Primal Decomposition of the Optimal Coordination of Vehicles at Traffic Intersections

Robert Hult, Mario Zanon, Sébastien Gros and Paolo Falcone

***Abstract*— In this paper we address the problem of coordinating automated vehicles at intersections, which we state as a constrained finite horizon optimal control problem. We present and study the properties of a primal decomposition of the optimal control problem. More specifically, the decomposition consists of an upper problem that allocates occupancy time-slots in the intersection, and lower-level problems delivering control policies for each vehicle. We investigate the continuity class of the upper problem, and show that it can be efficiently tackled using a standard sequential quadratic programming and that most computations can be distributed and performed by the participating vehicles. The paper is concluded with an illustrative numerical example.**

I. INTRODUCTION

Coordination of communicating automated vehicles at intersections, on-ramps, and in other traffic scenarios where roads merge or cross is a topic that has attracted a lot of attention in recent years, see, e.g., [3],[4], [8] or [12]. It is commonly claimed that, using vehicle automation and communication, the number of accidents in such scenarios can be reduced and both energy efficiency and infrastructure utilization can be improved. In particular, cooperatively decided control policies could be employed on the vehicle level to explicitly coordinate the use of the zone where collisions can occur (e.g., the inside of the intersection). In the fully automated case, such control systems would remove the need for traffic lights, signs and rules and thereby enable continuous flows of traffic.

The problem of finding the coordinating control policies can be interpreted as two interdependent sub-problems; first, the precedence order for the utilization of the coordination zone has to be established, i.e., the order in which the involved vehicles enter the intersection. Secondly, the control action for each vehicle needs to be computed such that the precedence order is respected and the vehicles cross the coordination zone without collision. Since the control actions taken by the cars depend on the precedence order, the two problems are coupled. Solving the overall coordination problem thus requires the solution of a combinatorial problem with dynamic constraints. It was shown in [3] that finding a feasible solution is NP-hard in general. Several works exist that address various aspects of the coordination problem at intersections. In [5], for instance, a heuristic scheme is presented where incoming vehicles requests access to the coordination zone from a central *Intersection Manager*, which accept or deny the request based on a set of rules. A road side coordinator is also proposed in [9], where collision free control actions are decided centrally and then disseminated to the nearby vehicles. In [1], on the other hand, a distributed algorithm is presented where a predefined interaction protocol is used to resolve predicted collisions between vehicles. Similarly, in [4] another distributed scheme is presented, where the precedence order is decided through sequential decision making and the control sub-problems solved using tools from optimal control.

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The authors are with the Department of Signals and Systems, Chalmers University of Technology, Göteborg, Sweden. e-mail: {robert.hult, mzanon, grosse, paolo.falcone}@chalmers.se

In this paper, we state the intersection coordination as an optimal control problem. We consider here that the precedence problem is solved separately, and assume that a precedence order is provided. The problem under consideration is then reduced to finding the control policies under a given precedence order. In particular, we first formulate the problem as a constrained, finite-time optimal control problem and propose a primal decomposition of this formulation. We then study the properties of the decomposed problem and show that the sensitivities of the objective and constraints are available at a low computational cost. Finally, we propose to solve the problem using a second order method and show that the decomposition allows for a large part of the computations to be distributed.

The approach taken in this paper builds partly on [8], where the coordination problem for intersections was formulated in an optimal control context, and a method for the approximate solution to both the precedence and control sub-problems were presented. As opposed to [8], this paper provides a study of the decomposed problem and details how the sensitivity information needed to solve it using a second order method can be computed in a distributed fashion.

A. Notation

In this paper we write scalars as x and vectors as $\mathbf{x}$. The gradient of a function $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$ with respect to $\mathbf{x} \in \mathbb{R}^n$ is written as $\nabla_{\mathbf{x}}\mathbf{f}(\mathbf{x}) \in \mathbb{R}^{n\times m}$, i.e. as the transpose of the Jacobian of the function with respect to $\mathbf{x}$. The Hessian of a scalar function $f(\mathbf{x})$ with respect to $\mathbf{x}$ is written $\nabla^2_{\mathbf{x}} f(\mathbf{x})$, and the total derivative with respect to x is written $\frac{\mathrm{d}}{\mathrm{d}x}$. Finally, the element on the i:th row and j:th column of a matrix A is denoted $(A)_{(i,j)}$.

II. PROBLEM FORMULATION

We consider the problem of coordinating M vehicles through an intersection under a fixed precedence order. The

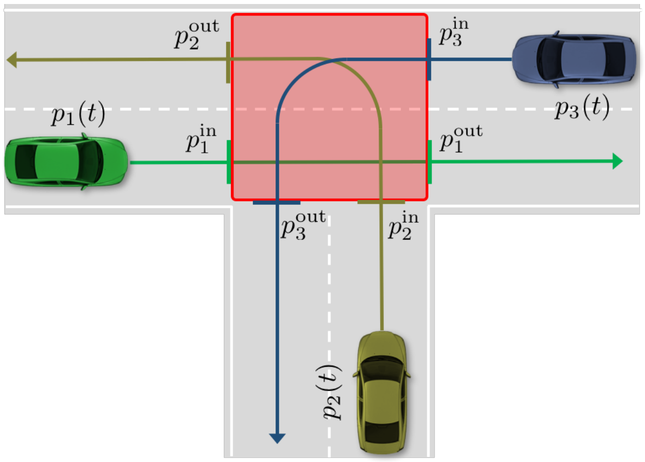

Fig. 1. Schematic illustration of the modelling of the interection scenarios considered in this paper. The red area corresponds to the zone where side collisions may occur, defined with p^{in} and p^{out} for each vehicle.

control action for each vehicle is therefore sought such that the precedence order is satisfied, collisions within the intersection are avoided and a prescribed performance criteria is optimized. We assume that there is one vehicle per road, and that each vehicle moves along a pre-determined path. The vehicles are indexed using $i = 1, \ldots, M$, and assumed to be ordered as they appear in the precedence order, i.e. so that vehicle i crosses the intersection before vehicle j if $i > j$.

A. Collision Avoidance

As shown in Fig. 1 the intersection can then be modeled as an interval, $[p_i^{\text{in}}, p_i^{\text{out}}]$, on the path of each vehicle i. A vehicle is considered to be inside the intersection at time t if the position along its path, $p_i(t)$, is such that $p_i(t) \in [p_i^{\text{in}}, p_i^{\text{out}}]$. If $p_i(t)$ is continuous and $\dot{p}_i(t) > 0, \forall t$, the times at which a vehicle enters (t_i^{in}) and exits (t_i^{out}) the intersection are implicitly and uniquely defined by

$$p_i(t_i^{\text{in}}) = p_i^{\text{in}}, \quad \text{and} \quad p_i(t_i^{\text{out}}) = p_i^{\text{out}}, \tag{1}$$

respectively. Provided that $p_i^{\text{in}}, p_i^{\text{out}}$ and $p_i(t)$ are defined such that the vehicle geometry can be ignored, a sufficient condition for collision avoidance is then

$$t_j^{\text{out}} \leq t_{j+1}^{\text{in}}, \ j = 1, \ldots, M-1. \tag{2}$$

B. Motion model

The dynamics along the paths are modeled as linear and discretized on a uniform time-grid with intervals of size T_s. Starting from the initial state $\mathbf{x}_{i,0} = \hat{\mathbf{x}}_{i,0}$, the state evolution of vehicle i is given by

