THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Supersymmetric Geometries in Type IIA Supergravity

Classification using the Spinorial Geometry Method

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Abstract

Supergravity theory is any supersymmetric theory with a local supersymmetry parameter. This thesis undertakes the study of type IIA supergravity, a supergravity theory in ten dimensions associated with type IIA string theory. In this framework we endeavour to classify all the classical geometries with minimal supersymmetry, where it should be noted that the constraints from minimal supersymmetry also apply to any solutions with enhanced supersymmetry.

This thesis, together with the appended papers, provides a complete classification of all geometries in standard and massive type IIA supergravity, that preserve one supersymmetry. Such supergravity backgrounds locally admit one of four types of Killing spinors, with different isotropy groups. The Killing spinor equations have been solved for all four types, identifying the geometry of spacetime and examining the conditions on the fluxes.

The picture that arises is that there are in fact three main cases, with isotropy groups Spin(7), SU(4) and $G_2 \ltimes \mathbb{R}^8$, each with a special case, covariantly characterised by the vanishing of a certain spinor bilinear. In the Spin(7) case, this results in an enhancement of the isotropy group to $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$.

In the present work, I introduce the concepts and methods involved in making such a classification using Spinorial Geometry. The Spinorial Geometry method exploits the linearity of the Killing spinor equations and an explicit basis in the space of spinors, as well as a gauge choice, to produce a linear system of equations in the fluxes and the spin connection. The thesis describes the steps involved, and the simplification of the resulting linear system. The results are discussed and compared with results in other supergravity theories, focusing on type IIB supergravity and eleven-dimensional supergravity.

This work has made heavy use of computer algebra, and a discussion of computer algebra is included.

This thesis is based on the following papers

Paper I

Supersymmetric geometries of IIA supergravity I
Ulf Gran, George Papadopoulos and Christian von Schultz
Journal of High Energy Physics, volume 2014, issue 5.
DOI: 10.1007/JHEP05(2014)024
Preprint available at arXiv:1401.6900 [hep-th]

My contribution: solving the Killing spinor equations and solving the associated linear system, arranging the solution in Spin(7) representations, analysing the geometry, participated in discussions and in writing the paper.

Paper II

Supersymmetric geometries of IIA supergravity II
Ulf Gran, George Papadopoulos and Christian von Schultz
Journal of High Energy Physics, volume 2015, issue 12.
DOI: 10.1007/JHEP12(2015)113
Preprint available at arXiv:1508.05006 [hep-th]

My contribution: solving the Killing spinor equations and solving the associated linear system, analysing the geometry, participated in discussions and in writing the paper.

Paper III

Supersymmetric geometries of IIA supergravity III
Ulf Gran, George Papadopoulos and Christian von Schultz
Preprint available at arXiv:1602.07934 [hep-th]

My contribution: verifying the solution in G_2 representations, analysing the geometry, participated in discussions and in writing the paper.

The papers all build heavily on the use of computer algebra. Here Ulf and I have worked independently, sharing no code — even using different computer algebra systems to do the work. I have written Maxima code for managing spinors and tensors, with support for Clifford algebra and more, for expanding the Killing spinor equations into a linear system and managing, solving and verifying the resulting system of equations in the varying representations.

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My supervisor, Ulf Gran, has supported me in more ways than I can count.

My collaborators, Ulf Gran and George Papadopoulos, were instrumental in bringing this ambitious project to fruition, tackling all the IIA geometries with minimal supersymmetry, without recourse to any ansatz or imposition of simplifying assumptions.

The thesis has greatly benefited from the suggestions and proofreading of Ulf Gran and Ingrid Strandberg, as well as Julie Gold who made helpful comments on the final draft.

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Contents

Chapter 1

Introduction

Physics can be about making some cool new gadget or device, or about saving the environment. But fundamental physics is, more than anything, research driven by curiosity — a quest to find the nature of reality on its most fundamental level, discovering the rules that govern it, unweaving the rainbow. In practice, much theoretical work in fundamental physics is done on theories and models with no known relation to the real world, or even theories that are known not to be phenomenologically viable — toy models exhibiting some interesting mathematical properties which might lead to new insights that could possibly, maybe, be useful in the future development of more realistic models of reality.

A distinction is often made between *proofs* and *evidence.* There may have been a time when people thought that physics consists of postulates, proven once and for all by experiments, and the various corollaries and theorems following from the postulates by pure logic. Several things have happened since — notably Einstein's theory of relativity overturning many postulates of Newtonian mechanics that were once thought rock solid. Now the focus is more on collecting *evidence.* Newtonian mechanics may have been disproved by Einstein, but the evidence for it remains solid — in its well-tested domain of applicability.

In a purely theoretical context, though, it is less clear that the *proof* concept is intractable. In experimental science we can't work everything out in terms of axioms and postulates, because no one knows exactly what those postulates would be like. Working things out from postulates can, on the other hand, be a fruitful exercise in mathematics and theoretical physics.

We don't always work like that, though. In fact, the theoretical physicist often works in a way very reminiscent of experimental science: by having a hypothesis about how different theoretical constructions relate to one another, and setting about gathering *evidence* for the hypothesis in question. Such a hypothesis is often called a *conjecture,* and though it may or may not be possible to prove the conjecture at some point (thus promoting it to a theorem), it is often possible to start by examining the conjecture for certain special cases. If it works out, that's evidence for the conjecture. If it doesn't work out, the conjecture has been disproved, and it may be revised or abandoned.

Supersymmetric theories are often used this way. At the time of writing, supersymmetry has not yet been found in the real world — and if supersymmetry exists, it is at least somewhat broken. That does not, however, stop supersymmetric theories — even manifestly unrealistic theories, such as maximally supersymmetric Yang–Mills theory — from being used quite successfully in examining various dualities and relationships between various theoretical concepts. The more symmetry you have, the easier it is to understand things, and understanding a simple problem is often a useful step towards understanding a hard one.

Supersymmetric solutions have historically been very important, for instance when going from weak coupling to strong coupling. Physics at strong coupling is normally beyond control, signifying the need to rethink the degrees of freedom chosen to describe the problem. Extrapolations between weak and strong coupling are normally impossible — unless you consider quantities protected by some symmetry. Treating objects protected by supersymmetry you can learn a surprising amount, without large quantum corrections getting in your way and spoiling the party. The present work would be useful for studying toy models, but it might also be possible to compactify and do deformations of supersymmetric geometries so that connection is made with phenomenologically viable models. By classifying the geometries we delineate the possibilities within the theoretical framework of type IIA supergravity. That should be of some help if you wish to construct new models in type IIA supergravity — whether realistic or not.

The concept of *symmetry* stands at the centre of fundamental physics, and is normally implemented using the mathematics of group theory and Lie algebras. A symmetry transformation is a mathematical transformation which leaves all measurable quantities intact. It can be an internal symmetry, such as changing the overall phase of a complex wavefunction, or it can be an external symmetry, involving angles and distances of spacetime itself, rather than just the components of the fields living in spacetime.

A symmetry transformation typically involves a number of scalar parameters; e.g., the phase shift α of a U(1) transformation of a complex field: $\psi(x) \mapsto \mathrm{e}^{\mathrm{i}\,\alpha}\,\psi(x)$. If the symmetry parameters, such as α, do not depend on the position x in spacetime, we call it a *global symmetry* or a *rigid symmetry.* If, on the other hand, the symmetry parameters depend on x, e.g. $\psi(x) \mapsto \mathrm{e}^{\mathrm{i}\,\alpha(x)}\,\psi(x)$, we talk about a *local symmetry* or *gauge symmetry.* Often, when a theory exhibits some global symmetry, it is useful to consider what would happen if the symmetry were local. (That is called *gauging* the symmetry.) Simply replacing α by $\alpha(x)$ would normally mean that the transformation is no longer a symmetry transformation, because of derivatives making trouble. Symmetry is then restored by replacing all derivatives with *covariant derivatives*, which differ from ordinary derivatives by some *connection* or *gauge potential* that you invent with its own transformation rule, created to restore the symmetry. The symmetry transformation, or *gauge transformation*, would then be done both to the original fields of the theory according to the original transformation rule, and simultaneously to the gauge potential according to the rule you made up to restore the symmetry. The commutator of the covariant derivative gives you the field strength associated to the gauge potential. According to the rules of quantum field theory, all renormalisable terms that you can construct (still respecting all the desired symmetries) must be added to the Lagrangian of the theory.

The importance of the gauging procedure to fundamental physics can hardly be overstated. For example, take the Dirac field $\psi(x)$, which may be used to describe electrons. The theory is invariant under the U(1) symmetry mentioned above, $\psi(x) \mapsto e^{i\alpha}\psi(x)$. Gauging the symmetry, i.e. making α a function on spacetime, requires us to have a covariant derivative with a gauge potential — the electromagnetic potential — and a corresponding field strength — the electromagnetic field strength, composed of the ordinary electric and magnetic fields. Then the rules for ordinary field theory give you Maxwell's equations. The rules for quantum field theory give you quantum electrodynamics, which is capable of describing all physical phenomena of everyday experience (except gravity and nuclear physics). The gauging procedure takes you from the existence of the electron to the full theory of electromagnetism. The standard model of particle physics does essentially the same, but with a larger symmetry group: $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, and describes all physical phenomena of everyday experience (except gravity). The corner stones of the standard model of cosmology — dark energy and dark matter — are still left out however, so the standard model of particle physics is not the end of the story.

One attractive solution to the dark matter problem is *supersymmetry.* As a consequence of supersymmetry, all fermion particles get their own boson superpartner, and all boson particles get their own fermion superpartner. From a mathematical point of view, supersymmetry is essentially the manifestation of the following idea: What if the symmetry parameters don't have to be Lorentz scalars? It turns out that it is possible to have symmetry transformations where the symmetry parameter is not a phase shift or some other such Lorentz scalar, but actually a *spinor.* The matter fields of the fermions are spinors, so when we take the symmetry parameter to be a spinor, the symmetry transformation necessarily relates the bosons to fermions, and vice versa. When talking about supersymmetry, one normally means *rigid* supersymmetry; i.e., a global symmetry, whose symmetry parameters do not depend on the point x in spacetime. As you may guess from the preceding discussion, one natural thing to ask when faced with such a global symmetry is if we can make it local — if we can *gauge* it.

It turns out we can, and moreover the resulting theory contains Einstein's theory of general relativity. For this reason, rather than talking about local supersymmetry or gauged supersymmetry, the established term is *supergravity.* It does not mean that the gravity is super-strong and that we are treating black holes or something (though black holes *are* interesting objects to study in supergravity theories); it simply means that there is local supersymmetry and there is gravity. Supergravity means that we have a symmetry whose symmetry parameter is a spinor that depends on the position x in the spacetime.

If the Lagrangian (or the action) of the theory is invariant under the symmetry transformation we say that the theory has the symmetry. (Otherwise the transformation wouldn't be a symmetry transformation.) If the theory has the symmetry, then the equations of motion (loosely speaking "the laws of physics") have that symmetry. This doesn't necessarily mean that the solution has that symmetry, however. There are a variety of ways to break a symmetry, and I won't go into the details here. Suffice to say, that the solution has a symmetry if it is invariant under the symmetry transformation. We are looking for supersymmetric geometries, and we get them by insisting that the solution is invariant under the supersymmetry transformation. If it is, we call the supersymmetry parameter a *Killing spinor.*

The most promising attempt at a quantum theory of gravity is widely regarded to be string theory, of which there are various types related by certain limits and dualities. In the limit where quantum gravity effects are small, these string theories give rise to different types of supergravity.

The focus of this work has been type IIA supergravity, and the classification of supersymmetric type IIA geometries. Type IIA supergravity is a ten-dimensional theory which can be obtained by taking a certain limit in type IIA string theory. It is the ten-dimensional theory whose supersymmetry parameter is a 32-component Majorana spinor (which may be viewed as two 16-component Majorana–Weyl spinors of opposite chirality); the ten-dimensional theory with two Majorana–Weyl spinors of the same chirality goes under the name IIB. Massless type IIA theory can also be obtained by doing a dimensional reduction of eleven-dimensional supergravity, which is the supergravity theory with the highest possible dimensionality.

Why study IIA supergravity? Because there was no systematic classification of type

IIA geometries yet. We start from one Killing spinor (minimal supersymmetry), and make no assumptions. From this we obtain the most general structure that *all* supersymmetric solutions must satisfy, since all supersymmetric solutions will have at least one Killing spinor.

Type IIA supergravity also has a two-form field strength, just like the ordinary electromagnetic field. That means that the intuition physicists have developed for Maxwell's theory applies to (at least part of) solutions of IIA supergravity. It is e.g. possible to have a black hole with some electric charge in this theory.

A systematic classification *has* been done before for eleven-dimensional supergravity [1, 2]. Some results in IIA supergravity can be obtained from known results in eleven dimensions, but that's not always feasible. One reason is the Romans cosmological constant of massive IIA supergravity: you don't get that from eleven dimensions, only the massless version of IIA supergravity; and some things that may be difficult to do in eleven dimensions, such as the study of black holes, may be easier to do in IIA theory directly, compared to doing the work in eleven dimensions and follow up by a dimensional reduction. The analysis in eleven dimensions would have to deal with all the higher Kaluza–Klein modes in type IIA.

The thesis has been organised as a compilation thesis, where these chapters serve as an introduction to the background and methods employed and a summary of the results, whereas the new results in their entirety are presented in the appended papers. Chapter 2 introduces the mathematics of fundamental physics and is intended to be readable by friends and family. Chapter 3 introduces spinors and how they may be viewed in terms of exterior algebra. Chapter 4 introduces the groups that will play a special role in the classification of supergravity solutions: Spin(7), SU(4) and G_2. Chapter 5 turns to supergravity and the search for supersymmetric solutions. Chapter 6 builds on this to give a concise description of the spinorial geometry method and its application to type IIA supergravity. Chapter 7 discusses the results: the classes of supergravity solutions with minimal supersymmetry in type IIA supergravity. This has all been based on extensive use of computer algebra, which is discussed in chapter 8. Finally, chapter 9 gives an outlook with ideas for further research.

The papers together comprise a complete classification of all the backgrounds of massive type IIA supergravity that preserve one supersymmetry. In Paper I we treat backgrounds that admit a Spin(7) invariant Killing spinor in the generic case and a $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ invariant Killing spinor as a special case; in Paper II we treat backgrounds that admit an SU(4) invariant Killing spinor; and in Paper III we treat the $G_2 \ltimes \mathbb{R}^8$ case. These three main cases, each with a special case that can be characterised covariantly by the vanishing of a spinor bilinear, exhaust all the possibilities, completing the classification of IIA backgrounds with minimal supersymmetry.

Chapter 2

Fundamentals

As you will remember from school, maths is all about numbers. Sometimes variables representing numbers, or matrices of numbers, or variables representing matrices of numbers. Maths is all about numbers and what you can do with numbers, which is fortunate, because computers are particularly good at numbers, which means that people don't need to do the maths themselves. The world is full of numbers, which is fortunate, because it makes the world understandable — at least to the computers. And since everything is just numbers...

Wait a moment. That's not actually true, is it? The world is made of real things, and the real things aren't actually numbers, are they? In fact, it is often far from easy to see how things from the real world could be mapped to the world of numbers. No, we need something more intuitive than numbers, something we can easily relate to the real world.

In other words, we need something more abstract. Now, to some people this may sound paradoxical — something more abstract is more intuitive? Perhaps you feel the urge to mentally close the curtains whenever things get too abstract, and refocus your attention on something more concrete, like what's for dinner. If so, I urge you not to do that, because abstraction really is a way to make things easier. Making an abstraction is essentially saying that you don't care about something. The world is an incredibly complex thing, with innumerable particles interacting all the time. It would be impossible to take in, even for a computer, if you had to take everything into account to understand anything. But we don't need to take everything into account. Rather than thinking about the table as a great collection of interacting atoms, I can for many purposes just model it as a rigid body. That's an abstraction of the real table. In the real table there's vibrations of atoms. But I don't have to care about that. The real table will break if subjected to large forces. But I don't have to care about that (I hope). The rigid body approximation is an abstraction where I can assume the table keeps its size and shape. The less you care the more abstract you are — but if you don't care at all you've abstracted too far.

One of the most abstract concepts the physicist uses is the concept of a set. We can study the set of all rotations, the set of molecules in a container, even sets of sets. A set is any well-defined collection of things, real or imagined. The concept of sets is easy to apply to the real world, because you can just point to any collection of things and say “that’s a set,” and you can point to some other collection of things and say “that’s another set,” and you can say whether the sets are disjoint, or if there is some overlap, or if they are in fact in some sense equivalent. You can do a lot of useful things already with sets, but for most practical purposes you do need something a bit more specific — something a bit less abstract.

The concept of *a group* is somewhat more specific. A group is a set together with a binary operation, sometimes called *group multiplication*, such that you can combine two elements of the group and get a result that is also in the group. The set of all rotations is a group, and a very important group too, where the group operation is simply performing both rotations one after the other.

A *vector space* or *linear space* is a group under addition (that is, the binary operation is the adding of vectors) that also has another operation: multiplication with a scalar (normally a real number, in some contexts a complex number).

An *algebra* is a vector space with a bilinear product: it is thereby a group both under addition *and* under multiplication, in addition to supporting multiplication by scalars. There are a number of axioms essentially ensuring that addition and multiplication work according to the accustomed rules of addition and multiplication. The most important algebras we shall encounter in this work are *Lie algebras* and *Clifford algebra* (also known as *geometric algebra*).

