Thesis for the Degree of Licentiate of Philosophy

A generalized finite element method for linear thermoelasticity

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Abstract

In this thesis we develop a generalized finite element method for linear thermoelasticity problems, modeling displacement and temperature in an elastic body. We focus on strongly heterogeneous materials, like composites. For classical finite element methods such problems are known to be numerically challenging due to the rapid variations in the data.

The method we propose is based on the local orthogonal decomposition technique introduced in [12]. In short, the idea is to enrich the classical finite element nodal basis function using information from the diffusion coefficient. Locally, these basis functions have better approximation properties than the nodal basis functions.

The papers included in this thesis first extends the local orthogonal decomposition framework to parabolic problems (Paper I) and to linear elasticity equations (Paper II). Finally, using the theory developed in these papers, we address the linear thermoelastic system (Paper III).

Keywords: Thermoelasticity, parabolic equations, linear elasticity, multiscale, composites, generalized finite element, local orthogonal decomposition, a priori analysis.

List of included papers

The following papers are included in this thesis:

- **Paper I.** Axel Målqvist and Anna Persson: *Multiscale techniques for parabolic equations*, Submitted.

- **Paper II.** Patrick Henning and Anna Persson: *A multiscale method for linear elasticity reducing Poisson locking*, Submitted.

- **Paper III.** Axel Målqvist and Anna Persson: *A generalized finite element method for linear thermoelasticity*, Submitted.

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Contents

Part 1

Introduction

Introduction

1. Background

In many applications the expansion and contraction of a material exposed to external forces and temperature changes are of great importance. For instance, it may be crucial when designing parts for aircrafts or when constructing a bridge.

In this thesis we study numerical solutions to linear thermoelastic systems, which consist of partial differential equations (PDEs) simulating displacement and temperature changes in materials over time. In particular, we are interested in applications where the material under consideration is strongly heterogeneous, e.g. composites. Composite materials are constructed using two or more different constituents. Typically, the material properties in composites vary on a very fine scale, as in, for instance, fiber reinforced materials. Modeling physical behavior in these materials results in equations with highly varying and oscillating coefficients. Such problems, that exhibit a lot of variations in the data, often on multiple scales, are commonly referred to as *multiscale* problems.

Classically, numerical solutions to thermoelasticity equations have been obtained using finite element methods (FEMs) based on continuous piecewise polynomials. These methods work well for homogeneous materials, or materials that are not varying too much in space. However, for strongly heterogeneous materials the classical FEMs struggle to approximate the solution accurately unless the mesh width is sufficiently small. Indeed, the mesh width must be small enough to resolve all the fine variations in the data. In practice, this leads to issues with computational cost and available memory.

Today's increasing interest in and usage of composite materials thus pose a demand for other types of numerical methods. Several such methods have been proposed over the last two decades, see, for instance, [9, 5, 1, 10]. However, the analysis of many of these methods require restrictive assumptions on the material, such as periodicity or separation of scales.

In [12] a generalized finite element method (GFEM), cf. [2], is proposed and rigorous analysis is provided. Convergence of the method is proven for an arbitrary positive and bounded coefficient, that is, no assumptions on periodicity or separation of scales are needed.

The purpose of this thesis is to generalize the method proposed in [12] to solve linear thermoelasticity equations with highly varying and oscillating coefficients. This is done in three steps. In Paper I we extend the method to linear parabolic problems, in Paper II we consider (stationary) linear elasticity equations and in Paper III we finally address the thermoelastic system. In all three papers we prove convergence of optimal order for highly varying coefficients and provide several numerical examples that confirm the analysis.

In the upcoming section we describe the system of equations used to model the displacement and temperature of an elastic material. In Section 2 the issue with applying the classical FEM to multiscale problems is described in more detail. In Section 3 we introduce the GFEM proposed in [12] for elliptic equations and discuss the main idea behind the extension to linear thermoelasticity. Finally, in Section 4 we summarize the appended papers and highlight the main results.

1.1. Linear thermoelasticity. Linear thermoelasticity refers to a coupled system of PDEs describing the displacement and temperature of an elastic body, see [3, 4]. To introduce the mathematical formulation of this system we let $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$, be a domain describing the initial configuration of an elastic medium. For a given simulation time $T > 0$, we let the vector valued function $u : [0, T] \times \Omega \to \mathbb{R}^d$ denote the displacement field and $\theta : [0, T] \times \Omega \to \mathbb{R}$ denote the temperature. To define boundary conditions for u we let Γ^u_D and Γ^u_N be two disjoint parts of the boundary such that $\Gamma^u_D \cup \Gamma^u_N = \partial\Omega$. On the part denoted Γ^u_D we impose Dirichlet boundary conditions corresponding to a clamped part of the material. On Γ^u_N, corresponding to the traction boundary, we impose Neumann boundary conditions. Similarly, we define Γ^θ_D and Γ^θ_N to be the drained and flux part of the boundary for the temperature θ.

Under the assumption that the displacement gradients are small, the strain tensor is given by the following linear relation

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^{\intercal}).$$

For isotropic materials the total stress tensor is given by

$$\bar{\sigma} = 2\mu\varepsilon(u) + \lambda(\nabla \cdot u)I - \alpha\theta I,$$

where I is the d-dimensional identity matrix and α is the thermal expansion coefficient. Furthermore, μ and λ denotes the Lamé coefficients satisfying

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},$$

where ν denotes Poisson's ratio and E denotes Young's elastic modulus. Poisson's ratio is a measure on the materials tendency to shrink (expand) when stretched (compressed) and Young's modulus describes the stiffness of the material. The coefficients α, λ, and μ are all material dependent and thus rapidly varying in space for strongly heterogeneous (multiscale) materials.