$$\mathbf{x}_{i,k+1} = A_i \mathbf{x}_{i,k} + B_i \mathbf{u}_{i,k}, \tag{3}$$

where $\mathbf{x}_{i,k}$ and $\mathbf{u}_{i,k}$ is the state and control at time $t_k = kT_s$ respectively, and $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$ for some n_i, m_i. In particular, the system matrices are defined as $A_i := \mathcal{A}_i(T_s) = \exp(A_{c,i}T_s)$ and $B_i := \mathcal{B}_i(T_s) = \int_0^{T_s} \exp(A_{c,i}(T_s - s))B_c \mathrm{d}s$, for the continuous time system matrices $A_{c,i}$ and $B_{c,i}$. The state and control of each vehicle i are additionally subject to the following constraints capturing e.g. actuator limitations and speed limits:

$$D_{i,k}\mathbf{x}_{i,k} + E_{i,k}\mathbf{u}_{i,k} \leq \mathbf{e}_{i,k}, \tag{4}$$

where $D_{i,k} \in \mathbb{R}^{q_i \times n_i}$, $E_{i,k} \in \mathbb{R}^{q_i \times m_i}$ and $\mathbf{e}_{i,k} \in \mathbb{R}^{q_i}$ for some q_i. For ease of notation we will in the following use

$$\mathbf{w}_{i,j} = \begin{bmatrix} \mathbf{x}_{i,j} \\ \mathbf{u}_{i,j} \end{bmatrix} \quad \text{and} \quad \mathbf{w}_i = \begin{bmatrix} \mathbf{w}_{i,0} \\ \vdots \\ \mathbf{w}_{i,N_i-1} \\ \mathbf{x}_{i,N_i} \end{bmatrix}, \tag{5}$$

where N_i is a time horizon associated with vehicle i.

The position of a vehicle along its path is a considered a linear function of $\mathbf{w}_i$, and is consequently only defined at the discrete times t_k. To use definition (1), and allow for continuous adjustment of t_i^{in} and t_i^{out}, a continuous representation of the position of a vehicle is needed. We therefore use the continuous time representation of the dynamics (3), to integrate the position at intermediate times $t \in]t_k, t_{k+1}[$. In particular, we define the continuous position piece-wise according to

$$p_i(t, \mathbf{w}_i) = C_i[\mathcal{A}_i(t - t_k), \mathcal{B}_i(t - t_k)]\mathbf{w}_{i,k}, \ t \in [t_k, t_{k+1}[, \tag{6}$$

where $C_i \in \mathbb{R}^{1 \times (n_i + m_i)}$. It can be shown that this implies that if $C_i A_{c,i}^j B_{c,i} = \mathbf{0}$ for $j = 0, \ldots, K$, i.e., that the relative degree of the continuous time system is $K+1$, $p_i(t, \mathbf{w}_i)$ is in $\mathcal{C}^K$ on the vector space spanned by the dynamics (3). We also assume that $\frac{\mathrm{d}}{\mathrm{d}t}p_i(t, \mathbf{w}_i) > 0, \forall t$ for $\mathbf{w}_i \in \mathbb{W}_i(\hat{\mathbf{x}}_{i,0})$, where

$$\mathbb{W}_i(\hat{\mathbf{x}}_{i,0}) = \left\{ \mathbf{w}_i \ \middle| \ \begin{array}{l} \mathbf{x}_{i,0} = \hat{\mathbf{x}}_{i,0} \\ \mathbf{x}_{i,k+1} = A_i\mathbf{x}_{i,k} + B_i\mathbf{u}_{i,k}, \ k \in \mathcal{N}_i \\ D_{i,k}\mathbf{x}_{i,k} + E_{i,k}\mathbf{u}_{i,k} \leq \mathbf{e}_{i,k}, \ k \in \mathcal{N}_i \end{array} \right\}, \tag{7}$$

using $\mathcal{N}_i = \{0, \ldots, N_i - 1\}$. Therefore, t_i^{in} and t_i^{out} are uniquely defined for a given $\mathbf{w}_i \in \mathbb{W}(\hat{\mathbf{x}}_{i,0})$ through

$$p_i(t_i^{\text{in}}, \mathbf{w}_i) = p_i^{\text{in}}, \quad \text{and} \quad p_i(t_i^{\text{out}}, \mathbf{w}_i) = p_i^{\text{out}}, \tag{8}$$

provided that $p_i(t_{N_i}, \mathbf{w}_i) > p^{\text{out}}$. Finally, we assume that the convex polyhedron $\mathbb{W}_i(\hat{\mathbf{x}}_{i,0})$ is bounded, and that the following technical assumption holds:

Assumption 1: For $0 \leq t_d \leq t_{N_i}$, $d \geq p_i(t_0, \mathbf{w}_i)$ let $S_i(t_d, d) = \{\mathbf{w}_i \mid p_i(t_d, \mathbf{w}_i) - d = 0\}$. Provided that $\mathbb{W}_i(\hat{\mathbf{x}}_{i,0}) \cap S_i(t_d, d) \neq \emptyset$, we assume that $\exists \underline{\mathbf{w}}_i, \bar{\mathbf{w}}_i \in \mathbb{W}_i(\hat{\mathbf{x}}_{i,0}) \cap S_i(t_d, d)$ satisfying

$$p_i(t, \underline{\mathbf{w}}_i) < p_i(t, \mathbf{w}_i) < p_i(t, \bar{\mathbf{w}}_i), \ \forall t > t_d, \tag{9}$$

for all $\mathbf{w}_i \in \mathbb{W}_i(\hat{\mathbf{x}}_{i,0}) \cap S_i(t_d, d)$ such that $\mathbf{w}_i \neq \underline{\mathbf{w}}_i$ and $\mathbf{w}_i \neq \bar{\mathbf{w}}_i$.

Assumption 1 states that if a vehicle passes a position d at time t_d, then there exists a unique control sequence that maximizes the position for $t > t_d$, and one that similarly minimizes it. In other words, the dynamics are thus such that there exists a notion of maximum and minimum control with respect to the position $p_i(t, \mathbf{w}_i)$.

C. Optimal Control Formulation

We restrict our attention to objective functions on the form

$$J_i(\mathbf{w}_i) = V_i(\mathbf{x}_{i,N_i}) + \sum_{k=0}^{N_i-1} l_{i,k}(\mathbf{w}_{i,j}), \tag{10}$$

where $V_i(\cdot)$ and all $l_{i,k}(\cdot)$ are convex, quadratic functions.

Writing $\mathbf{t}_i = [t_i^{\text{in}}, t_i^{\text{out}}]^\top$, $\mathbf{T} = [\mathbf{t}_1^\top, \dots, \mathbf{t}_M^\top]^\top$, $\mathbf{W} = [\mathbf{w}_1^\top, \dots, \mathbf{w}_M^\top]^\top$ and $\mathcal{M} = \{1, \dots, M\}$, the intersection coordination problem for a fixed order then reads as:

$$\begin{aligned}
\min_{\mathbf{T},\mathbf{W}} \quad & \sum_{i=1}^{M} J_i(\mathbf{w}_i) && \text{(11a)}\\
\text{s.t.} \quad & \mathbf{w}_i \in \mathbb{W}_i(\hat{\mathbf{x}}_{i,0}),\ i \in \mathcal{M} && \text{(11b)}\\
& p_i(t_i^{\text{in}}, \mathbf{w}_i) = p_i^{\text{in}},\ i \in \mathcal{M} && \text{(11c)}\\
& p_i(t_i^{\text{out}}, \mathbf{w}_i) = p_i^{\text{out}},\ i \in \mathcal{M} && \text{(11d)}\\
& t_i^{\text{out}} \le t_{i+1}^{\text{in}},\ i = 1, \dots, M-1 && \text{(11e)}
\end{aligned}$$

Problem (11) is a nonlinear program (NLP) with $\sum_{i=1}^{M}(2 + n_i + m_i)N_i$ decision variables, a convex quadratic objective, $\sum_{i=1}^{M} n_i N_i$ affine equality constraints, $M - 1 + \sum_{i=1}^{M} r_i N_i$ affine inequality constraints and the $2M$ nonlinear equality constraints (11c)-(11d).