These constructs are abstract enough that they are reasonably simple to apply to the real world. The standard device to map an abstract algebra to the world of numbers is called a *matrix representation*. A matrix representation is a map that to each element in the abstract algebra assigns a square matrix, such that any equation that holds true of elements in the abstract algebra, also holds true of the corresponding matrices (this is called a structure-preserving map, or a homomorphism). A matrix representation is not unique.

With this in mind, we next turn to Clifford algebras and spinors, which shall play a central role in what follows, establishing both the concepts and the conventions we use.

Chapter 3

Spinors

A physicist who confines himself to the study of fermions in three-dimensional space or four dimensional spacetime, will think of a spinor as a pair of complex numbers (sometimes called a *Pauli spinor*, $\psi \in \mathbb{C}^2$), or as a four-tuple of complex numbers (called a *Dirac spinor*, $\psi \in \mathbb{C}^4$) in four dimensions. These spinors are then acted upon by the *Pauli sigma matrices*,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

or the Dirac gamma matrices γ^μ, respectively. The Dirac gamma matrices look different depending on the choice of representation, for example

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix},$$

in the *Pauli–Dirac representation* used by e.g. [3], or

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix},$$

in the *Weyl representation* used by e.g. [4]. In either case, and for both σ_i and γ_μ, the important thing is that the matrices provide a matrix representation of the Clifford algebra in question. We will use a somewhat different representation of the Clifford algebra in this work, but the idea is the same. The spinors are what these matrices act on, which mathematicians may call the module of the algebra.

3.1 The geometric algebra

Geometric algebra is an algebra of scalars, vectors and multivectors. You may think of a vector as a directed line segment, a bivector as an oriented area, a trivector as an oriented volume, and so on. Multivectors of higher degree may be harder to visualise, but the idea is the same. We shall focus our attention on the vectors in what follows. There are two common ways of multiplying vectors: the scalar product and the exterior product (in three dimensions dual to the cross product). The *geometric product* combines these products into a single unified concept:

$$\vec{a} \diamond \vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}. \tag{3.1}$$

The symmetric part of the geometric product of two vectors $\vec{a}$ and $\vec{b}$ is

$$\tfrac{1}{2}\left(\vec{a} \diamond \vec{b} + \vec{b} \diamond \vec{a}\right) = \vec{a} \cdot \vec{b}. \tag{3.2}$$

Expressed in terms of the basis vectors $\vec{e}_A$ and the metric g_{AB}, (3.2) reads

$$\vec{e}_A \diamond \vec{e}_B + \vec{e}_B \diamond \vec{e}_A = 2\, g_{AB}. \tag{3.3}$$

The wedge product is the antisymmetric part of the geometric product:

$$\tfrac{1}{2}\left(\vec{a} \diamond \vec{b} - \vec{b} \diamond \vec{a}\right) = \vec{a} \wedge \vec{b}. \tag{3.4}$$

The geometric product is associative and distributive, and the action on higher multivectors follows from its action on the vectors.

As noted in chapter 2, the natural thing for a physicist, when faced with an associative algebra, is to consider a matrix representation of the algebra. Each basis vector $\vec{e}_A$ has an associated *gamma matrix* $\mathit{\Gamma}_A$, and the geometric product is implemented by matrix multiplication. The expression (3.3) is often rendered as

$$\mathit{\Gamma}^A \mathit{\Gamma}^B + \mathit{\Gamma}^B \mathit{\Gamma}^A = 2\, g^{AB} \tag{3.5}$$

in the matrix representation, where g^{AB} is the matrix inverse of the metric g_{AB}. The geometric algebra is often called Clifford algebra — in particular when its matrix representation is considered.

3.2 Abstract spinors and spinors from forms

As noted above, the spinors are the space that the gamma matrices act on. You may wonder, is it possible to define spinors without invoking a concrete matrix representation of the Clifford algebra? Perhaps, spinors could be seen as a part of the Clifford algebra itself? As it turns out, the answer is yes.

We demand of spinors that they be closed under addition, and closed under multiplication from the left by elements of the Clifford algebra. (In the matrix representation, this means that we can multiply a spinor with a gamma matrix from the left, and the result should still be a spinor.) If we to these conditions add that the spinors should be contained in the Clifford algebra, we have the definition of a *left ideal*. Though we are considering the special case of ten-dimensional spacetime below, the general construction more or less follows [5].

Let $V = \mathrm{span}_{\mathbb{R}} (\vec{e}_0, \vec{e}_1, \dots, \vec{e}_9)$ be the tangent space of ten-dimensional spacetime, with $\vec{e}_0 \cdot \vec{e}_0 = -1$, $\vec{e}_A \cdot \vec{e}_B = \delta_{AB}$ for A, $B > 0$. The complexification is $V_{\mathbb{C}} = \mathbb{C} \otimes V = \mathrm{span}_{\mathbb{C}} (\vec{e}_0, \vec{e}_1, \dots, \vec{e}_9)$.

We define the light-cone basis elements by

$$\boldsymbol{e}_+ = \frac{1}{\sqrt{2}} \left(\vec{e}_5 + \vec{e}_0 \right), \quad \boldsymbol{e}_- = \frac{1}{\sqrt{2}} \left(\vec{e}_5 - \vec{e}_0 \right). \tag{3.6}$$

Note that $\boldsymbol{e}_+ \cdot \boldsymbol{e}_+ = \boldsymbol{e}_- \cdot \boldsymbol{e}_- = 0$, while $\boldsymbol{e}_+ \cdot \boldsymbol{e}_- = 1$. We further define the holomorphic and anti-holomorphic basis elements $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}_{\bar{\alpha}}$, respectively.

$$\boldsymbol{e}_\alpha = \frac{1}{\sqrt{2}} \left(\vec{e}_\alpha - \mathrm{i}\, \vec{e}_{\alpha+5} \right), \quad \boldsymbol{e}_{\bar{\alpha}} = \frac{1}{\sqrt{2}} \left(\vec{e}_\alpha + \mathrm{i}\, \vec{e}_{\alpha+5} \right). \tag{3.7}$$

Note that $\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta = \boldsymbol{e}_{\bar{\alpha}} \cdot \boldsymbol{e}_{\bar{\beta}} = 0$, while $\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_{\bar{\beta}} = \delta_{\alpha\bar{\beta}}$.

This way, $V_{\mathbb{C}}$ naturally splits into two parts,

$$V_{\mathbb{C}} = W_{\mathrm{ho}} \oplus W_{\mathrm{aho}}$$

where $W_{\mathrm{ho}} = \mathrm{span}_{\mathbb{R}} (\boldsymbol{e}_-, \boldsymbol{e}_\alpha)$ and $W_{\mathrm{aho}} = \mathrm{span}_{\mathbb{R}} (\boldsymbol{e}_+, \boldsymbol{e}_{\bar{\alpha}})$: roughly speaking the holomorphic and anti-holomorphic parts, adjoined by $\boldsymbol{e}_-$ and $\boldsymbol{e}_+$, respectively. (This subdivision can be made for V too, if you impose a reality condition afterwards. Coefficients will be complex, with $X_\alpha{}^\star = X_{\bar{\alpha}}$.)

W_{ho} and W_{aho} are maximal totally singular subspaces of $V_{\mathbb{C}}$, meaning that the metric restricted to either vanishes:

$$\vec{x} \cdot \vec{y} = 0 \ \text{ for } \ \vec{x}, \vec{y} \in W_{\mathrm{ho}} \ \text{ and for } \ \vec{x}, \vec{y} \in W_{\mathrm{aho}}.$$

This is also called a totally isotropic subspace.

Totally isotropic subspaces play a special role in the theory of spinors [5, 6].

Let f be the geometric product of the elements of the base of W_{aho}. Then f is nilpotent ($f^2 = 0$), and $\mathcal{C}\ell\,(V) \diamond f$ is a minimal left ideal:

$$\mathcal{C}\ell\,(V) \diamond f = \mathcal{C}\ell\,(W_{\mathrm{ho}}) \diamond f.$$

This minimal left ideal can itself be regarded as the space of spinors, or we could equally well look to $\mathcal{C}\ell\,(W_{\text{ho}})$ and call *that* the space of spinors, since to any spinor $u \in \mathcal{C}\ell\,(W_{\text{ho}})$, there corresponds a spinor $u \diamond f \in \mathcal{C}\ell\,(V) \diamond f$ in the left ideal. Do note that the Clifford algebra $\mathcal{C}\ell\,(W_{\text{ho}})$ and the exterior algebra $\Lambda\,(W_{\text{ho}})$ are isomorphic vector spaces, and you may go from one to the other using a simple inclusion map. The spinors are acted upon by elements $v \in \mathcal{C}\ell\,(V)$ as $v \diamond u \diamond f = (\rho\,(v)\,u) \diamond f$, where $\rho\,(v)$ is the spin representation of v. Here, the action of ρ can be seen as acting on the exterior algebra, and we find that this construction requires e.g.

$$\rho\,(\boldsymbol{e}_1)\,u = \boldsymbol{e}_1 \wedge u \;\text{ and }\; \rho\left(\boldsymbol{e}_{\bar{1}}\right)u = 2\,\boldsymbol{e}_{\bar{1}} \lrcorner\, u.$$

Now, for practical reasons — and for compatibility with [7] — we are going to use a different representation, which we shall call $\mathit{\Gamma}$, which distributes that factor two a bit more symmetrically:

$$\mathit{\Gamma}\,(\boldsymbol{e}_1)\,u = \sqrt{2}\,\boldsymbol{e}_1 \wedge u \;\text{ and }\; \mathit{\Gamma}\left(\boldsymbol{e}_{\bar{1}}\right)u = \sqrt{2}\,\boldsymbol{e}_{\bar{1}} \lrcorner\, u. \tag{3.8}$$

We shall also employ the shorthand $\mathit{\Gamma}_A = \mathit{\Gamma}\left(\boldsymbol{e}_A\right)$. Note that $\mathit{\Gamma}$ is just another representation of the Clifford algebra, just like ρ — though it does make the mapping from the exterior algebra of spinor space to the left ideal of the Clifford algebra a bit more complicated than an inclusion map, owing to the need to account for factors of $\sqrt{2}$.

3.3 Dirac, Weyl and Majorana

A Dirac spinor is a complex spinor, which the $\mathit{\Gamma}$ matrices act on. In ten spacetime dimensions, this means they are 32 complex numbers.

A Majorana spinor is a real spinor — real with respect to a real structure on the geometric algebra side; real in the same sense that e.g. $\vec{e}_6$ is real, despite the fact that, through (3.8) and (3.7) we have

$$\mathit{\Gamma}\,(\vec{e}_6)\,u = \mathrm{i}\,\mathsf{e}_1 \wedge u - \mathrm{i}\,\mathsf{e}_1 \lrcorner\, u.$$

The fact that something that is real contains the imaginary unit in the matrix representation, indicates that something non-trivial has happened to the real structure: Complex conjugation on the geometric algebra side is not represented by simple complex conjugation in the matrix representation: There it is represented by the combination of complex conjugation and left-multiplication by the charge conjugation matrix C:

$$C = \mathit{\Gamma}_6\,\mathit{\Gamma}_7\,\mathit{\Gamma}_8\,\mathit{\Gamma}_9\,,$$

where the Γ matrices refer to the real basis rather than the complex basis defined in (3.7).

In other words, the reality condition that a Majorana spinor η would have to satisfy is

$$\eta = C\eta^{\star}. \tag{3.9}$$

A Majorana spinor in ten dimensions has real dimension 32.

The space of Dirac spinors Δ, may be further split into so called *Weyl spinors* of positive and negative chirality, Δ^{+} and Δ^{-}, respectively, transforming among themselves under $\mathrm{Spin}(9,1)$. This corresponds to forms of even and odd grade, when the spinor space is represented by the exterior algebra. Imposing the Majorana condition (3.9) as well gives us the Majorana–Weyl spinors Δ^{+}_{16} and Δ^{-}_{16} of positive and negative chirality, respectively.

The supersymmetry parameter of type IIA supergraivty is a Majorana spinor, containing two Majorana–Weyl spinors of opposite chirality.

3.4 Curved space

Handling scalars, vectors and higher tensors in curved space is not to difficult — any course in general relativity will cover the basics. Scalars are particularly simple, and the vectors (and tensors) are essentially handled by replacing ordinary derivatives, $\partial_M V^N$, by covariant derivatives,

$$\nabla_M V^N := \partial_M V^N + \Gamma^{N}_{ML}\, V^L$$

where Γ^{N}_{ML} are the Christoffel symbols, or connection coefficients.

Digging a little deeper, the concept of a vector requires a bit more thought compared to the case of flat space, since the common “it’s an arrow” intuition doesn’t really work all that well out of the box if the space is curved. The components V^M of a vector are properly seen to multiply the basis vectors, $\vec{V} = V^M\,\vec{e}_M$, but how do we actually make sense of the basis vectors $\vec{e}_M$? The answer lies in directional derivatives. Already in flat space there is a one-to-one correspondence between a vector $\vec{V}$ and the associated directional derivative $\vec{V}\cdot\nabla$ at a point. Nothing prevents us from taking the directional derivative to *be* the definition of a vector. Thus $\vec{V} \equiv V^M\,\vec{e}_M \equiv V^M\,\partial_M$. The directional derivative makes perfect sense even on curved manifolds. The vector space spanned by the partial derivatives ∂_M evaluated at a point p is called the tangent space of the manifold at that point, and may be visualised as a flat infinite space laying tangent to the manifold at the point, like a plane laying tangent on a sphere.

This formalism doesn't work for spinors. To work with a curved spacetime, we want to represent the basis vectors $\vec{e}_M$ by a derivative ∂_M, but to work with spinors, we want to represent the basis vectors $\vec{e}_A$ by some gamma matrix Γ_A. Clearly, we cannot do both at the same time. Clearly we *need* to do both at the same time.

In order to handle spinors on a curved manifold we need vielbeins, which essentially translate back and forth between curved indices as in $\vec{e}_M$ (which we identify with the derivative ∂_M) and flat indices as in $\vec{e}_A$ (which we identify with the gamma matrix Γ_A). Since we know how to handle spinors in flat space, and curved space is still locally flat, we assign a local frame with an orthonormal basis $\{\vec{e}_A\}$ at each point of the spacetime, related to the tangent space of the manifold by $\vec{e}_A = e_A^M \partial_M$, where A, B, ... are the flat indices and M, N, ... are the curved indices. They are related by the *vielbein* e_A^M. Instead of the Christoffel symbol, we have the spin connection, $\Omega_{M,AB}$ (which is antisymmetric in A and B). The expression for the covariant derivative of a vector expressed in flat indices is then

$$\nabla_M V^A = \partial_M V^A + \Omega_{M,}{}^A{}_B V^B.$$

But the real advantage is that with $\Omega_{M,AB}$, unlike the Γ_{ML}^N, we can act on a spinor ε:

$$\nabla_M \varepsilon = \partial_M \varepsilon + \frac{1}{4} \Omega_{M,AB} \Gamma^{AB} \varepsilon.$$

This will be necessary when we turn to supergravity and the Killing spinor equations.

Chapter 4

A closer look at Spin(7), SU(4) and G_2

Let's start by considering the *automorphism groups* of the *division algebras.*

A *homomorphism* is a structure-preserving map. If, for example, addition is defined for some objects x and y in a set X, and f is a homomorphism, then $f(x+y) = f(x) + f(y)$, and similarly for multiplication or division or any other structure that may be defined on X. An *isomorphism* is a *homomorphism* with an inverse. An *automorphism* is an *isomorphism* from a set to itself.

The division algebras are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. The real numbers have real dimension one and no imaginary units; the complex numbers have real dimension two, and one imaginary unit; the quaternions have real dimension four, and three imaginary units; and the octonions have real dimension eight, and seven imaginary units. Imaginary units square to -1 and anticommute among themselves. In each step in the Cayley–Dickson construction [8], you add a new, algebraically independent imaginary unit, and the other additional imaginary units are generated by multiplication with the new one. The division algebras lose an algebraic property at each step of the construction, and it is not possible to go beyond the octonions, since you then lose the property of being a division algebra at all.

Following [9], consider the vector spaces $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$ (we shall get back to the octonions $\mathbb{O}$ momentarily). What are the automorphism groups? If we want to preserve the metric, we get $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$. If we want to preserve a volume form too, we get $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, respectively. These groups all appear in Berger's list [10], and of these SU(4) ends up playing an important role in type IIA supergravity solutions with minimal supersymmetry. The other important groups we encounter, Spin(7) and G_2 are also on Berger's list, as the exceptional holonomy groups. These are related to the octonions. G_2 is the automorphism group of the octonions. In fact, if we split the octonions into a real and an imaginary part, $\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}\,(\mathbb{O})$, where $\mathrm{Im}\,(\mathbb{O}) \cong \mathbb{R}^7$, we see that G_2 leaves the $\mathbb{R}$ invariant; it is the automorphism group of the imaginary octonions. Spin(7) is also related to the octonions, but it is not quite an automorphism group, because it doesn't preserve the full algebraic structure of the octonions. It does however preserve *some* of the multiplicative structure of the octonions; for details, see e.g. [11].[1] G_2 is the smallest group, and we have $G_2 \subset \text{Spin}(7) \subset \text{SO}(8)$ and $\text{SU}(4) \subset \text{Spin}(7) \subset \text{SO}(8)$. (Note $G_2 \not\subset \text{SU}(4)$ and vice versa.) String theory has ten spacetime dimensions: one time dimension and nine spatial dimensions, or equivalently, two light-cone dimensions and eight spatial dimensions. The SO(8) in which G_2, SU(4) and Spin(7) are embedded, is associated with these eight spatial dimensions.