Now, Cauchy's equilibrium equations states that

$$-\nabla \cdot \bar{\sigma} = f,$$

where $f : \Omega \to \mathbb{R}^d$ denotes the external body forces. Furthermore, the temperature in the material can be described by the parabolic equation

$$\dot{\theta} - \nabla \cdot \kappa \nabla \theta + \alpha \nabla \cdot \dot{u} = g,$$

where $\kappa : \Omega \to \mathbb{R}^{d \times d}$ is the heat conductivity parameter and g denotes internal heat sources. Note that κ is material dependent and thus rapidly varying. To summarize, the linear thermoelastic system is given by the following system of equations

$$-\nabla \cdot (2\mu\varepsilon(u) + \lambda \nabla \cdot uI - \alpha\theta I) = f, \quad \text{in } (0,T] \times \Omega, \tag{1.1}$$

$$\dot{\theta} - \nabla \cdot \kappa \nabla \theta + \alpha \nabla \cdot \dot{u} = g, \quad \text{in } (0,T] \times \Omega, \tag{1.2}$$

$$u = 0, \quad \text{in } (0,T] \times \Gamma_D^u, \tag{1.3}$$

$$\bar{\sigma} \cdot n = 0, \quad \text{in } (0,T] \times \Gamma_N^u. \tag{1.4}$$

$$\theta = 0, \quad \text{on } (0,T] \times \Gamma_D^\theta, \tag{1.5}$$

$$\kappa \nabla \theta \cdot n = 0, \quad \text{on } (0,T] \times \Gamma_N^\theta. \tag{1.6}$$

$$\theta(0) = \theta_0, \quad \text{in } \Omega, \tag{1.7}$$

where we for simplicity assume homogeneous boundary conditions. Note that the equations (1.1)-(1.2) are coupled.

Remark 1.1. The system (1.1)-(1.7) is formally equivalent to a linear model for poroelasticity. In this case θ denotes the fluid pressure, κ the hydraulic conductivity, and α the Biot-Willis coupling-deformation coefficient. Hence, the results in this thesis also apply to the linear poroelastic system.

To define a FEM (and a GFEM) for (1.1)-(1.7) we define the corresponding variational (or weak) formulation. For this purpose we first need to introduce some notation and spaces. We use $(\cdot,\cdot)$ to denote the inner product in $L_2(\Omega)$ and $\|\cdot\|$ the corresponding norm. Let $H^1(\Omega) := W_2^1(\Omega)$ denote the classical Sobolev space with norm $\|v\|^2_{H^1(\Omega)} = \|v\|^2 + \|\nabla v\|^2$ and let $H^{-1}(\Omega)$ denote the dual space to H^1. Furthermore, let $L_p([0,T];X)$ denote the Bochner space with norm

$$\|v\|_{L_p([0,T];X)} = \Big(\int_0^T \|v\|_X^p \, \mathrm{dt} \Big)^{1/p}, \quad 1 \le p < \infty,$$
$$\|v\|_{L_\infty([0,T];X)} = \operatorname*{ess\,sup}_{0 \le t \le T} \|v\|_X,$$

where X is a Banach space equipped with the norm $\|\cdot\|_X$. The dependence on the interval $[0,T]$ and the domain Ω is frequently suppressed and we write, for instance, $L_2(L_2)$ for $L_2([0,T];L_2(\Omega))$. We also use the double-dot product notation to denote the Frobenius inner product of two matrices A and B

$$A : B = \sum_{i,j=1}^{d} A_{ij} B_{ij}, \quad A, B \in \mathbb{R}^{d \times d}.$$

Now, define the following spaces

$$V^1 := \{v \in (H^1(\Omega))^d : v = 0 \text{ on } \Gamma_D^u\}, \quad V^2 := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D^\theta\}.$$

Multiplying (1.1) with $v_1 \in V^1$ and (1.2) with $v_2 \in V^2$ and using Green's formula together with the boundary conditions (1.3)-(1.6) we arrive at the following variational formulation; find $u(t, \cdot) \in V^1$ and $\theta(t, \cdot) \in V^2$ such that, for a. e. $t > 0$,

$$(\sigma(u) : \varepsilon(v_1)) - (\alpha\theta, \nabla \cdot v_1) = (f, v_1), \quad \forall v_1 \in V^1, \tag{1.8}$$

$$(\dot{\theta}, v_2) + (\kappa\nabla\theta, \nabla v_2) + (\alpha\nabla \cdot \dot{u}, v_2) = (g, v_2), \quad \forall v_2 \in V^2, \tag{1.9}$$

and the initial value $\theta(0, \cdot) = \theta_0$ is satisfied. Here $\sigma(u) := 2\mu\varepsilon(u) + \lambda\nabla \cdot uI$ is the first part of $\bar{\sigma}$ involving only the displacement u, commonly referred to as the effective stress tensor.

Two functions u and θ are weak solutions if (1.8)-(1.9) are satisfied and $u \in L_2(V^1)$, $\nabla \cdot \dot{u} \in L_2(H^{-1})$, $\theta \in L_2(V^2)$, and $\dot{\theta} \in L_2(H^{-1})$. Existence and uniqueness of such weak solutions are proved in, e.g., [17, 16], and in [14] within the framework of linear degenerate evolution equations in Hilbert spaces. In [14] it is also proved that the system is of parabolic type, meaning that it is well posed for nonsmooth initial data with regularity estimates depending on negative powers of t.

2. Classical finite element

In this section we explain more carefully why the classical FEM fails to approximate the solution to problems with rapidly varying data. To simplify the discussion we start by considering elliptic equations.

2.1. Elliptic equations. Consider the elliptic equation

$$\begin{aligned} -\nabla \cdot A\nabla u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

with the variational formulation; find $u \in V$, such that

$$a(u, v) = (f, v), \quad \forall v \in V, \tag{2.1}$$

where $V = H_0^1(\Omega)$ and $a(u, v) := (A\nabla u, \nabla v)$. Here $A : \Omega \to \mathbb{R}^{d \times d}$ the diffusion coefficient is assumed to be rapidly oscillating.

To define a FEM we need a triangulation of the domain. Let $\{\mathcal{T}_h\}_{h>0}$ be a family triangulations of Ω with the mesh size $h_K := \operatorname{diam}(K)$, for $K \in \mathcal{T}_h$ and denote the largest diameter in the triangulation by $h := \max_{K \in \mathcal{T}_h} h_K$. Now let $V_h \subseteq V$ denote the space of continuous piecewise affine functions on the triangulation $\mathcal{T}_h$. The finite element formulation then reads; find $u_h \in V_h$, such that,

$$a(u_h, v) = (f, v), \quad \forall v \in V_h. \tag{2.2}$$

Classical a priori error analysis gives the bound

$$\|u_h - u\|_{H^1} \le Ch\|D^2u\|, \tag{2.3}$$

where D^2u denotes the second order (weak) derivatives of u. Not only does this bound require additional regularity of the solution, the norm $\|D^2u\|$ may also be very large if A is rapidly oscillating. Indeed, if A varies with frequency ϵ^{-1} for some $\epsilon > 0$, then $\|\nabla A\|_{L_\infty} = O(\epsilon^{-1})$. Estimating $\|D^2u\|$ with the problem data gives