III. Decomposition

To reduce the computational requirements of solving the coordination problem (11) with long horizons and large state spaces, a two-level decomposition of Problem (12) was presented in [8]. In this decomposition, the top level consists of a time-slot allocation problem and the low level consists of one optimal control problem for each of the M vehicles. More specifically, Problem (11) is equivalent to

$$\begin{aligned}
\min_{\mathbf{T}} \quad & \sum_{i=1}^{M} \Phi_i(\mathbf{t}_i, \hat{\mathbf{x}}_{i,0}) && \text{(12a)}\\
\text{s.t.} \quad & \mathbf{t}_i \in \mathcal{D}_i(\hat{\mathbf{x}}_{i,0}),\ \in \mathcal{M} && \text{(12b)}\\
& t_i^{\text{out}} \le t_{i+1}^{\text{in}},\ i = 1, \dots, M-1 && \text{(12c)}
\end{aligned}$$

where

$$\begin{aligned}
\Phi_i(\mathbf{t}_i, \hat{\mathbf{x}}_{i,0}) := \min_{\mathbf{w}_i} \quad & J_i(\mathbf{w}_i) && \text{(13a)}\\
\text{s.t.} \quad & \mathbf{w}_i \in \mathbb{W}_i(\hat{\mathbf{x}}_{i,0}) && \text{(13b)}\\
& p_i(t_i^{\text{in}}, \mathbf{w}_i) - p_i^{\text{in}} = 0 && \text{(13c)}\\
& p_i(t_i^{\text{out}}, \mathbf{w}_i) - p_i^{\text{out}} = 0 && \text{(13d)}
\end{aligned}$$

and

$$\mathcal{D}_i(\hat{\mathbf{x}}_{i,0}) = \operatorname{dom}(\Phi_i(\mathbf{t}_i, \hat{\mathbf{x}}_{i,0})). \tag{14}$$

Problem (13) is thus a parametric quadratic program, and $\mathcal{D}_i(\hat{\mathbf{x}}_{i,0})$ is the set of $\mathbf{t}_i$ for which (13) is feasible. Since we only consider the problem for given initial states, we drop the dependence on $\hat{\mathbf{x}}_{i,0}$ from for $\mathcal{D}_i(\hat{\mathbf{x}}_{i,0})$ and $\Phi_i(\mathbf{t}_i, \hat{\mathbf{x}}_{i,0})$ in the remainder of the paper.

We will in Section V present a method of solving (11) by applying a numerical algorithm to (12). To that end, we will first establish some properties of $\mathcal{D}_i$ and $\Phi_i(\mathbf{t}_i)$.

IV. Properties of Problem (12)

In this section, we discuss the properties of the objective function $\Phi_i(\mathbf{t}_i)$ of Problem (12) and constraint set $\mathcal{D}_i$ (14). In [8] it was shown that the function $\Phi_i(\mathbf{t})$ has a single unique minimum, but that it is not convex in general. We show here that $\Phi_i(\mathbf{t}_i)$ is $\mathcal{C}^0$, and piece-wise $\mathcal{C}^K$ on $\mathcal{D}_i$ if the position function $p_i(t, \mathbf{w}_i)$ is $\mathcal{C}^K$. We also show that $\mathcal{D}_i$ can be defined by two linear and two nonlinear inequalities. In particular, these inequalities can be evaluated by solving linear programs (LPs). Finally, we show that also the nonlinear constraints are $\mathcal{C}^0$, and piece-wise $\mathcal{C}^K$ if the position function $p_i(t, \mathbf{w}_i) \in \mathcal{C}^K$ when $\mathbf{w}_i \in \mathbb{W}_i$.

It will be useful to consider a condensed version of (13), where the state $\mathbf{x}_{i,k}$ is eliminated via the dynamics (3) using:

$$\mathbf{x}_{i,k+1}(\mathbf{u}_i) = A_i^{k+1}\hat{\mathbf{x}}_{i,0} + \sum_{j=0}^{k} A_i^{k-j} B_i \mathbf{u}_{i,j}, \tag{15}$$

so that $\mathbf{w}_i = G_i\mathbf{u}_i + \mathbf{b}_i\hat{\mathbf{x}}_{i,0}$, where $\mathbf{u}_i = [\mathbf{u}_{i,0}^\top, \dots, \mathbf{u}_{i,N_i-1}^\top]^\top$. The condensed objective is written as $J_i^{\text{c}}(\mathbf{u}_i^*)$ and the condensed constraints $\mathbb{W}_i(\hat{\mathbf{x}}_{i,0})$ as $\mathbf{h}_i(\mathbf{u}_i, \hat{\mathbf{x}}_{i,0}) := \mathbf{h}_i(\mathbf{u}_i) \le \mathbf{0}$, where $\mathbf{h}_i(\mathbf{u}_i) = [h_{i,1}(\mathbf{u}_i), \dots, h_{i,q_i N_i}(\mathbf{u}_i)]^\top$. Furthermore, we write the condensed position function as $p_i(t, \mathbf{w}_i) = p_i^{\text{c}}(t, \mathbf{u}_i)$. The following continuity result can then easily be shown:

Lemma 1: If $p_i(t, \mathbf{w}_i) \in \mathcal{C}^K$ for all $\mathbf{w}_i$ satisfying the dynamics (3), then $p_i^{\text{c}}(t, \mathbf{u}_i) \in \mathcal{C}^K$ and $\nabla_{\mathbf{u}_i} p_i^{\text{c}}(t, \mathbf{u}_i) \in \mathcal{C}^K$ for all $\mathbf{u}_i$.