4.1 The isotropy group of spinors

There are four orbits of spinors under the $\text{Spin}(9,1)$ gauge transformation. Two spinors are in the same orbit if they are related by some element in the symmetry group (up to normalisation, in this context). If two spinors are in different orbits, it is not possible to relate them to each other using the symmetry group.

Some subset of the $\text{Spin}(9,1)$ group will relate a given spinor to all the spinors in the same orbit. Some other subset of $\text{Spin}(9,1)$ will leave the spinor invariant. That subset is called the *stability subgroup* (also called the *isotropy group*). It turns out that the four orbits have different stability subgroups, and may be characterised by them: $\text{Spin}(7) \ltimes \mathbb{R}^8$, Spin(7), SU(4) and $G_2 \ltimes \mathbb{R}^8$. In each case we can choose a representative spinor to use in the Killing spinor equations; instead of treating a generic 32-component spinor, we consider

$$\varepsilon = f\ (\mathbb{1} + e_{1234}) + g\ (e_5 + e_{12345}) \tag{4.1}$$

with $g \neq 0$ in the Spin(7) case and $g = 0$ in the $\text{Spin}(7) \ltimes \mathbb{R}^8$ case,

$$\varepsilon = f\ (\mathbb{1} + e_{1234}) + g_1\ (e_5 + e_{12345}) + \mathrm{i}\, g_2\ (e_5 - e_{12345}) \tag{4.2}$$

with $g_2 \neq 0$ in the SU(4) case, and

$$\varepsilon = f\ (\mathbb{1} + e_{1234}) + g\ (e_1 + e_{234}) \tag{4.3}$$

with $f \neq 0$, $g \neq 0$ in the $G_2 \ltimes \mathbb{R}^8$ case.

The Spin(7) case and the $\text{Spin}(7) \ltimes \mathbb{R}^8$ case are treated in Paper I; the SU(4) case is treated in Paper II; and the $G_2 \ltimes \mathbb{R}^8$ case is treated in Paper III.

To find the stability subgroups and corresponding representative spinors, we follow the procedure outlined in [7] for type IIB supergravity. (A discussion of the stability subgroup of spinors in eleven-dimensional supergravity can be found in [12].) The first

[1] Though the octonions tell us something about where the groups come from and how they relate to algebra, we have not made use of them for actual computations.

part of [7] treats the stability subgroup of a Majorana–Weyl spinor of positive chirality. (The Majorana–Weyl spinors of positive chirality will be denoted by Δ_{16}^{+}.) That part of the discussion applies just as well to type IIA as to type IIB supergravity, and we will obtain the same result: a spinor of the form $\varepsilon = f\,(\mathbb{1} + e_{1234})$ with stability subgroup $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$, and that's all there is when considering a single Majorana–Weyl spinor of positive chirality. To get there, we start from the simplest spinor in some sense, the spinor $\mathbb{1}$. We see that this is not a Majorana spinor, and that it transforms into e_{1234} under charge conjugation. This leads us to consider the spinor $\mathbb{1} + e_{1234} \in \Delta_{16}^{+}$, and we find the stability subgroup by acting on it with all the generators of $\mathrm{Spin}(9,1)$ and see which linear combinations of such annihilate the spinor. This way, we obtain the stability subgroup of this spinor: $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$. The other linearly independent spinors in Δ_{16}^{+} are the eight Majorana–Weyl spinors containing e_5, spanning

$$\begin{aligned}\Delta_8 = \mathrm{span}_{\mathbb{R}}\{\; & e_{15} + e_{2345}, \quad e_{25} - e_{1345}, \quad e_{35} + e_{1245}, \quad e_{45} - e_{1235}, \\ & \mathrm{i}\,(e_{15} - e_{2345}),\; \mathrm{i}\,(e_{25} + e_{1345}),\; \mathrm{i}\,(e_{35} - e_{1245}),\; \mathrm{i}\,(e_{45} + e_{1235}) \;\},\end{aligned}$$

and the remaining seven Majorana–Weyl spinors spanning

$$\begin{aligned}\Lambda^1(\mathbb{R}^7) = \mathrm{span}_{\mathbb{R}}\{\; & e_{12} - e_{34}, \quad e_{13} + e_{24}, \quad e_{14} - e_{23}, \\ & \mathrm{i}\,(e_{12} + e_{34}),\; \mathrm{i}\,(e_{13} - e_{24}),\; \mathrm{i}\,(e_{14} + e_{23}),\; \mathrm{i}\,(\mathbb{1} - e_{1234})\}.\end{aligned}$$

It is easy to see that Δ_8 and $\Lambda^1\left(\mathbb{R}^7\right)$ transform among themselves under $\mathrm{Spin}(7)$, which doesn't touch the e_5 direction, and closer inspection allows us to identify Δ_8 and $\Lambda^1\left(\mathbb{R}^7\right)$ as the spin representation and the vector representation of $\mathrm{Spin}(7)$, respectively. Δ_{16}^{+} splits under $\mathrm{Spin}(7)$ as

$$\Delta_{16}^{+} = \mathrm{span}_{\mathbb{R}}\,(\mathbb{1} + e_{1234}) \oplus \Lambda^1\left(\mathbb{R}^7\right) \oplus \Delta_8, \text{ or equivalently} \tag{4.4}$$

$$\eta_+ = a\,(\mathbb{1} + e_{1234}) + \theta_1 + \theta_2 \text{ where } \theta_1 \in \Lambda^1\left(\mathbb{R}^7\right) \text{ and } \theta_2 \in \Delta_8. \tag{4.5}$$

This is the most general spinor in Δ_{16}^{+}.

Rather than working with the most general spinor, we would like to make a *gauge choice:* η_+ transforms under the gauge group $\mathrm{Spin}(9,1)$, and it is quite sufficient to study one spinor representative for each orbit of Δ_{16}^{+} under $\mathrm{Spin}(9,1)$. It turns out there is only one orbit of Δ_{16}^{+} in $\mathrm{Spin}(9,1)$, and any η_+ of the form (4.5) can be brought to the form

$$\eta_+ = f\,(\mathbb{1} + e_{1234}) \tag{4.6}$$

for some f by a $\mathrm{Spin}(9,1)$ transformation. To see this, we first use Spin(7) transformations to bring θ_1 and θ_2 to a simpler form, after which finding an explicit $\mathrm{Spin}(9,1)$ transformation leading us to $\eta_+ = f\,(\mathbb{1} + e_{1234})$ is relatively easy.

In type IIB supergravity, we have two Majorana–Weyl spinors of the same chirality, but in type IIA supergravity, we have two Majorana–Weyl spinors of the opposite chirality, which we gather into a single Majorana spinor with 32 components. The next steps therefore differ a bit from [7], but the underlying idea is the same.

We go between spinors of odd and even chirality by introducing or removing e_5, as the case may be. This way the spinor η_+ in (4.6) maps to a spinor of the form $\eta_- = a\,(e_5 + e_{12345})$, which has the same stability subgroup as η_+: $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$. Under Spin(7), the Majorana–Weyl spinors Δ_{16}^- split into $\Gamma^+\Lambda^1\left(\mathbb{R}^7\right)$ and $\Gamma^-\Delta_8$, just like Δ_{16}^+ splits into $\Lambda^1\left(\mathbb{R}^7\right)$ and Δ_8 above, only with e_5 added and removed, respectively. The most general Majorana–Weyl spinor of negative chirality is therefore

$$\eta_- = a\,(e_5 + e_{12345}) + \theta_1 + \theta_2 \quad \text{where } \theta_1 \in \Gamma^+\Lambda^1\left(\mathbb{R}^7\right) \text{ and } \theta_2 \in \Gamma^-\Delta_8. \tag{4.7}$$

As above, we may use Spin(7) transformations to choose a simple form of θ_1 and θ_2.

In type IIA supergravity, we deal with a Majorana spinor containing both η_+ and η_-: $\varepsilon = \eta_+ + \eta_-$. Any transformation on ε will act on both η_+ and η_-. Having invoked a $\mathrm{Spin}(9,1)$ transformation to simplify η_+, we can't invoke that trick again for η_- without potentially undoing the simple form of η_+. For this reason, we must confine ourselves to the stability subgroup of η_+ when simplifying η_-. This results in the cases (4.1), (4.2) and (4.3) above.

4.2 A closer look at SU(n)

SU(n) is the group of unitary $n \times n$ matrices with unit determinant. There are $n^2 - 1$ infinitesimal generators $\tilde{M}_{\alpha\bar{\beta}}$, where the tilde is used to denote the traceless part ($\tilde{M}_\alpha{}^\alpha = 0$).

SU(4) has a special role to play, not only when the spinor has isotropy group SU(4), but also in the Spin(7) case, as SU(4) is a proper subgroup of Spin(7). The Γ matrix representation (3.8) will naturally give us equations in terms of irreducible SU(4) representations, which we will then have to manually assemble into Spin(7) representations. SU(3), being a proper subgroup of G_2, plays the corresponding role when studying Killing spinors invariant under G_2.

The invariant forms of SU(n) are the Kähler form and the holomorphic volume form.

We will take the Kähler form as

$$\omega = -\left(e^1 \wedge e^6 + e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9\right) \tag{4.8}$$

in the case of SU(4), and

$$\hat{\omega} = -\left(e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9\right) \tag{4.9}$$

in the case of SU(3). The holomorphic volume form is

$$\chi = \left(e^1 + \mathrm{i}\, e^6\right) \wedge \left(e^2 + \mathrm{i}\, e^7\right) \wedge \left(e^3 + \mathrm{i}\, e^8\right) \wedge \left(e^4 + \mathrm{i}\, e^9\right) \tag{4.10}$$

in the case of SU(4), and

$$\hat{\chi} = \left(e^2 + \mathrm{i}\, e^7\right) \wedge \left(e^3 + \mathrm{i}\, e^8\right) \wedge \left(e^4 + \mathrm{i}\, e^9\right) \tag{4.11}$$

in the case of SU(3). In the Hermitian basis (3.7) we have

$$\omega_{\alpha\bar{\beta}} = -\mathrm{i}\,\delta_{\alpha\bar{\beta}}, \quad \chi_{\alpha_1\alpha_2\alpha_3\alpha_4} = 4\,\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}, \tag{4.12}$$

and similarly for the SU(3) forms:[2]

$$\hat{\omega}_{\alpha\bar{\beta}} = -\mathrm{i}\,\delta_{\alpha\bar{\beta}}, \quad \hat{\chi}_{\alpha_1\alpha_2\alpha_3} = 2\sqrt{2}\,\epsilon_{\alpha_1\alpha_2\alpha_3}. \tag{4.13}$$

4.3 A closer look at Spin(7)

The Spin(n) group is known as the double cover of the SO(n) group, and may be defined in terms of the Clifford algebra $\mathcal{C}\ell\,(n)$ as

$$\mathrm{Spin}(n) \coloneqq \left\{s \in \mathcal{C}\ell^+\,(n)\,, s \diamond s^t = 1, \forall\left(\vec{x} \in \mathbb{R}^n, s \diamond \vec{x} \diamond s^{-1} \in \mathbb{R}^n\right)\right\}$$

Thus Spin(7) seems closely linked to seven-dimensional space — and yet Spin(7) often pops up in the study of eight-dimensional manifolds. Indeed, the study of (4.1) yields an eight-dimensional submanifold with Spin(7) structure, and two orthogonal directions (one space, one time).

This may seem surprising at first. How can Spin(7) be embedded into an eight-dimensional setting? The naive answer would be to see Spin(7) as a subgroup of Spin(8), obtained by simply taking the generators of $\mathcal{C}\ell\,(7)$ from a seven-dimensional subspace of $\mathbb{R}^8$. We might simply think of Spin(7) as a subgroup of Spin(8) that leaves a certain vector, say $\vec{e}_8$, invariant.

[2] Note that the sign of ω differs from the convention used in [13].

But alas, here we talk of a Spin(7) which doesn't leave *any* vector in $\mathbb{R}^8$ invariant. Instead, it leaves a four-form ϕ invariant. In the naive construction, that seems impossible. This is not your naive Spin(7).

An element s of the Spin(n) group acts on a vector $\vec{x} \in \mathbb{R}^n$ by $s \diamond \vec{x} \diamond s^{-1}$, producing some rotation. However, our Spin(7) acts on a vector $\vec{x} \in \mathbb{R}^8$ as $\vec{x} \mapsto s \diamond \vec{x}$: both the action and the vector space are different from the usual Spin(n) case. Our Spin(7) acts only on the left, treating vectors as if they were spinors. Our Spin(7) is a subgroup of O(8). It is a subgroup of O(8) which leaves a four-form ϕ invariant; or in other words, it leaves the ternary cross product in eight dimensions invariant. The invariant tensor of Spin(7) may be defined as

$$\phi \coloneqq \mathrm{Re}\,(\chi) - \tfrac{1}{2}\,\omega \wedge \omega \tag{4.14}$$

where ω is the Kähler form (4.8) and χ the holomorphic volume form (4.10), both of which are invariant under the SU(4) subgroup of Spin(7). ϕ is called the Cayley form and was defined as (4.14) in [14] by identifying $\mathbb{R}^8$ with the octonions (which sometimes are called the *Cayley numbers*).

To be precise, the generators of our Spin(7) are the fifteen SU(4) generators $\tilde{M}_{\alpha\bar{\beta}}$ from section 4.2, and the six generators of the form

$$M^{+}{}_{\alpha\beta} \coloneqq M_{\alpha\beta} + \tfrac{1}{2}\,\epsilon_{\alpha\beta}\,M_{\bar{\gamma}\bar{\delta}}\,.$$

M_{AB} is represented by S_{AB} when acting on spinors, and by J_{AB} when acting on vectors and tensors, where

$$S^{AB} = \frac{\mathrm{i}}{4}\,\left[\Gamma^A, \Gamma^B\right], \quad \left(J^{AB}\right)_{CD} = \mathrm{i}\,\left(\delta^A{}_C\,\delta^B{}_D - \delta^A{}_D\,\delta^B{}_C\right).$$

Here, we are using the Hermitian basis defined in (3.7) above, with holomorphic indices taking values in $\{1, 2, 3, 4\}$. This basis will be especially useful when we solve the Killing spinor equations in terms of irreducible SU(4) representations in section 6.1. When working with the Spin(7) invariant spinor (4.1) we will need to make the connection to the invariant four-form ϕ, using (4.14) and (4.12). We will also want to know the contractions of ϕ and the covariant derivative on ϕ (which is constant in the local Lorentz frame):

$$\phi_{il_1l_2l_3}\,\phi^{jl_1l_2l_3} = 42\,\delta^j{}_i, \tag{4.15}$$

$$\phi_{i_1i_2l_1l_2}\,\phi^{j_1j_2l_1l_2} = -4\,\varphi_{i_1i_2}{}^{j_1j_2} + 12\,\delta^{j_1j_2}_{i_1i_2}, \tag{4.16}$$

$$\phi_{i_1i_2i_3l}\,\phi^{j_1j_2j_3l} = -9\,\delta^{j_1}{}_{[i_1}\,\phi_{i_2i_3]}{}^{j_2j_3} + 6\,\delta^{j_1j_2j_3}_{i_1i_2i_3}, \tag{4.17}$$

$$\nabla_A\phi_{B_1B_2B_3B_4} = 4\,\Omega_{A,[B_1}{}^{C}\,\phi_{|C|B_2B_3B_4]}, \tag{4.18}$$

where i, j, l denote eight-dimensional indices, and A, B, C denote ten-dimensional indices, taking values 0 and 5 (or + and −) in addition to the eight of i, j, l.

What we are most intrested in are all the possible contractions of all the possible derivatives on ϕ, since that gives us expressions in terms of the spin connection Ω, and relates them to covariant things. When solving the Killing spinor equations we get everything in terms of the spin connection, but it looks a bit nicer to express the result in terms of covariant quantities. A list of these contractions involving derivatives on ϕ may be found in appendix B.1 of Paper I.

When expressing the Killing spinor equations in terms of Spin(7) representations, we will need to know how two-forms, three-forms and four-forms split into Spin(7) representations (the one-forms, as noted above, transform in the spinor representation of Spin(7).) We have

$$\Lambda^2\left(\mathbb{R}^8\right) = \Lambda^2_{\mathbf{7}} \oplus \Lambda^2_{\mathbf{21}}, \tag{4.19}$$

$$\Lambda^3\left(\mathbb{R}^8\right) = \Lambda^3_{\mathbf{8}} \oplus \Lambda^3_{\mathbf{48}}, \tag{4.20}$$

$$\Lambda^4\left(\mathbb{R}^8\right) = \Lambda^4_{\mathbf{1}} \oplus \Lambda^4_{\mathbf{7}} \oplus \Lambda^4_{\mathbf{27}} \oplus \Lambda^4_{\mathbf{35}}. \tag{4.21}$$

What is $\Lambda^2_{\mathbf{7}}$? We have seen that Spin(7) can act on $\mathbb{R}^8$ with the spinor representation, but it can of course also act on $\mathbb{R}^7$ with the vector representation — it is after all a double cover of SO(7). As it happens, $\mathbb{R}^7$ can be mapped into $\Lambda^2\left(\mathbb{R}^8\right)$ using the Spin(7) gamma matrices. A vector is a spinor squared. The elements of $\Lambda^2_{\mathbf{7}}$ are the seven-dimensional vectors of Spin(7).

$\Lambda^2_{\mathbf{21}}$ is the adjoint representation of Spin(7). There are 21 generators of Spin(7), which were given above. They can be identified with $\Lambda^2_{\mathbf{21}}$ and are acted upon by the adjoint action of Spin(7).