$$\begin{aligned}\|D^2u\| &\le C\|\Delta u\| \le C\|A\nabla \cdot (\nabla u)\| = C\|\nabla \cdot (A\nabla u) - \nabla A \nabla u\| \\ &\le C\|\nabla \cdot (A\nabla u)\| + \|\nabla A\|_{L_\infty}\|\nabla u\| \le C(1 + \epsilon^{-1})\|f\|,\end{aligned}$$

where we used elliptic regularity in the first inequality and the bound $\|u\|_{H^1} \le C\|f\|$, derived from (2.1), in the last inequality. Furthermore, we can derive the bound $\|u_h\|_{H^1} \le C\|f\|$ from (2.2), which gives the following upper bound of the error; $\|u_h - u\|_{H^1} \le C\|f\|$. Hence, the error bound (2.3) takes the form

$$\|u_h - u\|_{H^1} \le C \min\left\{h + \frac{h}{\epsilon}, 1\right\}\|f\|,$$

and convergence does not take place unless $h < \epsilon$. If ϵ is small, the condition $h < \epsilon$, can be devastating considering computational cost and available memory.

2.2. Linear thermoelasticity. As in the previous section we define a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ and we let $V_h^1 \subseteq V^1$ and $V_h^2 \subseteq V^2$ denote finite element spaces consisting of continuous piecewise linear functions on this triangulation. Furthermore, we let $0 = t_0 < t_1 < ... < t_N = T$ be a uniform discretization of the time interval such that $t_j - t_{j-1} = \tau > 0$ for $j = 1, ..., N$. The classical FEM with a backward (implicit) Euler discretization in time for (1.8)-(1.9) reads; for $n \in \{1, ..., N\}$ find $u_h^n \in V_h^1$ and $\theta_h^n \in V_h^2$, such that

$$(\sigma(u_h^n) : \varepsilon(v_1)) - (\alpha\theta_h^n, \nabla \cdot v_1) = (f^n, v_1), \quad \forall v_1 \in V_h^1, \tag{2.4}$$

$$(\bar{\partial}_t\theta_h^n, v_2) + (\kappa\nabla\theta_h^n, \nabla v_2) + (\alpha\nabla \cdot \bar{\partial}_t u_h^n, v_2) = (g^n, v_2), \quad \forall v_2 \in V_h^2, \tag{2.5}$$

where $\bar{\partial}_t\theta_h^n := (\theta_h^n - \theta_h^{n-1})/\tau$ and similarly for $\bar{\partial}_t u_h^n$. Here $u_h^0 = u_{h,0}$ and $\theta_h^0 = \theta_{h,0}$, where $u_{h,0} \in V_h^1$ and $\theta_{h,0} \in V_h^2$ denote suitable initial conditions. The right hand sides are evaluated at time t_n, that is, $f^n := f(t_n)$ and $g^n := g(t_n)$.

A priori analysis for the system (2.4)-(2.5) can be found in [7]. It follows that the error is bounded by

$$\|u_h^n - u^n\|_{H^1} + \|\theta_h^n - \theta^n\| + \left(\sum_{j=1}^n \tau\|\theta_h^j - \theta^j\|_{H^1}^2\right)^{1/2} \le C_{\epsilon^{-1}}h + C\tau,$$

where the constant $C_{\epsilon^{-1}}$ depends on both $\|u(t_n)\|_{H^2}$ and $\|\theta(t_n)\|_{H^2}$. By arguments similar to the ones used for the elliptic equation in Section 2.1, we get that

$\|u(t_n)\|_{H^2} = O(\epsilon^{-1})$ and $\|\theta(t_n)\|_{H^2} = O(\epsilon^{-1})$, if the material has variations on a scale of size ϵ.

3. A generalized finite element method

In [12] a GFEM, often referred to as *local orthogonal decomposition*, is proposed and analyzed for elliptic equations of the form (2.1). In Section 3.1 below we describe this method and the main ideas used in the analysis. Finally, in Section 3.4 we describe how this method can be generalized to define a GFEM for linear thermoelasticity, which is the main objective of this thesis.

3.1. Elliptic equations. The method proposed in [12] builds on the ideas from the variational multiscale method [10, 11], where the solution space is decomposed to into a coarse and a fine part. The nodal basis functions in the coarse space is then modified by adding a *correction* from the fine space.

We begin by assuming that the mesh size h used in the classical FEM in (2.2) is fix and sufficiently small, that is $h < \epsilon$, such that the error (2.3) is small. In this case, the solution u_h and the space V_h are referred to as the reference solution and the reference space, respectively. Now define V_H similarly to V_h but with a larger mesh size $H > h$, such that $V_H \subseteq V_h$. Note that the classical FEM solution u_H in the coarse space V_H is not a good approximation to u. It is, however, cheaper to compute than u_h since $\dim(V_H) < \dim(V_h)$. The aim is now to define a new multiscale space V_{ms} with the same dimension as the coarse space V_H, but with better approximations properties.

To define such a space, we need an interpolation operator $I_H : V_h \to V_H$ with the properties $I_H \circ I_H = I_H$ and for $K \in \mathcal{T}_H$

$$H_K^{-1}\|v - I_H v\|_{L_2(K)} + \|\nabla I_H v\|_{L_2(K)} \leq C_I \|\nabla v\|_{L_2(\omega_K)}, \quad v \in V_h, \tag{3.1}$$

where $\omega_K := \cup\{\hat{K} \in \mathcal{T}_H : \hat{K} \cap K \neq \emptyset\}$. For a quasi-uniform mesh, the bounds in (3.1) can be summed to achieve a global bound

$$H^{-1}\|v - I_H v\| + \|\nabla I_H v\| \leq C\|\nabla v\|, \tag{3.2}$$

There are many interpolations operators that satisfy these conditions, for instance, the global L_2-projection. In Paper II and Paper III we use an interpolation of the form $I_H = E_H \circ \Pi_H$, where Π_H is the L_2-projection onto $P_1(\mathcal{T}_H)$, the space of functions that are affine on each triangle $K \in \mathcal{T}_H$ and $E_H : P_1(\mathcal{T}_H) \to V_H$ is an averaging operator. We refer to [13, 6] for further details and possible choices of I_H.