A. Properties of Φ_i

In this subsection we discuss the continuity class of the objective functions $\Phi_i(\mathbf{t}_i)$. For notational convenience, the sub-system indices are dropped in the remainder of this section. In its condensed form, (13) reads as

$$\begin{aligned}
\Phi(\mathbf{t}) = \min_{\mathbf{u}} \quad & J^{\text{c}}(\mathbf{u}) && \text{(16a)}\\
\text{s.t.} \quad & \mathbf{h}(\mathbf{u}) \le 0 && \text{(16b)}\\
& \mathbf{p}^{\text{c}}(\mathbf{t}, \mathbf{u}) = 0 && \text{(16c)}
\end{aligned}$$

where

$$\mathbf{p}^{\text{c}}(\mathbf{t}, \mathbf{u}) = \begin{bmatrix} p^{\text{c}}(t^{\text{in}}, \mathbf{u}) - p^{\text{in}} \\ p^{\text{c}}(t^{\text{out}}, \mathbf{u}) - p^{\text{out}} \end{bmatrix}. \tag{17}$$

Under the assumption that Linear Independence Constraint Qualification (LICQ) [11] holds, the primal-dual solution $\mathbf{z}^*$ to (16) for a given $\mathbf{t}$ satisfies the KKT conditions

$$\mathbf{r}(\mathbf{z}^*, \mathbf{t}) = \begin{bmatrix} \nabla_{\mathbf{w}} \mathcal{L}_v(\mathbf{z}^*, \mathbf{t}) \\ \mathbf{h}_{\mathbb{A}}(\mathbf{u}^*) \\ \mathbf{p}^{\text{c}}(\mathbf{t}, \mathbf{u}^*) \end{bmatrix} = \mathbf{0}. \tag{18}$$

where

$$\mathcal{L}_v = J^{\text{c}}(\mathbf{u}) + \boldsymbol{\mu}_{\mathbb{A}}^\top \mathbf{h}_{\mathbb{A}}(\mathbf{u}) + \boldsymbol{\nu}^\top \mathbf{p}^{\text{c}}(\mathbf{t}, \mathbf{u}). \tag{19}$$

Here $\mathbf{z} = [\mathbf{u}^\top, \boldsymbol{\mu}_{\mathbb{A}}^\top, \boldsymbol{\nu}^\top]^\top$ and $\mathbb{A}$ is the index set of the strictly active constraints $\mathbf{h}(\mathbf{u}^*)$ at the solution, i.e. $\mathbb{A} = \{i \quad \text{s.t.} \quad h_i(\mathbf{u}^*) = 0, \quad \mu_i^* > 0\}$. If in addition, the Second Order Sufficient Condition (SOSC) [11] holds at the solution,

$\mathbf{u}^*$ is the unique minimizer of Problem (16), and $\boldsymbol{\mu}_{\mathbb{A}}^*, \boldsymbol{\nu}^*$ are the corresponding unique Lagrange multipliers.

Proposition 1: If LICQ holds for Problem (16) for every parametric solution $\mathbf{u}^*(\mathbf{t})$, then $\Phi(\mathbf{t})$ is continuous on $\mathcal{D}$. Furthermore, if $\mathbf{p}^{\mathrm{c}}(\mathbf{t},\mathbf{u}) \in \mathcal{C}^K$ and SOSC hold then $\Phi(\mathbf{t}) \in \mathcal{C}^K$ at every $\mathbf{t}$ where no constraint is weakly active.

Proof: Since LICQ holds $\forall \mathbf{t} \in \mathcal{D}$, since $J^{\mathrm{c}}(\mathbf{u})$ is continuous and since the feasible set of (16) is compact, we have from [6] that $\Phi(\mathbf{t})$ is continuous on $\mathcal{D}$.

Furthermore, Lemma 1 guarantees that $p^{\mathrm{c}}(t,\mathbf{u}) \in \mathcal{C}^K$ and $\nabla_{\mathbf{u}} p^{\mathrm{c}}(t,\mathbf{u}) \in \mathcal{C}^K$. Since $\mathbf{r}(\mathbf{z},\mathbf{t}) \in \mathcal{C}$ inherits the continuity class of $p^{\mathrm{c}}(t,\mathbf{u})$ and $\nabla_{\mathbf{u}} p^{\mathrm{c}}(t,\mathbf{u})$, we have that $\mathbf{r}(\mathbf{z},\mathbf{t}) \in \mathcal{C}^K$. Since SOSC and LICQ also hold by assumption, $\nabla_{\mathbf{z}} \mathbf{r}(\mathbf{z}^*(\mathbf{t}),\mathbf{t})$ is full rank [11]. The implicit function theorem (IFT), as stated in, e.g., [10], therefore guarantees the existence of a function $\mathbf{z}^*(\mathbf{t}) \in \mathcal{C}^K$ such that $\mathbf{r}(\mathbf{z}^*(\mathbf{t}),\mathbf{t}) = 0$ in an open neighborhood of $\mathbf{t}$. Since $J^{\mathrm{c}}(\mathbf{u}^*)$ is quadratic and $\Phi(\mathbf{t}) = J^{\mathrm{c}}(\mathbf{u}^*(\mathbf{t}))$, it follows that $\Phi(\mathbf{t}) \in \mathcal{C}^K$ in a neighborhood of $\mathbf{t}$. ■

It should be observed that as a consequence of Proposition 1, $\Phi(\mathbf{t})$ is only $\mathcal{C}^0$ at entry and exit times $\mathbf{t}$ where a change in the active set $\mathbb{A}$ occurs.

B. Properties of the Constraint Set $\mathcal{D}$

In this subsection, we show that $\mathcal{D} = \mathrm{dom}(\Phi(\mathbf{t}))$ can be described by two linear and two nonlinear inequalities. We discuss the continuity class of the nonlinear inequalities, and show that they can be evaluated by solving linear programs.

Proposition 2: $\mathcal{D} = \{\mathbf{t} \mid \mathbf{F}(\mathbf{t}) \leq 0\}$ with

$$\mathbf{F}_i(\mathbf{t}) = \begin{bmatrix} b_{\mathrm{L}}^{\mathrm{in}} - t^{\mathrm{in}} \\ t^{\mathrm{in}} - b_{\mathrm{U}}^{\mathrm{in}} \\ t^{\mathrm{out}} - b_{\mathrm{U}}^{\mathrm{out}}(t^{\mathrm{in}}) \\ b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}}) - t^{\mathrm{out}} \end{bmatrix}, \tag{20}$$

where the upper bound on t^{in}, $b_{\mathrm{U}}^{\mathrm{in}}$, is given by

$$b_{\mathrm{U}}^{\mathrm{in}} := \max_{\mathbf{u},t} \quad t \tag{21a}$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{u}) \leq 0 \tag{21b}$$
$$p^{\mathrm{c}}(t,\mathbf{u}) - p^{\mathrm{in}} = 0 \tag{21c}$$

and the lower bound $b_{\mathrm{L}}^{\mathrm{in}}$ through the corresponding minimization. Similarly, the upper bound on t^{out}, $b_{\mathrm{U}}^{\mathrm{out}}(t^{\mathrm{in}})$, is given by

$$b_{\mathrm{U}}^{\mathrm{out}}(t^{\mathrm{in}}) := \max_{\mathbf{u},t} \quad t \tag{22a}$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{u}) \leq 0 \tag{22b}$$
$$p^{\mathrm{c}}(t^{\mathrm{in}},\mathbf{u}) - p^{\mathrm{in}} = 0 \tag{22c}$$
$$p^{\mathrm{c}}(t,\mathbf{u}) - p^{\mathrm{out}} = 0 \tag{22d}$$

and the lower bound $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$ as the corresponding minimization.

Proof: By definition, minimization and maximization of (21) defines bounds $b_{\mathrm{L}}^{\mathrm{in}} \leq t^{\mathrm{in}} \leq b_{\mathrm{U}}^{\mathrm{in}}$. Similarly, by definition the minimization and maximization of (22) defines bounds on t^{out}, given t^{in}, i.e., $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}}) \leq t^{\mathrm{out}} \leq b_{\mathrm{U}}^{\mathrm{out}}(t^{\mathrm{in}})$. We thus have that $\mathbf{t} \in \mathrm{dom}(\Phi(\mathbf{t})) \Rightarrow \mathbf{F}(\mathbf{t}) \leq 0$.