A one-form α in $\Lambda^1\left(\mathbb{R}^8\right)$ can be mapped to a three-form $\star\left(\alpha \wedge \phi\right)$. All such three-forms form $\Lambda^3_{\mathbf{8}}$, and the rest form $\Lambda^3_{\mathbf{48}}$.

$\Lambda^4_{\mathbf{1}}$ are all the four-forms proportional to the Spin(7) fundamental form ϕ. $\Lambda^4_{\mathbf{7}}$ is again the vector representation of Spin(7). You may go from $\Lambda^2_{\mathbf{7}}$ to $\Lambda^4_{\mathbf{7}}$ by simply contracting one index with the four-form ϕ. As for $\Lambda^4_{\mathbf{27}}$, we note that $\mathbf{27} = \frac{\mathbf{7}\times(\mathbf{7}+1)}{2} - 1$: these are the symmetric traceless bi-vectors. Similarly for $\Lambda^4_{\mathbf{35}}$, we have $\mathbf{35} = \frac{\mathbf{8}\times(\mathbf{8}+1)}{2} - 1$, indicative of symmetric traceless bi-spinors. This is also the part that is composed of anti-self dual four-forms.

4.4 A closer look at G_2

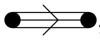
This section is partly intended as a refresher for people at least somewhat familiar with Lie algebras.

First, why the name?

There is a classification of all simple Lie algebras. There are some conditions that the Cartan matrix of a simple Lie algebra must fulfil, there is some combinatorics involved, but we are really just interested in the result. There are some infinite series of algebras, called A_r, B_r, C_r and D_r, and there are five isolated cases called E_r (for $\mathrm{r} \in \{6, 7, 8\}$), F_4 and G_2. Thus the G in G_2 is not an abbreviation — it's the seventh letter in an alphabetic enumeration. G_2 has more to do with seven than it does with words starting with G. For instance, the cross product in seven dimensions is intimately associated with the G_2 group.

The cross product of two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$ is a vector $\vec{c} = \vec{a} \times \vec{b}$ that is (1) orthogonal to both $\vec{a}$ and $\vec{b}$ and (2) has its length given by the area of a parallelogram spanned by $\vec{a}$ and $\vec{b}$. It turns out this is only possible for vectors in $\mathbb{R}^3$ and (perhaps somewhat surprisingly) $\mathbb{R}^7$. In $\mathbb{R}^3$, the cross product may be seen as an antisymmetric product of purely imaginary quaternions, while in $\mathbb{R}^7$, the cross product may be seen as an antisymmetric product of purely imaginary octonions. To get the cross product, we contract the two vectors with a three-form. In $\mathbb{R}^3$ the volume form will do the trick, yielding a cross product that is invariant under SO(3). In $\mathbb{R}^7$ we need another three-form, which we may call φ, and the cross product will not be invariant under SO(7), but rather under $\mathrm{G}_2 \subset \mathrm{SO}(7)$.

The number 2 in G_2 refers to the *rank* of the Lie algebra. The rank is the dimension of the Cartan subalgebra, the number of dots in the Dynkin diagram, as well as the number of simple positive roots in the root diagram.

The Dynkin diagram of G_2 is , encoding the Cartan matrix

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

The Cartan matrix gives a metric among the roots, allowing us to draw the root diagram in figure 4.1.

Each root corresponds to a generator. We have the six long roots of $\mathfrak{su}(3) \subset G_2$, six short roots, and the two generators of the Cartan subalgebra: fourteen generators in total. The generators have been written next to the roots in the root diagram in figure 4.1, with $\Gamma^{\bar{3}3} - \Gamma^{\bar{2}2}$ and $\Gamma^{\bar{4}4} - \Gamma^{\bar{2}2}$ spanning the Cartan subalgebra. The six gen-

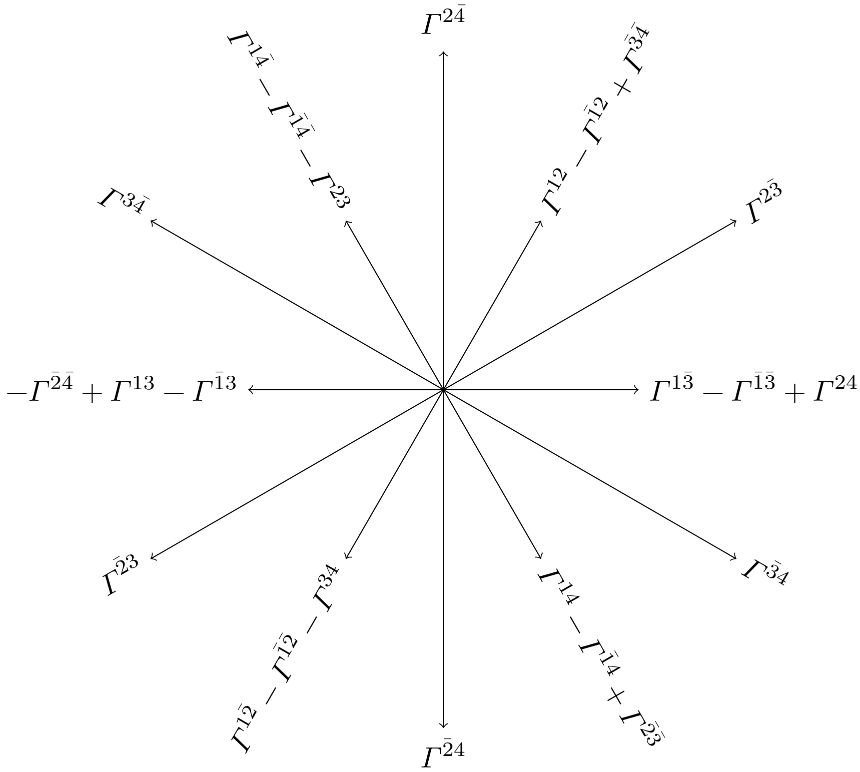
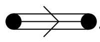


Figure 4.1: Root diagram of G_2. The Cartan subalgebra spanned by $\Gamma^{\bar{3}3} - \Gamma^{\bar{2}2}$ and $\Gamma^{\bar{4}4} - \Gamma^{\bar{2}2}$ is located in the centre. The diagram was obtained by considering what generators annihilate the spinor (4.3), which gives us the generators above in addition to the $\mathbb{R}^8$ generators Γ^{I-} for transverse spacelike I. From these generators we obtain the Killing form, which allows us to calculate the length and relative angles of the corresponding roots. We note that the root diagram thus obtained is precisely the same as the root diagram of G_2 obtained from the Cartan matrix encoded in the Dynkin diagram .

erators with long roots and the Cartan subalgebra together form the $\mathfrak{su}(3)$ subalgebra, whose eight generators are simply the traceless $\Gamma^{\alpha\bar{\beta}}$ where the holomorphic index α (and the antiholomorphic index $\bar{\beta}$) takes values in $\{2,3,4\}$ (and $\{\bar{2},\bar{3},\bar{4}\}$, respectively). This $\mathfrak{su}(3) \subset G_2$ will play a special role in our calculation.

We note that the holomorphic index 1 and the antiholomorphic index $\bar{1}$ (corresponding to real indices 1 and 6 according to (3.7)) only appear in the combination associated with 6:

$$\sqrt{2}\,\mathrm{i}\,\Gamma^{6\alpha} = \Gamma^{1\alpha} - \Gamma^{\bar{1}\alpha}, \quad \sqrt{2}\,\mathrm{i}\,\Gamma^{6\bar{\beta}} = \Gamma^{1\bar{\beta}} - \Gamma^{\bar{1}\bar{\beta}}. \tag{4.22}$$

A G_2 index i can therefore be taken to go over the three holomorphic and three antiholomorphic indices of $\mathfrak{su}(3) \subset G_2$ as well as the real index 6; or equivalently, a G_2 index goes over real dimensions 2 through 8, and is associated with a seven-dimensional subspace.

Using (4.22) we can list the fourteen G_2 generators in figure 4.1 as

$$\text{traceless } \Gamma^{\alpha\bar{\beta}}, \quad \sqrt{2}\,\mathrm{i}\,\Gamma^{6\alpha} + \epsilon^{\alpha}{}_{\bar{\gamma}_1\bar{\gamma}_2}\Gamma^{\bar{\gamma}_1\bar{\gamma}_2}, \quad \sqrt{2}\,\mathrm{i}\,\Gamma^{6\bar{\beta}} - \epsilon^{\bar{\beta}}{}_{\gamma_1\gamma_2}\Gamma^{\gamma_1\gamma_2}. \tag{4.23}$$

The invariant three-form of G_2 may be defined as

$$\varphi := \mathrm{Re}\,(\hat{\chi}) + e^6 \wedge \hat{\omega} \tag{4.24}$$

where $\hat{\chi}$ is the holomorphic volume form (4.11) and $\hat{\omega}$ is the Kähler form (4.9), both invariant under the SU(3) subgroup of G_2. The Hodge dual $\star\varphi$ is taken with respect to the volume form $e^2 \wedge e^3 \wedge e^4 \wedge e^6 \wedge e^7 \wedge e^8 \wedge e^9$, and is a G_2 invariant four-form.

When expressing the Killing spinor equations in terms of G_2 representations, we will need to know how two-forms, three-forms and four-forms split into G_2 representations. In the G_2 case, we have

$$\Lambda^2\left(\mathbb{R}^7\right) = \Lambda^2_{\mathbf{7}} \oplus \Lambda^2_{\mathbf{14}}, \tag{4.25}$$

$$\Lambda^3\left(\mathbb{R}^7\right) = \Lambda^3_{\mathbf{1}} \oplus \Lambda^3_{\mathbf{7}} \oplus \Lambda^3_{\mathbf{27}}, \tag{4.26}$$

$$\Lambda^4\left(\mathbb{R}^7\right) = \Lambda^4_{\mathbf{1}} \oplus \Lambda^4_{\mathbf{7}} \oplus \Lambda^4_{\mathbf{27}}. \tag{4.27}$$

Here $\Lambda^2_{\mathbf{7}}$ can be formed by taking the one-forms $\Lambda^1\left(\mathbb{R}^7\right)$ and contracting with the fundamental G_2 three-form φ. $\Lambda^2_{\mathbf{14}}$ is the adjoint representation of G_2; G_2 has 14 generators.

$\Lambda^3_{\mathbf{1}}$ are the three-forms that are proportional to φ. $\Lambda^3_{\mathbf{7}}$ can be formed by taking the one-forms $\Lambda^1\left(\mathbb{R}^7\right)$ and contracting with $\star\varphi$, the Hodge dual of the fundamental three-form φ. We have $\mathbf{27} = \frac{\mathbf{7}\times(\mathbf{7}+1)}{2} - 1$, so $\Lambda^3_{\mathbf{27}}$ are the traceless symmetric bi-vectors.

The four-forms split in the same way as the three-forms, and you can go between the three-forms and the four-forms using Hodge dualisation.

Chapter 5

Supergravity

5.1 Supersymmetric solutions and the Killing spinor equations

The fields of type IIA supergravity are (bosonic) the graviton g_{MN}, the NSNS 2-form potential B_{MN}, the RR 1-form potential C_M, the RR 3-form potential C_{MNP}; and (fermionic) one Majorana non-chiral gravitino ψ_M, and one Majorana non-chiral dilatino λ. We use H for the NSNS 3-form field strength, and $\tilde{S}$, $\tilde{F}$ and $\tilde{G}$ for the RR k-form field strength. The latter all tend to come with the dilaton as $\mathrm{e}^{\Phi}\,\tilde{S}$, $\mathrm{e}^{\Phi}\,\tilde{F}$ and $\mathrm{e}^{\Phi}\,\tilde{G}$, so we will absorb a factor of e^{Φ} into them and drop the tilde.

There are other supergravity theories too, beside type IIA. They have different field contents, and may require a different number of spacetime dimensions. The fermionic fields include at least the gravitino, in addition to the fields specific to the supergravity theory in question.

The focus is on bosonic solutions, where the fermionic fields of the theory are put to zero, in order to obtain classical solutions.[1]

A solution is called *supersymmetric* if the supersymmetry variations of all the fields vanish. Supersymmetry transformations relate bosons to fermions and fermions to bosons. The supersymmetry variation of a boson will be given by the fermionic fields and the supersymmetry parameter (which is a spinor), possibly multiplied by some Γ_M and numerical factors. For example

$$\delta_\varepsilon \Phi = \frac{1}{2}\,\bar{\varepsilon}\lambda,$$
$$\delta_\varepsilon e^A_M = \bar{\varepsilon}\Gamma^A\psi_M.$$

[1] Purely gravitational supergravity solutions, with *all* of the fields put to zero, have also been studied; e.g. [12]. Then the Killing spinor equation from the supersymmetry variation of the dilatino λ, $\mathcal{A}\varepsilon = 0$, becomes trivial, and the Killing spinor equation from the supersymmetry variation of the gravitino ψ_M, $\mathcal{D}_M\varepsilon = 0$ reduces to the condition that ε be a so called *parallel spinor*, satisfying $\nabla_M\varepsilon = 0$. Then you can apply Berger's list of holonomy groups [10] to classify the resulting geometries (see [15] and [9]).

These, and the other bosonic supersymmetry variations, can be found e.g. in [16]. The precise expressions do not concern us, only that all of them are build from the fermionic fields: the gravitino ψ_M and the dilatino λ. Since classical solutions have $\psi_M = 0$ and $\lambda = 0$, we know that the supersymmetry variations of the bosons vanish automatically. When looking for classical supergravity solutions, we will not get any constraints from the variation of the bosons.

The supersymmetry variation of the fermionic fields involve the bosonic fields:

$$\begin{aligned}\delta_\varepsilon \psi_M &= \nabla_M \varepsilon + \frac{1}{8} H_{MP_1P_2}\, \Gamma^{P_1P_2}\Gamma_{11}\,\varepsilon + \frac{1}{8} S\,\Gamma_M\,\varepsilon + \\ &\quad + \frac{1}{16} F_{P_1P_2}\, \Gamma^{P_1P_2}\Gamma_M\,\Gamma_{11}\,\varepsilon + \frac{1}{8\times 4!} G_{P_1P_2P_3P_4}\, \Gamma^{P_1P_2P_3P_4}\Gamma_M\,\varepsilon, \end{aligned} \tag{5.1}$$

$$\begin{aligned}\delta_\varepsilon \lambda &= \partial_P \Phi\, \Gamma^P \varepsilon + \frac{1}{12} H_{P_1P_2P_3}\, \Gamma^{P_1P_2P_3}\Gamma_{11}\,\varepsilon + \frac{5}{4} S\,\varepsilon + \\ &\quad + \frac{3}{8} F_{P_1P_2}\, \Gamma^{P_1P_2}\Gamma_{11}\,\varepsilon + \frac{1}{4\times 4!} G_{P_1P_2P_3P_4}\, \Gamma^{P_1P_2P_3P_4}\varepsilon. \end{aligned} \tag{5.2}$$

Unlike the bosonic case, these variations will not automatically vanish. These expressions involve the bosonic fields, which are otherwise unconstrained. We need to set the variations to zero, and the resulting equations need to be solved.

The supersymmetry variation of the gravitino, $\delta_\varepsilon \psi_M$, takes the form of a differential operator acting on the supersymmetry parameter ε: $\delta_\varepsilon \psi_M = \mathcal{D}_M \varepsilon$. This much is expected in any supergravity theory, though the precise expression for $\mathcal{D}_M$ will vary. There may or may not be other fermionic fields to consider; in type IIA we have the dilatino λ, whose supersymmetry variation is an algebraic condition on ε: $\delta_\varepsilon \lambda = \mathcal{A}\varepsilon$. The conditions $\mathcal{D}_M \varepsilon = 0$, $\mathcal{A}\varepsilon = 0$ are called the *Killing spinor equations*, and a spinor ε satisfying them is called a *Killing spinor*. The Killing spinor equations imply some conditions on the geometry and the bosonic fluxes. The integrability conditions $[\mathcal{D}_M, \mathcal{D}_N]\,\varepsilon = 0$ and $[\mathcal{D}_M, \mathcal{A}]\,\varepsilon = 0$ may imply some field equations or relate them to Bianchi identities. (See section 5.3.) However, to fully specify a solution you would need to solve all the equations of motion for the fluxes; leaving that out, we obtain classes of solutions.

5.2 Killing spinors and Killing vectors

The concept of a Killing vector will be familiar to anyone who has studied differential geometry or general relativity. A Killing vector is a coordinate independent way of describing a bosonic symmetry. There is a certain maximal amount of symmetry that the geometry can have in a given number of dimensions, and the number of (linearly independent) Killing vectors tells you what amount of symmetry you have.

Fewer will be familiar with the concept of a Killing spinor. Similarly to the bosonic case, there is a maximum amount of possible supersymmetry, and the number of Killing spinors tells you how much of that supersymmetry is realised for a given solution. One Killing spinor means *minimal supersymmetry*; maximal supersymmetry depends on what the underlying supergravity theory is — in type IIA maximal supersymmetry means having 32 Killing spinors. We shall assume the existence of one Killing spinor, but there may well be more. Minimal supersymmetry is the most general case, in which solutions with more supersymmetry can be found as potentially interesting special cases. All the requirements we find here will apply to any and all bosonic supergravity solutions, all the way up to maximal supersymmetry.