Now let V_{f} denote the kernel to the operator I_H

$$V_{\mathrm{f}} := \ker I_H = \{v \in V_h : I_H v = 0\}.$$

The space V_h can be decomposed as $V_h = V_H \oplus V_{\mathrm{f}}$, meaning that $v_h \in V_h$ can be decomposed into

$$v_h = v_H + v_{\mathrm{f}}, \quad v_H \in V_H,\ v_{\mathrm{f}} \in V_{\mathrm{f}}. \tag{3.3}$$

The kernel V_{f} is a fine scale (detail) space in the sense that it captures all features that are not captured by the coarse space V_H. Let $R_{\mathrm{f}} : V_h \to V_{\mathrm{f}}$ denote the Ritz projection onto V_{f}, that is,

$$a(R_{\mathrm{f}}v, w) = a(v, w), \quad \forall w \in V_{\mathrm{f}},\ v \in V_h. \tag{3.4}$$

Because of the decomposition (3.3) we have the identity

$$v_h - R_{\mathrm{f}}v_h = v_H + v_{\mathrm{f}} + R_{\mathrm{f}}(v_H + v_{\mathrm{f}}) = v_H - R_{\mathrm{f}}v_H,$$

since $v_{\mathrm{f}} \in V_{\mathrm{f}}$. Using this we can define the multiscale space V_{ms}

$$V_{\mathrm{ms}} := V_h - R_{\mathrm{f}}V_h = V_H - R_{\mathrm{f}}V_H. \tag{3.5}$$

Note that V_{ms} is the orthogonal complement to V_{f} with respect to the inner product $a(\cdot,\cdot)$ and must have the same dimension as V_H. Indeed, with $\mathcal{N}$ denoting the inner nodes in $\mathcal{T}_H$ and λ_z the basis function at node z, a basis for V_{ms} is given by

$$\{z \in \mathcal{N} : \lambda_z - R_{\mathrm{f}}\lambda_z\}.$$

Hence, that basis functions are the classical nodal basis functions modified by corrections $R_{\mathrm{f}}\lambda_z$ computed in the fine scale space.

Replacing V_h with V_{ms} in (2.2) we can now define the GFEM; find $u_{\mathrm{ms}} \in V_{\mathrm{ms}}$, such that,

$$a(u_{\mathrm{ms}}, v) = (f, v), \quad \forall v \in V_{\mathrm{ms}}. \tag{3.6}$$

The following theorem gives an a priori bound for the GFEM and can be found in [12]. We include the proof here since it is short and highlights the main ideas used in the analysis.

Theorem 3.1. *Let u_h be the solution to* (2.2) *and u_{ms} the solution to* (3.6). *Then*

$$\|u_{\mathrm{ms}} - u_h\|_{H^1} \le CH\|f\|,$$

where C does not depend on the derivatives of A.

Proof. Define $e := u_{\mathrm{ms}} - u_h$ and note that $e \in V_{\mathrm{f}}$. Hence, $I_He = 0$. Furthermore we have due to Galerkin orthogonality $a(e, v_{\mathrm{ms}}) = 0$ for $v_{\mathrm{ms}} \in V_{\mathrm{ms}}$. Using this together with the interpolation bound (3.2) we have

$$a(e,e) = -a(e,u_h) = -(f,e) \le \|f\|\|e\| = \|f\|\|e - I_He\| \le CH\|f\|\|\nabla e\|,$$

and the bound follows by using equivalence of the energy norm induced by $a(\cdot,\cdot)$ and the H^1-norm. □

From Theorem 3.1 we have that the solution given by the GFEM converges to u_h, with optimal order, independently of the derivatives (variations) of A. We emphasize that the total error is bounded by

$$\|u_{\mathrm{ms}} - u\|_{H^1} \le \|u_{\mathrm{ms}} - u_h\|_{H^1} + \|u_h - u\|_{H^1},$$

where the error in the second term is due to the classical FEM and assumed to be of reasonable size, since h is assumed to be sufficiently small.

Although the a priori analysis seems promising, the GFEM as suggested above suffers from some drawbacks. The problem of finding the corrections $R_\mathrm{f}\lambda_z$, which are needed to construct the basis, are posed in the entire fine scale space V_f which has the same dimension as V_h. Furthermore, the corrections generally have global support and therefore destroys the sparsity of the resulting discrete system. Both issues are resolved by performing a localization of the corrections. The localization is motivated by the observation that the correction $R_\mathrm{f}\lambda_z$ decay exponentially away from node z.

3.2. Localization. In [12] it is proved that the corrections decay exponentially and a localization procedure is proposed. However, in [8] a different localization technique is proposed which allows for smaller patches to be used. We describe the procedure in [8] here, which is also the procedure that is used in the appended papers.

We define patches of size k in the following way; for $K \in \mathcal{T}_H$

$$\omega_0(K) := \mathrm{int}\ K,$$
$$\omega_k(K) := \mathrm{int}\ \big(\cup \{\hat{K} \in \mathcal{T}_H : \hat{K} \cap \overline{\omega_{k-1}(K)} \neq \emptyset\}\big), \quad k = 1, 2, ...,$$

and let $V_\mathrm{f}(\omega_k(K)) := \{v \in V_\mathrm{f} : v(z) = 0 \text{ on } \overline{\Omega} \setminus \omega_k(K)\}$ be the restriction of V_f to the patch $\omega_k(K)$.

We proceed by noting that R_f in (3.4) can be written as the sum

$$R_\mathrm{f} = \sum_{K \in \mathcal{T}_H} R_\mathrm{f}^K,$$

where $R_\mathrm{f}^K : V_h \to V_\mathrm{f}$ and fulfills

$$a(R_\mathrm{f}^K v, w) = a(v, w)_K, \quad \forall w \in V_\mathrm{f}, \quad v \in V_h, \quad K \in \mathcal{T}_H, \tag{3.7}$$

where we define

$$a(v, w)_K := (A\nabla v, \nabla w)_{L_2(K)}, \quad K \in \mathcal{T}_H.$$

The aim is to localize these computations by replacing V_f with $V_\mathrm{f}(\omega_k(K))$. Define $R_{\mathrm{f},k}^K : V_h \to V_\mathrm{f}(\omega_k(K))$ such that

$$a(R_{\mathrm{f},k}^K v, w) = a(v, w)_K, \quad \forall w \in V_\mathrm{f}(\omega_k(K)), \quad v \in V_h, \quad K \in \mathcal{T}_H,$$

and set $R_{\mathrm{f},k} := \sum_{K \in \mathcal{T}_H} R_{\mathrm{f},k}^K$. We can now define the localized multiscale space

$$V_{\mathrm{ms},k} = \{v_H - R_{\mathrm{f},k} v_H : v_H \in V_H\}. \tag{3.8}$$

By replacing V_ms with $V_{\mathrm{ms},k}$ in (3.6) a localized GFEM can be defined; find $u_{\mathrm{ms},k} \in V_{\mathrm{ms},k}$ such that

$$a(u_{\mathrm{ms},k}, v) = (f, v), \quad \forall v \in V_{\mathrm{ms},k}. \tag{3.9}$$