We then need to show that $F(\mathbf{t}) \leq 0 \Rightarrow \mathbf{t} \in \mathrm{dom}(\Phi(\mathbf{t}))$. Let $b_{\mathrm{L}}^{\mathrm{in}}$ and $b_{\mathrm{U}}^{\mathrm{in}}$ denote the optimal cost of the maximization and minimization of (21) and $\mathbf{u}^{\max}$ and $\mathbf{u}^{\min}$ the primal solutions respectively, and let $u^\theta = \theta \mathbf{u}^{\max} + (1-\theta)\mathbf{u}^{\min}$. Due to convexity of $\mathbf{h}(\mathbf{w})$, we then have $\mathbf{h}(\mathbf{u}^\theta) \leq 0$ for $\theta \in [0,1]$. Due to the continuity of $p^{\mathrm{c}}(t,\mathbf{u})$, $\exists \theta \in [0,1]$ such that $p^{\mathrm{c}}(t,\mathbf{u}^\theta) - p^{\mathrm{in}} = 0$, $\forall t \in [b_{\mathrm{L}}^{\mathrm{in}}, b_{\mathrm{L}}^{\mathrm{out}}]$. For a given $t^{\mathrm{in}} \in [b_{\mathrm{L}}^{\mathrm{in}}, b_{\mathrm{U}}^{\mathrm{in}}]$, the same argument holds for $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$ and $b_{\mathrm{U}}^{\mathrm{out}}(t^{\mathrm{in}})$. It follows that $F(\mathbf{t}) \leq 0 \Leftrightarrow \mathbf{t} \in \mathrm{dom}(\Phi(\mathbf{t}))$. ■

As a consequence of Proposition 2, $b_{\mathrm{L}}^{\mathrm{in}}$ can be computed independently of $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$, while $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$ is a nonlinear function of t^{in} and requires $b_{\mathrm{L}}^{\mathrm{in}}$. We show next that the solutions to problems (21) and (22) can be computed by solving Linear Programs (LPs).

Proposition 3: The lower bound $b_{\mathrm{L}}^{\mathrm{in}}$, can be computed as the solution to

$$p^{\mathrm{c}}(t,\mathbf{u}^{\mathrm{in}}) - p^{\mathrm{in}} = 0, \tag{23}$$

where $\mathbf{u}^{\mathrm{in}}$ is the solution to the linear program (LP)

$$\max_{\mathbf{u}} \quad p^{\mathrm{c}}(t_N,\mathbf{u}) \tag{24a}$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{u}) \leq 0. \tag{24b}$$

Similarly, $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$ can be computed as $p^{\mathrm{c}}(t,\mathbf{u}^{\mathrm{out}}(t^{\mathrm{in}})) - p^{\mathrm{out}} = 0$, where $\mathbf{u}^{\mathrm{out}}(t^{\mathrm{in}})$ is the solution to the LP

$$\max_{\mathbf{u}} \quad p^{\mathrm{c}}(t_N,\mathbf{u}) \tag{25a}$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{u}) \leq 0 \tag{25b}$$
$$p^{\mathrm{c}}(t^{\mathrm{in}},\mathbf{u}) = 0 \tag{25c}$$

The upper bounds $b_{\mathrm{U}}^{\mathrm{in}}$ and $b_{\mathrm{U}}^{\mathrm{out}}(t^{\mathrm{in}})$ can be computed via the corresponding minimizations.

Proof: Consider the case of $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$. For a given t^{in}, we have that $p^{\mathrm{c}}(t_N,\mathbf{u}^{\mathrm{out}}) \geq p^{\mathrm{c}}(t_N,\mathbf{u})$ for all $\mathbf{u}$ feasible in (25). From Assumption (1), it then follows that $\mathbf{u}^{\mathrm{out}} = \bar{\mathbf{u}}$, where $\bar{\mathbf{u}}$ is the condensed form of $\bar{\mathbf{w}}$.

Then, by Assumption (1) $p^{\mathrm{c}}(t,\mathbf{u}^{\mathrm{out}}) > p^{\mathrm{c}}(t,\mathbf{u})$, $\forall t > t^{\mathrm{in}}$ and all feasible $\mathbf{u} \neq \mathbf{u}^{\mathrm{out}}$. With $p^{\mathrm{c}}(t^*,\mathbf{u}^{\mathrm{out}}) - p^{\mathrm{out}} = 0$ and $p^{\mathrm{c}}(t^{\mathrm{out}},\mathbf{u}) - p^{\mathrm{out}} = 0$, we consequently have $t^* < t^{\mathrm{out}}$, for all feasible $\mathbf{u} \neq \mathbf{u}^{\mathrm{out}}$. Consequently, $t^* = b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$. The result for $b_{\mathrm{L}}^{\mathrm{in}}, b_{\mathrm{U}}^{\mathrm{in}}$ and $b_{\mathrm{U}}^{\mathrm{out}}(t^{\mathrm{in}})$ follows. ■

We finally state the continuity class of the nonlinear functions $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$ and $b_{\mathrm{U}}^{\mathrm{out}}(t^{\mathrm{in}})$ in the following Proposition:

Proposition 4: If LICQ and SOSC hold at the solution to LP (24) for all $t^{\mathrm{in}} \in [b_{\mathrm{L}}^{\mathrm{in}}, b_{\mathrm{L}}^{\mathrm{out}}]$, then the functions $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$ and $b_{\mathrm{U}}^{\mathrm{out}}(t^{\mathrm{in}})$ are continuous on $[b_{\mathrm{L}}^{\mathrm{in}}, b_{\mathrm{U}}^{\mathrm{in}}]$. If in addition $p^{\mathrm{c}}(t,\mathbf{u}) \in \mathcal{C}^K$, the functions $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$ and $b_{\mathrm{U}}^{\mathrm{out}}(t^{\mathrm{in}})$ are piece-wise $\mathcal{C}^K$ on open subsets of $[b_{\mathrm{L}}^{\mathrm{in}}, b_{\mathrm{U}}^{\mathrm{in}}]$.

Proof: Let $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}}) = f(\mathbf{u}^*(t^{\mathrm{in}}))$ be a solution to (23), where $\mathbf{u}^*(t^{\mathrm{in}})$ is the minimizer of (24). By assumption, $\frac{\partial}{\partial t} p^{\mathrm{c}}(t,\mathbf{u}) > 0$, due to which the IFT gives that $f \in \mathcal{C}^K$ when $p^{\mathrm{c}}(t,\mathbf{u}) \in \mathcal{C}^K$.

It follows that the composition $f(\mathbf{u}^*(t^{\mathrm{in}}))$ is of continuity class $\mathcal{C}^{\min(K,K_{\mathbf{u}})}$, where $K_{\mathbf{u}}$ is the continuity class of $\mathbf{u}^*(t^{\mathrm{in}})$. Similar to Proposition 1, when there are no weakly active set at the solution and both LICQ and SOSC hold for Problem (24), then , $\mathbf{u}^*(t^{\mathrm{in}})$ is of continuity class $K_{\mathbf{u}} = K$. Hence, $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}}) \in \mathcal{C}^0$ and picewise in $\mathcal{C}^K$ on the open subsets of

$[b_{\mathrm{L}}^{\mathrm{in}}, b_{\mathrm{L}}^{\mathrm{out}}]$ for which the solution to (24) has only strictly active constraints. ■

Note that as with $\Phi(\mathbf{t})$, Proposition 4 entails that $\mathbf{F}(\mathbf{t})$ is only $\mathcal{C}^0$ when t^{in} is such that the solution to any of the LPs (25) has weakly active constraints.

