not affect the reconstruction of the permittivity of the three inclusions significantly. Indeed it is possible to remove the regularization term of the Tikhonov functional altogether, and still obtain a good reconstruction of the dielectric permittivity. This does not necessarily imply that the reconstruction problem is well-posed, since the mesh parameter h of the finite-dimensional approximation may act as an additional regularization parameter.

It should be noted that regularization may still improve the reconstruction of dielectric permittivity if an initial approximation ε_0, which is better than the homogeneous background approximation used here, is available. Such results are presented in [1, 2]. Moreover, it was demonstrated in [8] that iterative regularization improves the performance of adaptive reconstruction of magnetic permeability.

Acknowledgement

This research is supported by the Swedish Research Council (VR). The computations were performed on resources at Chalmers Centre for Computational Science and Engineering (C3SE) provided by the Swedish National Infrastructure for Computing (SNIC).

REFERENCES

[1] L. Beilina, T. Nguyen, M. Klibanov, and J. Malmberg, Reconstruction of shapes and refractive indices from backscattering experimental data using the adaptivity, *Inverse Probl.* **30**, 105007, 2014.

[2] L. Beilina, T. Nguyen, M. Klibanov, and J. Malmberg, Globally convergent and adaptive finite element methods in imaging of buried objects from experimental backscattering radar measurements, *J. Comput. Appl. Math.* **289**, 371–391, 2015.

[3] J. Malmberg, and L. Beilina, An adaptive finite element method in quantitative reconstruction of small inclusions from limited observations, preprint is available in arXiv:1603.04870, 2016.

[4] A. Bakushinsky, M. Kokurin, and A. Smirnova, *Iterative Methods for Ill-Posed Problems: An Introduction*, De Gruyter, Berlin, 2011.

[5] L. Beilina, M. Klibanov, and M. Kokurin, Adaptivity with relaxation for ill-posed problems and global convergence for a coefficient inverse problem, *J. Math. Sci.* **167**, 279–325, 2010.

[6] H. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, vol. 375 of *Mathematics and its Applications*, Kluwer Academic Publishers Group, Dordrecht, 1996.

[7] A. Tikhonov, A. Goncharsky, V. Stepanov, and A. Yagola, *Numerical Methods for the Solution of Ill-Posed Problems*, Kluwer Academic Publishers, Dordrecht, 1995.

[8] S. Hosseinzadegan, *Iteratively regularized adaptive finite element method for reconstruction of coefficients in Maxwell's system*, Master's thesis, Department of Mathematical Sciences, Chalmers University of Technology and Gothenburg University, 2015.