Since the dimension of $V_\mathrm{f}(\omega_k(K))$ can be made significantly smaller than the dimension of V_f (depending on k), the problem of finding $R_{\mathrm{f},k}\lambda_z$ is computationally cheaper than finding $R_\mathrm{f}\lambda_z$. Moreover, the resulting discrete system is sparse. It should also be noted that the computation of $R_{\mathrm{f},k}\lambda_z$ for all nodes z is suitable for parallelization, since they are independent of each other.

The convergence of the method (3.9) depends on the size of the patches. In [12, 8] the following Theorem is proved.

THEOREM 3.2. *Let* u_h *be the solution to* (2.2) *and* $u_{\mathrm{ms},k}$ *the solution to* (3.9)*. Then there exists* $\xi \in (0,1)$ *such that*

$$\|u_{\mathrm{ms},k} - u_h\|_{H^1} \leq C(H + k^{d/2}\xi^k)\|f\|,$$

where C *does not depend on the derivatives of* A.

To achieve linear convergence k should be chosen proportional to $\log H^{-1}$, that is, $k = c \log H^{-1}$, for some constant c.

3.3. Parabolic equations. A natural first step in generalizing the GFEM to linear thermoelasticity is to first extend it to a time dependent problem of parabolic type. Recall that the thermoelastic system (1.8)-(1.9) is parabolic [14]. This is the subject of Paper I.

We consider a parabolic problem on the following weak form; find $u(t) \in V$, such that, $u(0) = u_0$ and

$$(\dot{u}, v) + a(u, v) = (f, v), \quad \forall v \in V, \tag{3.10}$$

where $a(u, v) = (A\nabla u, \nabla v)$ as in the elliptic equation (2.1). The diffusion coefficient $A : \Omega \to \mathbb{R}^{d\times d}$ is assumed to not depend on time.

The classical FEM for (3.10) with a backward Euler discretization reads; for $n \in \{1, ..., N\}$ find $u_h^n \in V_h$, such that, $u_h^0 = u_{h,0}$

$$(\bar{\partial}_t u_h^n, v) + a(u_h^n, v) = (f^n, v), \quad \forall v \in V_h, \tag{3.11}$$

with the notation and time discretization as in Section 2.2 and $u_{h,0}$ a suitable approximation of u_0. It is well known, see, e.g., [15], that the following error estimate holds for the parabolic equation

$$\|u_h^n - u(t_n)\|_{H^1} \leq C_{\epsilon^{-1}} h + C\tau,$$

where $C_{\epsilon^{-1}}$ is constant depending on, among other terms, $\|u(t_n)\|_{H^2}$ and is thus of size ϵ^{-1} if A varies on scale of size ϵ. Hence, parabolic problems suffers from the same issues as elliptic problems when using classical finite element.

In the error analysis of the classical FEM, the error is usually split into the two parts

$$u_h^n - u(t_n) = u_h^n - R_h u(t_n) + R_h u(t_n) - u(t_n) =: \theta^n + \rho^n,$$

where $R_h : V \to V_h$ is the Ritz projection given by

$$a(R_h v, w) = a(v, w), \quad \forall w \in V_h,\ v \in V.$$

The error of the Ritz projection is given by the analysis of the elliptic problem

$$\|R_h v - v\|_{H^1} \leq Ch\|D^2 v\|. \tag{3.12}$$

This directly gives the error of ρ^n. Indeed, $\|\rho^n\|_{H^1} \leq Ch\|D^2 u(t_n)\|$, where $\|D^2 u(t_n)\| \leq C_{\epsilon^{-1}}\|\nabla \cdot A\nabla u(t_n)\| = C_{\epsilon^{-1}}\|f^n - \dot{u}(t_n)\|$ and $C_{\epsilon^{-1}}$ depend on the derivatives of A. Furthermore, to bound $\|\theta^n\|_{H^1}$ we put θ^n into (3.11), which gives

$$(\bar{\partial}_t\theta^n, v) + a(\theta^n, v) = -((R_h - I)\bar{\partial}_t u(t_n) + (\bar{\partial}_t u(t_n) - \dot{u}(t_n)), v)$$
$$=: -(\bar{\partial}_t \rho^n + \omega, v),$$

where the error of $\bar{\partial}_t\rho^n$ follows from (3.12) and the error of ω follows from Taylor's formula. In order to bound θ^n in the H^1-norm we can choose $v = \bar{\partial}_t\theta^n$.

Inspired by this we propose the following GFEM for the parabolic problem, where the space V_h in (3.11) is simply replaced by the multiscale space V_{ms} defined in Section 3.1; for $n \in \{1, ..., N\}$ find $u^n_{\mathrm{ms}} \in V_{\mathrm{ms}}$, such that, $u^0_{\mathrm{ms}} = u_{\mathrm{ms},0}$

$$(\bar{\partial}_t u^n_{\mathrm{ms}}, v) + a(u^n_{\mathrm{ms}}, v) = (f^n, v), \quad \forall v \in V_{\mathrm{ms}}, \tag{3.13}$$

with $u_{\mathrm{ms},0}$ a suitable approximation of $u_{h,0}$. Now, because of the choice of the space V_{ms} we can define a Ritz projection $R_{\mathrm{ms}} : V_h \to V_{\mathrm{ms}}$ by

$$a(R_{\mathrm{ms}}v, w) = a(v, w) = (\mathcal{A}_h v, w), \quad \forall v \in V_{\mathrm{ms}},$$

where $\mathcal{A}_h : V_h \to V_h$ is the operator defined by

$$(\mathcal{A}_h v, w) = a(v, w), \quad \forall w \in V_h.$$

The error analysis for the elliptic problem in [12] gives the bound

$$\|R_{\mathrm{ms}}v - v\|_{H^1} \le CH\|\mathcal{A}_h v\|, \quad \forall v \in V_h, \tag{3.14}$$

where C is independent of the derivatives of A. The assumption that A does not depend on time is crucial here. Otherwise, we would have to define a new space and compute a new set of basis functions at each time step t_n.