V. Solution approach

In this section we propose a numerical algorithm to solve the coordination problem. We detail how the needed derivatives of $\Phi_i(\mathbf{t}_i)$ and $\mathbf{F}_i(\mathbf{t}_i)$ are computed, and show that these are available to a small additional cost when $\Phi_i(\mathbf{t}_i)$ and $\mathbf{F}_i(\mathbf{t}_i)$ are evaluated. Due to availability of second order sensitivities and fast convergence, a second-order method is proposed.

We propose to solve the NLP (12) using sequential quadratic programming (SQP) [11]. The solution is then obtained through the Newton iterations

$$\mathbf{T}^+ = \mathbf{T} + \alpha \Delta \mathbf{T} \tag{26}$$

$$\boldsymbol{\gamma}^+ = \boldsymbol{\gamma} + \alpha(\boldsymbol{\gamma}^{\mathrm{s}} - \boldsymbol{\gamma}) \tag{27}$$

$$\boldsymbol{\lambda}^+ = \boldsymbol{\lambda} + \alpha(\boldsymbol{\lambda}^{\mathrm{s}} - \boldsymbol{\lambda}) \tag{28}$$

where $\boldsymbol{\gamma} = [\boldsymbol{\gamma}_1^\top, \dots, \boldsymbol{\gamma}_M^\top]^\top$ and $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_1^\top, \dots, \boldsymbol{\lambda}_{M-1}^\top]^\top$ are the multipliers associated with the constraints (12b) and (12c) respectively, $\alpha \in (0, 1]$ the stepsize and the Newton directions $\Delta\mathbf{T} = [\Delta\mathbf{t}_1, \dots, \Delta\mathbf{t}_M], \boldsymbol{\gamma}^{\mathrm{s}}, \boldsymbol{\lambda}^{\mathrm{s}}$ are obtained as the primal-dual solution to the QP problem

$$\min_{\Delta\mathbf{T}} \quad \frac{1}{2}\Delta\mathbf{T}^\top H \Delta\mathbf{T} + \sum_{i=1}^{M} \nabla^\top \Phi_i(\mathbf{t}_i)\Delta\mathbf{t}_i \tag{29a}$$

$$\text{s.t.} \quad \mathbf{F}_i(\mathbf{t}_i) + (\nabla \mathbf{F}_i(\mathbf{t}_i))^\top \Delta\mathbf{t}_i \le 0, \;\; i = 1, \dots, M \tag{29b}$$

$$t_i^{\mathrm{out}} + \Delta t_i^{\mathrm{out}} \le t_{i+1}^{\mathrm{in}} + \Delta t_{i+1}^{\mathrm{in}}, \;\; i = 1, \dots, M-1. \tag{29c}$$

Here, H is the Hessian $\nabla^2_{\mathbf{T}} \mathcal{L}(\mathbf{T}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$ of the Lagrange function of problem (12), defined as

$$\mathcal{L}(\mathbf{T}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) = \boldsymbol{\lambda}^\top C\mathbf{T} + \sum_{i=0}^{M} \Phi_i(\mathbf{t}_i) + \boldsymbol{\gamma}_i^\top \mathbf{F}_i(\mathbf{t}_i), \tag{30}$$

and $C\mathbf{T} \le 0$ is a stack of the collision avoidance constraints (12c). Since the couplings between the vehicle sub-problems (13) are linear, $\nabla^2_{\mathbf{T}} \mathcal{L}(\mathbf{T}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$ is block diagonal and the objective (29a) is separable and according to

$$\sum_{i=1}^{M} \frac{1}{2}\Delta\mathbf{t}_i H_i \Delta\mathbf{t}_i + \nabla\Phi_i(\mathbf{t}_i)^\top \Delta\mathbf{t}_i \tag{31}$$

where

$$\begin{aligned} H_i &= \nabla^2 \left(\Phi_i(\mathbf{t}_i) + \boldsymbol{\gamma}_i^\top \mathbf{F}_i(\mathbf{t}_i)\right) \\ &= \nabla^2 \Phi_i(\mathbf{t}_i) + \nabla^2_{\mathbf{t}_i} b^{\mathrm{out}}_{\mathrm{L}\;i}(t_i^{\mathrm{in}})\gamma_i^L + \nabla^2_{\mathbf{t}_i} b^{\mathrm{out}}_{\mathrm{U}\;i}(t_i^{\mathrm{in}})\gamma_i^U. \end{aligned} \tag{32}$$

Here, γ_i^U, γ_i^L are the multipliers associated with $t_i^{\mathrm{out}} - b^{\mathrm{out}}_{\mathrm{U}\;i}(t_i^{\mathrm{in}}) \le 0$ and $b^{\mathrm{out}}_{\mathrm{L}\;i}(t_i^{\mathrm{in}}) - t_i^{\mathrm{out}} \le 0$ respectively. Regularization is used when H_i is not positive definite and the step size α is adjusted via a suitable line-search procedure.

Remarks:

- It should be observed that in the iterations of the SQP procedure, evaluations of the objective might be required at some $\mathbf{t}_i \notin \mathcal{D}_i$, since $\mathcal{D}_i$ is not polyhedral in general. Consequently, evaluation might be necessary at some $\mathbf{t}_i$ for which some of the lower-level problems (13) are infeasible. This issue can be addressed by a suitable relaxation of the exit time constraint (13d) using exact penalty functions [11].
- It is important to observe that the objective function of the upper Problem (12) is continuous but non-differentiable when some constraints are weakly active in one of the lower-level problem (13). It follows that the proposed SQP strategy ought to be deployed using tools from non-smooth Newton schemes, where globalization and possible regularizations are used to guarantee convergence, e.g. as in [7].

A. Computation of derivatives

To compute the required derivatives of the objective and constraints, we deploy tools from parametric optimization. For ease of notation the sub-system index is dropped.

Derivatives of the objective function: The first-order derivative of $\Phi(\mathbf{t})$ with respect to t^{in} is [2]:

$$\frac{\mathrm{d}\Phi(\mathbf{t})}{\mathrm{d}t^{\mathrm{in}}} = \frac{\partial \mathcal{L}^{\mathrm{QP}}}{\partial t^{\mathrm{in}}} = \nu^{\mathrm{QP}} \frac{\partial p\left(t^{\mathrm{in}}, \mathbf{w}^{\mathrm{QP}}\right)}{\partial t^{\mathrm{in}}}, \tag{33}$$

Here, $\mathbf{w}^{\mathrm{QP}}$ is the primal solution of the QP (13) given $\mathbf{t}$, ν^{QP} is the Lagrange multiplier associated to the in time constraint (13c), and $\mathcal{L}^{\mathrm{QP}}$ is the corresponding Lagrange function. Differentiating (33) gives

