THE SIGNIFICANCE OF SYSTEM PRE-LOAD AT MODAL ANALYSIS OF LOW-RESONANT MECHANICAL SYSTEMS

Version 2

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Introduction

The objective of this paper is to call attention to a significant physical phenomenon, which is mostly disregarded in linear vibration analysis up to date. The full analysis of that phenomenon is beyond the limits of this paper, but may be found in [1,2].

There are no upper limits for making the analysis of vibration increasingly more complicated. It may be considered an art, however, to make the analysis as simple as possible, but still retain the ability to monitor significant phenomena.

The most simple kind of analysis of mechanical vibration in discrete systems comprises undamped, linear vibration. The system considered is then described by the very well known set of linear, second order ordinary differential equations, which, by using matrix notation, could be written very compactly:

$$M\ddot{d} + Kd = 0 \qquad \text{....Eq. 1}$$

where d and its second time derivative $\ddot{d}$ refer to an arbitrarily chosen displacement vector with as many, say N, components, as there are degrees of freedom in the system, M is the corresponding symmetric inertia matrix, and K is a symmetric restoring matrix (whereby the word restoring is used deliberately instead of the more commonly used word stiffness, in order to remove a mental obstacle, as will be discussed later).

Complex systems with many degrees of freedom may be difficult to overview, if their motion is described in an arbitrary frame of reference, which is the case for Eq. 1.

However, nowadays a standard procedure for easier understanding and handling of the problem is to transform the description of the motion to a special, unique frame of reference, by using normal or modal displacement coordinates, q_i, which will result in simple, uncoupled equations of motion:

$$\left.\begin{aligned} &m_1\ddot{q}_1 + k_1 q_1 = 0 \\ &\cdots\cdots\cdots \\ &m_i\ddot{q}_i + k_i q_i = 0 \\ &\cdots\cdots\cdots \\ &m_N\ddot{q}_N + k_N q_N = 0 \end{aligned}\right\} \qquad \text{....Eq. 2}$$

What has been stated so far, is basic knowledge in any modern textbook on vibration.

There is, however, a hidden difficulty i finding a simple but still adequate restoring matrix K for arbitrary, discrete mechanical systems. A review of textbooks, handbooks and scientific reports will reveal - with very few exceptions - that procedures given for composing K result in matrices, where each individual component is directly proportional to some material stiffness/flexibility property, e.g., spring stiffness, modulus of elasticity, etc. If such a matrix would be fully correct, then all vibration will vanish if the material stiffness properties will decrease to zero (because K will vanish). Also, the vibrational behavior will be nonsensitive to the loads acting upon the system at equilibrium, i.e., the pre-loads (because the content of mass and the material stiffness properties do not change with the pre-loads).

Such conclusions are, however, in conflict with some basic pieces of experience:

* a simple pendulum may oscillate without containing any elasticity,
* a string may oscillate, whereby its natural frequencies are controlled by the pre-tension, but not the elasticity.

We must conclude, therefore, that there are cases, where the restoring matrix K must contain more information than just data comprising geometry and stiffness/flexibility, in order to be able to monitor some well known physical phenomena.

It has been shown [1,2] that the basic equation, Eq. 1, could generally be given a more precise formulation:

$$M\ddot{d}+Kd=M\ddot{d}+(E+P)d=0 \qquad \Rightarrow \qquad K=E+P \qquad \text{....Eq. 3}$$

where E is the elasticity/stiffness/flexibility dependent contribution and P is the pre-load dependent contribution to the total restoring matrix K, whereby the action of gravity and other similar body loads can be expressed uniquely in terms of support pre-loads, and do not have to appear explicitly in the equations of motion.

A proper modal analysis of discrete mechanical systems must thus be based on such governing equations, where both types of the restoring effect, i.e., due to **stiffness** and **pre-loads**, are initially considered, even if one of them later on, due to special pre-conditions, may be excluded from the analysis.

A Conceptual Explanation

The derivation of the equations of motion, Eq. 1, might be based on equilibrium considerations according to Newton-Euler (contrary to energy considerations according to Lagrange). The first term, $M\ddot{d}$, accounts then for the inertial resistance to change of motion, and Kd accounts for the restoring effects, i.e., the forces that - if positive - try to return the system to its equilibrium configuration.

It is supposed that the equilibrium state of the system is known, both regarding geometry and loads, the latter here called pre-loads. The restoring effects according to Newton-Euler are originated from disturbed equilibrium of the system. The equilibrium itself, EQ, depends on the two abovementioned main groups of system parameters: loads, L, and geometry, G. If the operator **&** is introduced to indicate the combined action of two groups of parameters, then this dependence could be written as:

$$EQ=L\,\&\,G \qquad \text{....Eq. 4}$$

The disturbed equilibrium could formally be written as $\partial(EQ)$, which in a first, linear approximation could be evaluated as:

$$Kd = \partial(EQ) = \partial(L)\&G + \partial(G)\&L \qquad \text{....Eq. 5}$$

where the two right hand terms may be interpreted as:

$\partial(L)\&G =$ the action of incremental load changes upon the initial geometry of the system (assuming that the incremental load change is linearly related to the incremental change of geometry),

$\partial(G)\&L =$ the action of incremental changes in geometry upon the initial loading state (pre-loads) in the system.

The first term, $\partial(L)\&G$, corresponds to Ed in classical structural mechanics, and is well understood and consistently handled in text-books and handbooks.

The second term, $\partial(G)\&L$, corresponds to Pd, and is missing in most texts on arbitrary, discrete mechanical systems, although it may be found as the only term for restoring action in the specialized equations valid for specific problems, e.g., pendulums and pre-tensioned strings.