As for the elliptic equation we assume that h is sufficiently small to resolve the variations in A. This means that the reference solution u_h given by (3.11) approximates u in (3.10) sufficiently well. In the error analysis we can thus split

$$\|u^n_{\mathrm{ms}} - u(t_n)\|_{H^1} \le \|u^n_{\mathrm{ms}} - u^n_h\|_{H^1} + \|u^n_h - u(t_n)\|_{H^1},$$

where the second part is bounded by classical FEM error analysis. For the first part we can use a similar analysis, but with the new Ritz projection R_{ms}. We split the error into the parts

$$u^n_{\mathrm{ms}} - u^n_h = u^n_{\mathrm{ms}} - R_{\mathrm{ms}}u^n_h + R_{\mathrm{ms}}u^n_h - u^n_h =: \theta^n_{\mathrm{ms}} + \rho^n_{\mathrm{ms}},$$

where the error of ρ^n_{ms} is given by (3.14) and $\mathcal{A}_h u^n_h = P_h f^n - \bar{\partial}_t u^n_h$ with P_h denoting the L_2-projection onto V_h. For θ^n_{ms} we get by plugging θ^n_{ms} into (3.13)

$$(\bar{\partial}_t\theta^n_{\mathrm{ms}}, v) + a(\theta^n_{\mathrm{ms}}, v) = -(\bar{\partial}_t\rho^n_{\mathrm{ms}}, v), \quad \forall v \in V_{\mathrm{ms}}.$$

Naturally, the error bound in this case depends on the regularity of the (discrete) time derivative of the reference solution. Since the initial data is not in H^2 we expect, for instance, $\|\bar{\partial}_t u^n_h\|$ to depend on negative powers of t_n. This is possible since the backward Euler scheme preserves the smoothing effect of parabolic problems. In Paper I this is thoroughly investigated and error bounds involving negative powers of t_n are derived.

To utilize the localization introduced in Section 3.1 we can replace V_{ms} by $V_{\mathrm{ms},k}$, define a new Ritz projection $R_{\mathrm{ms},k} : V_h \to V_{\mathrm{ms},k}$, and perform similar splits of the error.

3.4. Linear thermoelasticity. In the classical finite element error analysis for linear thermoelasticity, a Ritz projection related to the stationary form of the problem is used to split the error into two terms. This Ritz projection is defined by the following $R_h(v_1, v_2) : V^1 \times V^2 \to V_h^1 \times V_h^2$, such that, $R_h(v_1, v_2) = (R_h^1(v_1, v_2), R_h^2 v_2)$ and for all $(v_1, v_2) \in V^1 \times V^2$,

$$\begin{aligned}(\sigma(v_1 - R_h^1(v_1, v_2)) : \varepsilon(w_1)) - (\alpha(v_2 - R_h^2 v_2), \nabla \cdot w_1) = 0, \quad \forall w_1 \in V_h^1,\\ (\kappa\nabla(v_2 - R_h^2 v_2), \nabla w_2) = 0, \quad \forall w_2 \in V_h^2.\end{aligned}$$

with error estimates (see [7, Lemma 2.2])

$$\|v_1 - R_h^1(v_1, v_2)\|_{H^1} \leq Ch\|D^2 v_1\| + C\|v_2 - R_h^2 v_2\|, \tag{3.15}$$

$$\|v_2 - R_h^2 v_2\|_{H^1} \leq Ch\|D^2 v_2\|. \tag{3.16}$$

The error can now be split according to

$$\begin{aligned}u_h^n - u(t_n) &= u_h^n - R_h^1(u(t_n), \theta(t_n)) + R_h^1(u(t_n), \theta(t_n)) - u(t_n) =: \eta_{h,u}^n + \rho_{h,u}^n,\\ \theta_h^n - \theta(t_n) &= \theta_h^n - R_h^2\theta(t_n) + R_h^2\theta(t_n) - \theta(t_n) =: \eta_{h,\theta}^n + \rho_{h,\theta}^n,\end{aligned}$$

where the error of $\rho_{h,u}^n$ and $\rho_{h,\theta}^n$ follows from (3.15)-(3.16). The first parts $\eta_{h,u}^n$ and $\eta_{h,\theta}^n$ can be plugged into the equation (2.4)-(2.5) to derive error estimates for these. Compare to the parabolic problem in Section 3.3. For the details we refer to [7].

To derive a GFEM for the thermoelasticity problem (1.8)-(1.9) we need to decompose two different spaces; V_h^1 and V_h^2. The decomposition of V_h^1 is performed with respect to the bilinear form $(\sigma(\cdot) : \varepsilon(\cdot))$ and the decomposition of V_h^2 with respect to $(\kappa\nabla\cdot, \nabla\cdot)$. This is done by mimicking the procedure described in Section 3.1. First define two interpolations $I_H^1 : V_h^1 \to V_H^1$ and $I_H^2 : V_h^2 \to V_H^2$ into the coarse finite element spaces $V_H^1 \subseteq V_h^1$ and $V_H^2 \subseteq V_h^2$. Now, the corresponding kernels are $V_{\mathrm{f}}^1 := \ker I_H^1$ and $V_{\mathrm{f}}^2 := \ker I_H^2$, and we can define the Ritz projections onto the these $R_{\mathrm{f}}^1 : V_h^1 \to V_{\mathrm{f}}^1$ and $R_{\mathrm{f}}^2 : V_h^2 \to V_{\mathrm{f}}^2$ given by

$$\begin{aligned}(\sigma(v_1 - R_{\mathrm{f}}^1 v_1) : \varepsilon(w_1)) = 0, \quad \forall w_1 \in V_{\mathrm{f}}^1,\ v_1 \in V_h^1\\ (\kappa\nabla(v_2 - R_{\mathrm{f}}^2 v_2), \nabla w_2) = 0, \quad \forall w_2 \in V_{\mathrm{f}}^2,\ v_2 \in V_h^2.\end{aligned}$$

The multiscale spaces are finally defined as

$$V_{\mathrm{ms}}^1 := V_H^1 - R_{\mathrm{f}}^1 V_H^1, \quad V_{\mathrm{ms}}^2 := V_H^2 - R_{\mathrm{f}}^2 V_H^2,$$

as in (3.5). With these spaces we can now define a Ritz projection corresponding to the stationary system. Define $R_{\mathrm{ms}}(v_1, v_2) : V_h^1 \times V_h^2 \to V_{\mathrm{ms}}^1 \times V_{\mathrm{ms}}^2$, such that,