Once we have a spinor, we can construct spacetime form bilinears. A one-form corresponds to a vector, and so it turns out that we can get a Killing vector from a Killing spinor. A vector constructed from spinors this way is quadratic in the spinors. The Killing vector is in some sense the square of a Killing spinor.

To square a spinor, either the Dirac or Majorana inner product may be used. We shall use the Dirac inner product here: $D(\eta, \theta) = \langle \Gamma_0 \eta, \theta \rangle$, where $\langle \cdot , \cdot \rangle$ is the natural inner product on a complex vector space, antilinear in its first argument. The Hermitian conjugate of the Γ matrices is, in the real basis, ${\Gamma_i}^\dagger = \Gamma_i$ for $i \in \{1, ..., 9\}$ and ${\Gamma_0}^\dagger = -\Gamma_0$, which implies ${\Gamma_A}^\dagger \Gamma_0 = -\Gamma_0 \Gamma_A$ for $A \in \{0, 1, ..., 9\}$. For a general operator $\not{\phi}$ in the real Clifford algebra, we get $\not{\phi}^\dagger \Gamma_0 = \Gamma_0 \overline{\not{\phi}}$, where $\overline{\not{\phi}}$ is the Clifford conjugate of $\not{\phi}$ (which means the reversal of the grade involution). For a scalar S, we have $\overline{S} = S$; for a four-form G, we have $\overline{\not{G}} = \not{G}$; for a two-form F, we have $\overline{\not{F}\Gamma_{11}} = \not{F}\Gamma_{11}$.[2] Expressed in terms of Clifford conjugation, we have

$$D(\not{\phi}\eta, \theta) = D\left(\eta, \overline{\not{\phi}}\theta\right). \tag{5.3}$$

The inner product $D(\varepsilon, \varepsilon)$ gives a scalar. To get a vector (or higher-degree forms), you insert a Γ matrix (or more):

$$\kappa_M = D(\varepsilon, \Gamma_M \varepsilon). \tag{5.4}$$

To show that κ is a Killing vector if ε is a Killing spinor, we need $\nabla_{(M}\kappa_{N)} = 0$. First, note that

$$\nabla_M \kappa_N = D(\nabla_M \varepsilon, \Gamma_N \varepsilon) + D(\varepsilon, \Gamma_N \nabla_M \varepsilon). \tag{5.5}$$

If ε is Killing, $\mathcal{D}_M \varepsilon = 0$, which according to (5.1) means that $\nabla_M \varepsilon$ is of the form

[2] The slash is Feynman slash notation, and means that all spacetime indices have been contracted with Γ matrices.

$\not{H}_M \Gamma_{11} \varepsilon + \not{\phi} \Gamma_M \varepsilon$ for an operator $\not{\phi}$ satisfying $\overline{\not{\phi}} = \not{\phi}$. We get four types of terms:

$$\begin{aligned} \nabla_M \kappa_N = \quad & D\left(\not{H}_M \Gamma_{11} \varepsilon, \Gamma_N \varepsilon\right) + D\left(\varepsilon, \Gamma_N \not{H}_M \Gamma_{11} \varepsilon\right) + \\ & + D\left(\not{\phi} \Gamma_M \varepsilon, \Gamma_N \varepsilon\right) + D\left(\varepsilon, \Gamma_N \not{\phi} \Gamma_M \varepsilon\right). \end{aligned} \tag{5.6}$$

The last two terms vanish upon symmetrization, because

$$D\left(\not{\phi} \Gamma_M, \Gamma_N \varepsilon\right) = -D\left(\varepsilon, \Gamma_M \overline{\not{\phi}} \Gamma_N \varepsilon\right) \tag{5.7}$$

and $\overline{\not{\phi}} = \not{\phi}$. This yields

$$\nabla_{(M} \kappa_{N)} = D\left(\varepsilon, \left(-\not{H}_M \Gamma_N \Gamma_{11} + \Gamma_N \not{H}_M \Gamma_{11}\right) \varepsilon\right). \tag{5.8}$$

$\not{H}_M$ is, up to numeric factors, $H_M{}^{P_1 P_2} \Gamma_{P_1 P_2}$, which when multiplied by Γ_N yields terms containing $H_{MNP} \Gamma^P$, vanishing upon symmetrization, and a $H_M{}^{P_1 P_2} \Gamma_{P_1 P_2 N}$ term, which is the same whether Γ_N comes from the left or the right. This shows $\nabla_{(M} \kappa_{N)} = 0$, and thus a Killing spinor gives a Killing vector.

Going beyond minimal supersymmetry gives us several Killing spinors, and you can form vector bilinears from all possible combinations. Though we have not shown it here, all vector fields thus obtained are Killing vectors.

5.3 Integrability conditions

In eleven-dimensional supergravity, which is closely related to type IIA supergravity, the equation of motion for the gravitino is [17]

$$\Gamma^{MNP} \mathcal{D}_N \psi_P = 0.$$

Taking the supersymmetry variation of that, we obtain

$$\delta_\varepsilon \left(\Gamma^{MNP} \mathcal{D}_N \psi_P\right) = \Gamma^{MNP} \mathcal{D}_N \left(\delta_\varepsilon \psi_P\right)$$

because while $\mathcal{D}$ contains some fluxes, they are bosonic fields, which means their supersymmetry variation is expressed in terms of the fermionic fields, which we put to zero. Now, $\delta_\varepsilon \psi_M$ happens to be precisely $\mathcal{D}_M \varepsilon$, which lands us in

$$\Gamma^{MNP} \mathcal{D}_N \mathcal{D}_P \varepsilon = 0.$$

In Paper I, we use

$$\mathcal{I}_M = \Gamma^N \left[\mathcal{D}_M, \mathcal{D}_N\right] \varepsilon$$

instead, which amounts to the same thing, since you can convert between one gamma and three gammas using the Clifford algebra:

$$\left(\mathit{\Gamma}^{MN} - g^{MN}\right)\mathcal{I}_N = \mathit{\Gamma}^{MNP}\mathcal{D}_N\mathcal{D}_P\varepsilon.$$

Type IIA supergravity (without the Romans mass) can be obtained by dimensional reduction from eleven dimensions, so the previous discussion carries through essentially unchanged. However, we do get one more thing to think about: the dilatino λ and the associated Killing spinor equation $\mathcal{A}\varepsilon = 0$. In IIA, we therefore construct both $\mathcal{I}\varepsilon = \mathit{\Gamma}^M\left[\mathcal{D}_M, \mathcal{A}\right]\varepsilon$ and $\mathcal{I}_M\varepsilon = \mathit{\Gamma}^N\left[\mathcal{D}_M, \mathcal{D}_N\right]\varepsilon$. Naturally, for a Killing spinor ε, which satisfies $\mathcal{A}\varepsilon = 0$ and $\mathcal{D}_M\varepsilon = 0$, we must have $\mathcal{I}\varepsilon = 0$ and $\mathcal{I}_M\varepsilon = 0$.

The supersymmetry variation of an equation of motion can be expressed in terms of the equations of motion of the theory. In other words, we expect it to be possible to express both $\mathcal{I}\varepsilon = 0$ and $\mathcal{I}_M\varepsilon = 0$ in terms of the field equations and Bianchi identities of the theory, and in Paper I we show that this is indeed the case. (That is also how it works out in eleven-dimensional supergravity [1], [18] and in type IIB supergravity [19].) In Paper I we obtain

$$\begin{aligned}
\mathcal{I}\varepsilon = {} & \left(\mathrm{F}\mathit{\Phi} - \mathrm{F}G_{(3)}\,\mathit{\Gamma}^{(3)} + \mathrm{B}G_{(5)}\,\mathit{\Gamma}^{(5)}\right)\varepsilon + \\
& + \left(-3\,\mathrm{F}F_{(1)}\,\mathit{\Gamma}^{(1)} + \mathrm{F}H_{(2)}\,\mathit{\Gamma}^{(2)} + \mathrm{B}F_{(3)}\,\mathit{\Gamma}^{(3)} + 2\,\mathrm{B}H_{(4)}\,\mathit{\Gamma}^{(4)}\right)\mathit{\Gamma}_{11}\,\varepsilon, \\
\mathcal{I}_M\varepsilon = {} & \left(-\frac{1}{2}\,E_{M(1)}\,\mathit{\Gamma}^{(1)} - \frac{1}{4}\,E_P{}^P\,\mathit{\Gamma}_M + \frac{1}{2}\,\mathrm{F}\mathit{\Phi}\,\mathit{\Gamma}_M + \mathrm{F}G_{(3)}\,\mathit{\Gamma}^{(3)}{}_M - 5\,\mathrm{B}G_{M(4)}\,\mathit{\Gamma}^{(4)}\right)\varepsilon + \\
& + \Bigg(\mathrm{F}H_{M(1)}\,\mathit{\Gamma}^{(1)} + \mathrm{F}F_{(1)}\,\mathit{\Gamma}^{(1)}{}_M - \mathrm{B}F_{M(2)}\,\mathit{\Gamma}^{(2)} + \\
& \qquad + \frac{1}{3}\,\mathrm{B}H_{M(3)}\,\mathit{\Gamma}^{(3)} + \mathrm{B}H_{(4)}\,\mathit{\Gamma}^{(4)}{}_M\Bigg)\mathit{\Gamma}_{11}\,\varepsilon,
\end{aligned}$$

where F stands for field equation and B stands for Bianchi identity:

$$\begin{aligned}
E_{MN} = {} & R_{MN} - \frac{1}{12}\,G_{M(3)}\,G_N{}^{(3)} + \frac{1}{96}\,g_{MN}\,G_{(4)}\,G^{(4)} + \frac{1}{4}\,g_{MN}\,S^2 - \\
& - \frac{1}{4}\,H_{M(2)}\,H_N{}^{(2)} - \frac{1}{2}\,F_{MP}\,F_N{}^P + \frac{1}{8}\,g_{MN}\,F_{(2)}\,F^{(2)} + 2\,\nabla_M\partial_N\mathit{\Phi}, \\
\mathrm{F}\mathit{\Phi} = {} & \Box\mathit{\Phi} - 2\,(\partial\mathit{\Phi})^2 - \frac{3}{8}\,F_{(2)}\,F^{(2)} - \frac{1}{96}\,G_{(4)}\,G^{(4)} + \\
& + \frac{1}{12}\,H_{(3)}\,H^{(3)} - \frac{5}{4}\,S^2, \\
\mathrm{F}H_{MN} = {} & \frac{1}{4}\Bigg(\nabla^P H_{MNP} - 2\left(\partial^P\mathit{\Phi}\right)H_{MNP} - \frac{1}{2}\,G_{MN(2)}\,F^{(2)} - F_{MN}\,S + \\
& + \frac{1}{1152}\,\epsilon_{MN(4)(\tilde{4})}\,G^{(4)}\,G^{(\tilde{4})}\Bigg),
\end{aligned}$$

$$
\begin{aligned}
\mathrm{F}F_{M} &= \frac{1}{4}\left(\nabla^{P} F_{MP} - \left(\partial^{P}\varPhi\right) F_{MP} + \frac{1}{6} G_{M(3)} H^{(3)}\right), \\
\mathrm{F}G_{M_1M_2M_3} &= \frac{1}{4!}\Big(\nabla^{P} G_{M_1M_2M_3P} - \left(\partial^{P}\varPhi\right) G_{M_1M_2M_3P} - \\
&\qquad - \frac{1}{144}\,\epsilon_{M_1M_2M_3(3)(4)} H^{(3)} G^{(4)}\Big), \\
\mathrm{B}H_{M_1M_2M_3M_4} &= \frac{1}{4!}\,\nabla_{[M_1} H_{M_2M_3M_4]}, \\
\mathrm{B}F_{M_1M_2M_3} &= \frac{3}{8}\left(\nabla_{[M_1} F_{M_2M_3]} - \left(\partial_{[M_1}\varPhi\right) F_{M_2M_3]} - \frac{1}{3} H_{M_1M_2M_3}\, S\right), \\
\mathrm{B}G_{M_1M_2M_3M_4M_5} &= \frac{1}{4\times 4!}\Big(\nabla_{[M_1} G_{M_2M_3M_4M_5]} - \left(\partial_{[M_1}\varPhi\right) G_{M_2M_3M_4M_5]} - \\
&\qquad - 2\, F_{[M_1M_2}\, H_{M_3M_4M_5]}\Big).
\end{aligned}
$$

Note that though both $[\mathcal{D}_M, \mathcal{D}_N]\,\varepsilon$ and $[\mathcal{D}_M, \mathcal{A}]\,\varepsilon$ contain derivatives, the commutator ensures that no derivatives end up acting on ε after simplification: the derivatives act on the fluxes hidden in $\mathcal{D}_M$ and $\mathcal{A}$. The resulting differential expressions with the fluxes are sorted according to number of gamma matrices acting on ε or $\varGamma_{11}\,\varepsilon$ — it is useful to think of them as linearly independent. Unfortunately, and here's the rub, this simple procedure does *not* directly give us the field equations as they were given above. In order to arrive at the expressions given, you need to add zero, and there are a number of algebraic expressions evaluating to zero to choose from. The full computation involves a linear combination of all kinds of fluxes, with the appropriate gamma matrices, multiplying $\mathcal{A}\varepsilon$.

The Bianchi identities are first order equations, and therefore easier to solve than the field equations. If we can get a field equation expressed in terms of Bianchi identities, that tends to simplify things.

This far we get without saying anything about ε. Now, for a specific spinor ε some gamma matrices will annihilate ε, and then the corresponding coefficient (field equation or Bianchi identitiy) drops out of the expression, and that particular representation will be unconstrained by the integrability conditions of the Killing spinor equations. This is why we talk about classifying *geometries* rather than solutions — this approach may leave some field equations that still have to be solved in order to have the full solution. As for those representations that aren't annihilated, we get some field equations that are automatically satisfied, or that are given in terms of the Bianchi identities.

So while we fix many general aspects of the solution, there are still some unconstrained parameters to play with, which is natural, since among others, all solutions with enhanced supersymmetry would be special cases of the geometries we present here.

Chapter 6

Spinorial geometry and solving the KSEs

With the background you now have on spinors, the spinorial geometry approach should feel fairly natural: Choose a representation for the Clifford Γ matrices in terms of creation and annihilation opreators, inducing a natural basis in the space of spinors; see section 3.2. Take the most general spinor ε, expressed in terms of exterior algebra forms, imposing any Majorana or Weyl conditions as appropriate for the supergravity theory under study. Choose a gauge as in section 4.1, and solve and interpret the linear system of equations given by the Killing spinor equations. We also *construct* all the spinor bilinears, which correspond to differential forms on the spacetime, and we are able to give explicit expressions for them in terms of the invariant tensors associated with the isotropy group of the spinors.

Though in many ways straightforward and natural, significant work in classifying supersymmetric geometries has been done without using the spinorial geometry approach — for a review of this earlier work, see [20]. Instead of working with the spinors directly, focus is directed at the spinor bilinears. The complete set of spinor bilinears contain the same information as the spinors — though you made a linear problem quadratic in the process. The bilinears are not algebraically independent, however, and information about the symmetry group of the spinors is encoded in algebraic relations between bilinears. (Algebraic relations between the bilinears necessitate the use of Fierz rearrangements, which the more explicit spinorial geometry approach simply doesn't need.)

6.1 Linear system in $\mathrm{SU}(n)$ indices

The spinors can be written in terms of gamma matrices acting on the $\mathbb{1}$ form. In the case of the Spin(7) invariant spinor of (4.1), we have

$$\varepsilon = f\left(1 + \frac{1}{4}\,\Gamma^{\bar{1}\bar{2}\bar{3}\bar{4}}\right)\mathbb{1} + \frac{1}{\sqrt{2}}\,g\left(1 + \frac{1}{4}\,\Gamma^{\bar{1}\bar{2}\bar{3}\bar{4}}\right)\Gamma^{+}\mathbb{1},$$

or equivalently

$$\varepsilon = f\left(1+\frac{1}{4}\frac{1}{4!}\,\epsilon_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\,\Gamma^{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\right)\mathbb{1} + \frac{1}{\sqrt{2}}\,g\left(1+\frac{1}{4}\times\frac{1}{4!}\,\epsilon_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\,\Gamma^{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\right)\Gamma^{+}\mathbb{1}. \tag{6.1}$$

Note that the expression is manifestly SU(4) invariant: The SU(4) generators $\tilde{M}_{\alpha\bar{\beta}}$ all commute with Γ^+, f, g, and numeric constants, they annihilate the Clifford vaccuum $\mathbb{1}$, and $\epsilon_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\,\Gamma^{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}$ is just the SU(4) invariant anti-holomorphic volume form $\chi^\star$ in the Clifford Γ representation (up to numeric factors).

We will focus on the Spin(7) case for now; the $G_2 \ltimes \mathbb{R}^8$ proceeds similarly, though manifestly invariant under $SU(3) \subset G_2$ rather than SU(4). The SU(4) case is similar too, but it spares us the extra complication of reassembling the results in Spin(7) or G_2 representations.

For even if ε in (6.1) is invariant under the whole of Spin(7), we will obtain the linear system in terms of $SU(4) \subset Spin(7)$ representations.