The reason for the absence of the second term, Pd, in most texts on arbitrary, discrete mechanical systems (as well as in structural mechanics in general) seems to be the desire to linearize the problem around the unloaded state, where, by definition, the loads are zero and thus also $P \equiv 0$. Doing so, the influence of pre-loads could subsequently be treated as a kind of second order effect for an initially unloaded system. Such an approach will also work, but is less clever than a linearization about the pre-loaded equilibrium configuration, for which case Eq. 5 is valid.

It is obvious that both the restoring effects exist in parallel on the conceptual level of the analysis. Due to special system characteristics, e.g., symmetry, zero pre-load, zero flexibility, etc., one of them may vanish completely. In most cases in practice, both terms exist, but one is quantitatively dominating, why the other could be numerically neglected. There are, however, very realistic applications, where both the terms contribute to the restoring effect with the same order of magnitude. This is observed to be the case with multi-dimensional, low-resonant systems, which are common at resiliant mounting of machines, vehicles and apparatus.

The Key Issue: Model of Springs

Any arbitrary undamped discrete mechanical system consists of:

* rigid bodies
* massless supports and connectors

The second of the abovementioned groups could generally be modeled as massless springs of zero/finite/infinite multi-dimensional stiffness. Each spring is considered to belong to one specific body, the **base body**, and the "free" or "upper" end of that spring is attached to another rigid body, the **connected body**. The "free" or "upper" end of the spring constitutes a unique **interconnection point**, identified by its location as well as by the orientation of a thought, locally attached frame of reference. At equilibrium, all interconnection points, as well as the loads (forces, $\vec{F}^o$, and moments, $\vec{M}^o$) transmitted through these points, are assumed to be known.

The interconnection point for an arbitrary spring could be observed from the related base body, which constitutes a logical frame of reference. At the initial, equilibrium configuration the spring is loaded by the pre-load. If then the interconnecting end of the spring is given an incremental "elastic" displacement (both in translation, $\vec{\Delta}$, and in rotation, $\vec{\phi}$), the spring load at the displaced interconnection point is changed to magnitude and orientation with reference to the base body.

The new vectorial load is preferably defined and visualized as the sum of the initial or pre-load vector, and the "elastic" load change vector ($\vec{F}^{el}$ and $\vec{M}^{el}$), cf. Fig. 1:

$$\vec{F} = \vec{F}^{o} + \vec{F}^{el}; \qquad \vec{M} = \vec{M}^{o} + \vec{M}^{el} \qquad \text{....Eq. 6}$$

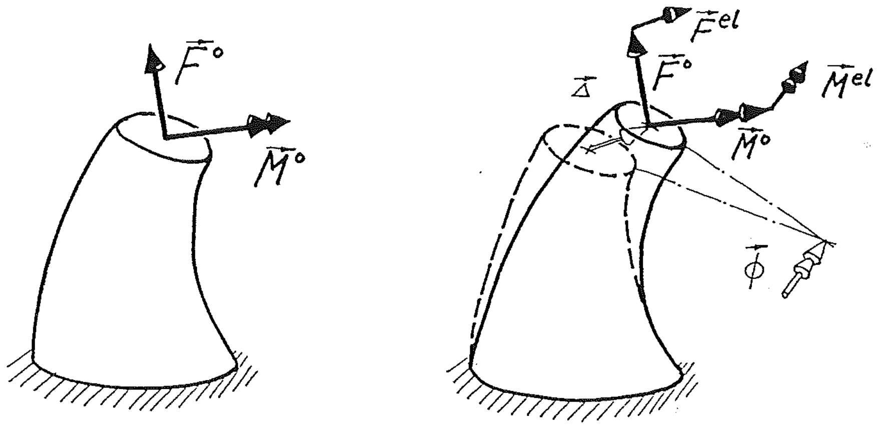


Fig. 1: Spring in initial and displaced/deformed configuration

For a sufficiently small spring displacement from its original equilibrium state (the initial deformation may, however, be large!), a linear relationship will be found between the incremental displacement and load change vectors, expressed as a six by six element symmetric stiffness matrix S, if the vectors consist of three components each:

$$\begin{pmatrix} \vec{F}^{el} \\ \vec{M}^{el} \end{pmatrix} = S \begin{pmatrix} \vec{\Delta} \\ \vec{\phi} \end{pmatrix} \qquad \text{....Eq. 7}$$

Simple and reliable theoretical procedures for the evaluation of the complete stiffness matrix S are available only for a limited number of types of pre-loaded springs, e.g., **axially** pre-loaded strings, beams and helical springs. For other types and/or differnetly pre-loaded springs the stiffness matrix may be obtained either experimentally or analytically by using advanced finite element techniques.

The pre-loads, $\vec{F}^{o}$ and $\vec{M}^{o}$, are generally finite (but may be zero as a special case). They maintain their magnitude, but rotate with the rotation of the base body. They act at the interconnection points between two bodies.

Displaced from their original points of application, the pre-loads, sized and oriented as stated in the preceding paragraph, will produce the positive or negative restoring action, which is reflected in the matrix P.

The "elastic" load changes could be kept arbitrarily small, when only vibration of small amplitude is studied. Such "infinitely small" load changes according to Eq. 7, acting at the interconnection points (no matter, if initial or displaced), will produce the "elastic" restoring action, which is reflected in the matrix E.