$R_{\text{ms}}(v_1, v_2) = (R^1_{\text{ms}}(v_1, v_2), R^2_{\text{ms}} v_2)$ and for all $(v_1, v_2) \in V^1_h \times V^2_h$,

$$(\sigma(v_1 - R^1_{\text{ms}}(v_1, v_2)) : \varepsilon(w_1)) - (\alpha(v_2 - R^2_{\text{ms}} v_2), \nabla \cdot w_1) = 0, \quad \forall w_1 \in V^1_{\text{ms}},$$
$$(\kappa \nabla(v_2 - R^2_{\text{ms}} v_2), \nabla w_2) = 0, \quad \forall w_2 \in V^2_{\text{ms}}.$$

The spaces V^1_{ms} and V^2_{ms} are designed to handle multiscale behavior in the coefficients μ, λ, and κ respectively. However, α is also material dependent and can be expected to vary at the same scale. For this reason, we shall add an extra correction to the solution $R_{\text{ms}}(v_1, v_2)$ inspired by the techniques in [11, 8]. This additional correction is defined as $\tilde{R}_{\text{f}} : V^2_h \to V^1_{\text{f}}$, such that,

$$(\sigma(\tilde{R}_{\text{f}} v_2) : \varepsilon(w_1)) = (\alpha R^2_{\text{ms}} v_2, \nabla \cdot w_1), \quad \forall w_1 \in V^1_{\text{f}},$$

and we define $\tilde{R}^1_{\text{ms}}(v_1, v_2) = R^1_{\text{ms}}(v_1, v_2) + \tilde{R}_{\text{f}} v_2$. Using the two operators $\mathcal{A}_1 : V^1_h \times V^2_h \to V^1_h$ and $\mathcal{A}_2 : V^2_h \to V^2_h$ defined by

$$(\mathcal{A}_1(v_1, v_2), w_1) = (\sigma(v_1) : \varepsilon(w_1)) - (\alpha v_2, \nabla \cdot w_1), \qquad \forall w_1 \in V^1_h,$$
$$(\mathcal{A}_2 v_2, w_2) = (\kappa \nabla v_2, \nabla w_2), \qquad \forall w_2 \in V^2_h,$$

we prove, in Paper III, that the following error bounds hold for any $(v_1, v_2) \in V^1_h \times V^2_h$

$$\|v_1 - \tilde{R}^1_{\text{ms}}(v_1, v_2)\|_{H^1} \leq CH \|\mathcal{A}_1(v_1, v_2)\| + C\|v_2 - R^2_{\text{ms}} v_2\|, \tag{3.17}$$

$$\|v_2 - R^2_{\text{ms}} v_2\|_{H^1} \leq CH \|\mathcal{A}_2 v_2\|, \tag{3.18}$$

where C is independent of the variations in μ, λ, α, and κ.

The following system defines a GFEM for the time dependent problem (2.4)-(2.5). For $n \in \{1, ..., N\}$ find $\tilde{u}^n_{\text{ms}} = u^n_{\text{ms}} + u^n_{\text{f}}$, with $u^n_{\text{ms}} \in V^1_{\text{ms}}$ and $u^n_{\text{f}} \in V^1_{\text{f}}$, and $\theta^n_{\text{ms}} \in V^2_{\text{ms}}$, such that

$$(\sigma(\tilde{u}^n_{\text{ms}}) : \varepsilon(v_1)) - (\alpha \theta^n_{\text{ms}}, \nabla \cdot v_1) = (f^n, v_1), \quad \forall v_1 \in V^1_{\text{ms}}, \tag{3.19}$$

$$(\bar{\partial}_t \theta^n_{\text{ms}}, v_2) + (\kappa \nabla \theta^n_{\text{ms}}, \nabla v_2) + (\alpha \nabla \cdot \bar{\partial}_t \tilde{u}^n_{\text{ms}}, v_2) = (g^n, v_2), \quad \forall v_2 \in V^2_{\text{ms}}, \tag{3.20}$$

$$(\sigma(u^n_{\text{f}}) : \epsilon(w_1)) - (\alpha \theta^n_{\text{ms}}, \nabla \cdot w_1) = 0, \qquad \forall w_1 \in V^1_{\text{f}}, \tag{3.21}$$

where $\tilde{u}^0_{\text{ms}} = \tilde{u}_{\text{ms},0}$ and $\theta^0_{\text{ms}} = \theta_{\text{ms},0}$ are suitable approximations of $u_{h,0}$ and $\theta_{h,0}$ (see Paper III). Here we have added an additional correction, u^n_{f}, on u^n_{ms} inspired by the correction in the stationary setting. Following the classical finite element analysis one can now split the error according to

$$\tilde{u}^n_{\text{ms}} - u^n_h = \tilde{u}^n_{\text{ms}} - \tilde{R}^1_{\text{ms}}(u^n_h, \theta^n_h) + \tilde{R}^1_{\text{ms}}(u^n_h, \theta^n_h) - u^n_h =: \tilde{\eta}^n_{\text{ms},u} + \rho^n_{\text{ms},u},$$
$$\theta^n_{\text{ms}} - \theta^n_h = \theta^n_{\text{ms}} - R^2_{\text{ms}} \theta^n_h + R^2_{\text{ms}} \theta^n_h - \theta^n_h =: \eta^n_{\text{ms},\theta} + \rho^n_{\text{ms},\theta},$$

where the error of $\tilde{\rho}^n_{\text{ms},u}$ and $\rho^n_{\text{ms},\theta}$ are bounded by (3.17)-(3.18). The error of $\tilde{\eta}^n_{u,\text{ms}}$ and $\eta^n_{\theta,\text{ms}}$ follows by plugging these into (3.19)-(3.21). However, in this case $\tilde{\eta}^n_{\text{ms},u} \notin V^1_{\text{ms}}$, which needs to be taken into account in the analysis.