When we act on the spinor ε with $\mathcal{A}$ and $\mathcal{D}_M$ we get terms with gamma matrices on the form $\Gamma^{(a)}\Gamma_{(b)}\Gamma^{(c)}$, where (a) denotes a ten dimensional indices; e.g. $\Gamma^{(a)} := \Gamma^{A_1A_2\ldots A_a}$ and $\Gamma^{(0)} \equiv 1$. A product of gamma matrices may be simplified using (3.5) — i.e. using the Clifford algebra — and the expression is brought to the form

$$\mathcal{D}_M\varepsilon = \sum_{a=0}^{5} X_{(a)}\,\Gamma^{(a)}\mathbb{1} \tag{6.2}$$

for some $X_{(a)}$. (Here $X_{(0)}$ would be a scalar, and again $\Gamma^{(0)} = 1$.) Since we are using what amounts to an oscillator basis for Γ, a familiar normal-ordering procedure with the anticommutator given by the Clifford algebra guarantees that $\Gamma^{(a)}$ can be written purely in terms of creation operators, i.e. as a product in the set of antiholomorphic $\Gamma^{\bar{\alpha}}$ and Γ^+. We have $\mathcal{D}_M\varepsilon = 0$ if and only if all $X_{(a)} = 0$ in said expression. A similar procedure works for the algebraic equation $\mathcal{A}\varepsilon = 0$.

We get a linear system in the fluxes, the spin connection $\Omega_{A,BC}$ and derivatives on the functions f, g appearing in the Killing spinor (4.1). When solving this system we organise it in terms of irreducible SU(4) representations. For example, table 6.1 is a subset of the equations we get from the Spin(7) invariant spinor (4.1) in the gauge where $f = g$.

Of course, with the Spin(7) invariant spinor (4.1), we don't really want SU(4) expressions. We want Spin(7) expressions, with eight-dimensional indices i, j, …, rather than the four holomorphic and four anti-holomorphic indices of SU(4). As table 6.1 shows, rewriting the SU(4) expressions is often very simple, or even trivial. All the

Table 6.1: Converting SU(4) expressions to Spin(7) expressions. (The complex conjugate of SU(4) expressions is implicitly implied.)

SU(4) expression		Spin(7) expression
$\partial_0 f = 0$	$\Leftrightarrow$	$\partial_0 f = 0$
$\partial_5 f = -\frac{1}{2} f\, \Omega_{0,05}$	$\Leftrightarrow$	$\partial_5 f = -\frac{1}{2} f\, \Omega_{0,05}$
$\partial_\alpha f = -\frac{1}{2} f\, \Omega_{0,0\alpha}$	$\Leftrightarrow$	$\partial_i f = -\frac{1}{2} f\, \Omega_{0,0i}$
$\Omega_{5,05} = 0$	$\Leftrightarrow$	$\Omega_{5,05} = 0$
$\Omega_{5,0\alpha} = -\Omega_{\alpha,05}$	$\Leftrightarrow$	$\Omega_{5,0i} = -\Omega_{i,05}$
$\Omega^{\gamma}{}_{,0\gamma} = -\Omega_{\gamma,0}{}^{\gamma}$, $\tilde{\Omega}_{\bar{\beta},0\alpha} = -\tilde{\Omega}_{\alpha,0\bar{\beta}}$, $\Omega_{(\alpha_1,\alpha_2)0} = 0$	$\Leftrightarrow$	$\Omega_{(i_1,i_2)0} = 0$

same, Spin(7) is a larger group than SU(4) and sometimes you need to piece together the Spin(7) representation using several of the SU(4) expressions, making the end result much more concise.

Then comes the part where you try to interpret the equations: What do they really say about the geometry and the fluxes? The spin connection is not a covariant quantity, so what the equations in table 6.1 say might not be entirely obvious. In this case, all the equations in table 6.1 are captured in

$$\nabla_A \kappa_B + \nabla_B \kappa_A = 0, \tag{6.3}$$

where $\kappa = f^2\, e^0$ is the spacetime one-form spinor bilinear mentioned in section 5.2. In other words, κ is a Killing one-form, and the associated vector field K is a Killing vector:

$$\mathcal{L}_K g = 0. \tag{6.4}$$

That we should get a Killing vector from our Killing spinor is entirely expected, and the equations in table 6.1 are the equations confirming that this is so.

Some of the equations we get from the linear system will be purely geometric constraints, like the equations in table 6.1 equivalent to (6.3). Others involve both the fluxes and the spin connection; then we choose the strategy to express the fluxes in terms of the geometry.

6.2 Rewriting in terms of Spin(7) expressions

Writing an SU(4) scalar or vector in terms of Spin(7) is straightforward, but the higher-degree forms require some more thinking. A two-form F decomposes as

$$\begin{aligned} F &= \frac{1}{2} F_{AB}\, e^A \wedge e^B, \\ &= F_{05}\, e^0 \wedge e^5 + F_{0i}\, e^0 \wedge e^i + F_{5i}\, e^5 \wedge e^i + \frac{1}{2} F_{ij}\, e^i \wedge e^j, \end{aligned} \tag{6.5}$$

where F_{ij} can be further decomposed into two distinct Spin(7) representations according to (4.19): $F_{ij} = F_{ij}^{(\mathbf{7})} + F_{ij}^{(\mathbf{21})}$, where the bold number denotes the number of degrees of freedom in the representation. It is possible for some of these parts to be determined by the geometry (i.e. you can solve for them in terms of $\Omega_{A,BC}$) while other parts can be unconstrained by the Killing spinor equations. Indeed, that is the case for F, where all parts except $F_{ij}^{(\mathbf{21})}$ are given in terms of the geometry and the Romans cosmological constant of the theory.[1] The exact expression may be found in Paper I appended to this thesis.

As in the example above, there is more than one irreducible SU(4) representation corresponding to $F_{ij}^{(\mathbf{7})}$, and more than one corresponding to $F_{ij}^{(\mathbf{21})}$. But even without knowing that F_{ij} decomposes as $F_{ij} = F_{ij}^{(\mathbf{7})} + F_{ij}^{(\mathbf{21})}$ under Spin(7) we are led to guess the right expressions from the corresponding SU(4) expressions. Consider for example the equation

$$F_{\alpha_1\alpha_2} - \frac{1}{2}\,\epsilon_{\alpha_1\alpha_2}{}^{\bar\beta_1\bar\beta_2}\, F_{\bar\beta_1\bar\beta_2} = -2\left(\Omega_{0,\alpha_1\alpha_2} - \frac{1}{2}\,\epsilon_{\alpha_1\alpha_2}{}^{\bar\beta_1\bar\beta_2}\,\Omega_{0,\bar\beta_1\bar\beta_2}\right). \tag{6.6}$$

The Levi-Civita tensor ϵ is not an invariant Spin(7) tensor, but rather an SU(4) object. The simple expedient of replacing ϵ by the Spin(7) invariant tensor ϕ results in an expression that reproduces (6.6) with an extra factor of two, as well as another SU(4) equation:

$$F_\gamma{}^\gamma = -2\,\Omega_{0,\gamma}{}^\gamma. \tag{6.7}$$

An SU(4) two-form has $\frac{\mathbf{4}\times(\mathbf{4}-1)}{2} = 6$ independent components, and a scalar only one. Thus, an equation that expands to (6.6) and (6.7) has **7** independent components.

[1] $F_{ij}^{(\mathbf{21})}$ does appear in the equations constraining the solution, as it is related to the four-form field strength G, but we shall treat it as the independent variable. That the **21** representation faces fewer constraints than other representations is entirely expected: it is the adjoint representation, given by the generators of Spin(7), and the spinor is Spin(7) invariant: $F_{ij}^{(\mathbf{21})}\,\Gamma^{ij}\varepsilon = 0$. This makes it drop out of the algebraic equation $\mathcal{A}\varepsilon = 0$, facing constraints only from the differential Killing spinor equation, where $F_{ij}^{(\mathbf{21})}\,\Gamma^{ij}$ does not act directly on the spinor.

Now, even though $F_{ij} - \frac{1}{2}\,\phi_{ijkl}\,F^{kl}$ *is* in the **7** representation of Spin(7), it doesn't mean that it is our $F^{(\mathbf{7})}_{ij}$ in the decomposition $F_{ij} = F^{(\mathbf{7})}_{ij} + F^{(\mathbf{21})}_{ij}$. We want

$$F^{(\mathbf{7})}_{ij} = \left(P^{(\mathbf{7})}\right)^{kl}_{i\,j}\,F_{kl} \tag{6.8}$$

for some projector $P^{(\mathbf{7})}$. Being a projector, we want $P^{(\mathbf{7})}$ to satisfy $\left(P^{(\mathbf{7})}\right)^2 = P^{(\mathbf{7})}$ — we need to fix the normalisation. The result is

$$F^{(\mathbf{7})}_{ij} = \frac{1}{4}\left(F_{ij} - \frac{1}{2}\,\phi_{ijkl}\,F^{kl}\right). \tag{6.9}$$

Similarly, $F^{(\mathbf{27})}_{ij}$ is given by

$$F^{(\mathbf{27})}_{ij} = \frac{1}{4}\left(3\,F_{ij} + \frac{1}{2}\,\phi_{ijkl}\,F^{kl}\right).$$

But we don't have to guess what the Spin(7) representations are from the SU(4) expressions. We can also start from the known Spin(7) decompositions of two-, three- and four-forms presented in section 4.3. For the three-forms, we have $\Lambda^3\left(\mathbb{R}^8\right) = \Lambda^3_{\mathbf{8}} \oplus \Lambda^3_{\mathbf{48}}$ where

$$\Lambda^3_{\mathbf{8}} = \left\{\star\left(\alpha\wedge\phi\right), \alpha\in\Lambda^1\left(\mathbb{R}^8\right)\right\}, \quad \Lambda^3_{\mathbf{48}} = \left\{\alpha\in\Lambda^3\left(\mathbb{R}^8\right), \alpha\wedge\phi = 0\right\}.$$

As noted in section 4.3, there is a map between one-forms and three-forms given by $\alpha \mapsto \star\left(\alpha\wedge\phi\right)$, which is effectively the same as contracting with ϕ, since ϕ is self-dual:

$$\begin{aligned}\left(\star\left(\alpha\wedge\phi\right)\right)_{i_1 i_2 i_3} &= 20\,\alpha_j\,\phi^j{}_{i_1 i_2 i_3} && \text{if } \alpha\in\Lambda^1\left(\mathbb{R}^8\right),\\ \left(\star\left(\alpha\wedge\phi\right)\right)_i &= 840\,\alpha_{j_1 j_2 j_3}\,\phi^{j_1 j_2 j_3}{}_i && \text{if } \alpha\in\Lambda^3\left(\mathbb{R}^8\right).\end{aligned}$$

This enables us to write the projector for the **8** representation as

$$\left(P^{(\mathbf{8})}\right)^{j_1 j_2 j_3}_{i_1 i_2 i_3} = \frac{1}{7}\times\frac{1}{3!}\,\phi^{k j_1 j_2 j_3}\,\phi_{k i_1 i_2 i_3} \tag{6.10}$$

and we can take our **48** to be

$$\left(P^{(\mathbf{48})}\right)^{j_1 j_2 j_3}_{i_1 i_2 i_3} = \delta^{j_1 j_2 j_3}_{i_1 i_2 i_3} - \left(P^{(\mathbf{8})}\right)^{j_1 j_2 j_3}_{i_1 i_2 i_3}.$$

We can let the projectors act on our forms, and then write the resulting expressions in irreducible SU(4) expressions, and then go hunt for them in the linear system.

6.3 Getting rid of the spin connection

There are a few finishing touches you would typically want to apply. One of them is getting rid of the spin connection $\Omega_{A,BC}$, in favour of covariant quantities. Taking the Spin(7) case as an example, we may use (4.18) in conjunction with equations (4.15), (4.16), (4.17) to obtain covariant quantities such as

$$\theta_i = -\frac{1}{36}\,\nabla^m \phi_{mk_1k_2k_3}\,\phi^{k_1k_2k_3}{}_i, \quad \theta_5 = -\frac{1}{42}\,\phi^{k_1k_2k_3k_4}\,\nabla_{k_1}\phi_{5k_2k_3k_4}. \tag{6.11}$$

It is also useful to write some expressions involving $\Omega_{A,BC}$ in terms of the exterior derivative on the directions given by the spinor bilinear one-forms, e.g. $\mathrm{d}e^0$, $\mathrm{d}e^5$ or $\mathrm{d}e^-$, as the case may be. Let us find $\mathrm{d}e^A$ in terms of $\Omega_{A,BC}$.

Since $e^A = e^A_\mu\,\mathrm{d}x^\mu$, we have $\mathrm{d}e^A = \partial_\nu e^A_\mu\,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$. Using that the connection and the vielbein are compatible, $\nabla_\mu e^A_\nu = 0$, we obtain

$$\partial_\mu e^A_\nu + \Omega_{\mu,}{}^A{}_B\, e^B_\nu - \Gamma^\sigma_{\mu\nu}\, e^A_\sigma = 0.$$

Taking the antisymmetric part we obtain $\partial_{[\mu} e^A_{\nu]} = -\Omega_{\mu,}{}^A{}_B\, e^B_\nu$, wherefore

$$\mathrm{d}e^A = -\Omega_{\mu,}{}^A{}_B\, e^B_\nu\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \quad \left(\mathrm{d}e^A\right)_{\mu\nu} = -\Omega_{\mu,}{}^A{}_B\, e^B_\nu + \Omega_{\nu,}{}^A{}_B\, e^B_\mu.$$

We arrive at

$$\left(\mathrm{d}e^A\right)_{BC} = \left(\mathrm{d}e^A\right)_{\mu\nu}\, e^\mu_B\, e^\nu_C = 2\,\Omega_{[B,C]}{}^A. \tag{6.12}$$

This allows us to relate the spin connection to $\mathrm{d}e^A$ as desired.

Note that what we are really doing here, both in (6.11) and in (6.12), is rewriting the spin connection in terms of forms that arise as spinor bilinears (up to normalisation). Both e^0, e^5 (or e^- as the case may be), appearing in (6.12), and ϕ, appearing in (6.11), are spinor bilinears. This way, the geometric meaning and origin become clearer, and our expressions manifestly covariant.

6.4 Adapting a metric to commuting vectors

If x and y are coordinates in some coordinate system, the corresponding vectors, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, will commute (as is made manifest by the notation — partial derivatives commute). Conversely, if two vectors X and Y commute, we can introduce coordinates x and y along X and Y, respectively, such that $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$. This allows us to adapt a metric to X and Y.

We do this explicitly in Paper II (though we do find commuting vectors in Paper I too). In Paper II the commuting vectors are

$$K = f^2\, e^0, \quad X = f^2\, e^5. \tag{6.13}$$

We name the corresponding coordinates τ and σ, i.e. $K = \partial_\tau$ and $X = \partial_\sigma$. Since K is a Killing vector, the components of the metric will be independent of τ. There may still be a dependence on σ and the remaining coordinates of the spacetime.

We shall use the coordinate names to label the corresponding indices; e.g. if x^M are the coordinates on spacetime, then $x^\sigma = \sigma$ by definition of notation. We have $X = X^M \partial_M = X^\sigma \partial_\sigma = \frac{\partial}{\partial\sigma}$, where we note that $X^\sigma = 1$ by definition, and in a similar fashion $K^\tau = 1$. Expressed in terms of the vielbeins e_M^A we have $X^A = X^M e_M^A = X^\sigma e_\sigma^A = e_\sigma^A$, and similarly $K^A = e_\tau^A$. Using (6.13) we get

$$e_\sigma^A = f^2 \delta_5^A, \quad e_\tau^A = -f^2 \delta_0^A. \tag{6.14}$$

The metric is

$$g_{MN} = e_M^A e_N^B g_{AB} = -e_M^0 e_N^0 + e_M^5 e_N^5 + e_M^i e_N^j \delta_{ij} \tag{6.15}$$

where i, j are flat indices corresponding to directions perpendicular to e^0 and e^5. Written out in components, (6.15) reads

$$\begin{aligned} g_{\sigma\sigma} &= \left(X^5\right)^2 = f^4, \\ g_{\tau\tau} &= -\left(K^0\right)^2 = -f^4, \\ g_{\sigma\tau} = g_{\tau\sigma} &= 0, \\ g_{\sigma\xi} &= e_\xi^5 = \text{unknown}, \\ g_{\tau\xi} &= -e_\xi^0 = \text{unknown}, \\ g_{\xi_1\xi_2} &= e_{\xi_1}^i e_{\xi_2}^j \delta_{ij} = \text{unknown}, \end{aligned}$$

where ξ denotes a coordinate apart from σ and τ. This can then be written as

$$\mathrm{d}s^2 = -f^4 \left(\mathrm{d}\tau + m\right)^2 + f^4 \left(\mathrm{d}\sigma + n\right)^2 + \mathrm{d}s_{(8)}^2,$$

where m and n are 1-forms, and $\mathrm{d}s_{(8)}^2$ is a metric in the directions transverse to X and K.

6.5 The finishing touches

After having written out the linear system first in terms of irreducible SU(4) representations, and then rewritten it in terms of Spin(7) representations, it remains to put it in the final form: to piece together the various representations and give the resulting expression for the fluxes in terms of the geometry and other fluxes. Naturally, some of these representations will go into the final expressions still undetermined, as the $F^{(\mathbf{21})}$

mentioned above, while others will be completely determined. The undetermined parts are not completely arbitrary, though, as they will still need to satisfy the field equations.

The $G_2 \ltimes \mathbb{R}^8$ case proceeds similarly to the Spin(7) case. Here, the linear system is first written in terms of irreducible SU(3) representations, and then rewritten in terms of G_2 representations. In the $G_2 \ltimes \mathbb{R}^8$ special case, we piece together all the representations and give the fluxes directly in terms of geometry and other fluxes, but in the generic case we found it better to write each separately, describing in text where each component representation may be found in the system of equations.