Finally it should be observed that the designation "elastic" intentionally is given in quotes. The **linear** relationships between incremental spring end displacements and load changes depend not only on elastic properties, but also on the pre-loads, acting on the springs considered. The pre-load influence on the stiffness is significant at almost all helical spring applications, but less important at structural members of the type beams and rods. Actually, when analyzing the stiffness of continua on the infinitesimal scale, there exists an parallel to the E and P matrices on the finite, discrete scale, which is well understood in the analysis of axially pre-loaded beams and helical springs, cf. Eq. 12.

Thus, the pre-load influence on the linearized analysis of the restoring action at discrete, mechanical systems is twofold: direct via the matrix P, and indirect via the matrix E.

Axially Pre-loaded Symmetric Springs

Up to date the most comprehensive stiffness model of axially pre-loaded symmetric springs is given by HARINGX [3]. It covers explicitly helical springs, but by simple transformations prismatic bars with invariant cross sections are covered, as well. The model includes the effects of axial pre-load, cross-sectional shear flexibility and axial variation of length.

The spring is treated as a one-dimensional continuum, which is originally straight and axially pre-loaded (positive in tension), Fig. 2a. The analysis of incremental **axial** loading is trivial. The result of incremental **lateral** loading is depicted in Fig. 2b.

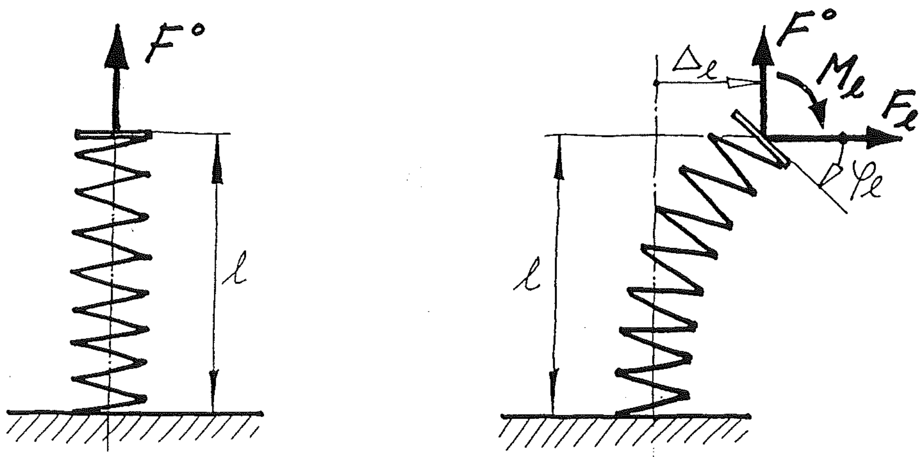


Fig. 2: Axially pre-tensioned (a) and laterally loaded (b) spring

The total spring stiffness is related to four independent cross-sectional stiffness properties: S_e for elongation (tension/compression), S_t for torsion, S_b for bending and S_s for shear. For a cylindrical helical spring the stiffness properties are evaluated from the following spring characteristics:

d = wire diameter
D = coil diameter
l_0 = length of unloaded spring (l = length of pre-loaded spring, ... cf. Eq. 10)
n = number of active coils
G = shear modulus
ν = Poisson's ratio
F^o = axial pre-load (positive in tension)

which yield:

$$S_e = \frac{Gld^4}{8nD^3}$$

$$S_t = \frac{(1+\nu)Gld^4}{32nD}$$

....Eq. 8

$$S_b = \frac{(1+\nu)Gld^4}{16(2+\nu)nD}$$

$$S_s = \frac{(1+\nu)Gld^4}{4nD^3}$$

The resulting **axial** spring stiffnesses (elongation/compression and torsion) are both elementary:

$$k_{\Delta a} = \frac{S_e}{l} = \frac{Gd^4}{8nD^3}$$

....Eq. 9

$$k_{\phi a} = \frac{S_t}{l} = \frac{(1+\nu)Gd^4}{32nD}$$

and the pre-loaded spring length at the axial tensioning load F^o is:

$$l = l_o + F^o/k_{\Delta a} = l_o + 8F^o nD^3/Gd^4$$ ....Eq. 10

The resulting **lateral** spring stiffnesses (lateral deflection and rotation, cf. Fig.2) are mutually coupled and are defined as:

$$F_l = +k_{\Delta l}\Delta_l - k_{\Delta\phi}\phi_l$$

....Eq. 11

$$M_l = -k_{\Delta\phi}\Delta_l + k_{\phi l}\phi_l$$

and introduced constants could be evaluated as functions of F^o from:

$$k_{\Delta l} = \frac{F^0}{l} \cdot \frac{1}{1-(1-F^o/S_s)\tanh\kappa/\kappa}$$

$$k_{\Delta\phi} = (k_{\Delta l}l - F^o)/2$$ ....Eq. 12

$$k_{\phi l} = \frac{k_{\Delta\phi}l}{2} \cdot \left(1 - \left(1 - \frac{F^o}{S_s}\right) \cdot \frac{1}{\kappa\tanh\kappa} + \frac{1}{\tanh^2\kappa}\right)$$