To proceed we need to perform a localization of both spaces V^1_{ms} and V^2_{ms}. We use the patches $\omega_k(K)$ defined in Section 3.2 to define localized spaces $V^1_{\text{ms},k}$ and $V^2_{\text{ms},k}$, as in (3.8). To motivate this we need to show that the corrections $R^1_{\text{f}} \lambda_x$ and $R^2_{\text{f}} \lambda_y$ decay exponentially away from node x and y, where λ_x and λ_y

denotes the classical hat functions in V_H^1 and V_H^2 respectively. The correction $R_{\mathrm{f}}^2\lambda_y$ is based on the bilinear form $(\kappa\nabla\cdot,\nabla\cdot)$ of the same type as in Section 3.1 and the decay thus follows directly from [12, 8]. The correction $R_{\mathrm{f}}^1\lambda_x$ is based on the elasticity form $(\sigma(\cdot):\varepsilon(\cdot))$ and the decay does *not* follow directly from the earlier results. This is instead proven in Paper II.

The localized GFEM for (2.4)-(2.5) is now defined as; for $n\in\{1,...,N\}$ find

$$\tilde{u}^n_{\mathrm{ms},k}=u^n_{\mathrm{ms},k}+\sum_{K\in\mathcal{T}_H}u^{n,K}_{\mathrm{f},k},\ \text{with } u^n_{\mathrm{ms},k}\in V^1_{\mathrm{ms},k},\ u^{n,K}_{\mathrm{f},k}\in V^1_{\mathrm{f}}(\omega_k(K)),$$

and $\theta^n_{\mathrm{ms},k}\in V^2_{\mathrm{ms},k}$, such that

$$(\sigma(\tilde{u}^n_{\mathrm{ms},k}):\varepsilon(v_1))-(\alpha\theta^n_{\mathrm{ms},k},\nabla\cdot v_1)=(f^n,v_1),\quad \forall v_1\in V^1_{\mathrm{ms},k}, \tag{3.22}$$

$$\begin{aligned}(\bar{\partial}_t\theta^n_{\mathrm{ms},k},v_2)&+(\kappa\nabla\theta^n_{\mathrm{ms},k},\nabla v_2)\\&+(\alpha\nabla\cdot\bar{\partial}_t\tilde{u}^n_{\mathrm{ms},k},v_2)=(g^n,v_2),\quad \forall v_2\in V^2_{\mathrm{ms},k},\end{aligned} \tag{3.23}$$

$$(\sigma(u^{n,K}_{\mathrm{f},k}):\varepsilon(w_1))-(\alpha\theta^n_{\mathrm{ms},k},\nabla\cdot w_1)_K=0,\qquad \forall w_1\in V^1_{\mathrm{f}}(\omega_k(K)). \tag{3.24}$$

The main theorem in this thesis is Theorem 3.3 below and is proved in Paper III under certain conditions on the size of H. Here $C_{f,g}$ denotes a constant depending on f and g, see Paper III for details.

THEOREM 3.3. *Let* $\{u^n_h\}_{n=1}^N$ *and* $\{\theta^n_h\}_{n=1}^N$ *be the solutions to* (2.4)-(2.5) *and* $\{\tilde{u}^n_{\mathrm{ms},k}\}_{n=1}^N$ *and* $\{\theta^n_{\mathrm{ms},k}\}_{n=1}^N$ *the solutions to* (3.22)-(3.24). *For* $n\in\{1,...,N\}$ *we have*

$$\|u^n_h-\tilde{u}^n_{\mathrm{ms},k}\|_{H^1}+\|\theta^n_h-\theta^n_{\mathrm{ms},k}\|_{H^1}\le C(H+k^{d/2}\xi^k)\big(C_{f,g}+t_n^{-1/2}\|\theta^0_h\|_{H^1}\big),$$

where C *and* $C_{f,g}$ *are constants independent of the variations in* σ,λ,α, *and* κ.

4. Summary of papers

Paper I. In Paper I we propose and analyze the GFEM (3.13) for parabolic equations with highly varying and oscillating coefficients. We prove convergence of optimal (second) order in the L_2-norm to the reference solution assuming initial data only in L_2. We do not assume any structural conditions on the multiscale coefficient, such as, periodicity or scale separation. Furthermore, we show how to extend this method to semilinear parabolic problems, where the right hand side in (3.10) is replaced by $f(u)$.

Paper II. In Paper II we propose a GFEM for linear elasticity equations with applications in heterogeneous materials. In particular, we prove exponential decay of the corrections $R_{\mathrm{f}}^1\lambda_z$ in Section 3.4. Furthermore, we prove that the GFEM reduces the locking effect that occur for materials with large Lamé parameter λ when using classical continuous and piecewise linear finite elements.

Paper III. In Paper III we build on the theory developed in Paper I and Paper II (originating from [12]) to define a GFEM for linear thermoelasticity with highly varying coefficients describing a heterogeneous material. We prove linear convergence to the reference solution in the H^1-norm independent of the variations in the data, see Theorem 3.3 in Section 3.4.

5. Future work

In Paper I on parabolic equations we assume that the diffusion coefficient $A(x)$ is independent of time. A natural extension would be to include time dependent coefficients $A(t, x)$. However, the main idea of the paper, to replace V_h with the space V_{ms} in (3.11), then fails. We would need to have a new space V_{ms}^n for each time t_n, since the diffusion coefficient $A(t_n, \cdot)$ takes different values for different times t_n. This is considerably more expensive than the time independent case, since we need to compute new corrections at each time step. It is possible that a more refined strategy could be developed by working with the parabolic problem in a space-time framework and perform localization in both time and space.

In applications involving composite materials there may be uncertainties in the material parameters, such as position or rotation, coming from the assembly procedure. These uncertainties can, for instance, be modeled by letting the coefficients depend on a random variable ω. A first step in extending the GFEM framework to such problems could be to consider an elliptic problem of the form

$$-\nabla \cdot A(x, \omega)\nabla u(x, \omega) = f(x, \omega),$$

where $A(\cdot, \omega)$ is multiscale in space for a fix ω. This problem suffers from the same problem as the time dependent case, since $A(\cdot, \omega)$ now takes different values for different outcomes ω.

In the analysis of the localization the constant $\xi \in (0, 1)$, see e.g. Theorem 3.2, depends on the contrast β/α of A, that is, the ratio between the maximal and minimal value obtained by A. Also the constant C in Theorem 3.2 depends on this ratio. However, in available numerical examples, see Paper I and Paper II, but also, e.g., [12, 8], the size of the patches and the resulting convergence does not seem to be affected by large contrasts. Thus, the error bounds derived for the localization could be too crude. This should be further investigated to derive sharper error bounds for special classes of A.

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