Similarly, the SU(4) case has been presented in Paper II as a linear system for the component representations of the fluxes, and in the generic SU(4) case, we do not fully disentangle the linear system. The main complication arises in the $(0, 2)$ representation, where there are projectors

$$P^{\pm}\left(g_{\mathbb{C}}\, G_{\bar{\alpha}\bar{\beta}}\right) \equiv \frac{1}{2}\left(g_{\mathbb{C}}\, G_{\bar{\alpha}\bar{\beta}} \pm \frac{1}{2}\, g_{\mathbb{C}}^{\star}\, G^{\bar{\gamma}_1\bar{\gamma}_2}\, \epsilon_{\bar{\gamma}_1\bar{\gamma}_2\bar{\alpha}\bar{\beta}}\right), \tag{6.16}$$

where $g_{\mathbb{C}} = g_1 + \mathrm{i}\, g_2$. The problem is that $P^{\pm}$ acts both on the fluxes and on $g_{\mathbb{C}}$. While it is possible to solve equations involving $P^{\pm}$ anyway, we have found that it does not give a very illuminating answer. The linear system is probably more useful in the way has been given.

Chapter 7

Discussion of results

There are, generically, seven spinor bilinears that we can form from the Killing spinor ε and $\tilde{\varepsilon} = \Gamma_{11}\,\varepsilon$. There is one 0-form, $\sigma\,(\varepsilon, \tilde{\varepsilon})$; two 1-forms, $\kappa\,(\varepsilon, \varepsilon)$ and $\kappa\,(\varepsilon, \tilde{\varepsilon})$; one 2-form, $\omega\,(\varepsilon, \varepsilon)$; one 4-form, $\zeta\,(\varepsilon, \tilde{\varepsilon})$; and two 5-forms, $\tau\,(\varepsilon, \varepsilon)$ and $\tau\,(\varepsilon, \tilde{\varepsilon})$. In each of the main cases, Spin(7), SU(4) and $\mathrm{G}_2 \ltimes \mathbb{R}^8$, we find a special case characterized by the vanishing of a bilinear: the scalar $\sigma\,(\varepsilon, \tilde{\varepsilon})$ in the Spin(7) and SU(4), and the one-form $\kappa\,(\varepsilon, \tilde{\varepsilon})$ in the case of $\mathrm{G}_2 \ltimes \mathbb{R}^8$ (see table 7.1). The Spin(7) special case is remarkable both in that the isotropy group enhances to $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$, and that so many bilinears vanish. From seven non-vanishing bilinears, we go down to just two: κ and $\kappa \wedge \phi$. There is, however, significant simplifications of the bilinears in all the special cases. For instance, in the SU(4) special case, $\omega\,(\varepsilon, \varepsilon)$ becomes directly proportional to the Kähler form of the eight-dimensional subspace perpendicular to $\kappa\,(\varepsilon, \varepsilon)$ and $\kappa\,(\varepsilon, \tilde{\varepsilon})$, whereas the generic SU(4) case lacks this orthogonality. The bilinears in the $\mathrm{G}_2 \ltimes \mathbb{R}^8$ special case become κ, $\kappa \wedge \mathsf{e}^1$, $\kappa \wedge \varphi$, $\kappa \wedge \star\varphi$, and $\kappa \wedge \mathsf{e}^1 \wedge \varphi$, where $\kappa = -2\, f^2\, \mathsf{e}^-$.

In the cases of Spin(7) and SU(4) (special and generic), we get a timelike Killing vector from $\kappa\,(\varepsilon, \varepsilon)$ and in the cases of $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ and $\mathrm{G}_2 \ltimes \mathbb{R}^8$ (special and generic), we get a lightlike Killing vector from $\kappa\,(\varepsilon, \varepsilon)$. As noted in Paper III this lightlike Killing vector is always expected when the isotropy group of the Killing spinor is of the form $H \ltimes \mathbb{R}^m$ for some compact group H.

7.1 Relating the cases in eleven dimensions

Type IIA supergravity with vanishing Romans mass follows by dimensional reduction from eleven-dimnesional supergravity, which was treated in [21] using the same spinor conventions as the present work. In eleven dimensions, there are two orbits of spinors: SU(5) and $\left(\mathrm{Spin}(7) \ltimes \mathbb{R}^8\right) \times \mathbb{R}$. The SU(5) geometries are massive, in the sense of having a timelike Killing vector, and the $\left(\mathrm{Spin}(7) \ltimes \mathbb{R}^8\right) \times \mathbb{R}$ geometries are massless, in the sense of having a lightlike Killing vector.

Table 7.1: Spinor bilinears in the various cases. "⋆" denotes non-zero entries, "↑" means it's the same as the one above (up to sign), "0" means it vanishes.

Grade	**Form**	Spin(7)	Spin(7) ⋉ $\mathbb{R}^8$	SU(4) **generic**	SU(4) **special**	$G_2 \ltimes \mathbb{R}^8$ **generic**	$G_2 \ltimes \mathbb{R}^8$ **special**
0	$\sigma(\varepsilon, \tilde{\varepsilon})$	⋆	0	⋆	0	0	0
1	$\kappa(\varepsilon, \varepsilon)$	⋆	⋆	⋆	⋆	⋆	⋆
1	$\kappa(\varepsilon, \tilde{\varepsilon})$	⋆	↑	⋆	⋆	⋆	0
2	$\omega(\varepsilon, \varepsilon)$	⋆	0	⋆	⋆	⋆	⋆
4	$\zeta(\varepsilon, \tilde{\varepsilon})$	⋆	0	⋆	⋆	⋆	⋆
5	$\tau(\varepsilon, \varepsilon)$	⋆	⋆	⋆	⋆	⋆	⋆
5	$\tau(\varepsilon, \tilde{\varepsilon})$	⋆	↑	⋆	⋆	⋆	⋆

Using our eleven-dimensional intuition, the IIA geometries can be massive for two distinct reasons: either they are the dimensional reduction of a massive eleven-dimensional solution, or they come from a massless eleven-dimensional solution, but have some momentum running along the compact direction. Since the compact dimension is invisible in ten dimensions, any energy stored in there will be percieved as mass.[1]

Type IIA supergravity with a nonzero Romans mass cannot be obtained from eleven-dimensional supergravity, but the eleven-dimensional point of view can still help provide some feeling for what happens.

The spinors of the Spin(7), Spin(7) ⋉ $\mathbb{R}^8$ and $G_2 \ltimes \mathbb{R}^8$ cases (both the generic cases and the special cases) can be related using rotations and boosts involving the eleventh direction, represented in this context by Γ^{11}. A boost involving the eleventh direction takes us between the Spin(7) and Spin(7) ⋉ $\mathbb{R}^8$ spinors, and a rotation in the $(1, 11)$ plane takes us between the Spin(7) ⋉ $\mathbb{R}^8$ spinor and the $G_2 \ltimes \mathbb{R}^8$ spinor. When this rotation is a rotation by 90°, we obtain the $G_2 \ltimes \mathbb{R}^8$ special case.

The SU(4) case is the dimensional reduction of the SU(5) geometries of eleven dimensions, and inherits its massive quality from there.

[1] Not to be confused with the Romans cosmological constant, a.k.a. Romans mass parameter. Type IIA supergravity with nonzero Romans cosmological constant is sometimes called massive IIA supergravity, but these solutions are massive in a different sense.

7.2 Relating the cases to IIB supergravity

While type IIA supergravity has two Majorana–Weyl spinors of opposite chirality, which we write together as a 32 component Majorana spinor, type IIB supergravity has two Majorana–Weyl spinors of the same chirality, which we write together as a complex Weyl spinor. Comparing with the expressions in [7, 22], we see a direct correspondence between IIA Killing spinors and IIB Killing spinors: You multiply the odd chirality part by i and add or remove e_5 as the case may be. The IIB Killing spinors are

$$\varepsilon = (f + \mathrm{i}\, g)\ (\mathbb{1} + e_{1234})\,, \tag{7.1}$$

$$\varepsilon = (f - g_2 + \mathrm{i}\, g_1)\ \mathbb{1} + (f + g_2 + \mathrm{i}\, g_1)\ e_{1234}\,, \tag{7.2}$$

$$\varepsilon = f\ (\mathbb{1} + e_{1234}) - \mathrm{i}\, g\ (e_{15} + e_{2345})\,, \tag{7.3}$$

corresponding to (4.1), (4.2) and (4.3), respectively.

The gauge group of type IIB theory, $\mathrm{Spin}(9,1) \times \mathrm{U}(1)$, is almost the same as the $\mathrm{Spin}(9,1)$ gauge group of type IIA theory, but, importantly, its action on the spinors, and in particular on the functions appearing in the spinors, is noticeably different. Taking the G_2 invariant spinor for example, in type IIA theory $f = g$ is a special case, whereas in type IIB theory $f = g$ is a gauge choice. The type IIA special case is associated with the vanishing of a spinor bilinear, which is a covariant statement. In type IIB no obvious G_2 special case can be found.

Though this makes a one-to-one correspondence between the classes of solutions we found in type IIA supergravity and the IIB solutions presented in [7] and [22] rather unlikely, we still find a Spin(7) special case with just two non-vanishing bilinears: κ and $\kappa \wedge \phi$, where $\kappa = f^2\,\left(e^0 - e^5\right)$ yields the Killing vector (in both IIA and IIB theory). We also find an analogous SU(4) special case, which is the pure spinor case treated in [7].

It is interesting to note that the spinors (7.1), (7.2) and (7.3) are precisely the spinors you get from a naive application of T-duality. [23] gives the formula

$$\hat{\varepsilon}_{\mathrm{IIB}} = \hat{\varepsilon}^2 - \mathrm{i}\,\hat{\Gamma}^9\,\hat{\varepsilon}^1 \tag{7.4}$$

for going between a supersymmetry parameter in IIA to one in IIB, and it is an empirical fact that it maps the IIA spinors (4.1), (4.2), (4.3) to the IIB spinors (7.1), (7.2), and (7.3), respectively, if you take $\hat{\Gamma}^9 = \Gamma^0$, even though it is far from clear that (7.4) is actually applicable here. T-duality is based on a compactification of a Killing direction on a circle. You can get the same nine-dimensional system from compactifying either a IIA solution or a IIB solution, which allows us to relate the IIA and IIB solutions. This is normally done by compactifying a spacelike direction, which was done in the 9 direction to derive (7.4), but to relate our cases we would need to do T-duality along lightlike and timelike directions. The IIA SU(4) case has a timelike Killing vector, and the IIA G_2 case has a lightlike Killing vector — but in the IIB SU(4) and G_2 cases it is the other way around. This introduces some complications (and lightlike directions introduce division by zero in the Buscher rules given in [24]). Another complication is the fact that the Buscher rules involve the potentials of the fields, while we are only working with the field strengths. We know little about the potentials of the fields. We are working with whole classes of solutions, rather than any one explicitly known solution, and T-duality from the known IIB results of [7, 22] would not be a workable alternative to solving the IIA case on its own.

Chapter 8

Computer algebra

8.1 The need for a Computer Algebra System

We are solving the Killing spinor equations, $\mathcal{D}_\mu \varepsilon = 0$ and $\mathcal{A}\varepsilon = 0$ for various ε. Take a moment to consider the result, as expressed in appendices D and E of Paper I, appendices C and D of Paper II, and appendix D of Paper III. If you have the time, grab pen and paper and just copy down the system of equations, just to get a feeling for it.

This is just the result, mind you. We had to do all the work to get there first. In fact, both Ulf Gran and I have *independently* verified these equations, to ensure they are complete and correct. We had to insert the expressions for $\mathcal{D}_\mu$, $\mathcal{A}$ and ε, do the Clifford algebra, split the equations into a linear system sorted by irreducible representations of SU(4) or SU(3) $\subset$ G_2, and then, in the Spin(7) and G_2 cases, assemble the equations in Spin(7) $\supset$ SU(4) and $G_2 \supset$ SU(3) representations, respectively. Then comes simplification, analysis and interpretation (helped by George Papadopoulos). Doing all this by hand would have been a gargantuan project, bordering on madness.

Thus the need for a computer algebra system (CAS). A CAS can handle enormous systems of equations, and can perform elementary operations at tremendous speeds. It doesn't make typos. It doesn't make sign errors — unless you accidentally tell it to. Therein lies the main source of errors when using a CAS: the introduction of bugs, owing to the need to explain to the computer any non-elementary operations to be performed. A fair amount of programming is involved, and bugs can be introduced both from logical errors or misunderstandings of CAS features on our part, and indeed from bugs in the CAS itself. It was to mitigate these problems that it was decided that Ulf and I would work independently, sharing no code, not even using the same computer algebra system. Ulf chose to work in Mathematica, a popular proprietary CAS, for which he had previously written code suited to our purposes (such as the GAMMA package [25]), while I chose to work in Maxima (formerly Macsyma), the popular open source CAS. (No discussion of computer algebra systems would be complete without mentioning

Macsyma [26]. Stephen Wolfram was a heavy user of Macsyma [27], before he went on to create Mathematica. Maple was created as a replacement for Macsyma capable of running on cheaper hardware [28]. Maxima is a direct continuation of the 1982 version of Macsyma, though Macsyma development continued independently from Maxima for some years after that.)

Maxima provides a great environment for manipulating symbolic expressions — but isn't always good at it. Consider, for example, using Maxima 5.37.3:

(%i1) `declare(`n`, integer);`

(%o1) done

(%i2) $\int_0^{\pi} \sin(x) \sin(n\,x)\, dx$;

(%o2) 0

(%i3) n`: 1;`

(%o3) 1

(%i4) $\int_0^{\pi} \sin(x) \sin(n\,x)\, dx$;

(%o4) $\frac{\pi}{2}$

Here, Maxima silently ignores a special case when integrating $\sin(x)\sin(n\,x)$, for integer n: the case where $n = 1$ (see also [29]). The problems aren't restricted to integration either. When `realonly` is set to `true`, the solver `solve(`$x^2 + y^2 = 0$`, [`x`, `y` ])` returns no solutions, where $x = y = 0$ might have been found. Maxima misses special cases, and sometimes asks the user for more info. A lot of code had to be written from scratch.

8.2 How Maxima works

Mathematical expressions are quite naturally modelled as expression trees. An expression is either an atom or a composite expression. An atom can for instance be a number or a symbol — the smallest possible part of an expression. A composite expression is an operator acting on a list of arguments, each argument being an expression. Maxima stores composite expressions in Lisp lists, with the operator in the `car` and the arguments in the `cdr`; the operator is considered the zeroth part of the expression.[1]

[1] Lisp lists are singly-linked lists, consisting of several so called *cons cells*. Each cons cell holds two values, or pointers to values, called the `car` (holding the data to be stored at that position in the list) and the `cdr` (pointing to the next cons cell in the list). You can traverse the list by going from

Maxima is written in Common Lisp, but the Maxima language is distinct from it. You can write pure Maxima code, or extend Maxima using Common Lisp, or do parts in Maxima and parts in Lisp. Syntax wise, Lisp is simple and restrictive: always operator before arguments, always in parenthesis: `(+ 1 (* 2 3))` for $1+2\times3$. Maxima syntax is very flexible, and I ended up defining several new Unicode operators for programming purposes.

The fact that Maxima expressions are trees of singly linked lists has some consequences for programming. For example, such lists can only be traversed in the forward direction. This means that appending an element to the list requires the traversal of the entire list, going from `cdr` to `cdr` until you arrive at the last cons cell, and adjusting it to point to the newly created cons cell containing the last element. On the other hand, inserting an element at the beginning of the list only requires the creation of one cons cell, with the rest of the list in the `cdr`. Therefore you should always build the list starting from the back — if this makes the list come out backwards, it's cheaper to just reverse the list afterwards, compared to trying to build it in the desired order from the beginning. There are other programming considerations too, arising from the choice of data structure, but most of them don't really affect the mathematics.

One aspect of programming with lists *will* however directly impact the mathematical representation of certain operators: The choice between flat and deeply nested data structures illustrated by figure 8.1, where we consider the operator '+'. Mathematically speaking, V is a group under addition if it has a binary operation $+ : V \times V \to V$ satisfying the group axioms (feel free to think about addition of real numbers, if you prefer). Implementing this mathematical fact directly however, would mean encoding $a + b + c + d$ as a deeply nested list structure (the left expression tree in figure 8.1, assuming left associativity). For associative operators, this is unnecesssary. We can just as well represent $a + b + c + d$ as in the right expression tree in figure 8.1, in a flat list structure. The flat structure is simpler, and often less complicated to work with — and it reflects the design descisions the Maxima team have made in practice. Mathematically, this means that we no longer see '+' as a binary operator $+ : V^2 \to V$, but as a $n$ary operator $+ : V^n \to V$, for arbitrary n. (Strictly speaking, this means we have $+ : \{v \in V^n, n \in \mathbb{N}\} \to V$.) Taken with no arguments, + maps the empty list to the identity of addition in V (the identity of addition in $\mathbb{R}$ is zero). Any binary group operation uniquely defines such an $n$ary operation, given associativity. This was taken into account when defining a new wedge product operation for Maxima.

`cdr` to `cdr`, so in this sense the `cdr` "points to the rest of the list."