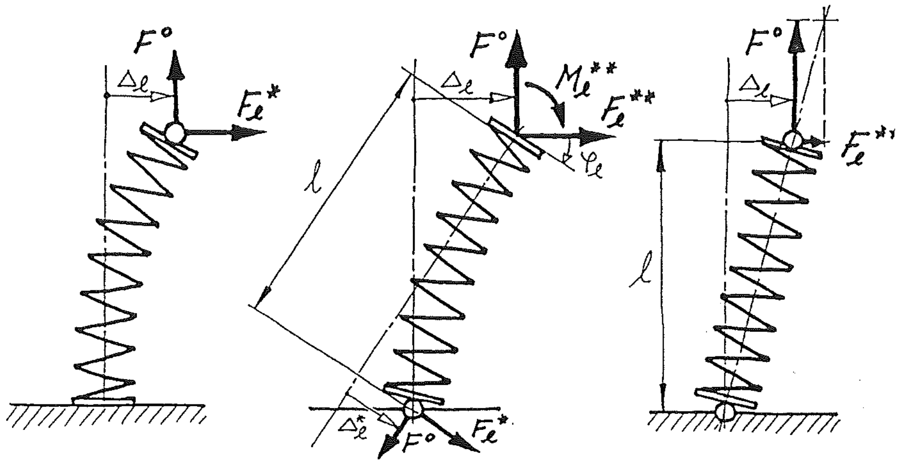
where

$$\kappa^2 = \left(1 - \frac{F^o}{S_s}\right) \cdot \frac{F^o}{S_b} \cdot \frac{l^2}{4}$$

The evaluation of the lateral stiffnesses requires special care, e.g., usage of series expansions and complex algebra, if the axial pre-load is absent or is compressive, respectively.

The stiffness values given above are directly applicable, when both the ends of the helical spring are clamped to the two attached bodies, i.e., the base body and the connected body. However, these basic stiffness values could easily be modified to cover also those cases, when either one or both of the spring ends are pinned, i.e., do not carry any moments.

The three possible cases containing pinned ends are defined and depicted in Fig. 3.



case * (a) case ** (b) case *** (c)

Fig. 3: Various combinations of pinned spring ends

Attached body pinned (case *):

In this case both M_a^* and M_l^* must be zero independent of the connecting end rotation, and both forces F_a^* and F_l^* must be related to the spring end translation, only. This is obtained if the transformed stiffness constants, k^*, as derived below, are used in Eq. 11:

$$M_l^* = -k_{\Delta\phi}\Delta_l + k_{\phi l}\phi_l = 0 \quad \Rightarrow \quad \phi_l = k_{\Delta\phi}/k_{\phi l}\cdot\Delta_l \quad \Rightarrow \quad F_l^* = (k_{\Delta l} - k_{\Delta\phi}^2/k_{\phi l})\Delta_l \quad \Rightarrow$$

$$k_{\Delta a}^* = k_{\Delta a}; \qquad k_{\Delta l}^* = (k_{\Delta l} - k_{\Delta\phi}^2/k_{\phi l}); \qquad k_{\phi a}^* = k_{\phi l}^* = k_{\Delta\phi}^* = 0 \qquad \text{....Eq. 13}$$

Base body pinned (case **):

Another set of modified stiffness constants, k^{**}, may be obtained for this case by making use of the previous result, Eq. 13. The deformed spring is studied, where the lateral deformations are assumed to be sufficiently small and the axial pre-load to be non-negligible, using the pinned base end as a frame of reference.

Geometric considerations then yield:

$\Delta_l^* = l\phi_l - \Delta_l$ (first order small lateral displacements)

and the equilibrium conditions require:

$F^o = F^{o*}$ (finite order axial forces)

$F_l^{**} = -F_l^* + \phi_l F^{o*}$ (first order small lateral forces)

$M_l^{**} = lF_l^* + \Delta_l^* F^{o*}$ (first order small lateral moments)

which are combined to

$$F_l^{**} = k_{\Delta l}^*\Delta_l - (k_{\Delta l}^* l - F^o)\phi_l = k_{\Delta l}^{**}\Delta_l - k_{\Delta\phi}^{**}\phi_l$$

$$M^{**} = -(k_{\Delta l}^* l - F^o)\Delta_l + (k_{\Delta l}^* l - F^o)l\phi_l = -k_{\Delta\phi}^{**}\Delta_l + k_{\phi l}^{**}\phi_l$$

In the axial direction the uncoupled stiffness equations read:

$$F_a^{**} = k_{\Delta a}\Delta_a; \qquad M_a^{**} \equiv 0$$

From the derived relationships is found by identification:

$$k_{\Delta a}^{**} = k_{\Delta a}; \quad k_{\phi a}^{**} = 0; \quad k_{\Delta l}^{**} = k_{\Delta l} - k_{\Delta\phi}^2/k_{\phi l} = k_{\Delta l}^{*} \qquad \text{....Eq. 14}$$

$$k_{\Delta\phi}^{**} = k_{\Delta l}^{*} l - F^o; \quad k_{\phi l}^{**} = (k_{\Delta l}^{*} l - F^o) l = k_{\Delta\phi}^{**} l$$

Both ends pinned (case *):**

Apparent lateral stiffness, k^{***}, is identified for this case, as well, as a consequence of the definition of lateral load increments, cf. Fig. 1.

From Fig. 3 (c) similar triangles yield:

$$\Delta_l/l = F_l^{***}/F^o \quad \Rightarrow \quad F_l^{***} = F^o/l \cdot \Delta_l = k_{\Delta l}^{***}\Delta_l; \quad F_a^{***} = k_{\Delta a}\Delta_a$$

and by identification is finally obtained:

$$k_{\Delta a}^{***} = k_{\Delta a}; \quad k_{\Delta l}^{***} = F^o/l; \quad k_{\phi a}^{***} = k_{\phi l}^{***} = k_{\Delta\phi}^{***} = 0 \qquad \text{....Eq. 15}$$

The lateral stiffness according to Eq. 15 is, for a specific spring, obviously negative, when the pre-load is compressive. Such a negative lateral stiffness makes sense only if the compressive load doesn't exceed the lowest Euler buckling load for the spring considered.