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}{t^{\mathrm{in}}}^2} = \frac{\mathrm{d}\nu^{\mathrm{QP}}}{\mathrm{d}t^{\mathrm{in}}}\frac{\partial p}{\partial t^{\mathrm{in}}} + \nu^{\mathrm{QP}}\left(\frac{\partial^2 p}{\partial {t^{\mathrm{in}}}^2} + \frac{\partial^2 p}{\partial t^{\mathrm{in}} \partial \mathbf{w}}\frac{\mathrm{d}\mathbf{w}^{\mathrm{QP}}}{\mathrm{d}t^{\mathrm{in}}}\right). \tag{34}$$

where the arguments have been dropped for brevity. The derivatives $\frac{\mathrm{d}\Phi}{\mathrm{d}t^{\mathrm{out}}}$, $\frac{\mathrm{d}\Phi}{\mathrm{d}t^{\mathrm{in}}\mathrm{d}t^{\mathrm{out}}}$ and $\frac{\mathrm{d}^2\Phi}{\mathrm{d}{t^{\mathrm{out}}}^2}$ are obtained similarly. Given $\mathbf{t} \in \mathcal{D}$, the primal-dual solution $\mathbf{z}^{\mathrm{QP}}$ of (13) satisfies the corresponding KKT conditions, here compactly written as $\mathbf{r}^{\mathrm{QP}}(\mathbf{z}^{\mathrm{QP}}(\mathbf{t}), \mathbf{t}) = 0$. For t^{in}, we then have

$$\frac{\mathrm{d}\mathbf{r}^{\mathrm{QP}}}{\mathrm{d}t^{\mathrm{in}}} = \frac{\partial \mathbf{r}^{\mathrm{QP}}}{\partial t^{\mathrm{in}}} + \left(\nabla_{\mathbf{z}} \mathbf{r}^{\mathrm{QP}}\right)^\top \frac{\mathrm{d}\mathbf{z}^{\mathrm{QP}}}{\mathrm{d}t^{\mathrm{in}}} = 0, \tag{35}$$

and similar expression holds for t^{out}. The second order derivatives of $\Phi(\mathbf{t})$ thus require the solution $\frac{\mathrm{d}\mathbf{z}^{\mathrm{QP}}}{\mathrm{d}t^{\mathrm{in}}}$ of the linear system (35).

Derivatives of the constraints: For the derivatives of the constraint $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})$, Proposition 3 ensures that $b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}}) = f(t^{\mathrm{in}}, \mathbf{w}^{\mathrm{LP}}(t^{\mathrm{in}}))$, where $f(t^{\mathrm{in}}, \mathbf{w}^{\mathrm{LP}}(t^{\mathrm{in}}))$ denotes the solution to $p(t^{\mathrm{out}}, \mathbf{w}^*_{\mathrm{LP}}(t^{\mathrm{in}})) - p^{\mathrm{out}} = 0$ and $\mathbf{w}^{\mathrm{LP}}(t^{\mathrm{in}})$ is the primal solution to the corresponding LP (25). It follows that

$$\frac{\mathrm{d}b_{\mathrm{L}}^{\mathrm{out}}(t^{\mathrm{in}})}{\mathrm{d}t^{\mathrm{in}}} = \left(\nabla_{\mathbf{w}} f\right)^\top \frac{\mathrm{d}\mathbf{w}^{\mathrm{LP}}(t^{\mathrm{in}})}{\mathrm{d}t^{\mathrm{in}}} \tag{36}$$

As in (35), the first-order derivative of the primal solution is obtained through the solution of the linear system

$$\left(\nabla_{\mathbf{z}} \mathbf{r}^{\mathrm{LP}}\right)^\top \frac{\mathrm{d}\mathbf{z}^{\mathrm{LP}}}{\mathrm{d}t^{\mathrm{in}}} = -\frac{\partial \mathbf{r}^{\mathrm{LP}}}{\partial t^{\mathrm{in}}} \tag{37}$$

where $\mathbf{r}^{\mathrm{LP}}(\mathbf{z}^{\mathrm{LP}}(t^{\mathrm{in}}), t^{\mathrm{in}}) = 0$ and $\mathbf{z}^{\mathrm{LP}}(t^{\mathrm{in}})$ are the KKT conditions and primal-dual solution of the LP for t^{in}, respectively. The second order derivatives $\frac{\mathrm{d}^2\mathbf{w}^{\mathrm{LP}}}{\mathrm{d}t^{\mathrm{in}2}}$ are thereafter given as the solution to

$$\left(\nabla_{\mathbf{z}}\mathbf{r}^{\mathrm{LP}}\right)^\top \frac{\mathrm{d}^2\mathbf{z}^{\mathrm{LP}}}{\mathrm{d}{t^{\mathrm{in}}}^2} = \frac{\partial^2\mathbf{r}^{\mathrm{LP}}}{\partial^2 t^{\mathrm{in}}} - \mathbf{R}^\top \frac{\mathrm{d}\mathbf{z}^{\mathrm{LP}}}{\mathrm{d}t^{\mathrm{in}}}, \tag{38}$$

where $\mathbf{R}$ is such that

$$(\mathbf{R})_{(i,j)} = \left(\nabla_{\mathbf{z}}\frac{\partial r_j^{\mathrm{LP}}}{\partial z_i}\right)^\top \frac{\mathrm{d}\mathbf{z}^{\mathrm{LP}}}{\mathrm{d}t^{\mathrm{in}}}, \tag{39}$$

using $\mathbf{r}^{\mathrm{LP}}(\mathbf{z}) = [r_1^{\mathrm{LP}}(\mathbf{z}), \ldots, r_p^{\mathrm{LP}}(\mathbf{z})]$ and $\mathbf{z} = [z_1, \ldots, z_p]$.

Efficient computation of derivatives: We emphasize here that the computations required to formulate the QP subproblem (29) are completely separable between the vehicles. In particular, for each iteration in the SQP procedure, each vehicle can separately solve the corresponding QP (13) to evaluate the objective $\Phi_i(\mathbf{t}_i)$, and the two LPs (25) to evaluate the constraints ${b^{\mathrm{out}}_{\mathrm{L}}}_i(t^{\mathrm{in}})$ and ${b^{\mathrm{out}}_{\mathrm{U}}}_i(t^{\mathrm{in}})$.

If a second-order method is used to solve (13), the last iterate of the QP solver will involve a factorization of $\nabla_{\mathbf{z}}\mathbf{r}^{\mathrm{QP}}(\mathbf{z}^{\mathrm{QP}}, \mathbf{t})$. The computation of $\nabla^2_{\mathbf{t}}\Phi$, is therefore cheap, as this factorization can be reused to solve (35) for $\frac{\mathrm{d}\mathbf{z}^{\mathrm{QP}}}{\mathrm{d}\mathbf{t}}$. The same applies to the sensitivities of the constraints, computed through the LPs (25).

VI. Numerical Example

We consider a scenario where all vehicles share the sampling time T_s , horizon length N and where their motion is modeled as a double integrator, i.e.,

$$\mathbf{x}_{i,k+1} = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}\mathbf{x}_{i,k} + \begin{bmatrix} \frac{1}{2}T_s^2 \\ T_s \end{bmatrix}\mathbf{u}_{i,k}, \ \forall i \in \mathcal{M}. \tag{40}$$

Here, $\mathbf{x}_{i,k} = [p_{i,k}, v_{i,k}]^\top$ contains the position and velocity and $\mathbf{u}_{i,k}$ is the acceleration. The position function is then

$$p_i(t, \mathbf{w}_i) = p_{i,k} + (t - t_k)v_{i,k} + \frac{1}{2}(t - t_k)^2\mathbf{u}_{i,k} \tag{41}$$

where $t_k = kT_s$ and $k = \mathrm{floor}(t/T_s)$. The constraints are $\underline{u}_i \le \mathbf{u}_{i,k} \le \bar{u}_i, \quad \epsilon \le v_k$, where $\underline{u}_i, \bar{u}_i \in \mathbb{R}$ and $\epsilon > 0$. The objective is

$$J_i(\mathbf{w}_i) = (v_i^{\mathrm{ref}} - v_{i,N})^2 Q_i + \sum_{k=0}^{N-1}(v_i^{\mathrm{ref}} - v_{i,k})^2 Q_i + \mathbf{u}_{i,k}^2 R_i \tag{42}$$

where v_i^{ref} is a constant desired speed and $Q_i > 0, R_i > 0$.