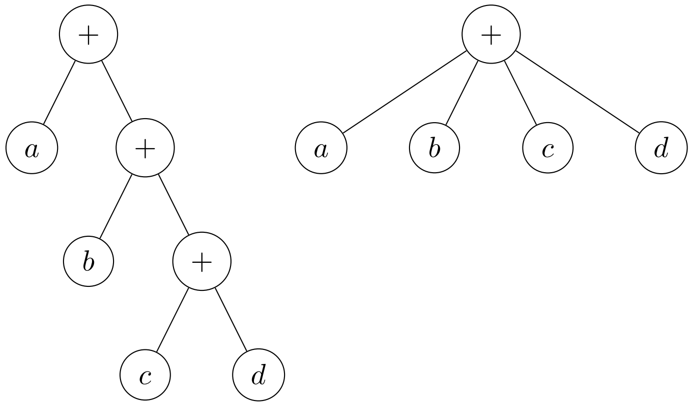


Figure 8.1: Deeply nested expression tree (left) versus flat expression tree (right).

8.3 Tensors for Maxima

There are three tensor packages for Maxima: `ctensor`, focusing on the components of tensors, suitable for e.g. calculating the Riemann curvature tensor for a given metric. That's not what we want — we want to say things about the irreducible representations of the tensors, avoiding statements about individual components if we can.

The package for tensor algebra, `atensor`, and the package for tensor index manipulation, `itensor`, are closer to what we want, though neither of them is in itself sufficient. In the end, I developed my own tensor package (available on request), with support for holomorphic and anti-holomorphic indices, G_2 indices and full spacetime indices, with support for Clifford algebra, contractions of Levi-Civita symbols, and more.

8.4 Importing LaTeX equations

We write our papers using LaTeX, a document preparation system very well suited to making high quality PDF documents. Writing LaTeX equations is, however, mostly a manual process, introducing the risk of typos. While the equations in Paper I and Paper II were manually checked for typos, it is obviously better to automate the checks, which was done with Paper III.

The problem with importing LaTeX expressions into a computer algebra system is that it is actually impossible. That doesn't mean that it hasn't been tried — Mathematica has such a function, though nowhere near powerful enough for our purposes (failing already at simple expressions such as `\tensor{g}{_a_b}` for g_{ab}). Faithfully converting generic LaTeX code to anything that isn't PDF or similar simply can't be done — you can't even parse it without evaluating it, since the code can change the rules of the language itself on the fly. You can't tell what a LaTeX command does without running it in the context it appears; you can't even tell if it is a command or not.

That being said, most people don't use the full capabilities of the language; they use a subset of the language and some standard packages, without changing the logic or syntax of the language on the fly. Support for importing such a subset into a CAS is very possible, if care is taken to avoid ambiguous constructs (is "`f (a + b)`" function application or multiplication?). A package for importing LaTeX equations into Maxima, including full support for tensors and features needed for the Paper III equations, has been written using `flex` and GNU Bison [30], and is available on request. Typos were found and corrected.

Chapter 9

Outlook

This was minimal supersymmetry in IIA supergravity. You can also do maximal supersymmetry and other fractions of supersymmetry. Maximal supersymmetry, treated in [31], naturally imposes the most stringent requirements on the geometry; it turns out all such solutions are locally isomorphic to flat Minkowski space with zero fluxes — and since the Romans mass parameter is set to zero, there are no massive solutions with maximal supergravity. Near-maximal supersymmetry can be treated by simply mirroring our approach to treat the orthogonal complement in the space of spinors, as has been done in type IIA supergravity in [32] and earlier in IIB supergravity in [33]. Near-maximal supersymmetry means preserving 31 out of 32 possible supersymmetries. Such states are sometimes called *preons*, and it was suggested in [34] in an eleven-dimensional context that these hypothetical states could be the primary constituents from which any BPS state could be built. As it turns out, any supergravity solution preserving at least 31 supersymmetries is actually maximally supersymmetric, meaning that preons don't exist as classical supergravity solutions. The close connection between IIA supergravity and eleven-dimensional supergravity means that the absence of preons in type IIA supergravity in itself severely restricts any possible preons in eleven-dimensional supergravity, and a deeper analysis using spinorial geometry, as was done in [35], shows that there are indeed no preons in eleven-dimensional supergravity.

A complete classification of supersymmetric geometries should ideally consider all possible fractions of supersymmetry, which has actually been done for heterotic supergravity in [36, 37] and type I supergravity in [37]. Treating all fractions of supersymmetry means postulating a certain number of Killing spinors (corresponding to the fraction of supersymmetry), and treating them, or their orthogonal complement, using the methods outlined here — the most challenging case would naturally be half-maximal supersymmetry, where we would need to treat the greatest number of Killing spinors.

Do note that the spinorial geometry approach scales linearly with the number of spinors. The earlier approach would center on the spinor bilinears, which already makes the problem quadratic in the number of spinors, and on algebraic relations between the bilinears, making the problem quartic in the number of spinors (see [1, 2] for 11D). For all but the smallest number of Killing spinors this threatens to make the problem intractable.

Given the classification of supergravity backgrounds, one could start looking for new interesting solutions. The special cases we found are much less intractable than the general case, and should warrant further study.

There are some known IIA results, but they may not be the most general ones. Often people start from an ansatz (see e.g. [38, 39, 40]), restricting their focus from the start to specific classes of solutions (which may be very interesting, with e.g. [40] considering compactifications to four-dimensional Minkowski space.) In our approach we make as few assumptions as we can — minimal supersymmetry — and study the implications in full generality. By plugging in known solutions into this classification, we can see where it is possible to deform the solution in various ways (for instance by turning on some new components of the fluxes). It should also be possible to look for new kinds of supersymmetric black holes, by adding the requirement that there is an horizon in the spacetime. (Outside a black hole there should be a timelike Killing vector field, which becomes a null Killing vector at the horizon. Some results for IIA are available in [41, 42]. For similar work in other supergravities, see e.g. [43] (heterotic), [44] (IIB), [45, 46] (11D).)

There is also another type of spacetime that captures the interest of the modern physicist: Asymptotically Anti-de Sitter spaces have a special role in the promising field of gauge/gravity duality, which relates a gravity theory to a strongly coupled field theory. To study asymptotically AdS spaces in our framework, one would start with an appropriate metric ansatz, and simplify the spinors using only transformations that keep the form of the metric intact, before inserting them into the Killing spinor equations. This may allow us to make the connection to Condensed Matter Theory, and other neighbouring areas of physics.

Bibliography

[1] Jerome P. Gauntlett and Stathis Pakis. The geometry of D = 11 Killing spinors. *Journal of High Energy Physics*, 2003(04):039, 2003. ISSN 1126-6708. doi:10.1088/1126-6708/2003/04/039. arXiv:hep-th/0212008.

[2] Jerome P. Gauntlett, Jan B. Gutowski, and Stathis Pakis. The geometry of D = 11 null Killing spinors. *Journal of High Energy Physics*, 2003(12):049, 2004. ISSN 1126-6708. doi:10.1088/1126-6708/2003/12/049. arXiv:hep-th/0311112.

[3] Anthony Zee. *Quantum field theory in a nutshell*. Princeton University Press, Princeton, N.J, 2003. ISBN 0691010196 (acid-free paper).

[4] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley Pub. Co., Reading, Mass, 1995. ISBN 0201503972.

[5] C. Chevalley and P. Cartier. *The Algebraic Theory of Spinors and Clifford Algebras: Collected Works*. Springer Berlin Heidelberg, 1996. ISBN 9783540570639.

[6] Elie Cartan. *Theory of spinors*. Hermann, Paris, 1966. Translation of Lecons sur la théorie des spineurs, v. 1, which was compiled from notes of the author's lectures by André Mercier.

[7] Ulf Gran, Jan Gutowski, and George Papadopoulos. The spinorial geometry of supersymmetric IIB backgrounds. *Classical and Quantum Gravity*, 22(12):2453, 2005. ISSN 0264-9381. doi:10.1088/0264-9381/22/12/010. arXiv:hep-th/0501177.

[8] L. E. Dickson. On Quaternions and Their Generalization and the History of the Eight Square Theorem. *Annals of Mathematics*, 20(3):155–171, 1919. ISSN 0003486X. doi:10.2307/1967865. `http://www.jstor.org/stable/1967865`.

[9] Dominic D. Joyce. *Compact manifolds with special holonomy*. Oxford mathematical monographs. Oxford University Press, Oxford [England], 2000. ISBN 0198506015 (alk. paper).

[10] Marcel Berger. Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés riemanniennes. *Bulletin de la Société Mathématique de France*, 83:279–330, 1955. http://www.numdam.org/item?id=BSMF_1955__83__279_0.

[11] John H. Conway and Derek A. Smith. *On quaternions and octonions : their geometry, arithmetic, and symmetry.* A K Peters Ltd, Natick, Massachusetts, 2003. ISBN 1568811349; 9781568811345.

[12] Jose Miguel Figueroa-O'Farrill. Breaking the M-waves. *Classical and Quantum Gravity*, 17(15):2925, 2000. ISSN 0264-9381. doi:10.1088/0264-9381/17/15/306. arXiv:hep-th/9904124.

[13] Mikio Nakahara. *Geometry, topology, and physics.* Graduate student series in physics. Institute of Physics Publishing, Bristol, 2nd ed. edition, 2003. ISBN 0750306068.

[14] Reese Harvey and H. Blaine Lawson. Calibrated geometries. *Acta Mathematica*, 148(1):47–157, 1982. ISSN 0001-5962.

[15] McKenzie Y. Wang. Parallel spinors and parallel forms. *Annals of Global Analysis and Geometry*, 7(1):59–68, 1989. ISSN 0232-704X. doi:10.1007/BF00137402.

[16] Eric A. Bergshoeff, Mees de Roo, Sven F. Kerstan, Tomás Ortín, and Fabio Riccioni. IIA ten-forms and the gauge algebras of maximal supergravity theories. *Journal of High Energy Physics*, 2006(07):018, 2006. ISSN 1126-6708. doi:10.1088/1126-6708/2006/07/018. arXiv:hep-th/0602280.

[17] Eugene Cremmer, Bernard Julia, and Joel Scherk. Supergravity in theory in 11 dimensions. *Physics Letters B*, 76(4):409–412, 1978. ISSN 0370-2693. doi:10.1016/0370-2693(78)90894-8.

[18] U. Gran, G. Papadopoulos, and D. Roest. Systematics of M-theory spinorial geometry. *Classical and Quantum Gravity*, 22(13):2701, 2005. ISSN 0264-9381. doi:10.1088/0264-9381/22/13/013. arXiv:hep-th/0503046.

[19] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest. Systematics of IIB spinorial geometry. *Classical and Quantum Gravity*, 23(5):1617, 2006. ISSN 0264-9381. doi:10.1088/0264-9381/23/5/012. arXiv:hep-th/0507087.

[20] Jerome P. Gauntlett. Classifying supergravity solutions. *Fortschritte der Physik*, 53(5–6):468–479, 2005. ISSN 1521-3978. doi:10.1002/prop.200510206. arXiv:hep-th/0501229.

[21] Joe Gillard, Ulf Gran, and George Papadopoulos. The spinorial geometry of supersymmetric backgrounds. *Classical and Quantum Gravity*, 22(6):1033, 2005. ISSN 0264-9381. doi:10.1088/0264-9381/22/6/009. arXiv:hep-th/0410155.

[22] U. Gran, J. Gutowski, and G. Papadopoulos. The G_2 spinorial geometry of supersymmetric IIB backgrounds. *Classical and Quantum Gravity*, 23(1):143, 2005. ISSN 0264-9381. doi:10.1088/0264-9381/23/1/009. arXiv:hep-th/0505074.

[23] Tomás Ortín. *Gravity and strings*. Cambridge University Press, Cambridge, UK; New York, 2004. ISBN 051118476X 0521824753.

[24] Eric Bergshoeff, Chris Hull, and Tomas Ortin. Duality in the type-II superstring effective action. *Nuclear Physics B*, 451(3):547–575, 1995. ISSN 0550-3213. doi:10.1016/0550-3213(95)00367-2. arXiv:hep-th/9504081.

[25] Ulf Gran. GAMMA: A Mathematica package for performing gamma-matrix algebra and Fierz transformations in arbitrary dimensions, 2001. arXiv:hep-th/0105086.

[26] William A. Martin and Richard J. Fateman. The MACSYMA system. In *Proceedings of the second ACM symposium on Symbolic and algebraic manipulation*, pages 59–75. ACM, 1971. doi:10.1145/800204.806267.

[27] Stephen Wolfram. There Was a Time before Mathematica…. `http://blog.stephenwolfram.com/2013/06/there-was-a-time-before-mathematica/`.

[28] Gaston Gonnet and Thomas Haigh. An interview with Gaston Gonnet. `http://history.siam.org/oralhistories/gonnet.htm`.

[29] Maxima Bug Tracker. Maxima bug #2652: integrate(sin(nx)sin(p*x),x,0,%pi) fail to consider n=p case. `http://sourceforge.net/p/maxima/bugs/2652/`.

[30] C. Donnelly and R. M. Stallman. *Bison: The YACC-compatible Parser Generator (November 1995, Bison Version 1.25)*. Free Software Foundation, 1995. ISBN 9781882114450.

[31] Jose Figueroa-O'Farrill and George Papadopoulos. Maximally supersymmetric solutions of ten-and eleven-dimensional supergravities. *Journal of High Energy Physics*, 2003(03):048, 2003. ISSN 1126-6708. doi:10.1088/1126-6708/2003/03/048. arXiv:hep-th/0211089.

[32] Igor A. Bandos, Jose A. De Azcarraga, and Oscar Varela. On the absence of BPS preonic solutions in IIA and IIB supergravities. *Journal of High Energy Physics*, 2006(09):009, 2006. ISSN 1126-6708. doi:10.1088/1126-6708/2006/09/009. arXiv:hep-th/0607060.

[33] Ulf Gran, Jan Gutowski, George Papadopoulos, and Diederik Roest. N=31 is not IIB. *Journal of High Energy Physics*, 2007(02):044, 2007. ISSN 1126-6708. doi:10.1088/1126-6708/2007/02/044. arXiv:hep-th/0606049.

[34] Igor A. Bandos, José A. De Azcárraga, Jose M. Izquierdo, and Jerzy Lukierski. BPS states in M theory and twistorial constituents. *Physical review letters*, 86 (20):4451, 2001. doi:10.1103/PhysRevLett.86.4451. arXiv:hep-th/0101113.

[35] Ulf Gran, Jan Gutowski, George Papadopoulos, and Diederik Roest. N=31, D=11. *Journal of High Energy Physics*, 2007(02):043, 2007. ISSN 1126-6708. doi:10.1088/1126-6708/2007/02/043. arXiv:hep-th/0610331.

[36] Ulf Gran, Philipp Lohrmann, and George Papadopoulos. The spinorial geometry of supersymmetric heterotic string backgrounds. *Journal of High Energy Physics*, 2006(02):063, 2006. ISSN 1126-6708. doi:10.1088/1126-6708/2006/02/063. arXiv:hep-th/0510176.

[37] Ulf Gran, George Papadopoulos, Peter Sloane, and Diederik Roest. Geometry of all supersymmetric type I backgrounds. *Journal of High Energy Physics*, 2007(08): 074, 2007. ISSN 1126-6708. doi:10.1088/1126-6708/2007/08/074. arXiv:hep-th/0703143.

[38] Dieter Lüst and Dimitrios Tsimpis. Supersymmetric AdS4 compactifications of IIA supergravity. *Journal of High Energy Physics*, 2005(02):027, 2005. ISSN 1126-6708. doi:10.1088/1126-6708/2005/02/027. arXiv:hep-th/0412250.

[39] Jo Bovy, Dieter Lüst, and Dimitrios Tsimpis. $\mathcal{N} = 1, 2$ supersymmetric vacua of IIA supergravity and SU(2) structures. *Journal of High Energy Physics*, 2005(08): 056, 2005. ISSN 1126-6708. doi:10.1088/1126-6708/2005/08/056. arXiv:hep-th/0506160.

[40] Klaus Behrndt and Mirjam Cvetič. General $\mathcal{N} = 1$ supersymmetric flux vacua of massive type IIA string theory. *Physical review letters*, 95(2):021601, 2005. doi:10.1103/PhysRevLett.95.021601. arXiv:hep-th/0403049.

[41] Ulf Gran, Jan Gutowski, Usman Kayani, and George Papadopoulos. Dynamical symmetry enhancement near IIA horizons. *Journal of High Energy Physics*, 2015 (6):1–27, 2014. ISSN 1029-8479. doi:10.1007/JHEP06(2015)139. arXiv:1409.6303 [hep-th].

[42] Ulf Gran, Jan Gutowski, Usman Kayani, and George Papadopoulos. Dynamical symmetry enhancement near massive IIA horizons. *Classical and Quantum Gravity*, 32(23):235004, 2015. ISSN 0264-9381. doi:10.1088/0264-9381/32/23/235004. arXiv:1411.5286 [hep-th].

[43] Jan Gutowski and George Papadopoulos. Heterotic black horizons. *Journal of High Energy Physics*, 2010(7):1–35, 2010. doi:10.1007/JHEP07(2010)011. arXiv:0912.3472 [hep-th].

[44] Ulf Gran, J. Gutowski, and G. Papadopoulos. IIB horizons. *Classical and Quantum Gravity*, 30(20):205004, 2013. ISSN 0264-9381. doi:10.1088/0264-9381/30/20/205004. arXiv:1304.6539 [hep-th].

[45] Jan Gutowski and George Papadopoulos. Static M-horizons. *Journal of High Energy Physics*, 2012(1):1–22, 2012. ISSN 1029-8479. doi:10.1007/JHEP01(2012)005. arXiv:1106.3085 [hep-th].

[46] Jan Gutowski and George Papadopoulos. M-horizons. *Journal of High Energy Physics*, 2012(12):1–36, 2012. ISSN 1029-8479. doi:10.1007/JHEP12(2012)100. arXiv:1207.7086 [hep-th].