Other types of springs and supports:

The stiffness of an axially pre-loaded helical spring with four different sets of spring end conditions is described quite well by the equations given in the preceding paragraphs. A number of other types of springs could be successfully modeled, as well, by using the equations originally derived for helical springs.

Elastic beams, e.g., including those carrying axial pre-loads, could be modeled by replacing the cross-sectional stiffnesses S with the proper beam quantities, e.g., AE in bending. Beams are then modeled beyond the second order Euler-Timoshenko-Berry theory, with the possibility to account for the changed length of the beam, as well.

Various types of **joints** and **bearings** can be modeled by assigning some spring stiffness components, k, zero values and others very high values, e.g., a spherical joint is modeled by $k_{\Delta a}$ & $k_{\Delta l} \Rightarrow \infty$ and $k_{\phi a} = k_{\phi l} = k_{\Delta l} = 0$.

An illustrative basic example

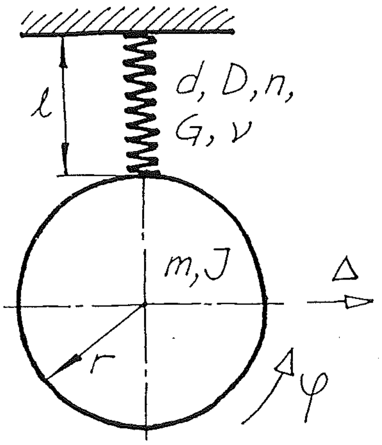
The basic ideas introduced in this paper as well as the powerfullness of related computational methods or algotithms proposed are demonstrated in the following basic example. The vibrating system investigated consists of a rigid homogeneous sphere, hanging vertically and symmetrically in an ordinary helical spring, the both ends of which are clamped to the body and the supporting frame, respectively, cf. Fig. 4.

The number of degrees of freedom and expected natural frequencies is generally six for one rigid body. In the special case studied, it might be understood from symmetry that the motion about and along the vertical axis of symmetry is uncoupled in each of the two modes of motion, and that the lateral motion is described by the same equations of motion regardless of the orientation of the lateral direction. The motion is described by two coordinates, e.g., the lateral translation of the center of the sphere, Δ, and the rotation about a horisontal axis, ϕ. The equations of motion will be coupled and yield two natural frequencies of the system.

The system will thus have four numerically different natural frequencies, the two axial uncoupled ones are independent of the axial pre-load in the spring and might be obtained by using elementary methods. The two lateral modes and natural frequencies are influenced considerably of the pre-load: partly due to the contribution from the matrix $\boldsymbol{P}$, cf. Eq. 3, partly via the pre-load dependent spring stiffnesses, c.f. Eq. 12.

The lateral motion is described by just one general set of two coupled equations, yielding two natural frequencies and mode vectors. The latter might be used to perform the modal transformation, decoupling the two equations of lateral motion. The general set of equations is easily modified to predict the natural frequencies when:

* either one or both of the spring ends are pinned,
* the pre-load is absent, e.g., as in space $(F^o \Rightarrow 0)$,
* the system is a rigid pendulum $(G \Rightarrow \infty;\ F^o = mg)$.

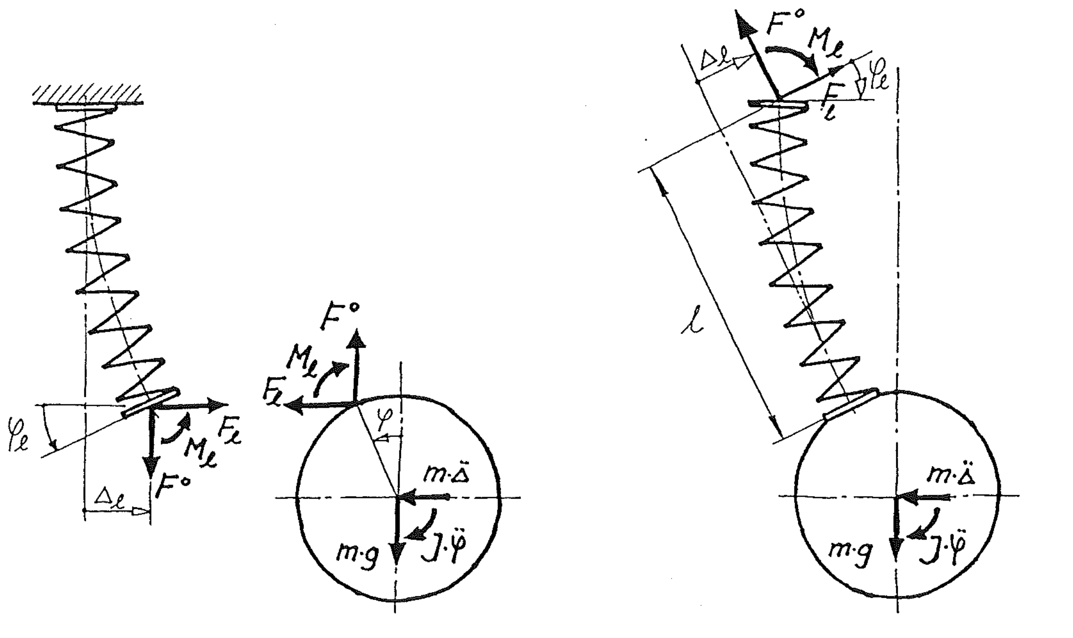


System data:

Spring:
free length = 0.1 (m)
diam. of wire = 0.004 (m)
diam. of coil = 0.034 (m)
number of coils = 8.7
material = steel

Sphere:
radius = 0.2 (m)
mass = 11.56 (kg)
moment of inertia = 0.231 (kg*m*m)

Fig. 4: Configuration of and data on an illustrative basic example



base = frame (a) base = sphere (b)

Fig. 5: Configuration of the laterally deformed system

Derivation of equations of motion

The equations of lateral motion are derived in two versions: (a) the supporting frame is considered to be the base body, and (b) the sphere is considered to be the base body.