In our implementation of the algorithm, the stepsize α is chosen through backtracking on a T_1 merit function [11] and the termination criterion is taken as the ∞-norm of the KKT-residual of the upper problem (12).

In the examined scenario, there are 4 vehicles, where $v_i^{\mathrm{ref}} = 80\ [km/h]$, $\underline{u}_i = -2\ [m/s^2]$, $\bar{u}_i = 2\ [m/s^2]$ and $Q_i = R_i = 1$, identical for all vehicles. Furthermore we set $N = 150$ and $T_s = 0.1$, and define the intersection with $p^{\mathrm{in}} = 0$ and $p^{\mathrm{out}} = 10$ for all vehicles. The initial states are set as in Table I, and are such that all vehicles collide if coordination is not performed. Finally, the algorithm is initialized at the optimal non-cooperative solution, i.e., at the timeslots given from the separate solution of the vehicle-level optimal control problems.

TABLE I

Initial States for vehicles in numerical example

i	1	2	3	4
$p_{i,0}\ [m]$	-160	-163	-166	-166
$v_{i,0}\ [km/h]$	70	75	80	85

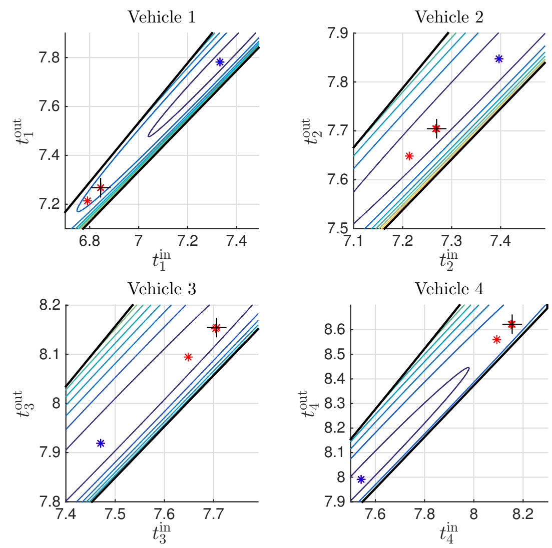

Fig. 2. The iterates of the upper problem (12). The pictures shows a subset of the feasible sets $\mathcal{D}_i$, demarkated in black, contours of $\Phi_i(\mathbf{t}_i)$, and the iterates of the SQP procedure. The blue markers is the initial guess used, the red markers the iterates of the algorithm and the black crosses the solution.

A. Results

For this particular example, the algorithm converges to a tolerance of 10^{-6}, with full steps ($\alpha = 1$) in 4 iterations. In the process it thus solves 4 QPs (Problem (13)) and 10 LPs (Two times the bounds on t^{in}(24), eight times (25)) per vehicle. The iterates $\mathbf{T}$ of the SQP algorithm, produced as in (26), are visualized together with the objective functions $\Phi_i(\mathbf{t}_i)$ and feasible sets $\mathcal{D}_i$ in Fig. 2. The position, velocity and acceleration trajectories of the vehicles produced by the solutions of (13), corresponding to the iterates in the upper problem (12), are shown in Fig. 3 and Fig. 4.

The numerical results demonstrate that from a non-cooperative starting point, the algorithm quickly reaches a practically acceptable solution. In particular, after two iterations no constraints are violated and the change in the primal variables of the upper Problem (12) is in the order of milliseconds, far beyond any practical requirements. As evident from Fig. 3 and Fig. 4, the corresponding change in the solution of the vehicle problems (13) is equally small.

VII. Conclusions and Future Work

In this paper, we first formulated the optimal intersection coordination for autonomous vehicles under a given precedence order. We proceeded with proposing a decomposition

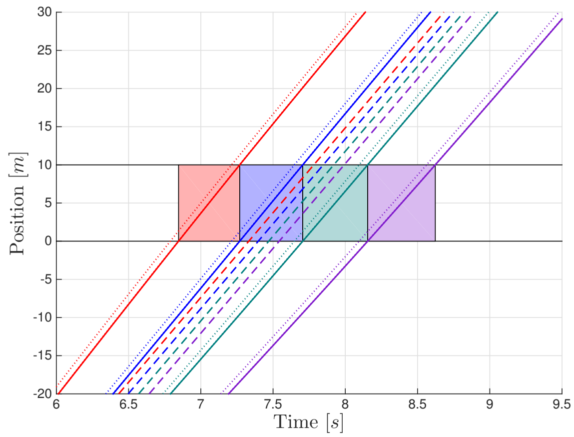

Fig. 3. The position trajectories of the example scenario. The horizontal black lines marks the beginning and end of the intersection and the colored fields marks timeslots during which a vehicle is inside the intersection, i.e., the solution to (12), with one color per vehicle. The correspondingly colored graphs are the trajectories obtained by solving (13) for the different vehicles. Here the dashed is what the algorithm starts from, the dotted the trajectories during the algorithm's iterations and the solid the solution.

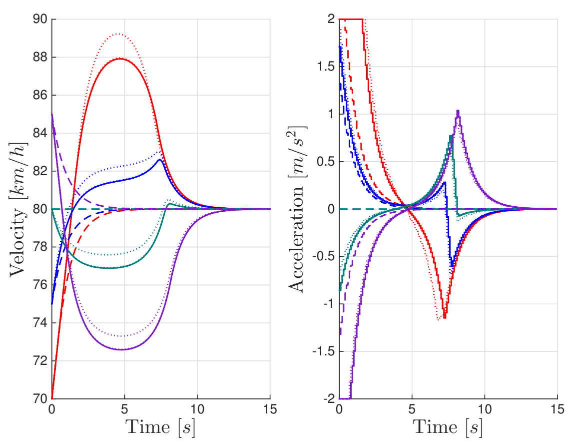

Fig. 4. The velocity (left) and acceleration (right) for the different vehicles. Color and linestyle as in Fig. 3

of the problem into a upper level time-slot allocation problem and multiple vehicle-level optimal control problems. We then showed the properties of the decomposed problem and proposed a solution using SQP. Although formal convergence guarantees are not presented, no issues have been observed in practice.

In a practical setting with a central coordination controller, the presented decomposition and SQP allows for the calculations of the cost, constraints and their derivatives to be done in parallel, on board the vehicles, and thereafter transmitted to the central controller over the wireless. Consequently, as the numerical example demonstrates, the method then allows the problem to be solved within a few rounds of communication (one round per SQP iteration in the case of full line search steps), exchanging only the evaluation of the objective and constraints and the associated sensitivity information each round. Since, in practice, the reliance on wireless communication would affect the algorithm, an interesting future research direction is a comparison with respect to communication usage between the method suggested herein and other methods. We also aim at investigating the extension of the coordination control scheme described in this paper to closed loop control, and in particular to address the combinatorial side of the problem, i.e., that of finding the precedence order we in this paper assume given. Finally, we are currently working on a distributed interior point method where the issue of piece-wise differentiability of the objective is resolved and where convergence guarantees can be given.

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