Case a:

The displacement of the sphere gives rise to the following lateral deformations of the spring, cf. Fig. 5a:

$$\Delta_l = \Delta - r\phi; \qquad \phi_l = \phi$$

which gives rise to spring forces according to Eqs 11 and 12 (where the obvious subscript l is omitted).

Combined with the inertial resistance to motion in terms of the generalized d'Alembert forces and the action of gravity, as is shown in Fig. 5a, the lateral equilibrium of the sphere yields the following equations of motion:

$$\begin{pmatrix} m & 0 \\ o & I \end{pmatrix}\begin{pmatrix} \ddot{\Delta} \\ \ddot{\phi} \end{pmatrix} + \begin{pmatrix} k_\Delta & -k_{\Delta\phi} + rk_\Delta \\ -k_{\Delta\phi} + rk_\Delta & k_\phi + 2k_{\Delta\phi}r + r^2k_\Delta + mgr \end{pmatrix}\begin{pmatrix} \Delta \\ \phi \end{pmatrix} = 0 \qquad \text{....Eq. 16}$$

Case b:

The displacement of the sphere gives here rise to other lateral deformations of the opposite end of the spring, cf. Fig. 5b:

$$\Delta_l = (r+l)\phi - \Delta; \qquad \phi_l = \phi$$

Proceeding in the same way as in the preceding case, another set of equations of motion is obtained for the description of the lateral motion:

$$\begin{pmatrix} m & 0 \\ o & I \end{pmatrix}\begin{pmatrix} \ddot{\Delta} \\ \ddot{\phi} \end{pmatrix} + \begin{pmatrix} k_\Delta & k_{\Delta\phi} - k_\Delta(r+l) + mg \\ k_{\Delta\phi} - k_\Delta(r+l) + mg & \begin{pmatrix} k_\phi - 2k_{\Delta\phi}(r+l) + \\ +k_\Delta(r+l)^2 - mg(r+l) \end{pmatrix} \end{pmatrix}\begin{pmatrix} \Delta \\ \phi \end{pmatrix} = 0 \qquad \text{....Eq. 17}$$

The second part of the left side of Eqs 16 and 17, which corresponds to the total restoring action in the system, cf. Eq. 3, looks different, when the equations are compared, although both equations are valid for the same physical system. They could, however, be brought to a uniform shape, if $k_{\Delta\phi}$, as given in Eq. 12, is eliminated. For the basic case (no ends pinned) is then obtained:

$$\begin{pmatrix} m & 0 \\ o & I \end{pmatrix}\begin{pmatrix} \ddot{\Delta} \\ \ddot{\phi} \end{pmatrix} + \begin{pmatrix} k_\Delta & -k_\Delta(r+l/2) + mg/2 \\ -k_\Delta(r+l/2) + mg/2 & k_\phi + k_\Delta r(r+l) \end{pmatrix}\begin{pmatrix} \Delta \\ \phi \end{pmatrix} = 0 \qquad \text{....Eq. 18}$$

Spring stiffnesses can be evaluated for modified spring end conditions according to Eqs 12 through 15 and the eigenvalue problem, as stated in either one of Eqs 16 or 17, can be solved, yielding two values for the circular eigenfrequency ω for each set of spring end conditions.

The basic sets of equations, i.e., Eq. 16 or 17 with four combinations of spring end conditions, can also be used to investigate some special conditions.

The gravity g may decrease continuously to zero from the terrestrial reference value g_0. Zero gravity simulates the celestal conditions at free orbiting in space and demonstrates also clearly the pre-load influence.

The stiffness of the helical spring may increase continuously to infinity by letting the modulus of elasticity (or the modulus of shear) increase from its normal value to infinity. If then at least one of the spring ends is pinned, the system transfers to a rigid pendulum.

Numerical values of the two related circular eigenfrequencies have been computed for all the various cases described in the preceding paragraphs. The values are given in Fig. 6, from which the following conclusions can be drawn:

I. The influence of the pre-load is considerable, and relatively stronger for the lower mode compared to the higher mode.

II. For the celestal case some eigenfrequencies decrease to zero, others remain finite. The number of vanishing eigenfrequencies is in agreement with the expected characteristics at various spring end conditions.

III. For the terrestrial pendulum case some eigenfrequencies increase to infinity, others remain finite. The number of finite eigenfrequencies, as well as their numerical values, is in agreement with the well known characteristics of single and double pendulums.

Complex discrete systems

The principles used in the analysis of the basic example in the preceding section have been developed further to cover any system comprising arbitrarily arranged rigid bodies and arbitrarily interconnected massless linear springs and/or various types of joints and supports. Details are given in [1,2]. For complex systems it is possible to predict theoretically all the effects demonstrated in the basic example, i.e., celestal conditions, rigid terrestrial pendulums, etc.

Experimental evidence

An experimental structure as shown in Fig. 7 has been developed to demonstrate the pre-load influence as an isolated phenomenon. Two rigid bodies are constrained to vibrate in one plane, only. Damping is eliminated almost completely by using aerostatic bearings for constraining the motion.

By tilting the supporting plane about a horisontal axis, the system pre-load is changed without changing any component of the system.

The system has six unconstrained modes of motion. Two of them are axial, and due to symmetry, the axial modes are decoupled from the remaining modes and are not influenced of the pre-load.

There are four degrees of freedom for lateral motion. All of these modes are mutually coupled.

The natural frequencies for the system have been determined experimentally by a sweeping excitation technique for various angles of tilt. Some significant results are shown in Fig. 8. The natural frequencies of axial motion were almost constant for the whole range of tilt.

Fig. 8 demonstrates a considerable influence of pre-load. The influence is strongest at low natural frequencies.

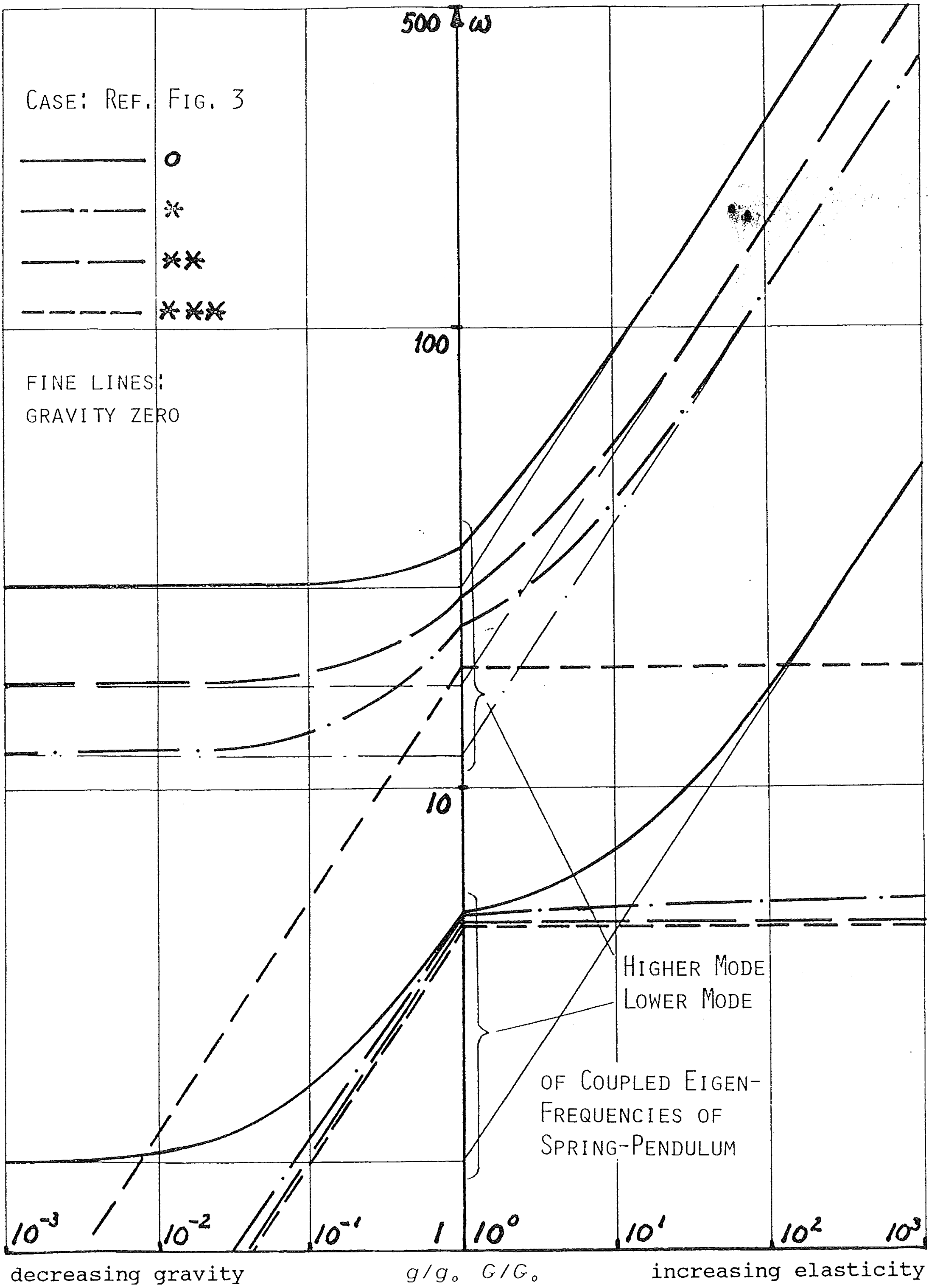


Fig. 6: Computed values of lateral frequencies

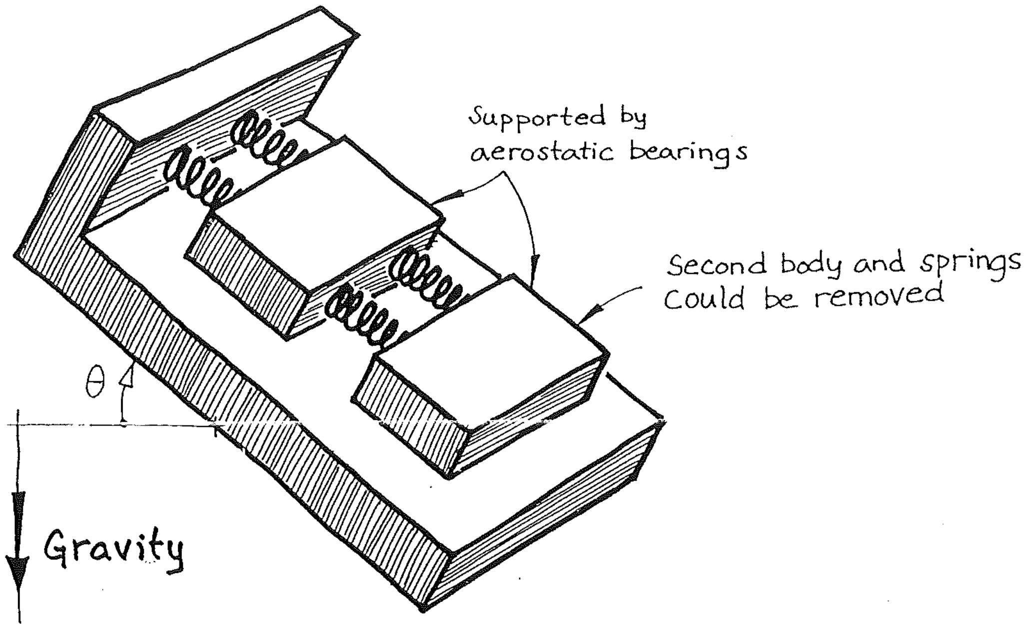


Fig. 7: Test rig

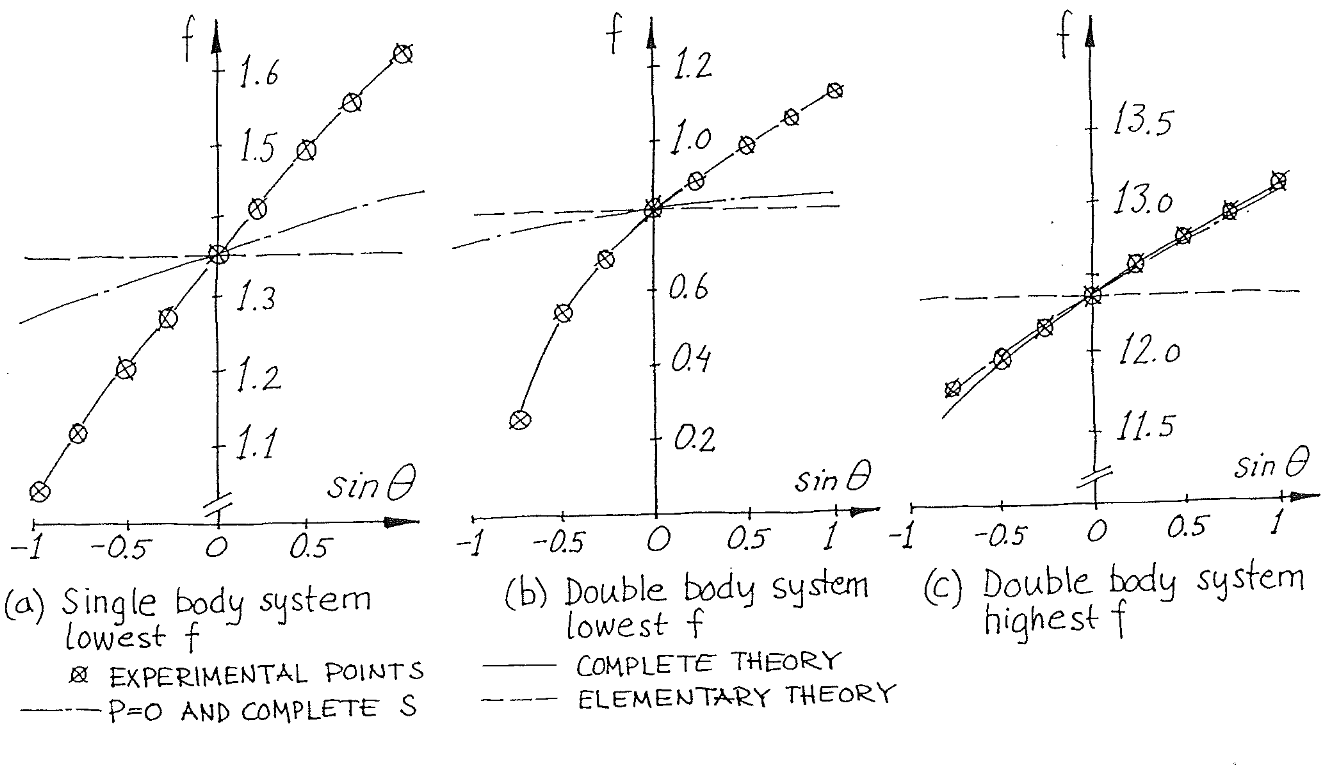


Fig. 8: Experimental data on lateral frequencies

The natural frequencies of the experimental system have also been predicted theoretically. Two different types of analysis have been used: one complete theory, including the influence from both the matrix *P* and the pre-load sensitive spring stiffness *S*, and the other disregarding the influence from the matrix *P*, but considering the pre-load sensitivity of the helical springs. The influence from the matrix *P* is dominant at low natural frequensies, whereas the spring stiffness sensitivity is dominant at higher natural frequencies.

The quoted experiments are reported in more detail in [4,5].

Conclusions

The significance of the pre-loads has been demonstrated clearly both theoretically and experimentally. The phenomenon has been given a conseptual explanation. Analytical procedures have been developed, which automatically take into account both the elastic and pre-load dependent restoring effects in linear multi-dimensional discrete systems, no matter which one that happens to be the quantitatively dominating restoring source.

References

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