THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

(2,0) Theory and Higher Spin

Twisting, Turning and Spinning Towards Higher Energies

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To my father,
for inspiring me to be curious about the world.
I miss you!

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Abstract

This thesis investigates an enigmatic six-dimensional quantum theory known as (2,0) theory and a three dimensional conformal theory of higher spin. The former has resisted an explicit construction as a quantum field theory, yet its existence can be inferred from string theory and M-theory where it plays a prominent role. Theories of higher spin, only recently emerging with consistent formulations, also have intricate connections with string theory where they might provide insight into the high energy behaviour and have recently played an important part in holographic dualities. A deeper understanding of these theories is therefore an important challange that promise to provide new insight into string theory and the mathematical framework of theoretical physics in general.

First the six dimensional (2,0) theory is investigated in terms of an explicit formulation of one free tensor multiplet on circle fibrations. The fibration geometry provides additional data in a compactification to five dimensions used to derive an interacting generalization. Topological twisting of the tensor multiplet is then carried out, resulting in an off-shell formulation making use of the Q-cohomology structure.

The second part of the thesis concerns conformal higher spin in three dimensions, constructed as an extension of the gauge theory formulation of gravity. Using a computer tensor algebra system developed for this purpose, the full non-linear system is solved at the spin 3 level.

Keywords: Supersymmetry, Yang-Mills theory, Topological field theory, Topological twisting, (2,0) theory, Compactification, Circle fibrations, Higher spin.

Appended papers

This thesis is based on the following papers henceforth referred to as Paper I - Paper IV.

I. *(2,0) theory on circle fibrations*,
Hampus Linander and Fredrik Ohlsson,
JHEP **01** (2012) 159, arXiv:1111.6045 [hep-th]

II. *The trouble with twisting (2,0) theory*,
Louise Anderson and Hampus Linander,
JHEP **03** (2014) 062, arXiv:1311.3300 [hep-th]

III. *Off-shell structure of twisted (2,0) theory*,
Ulf Gran, Hampus Linander and Bengt E.W. Nilsson, JHEP **11** (2014) 032, arXiv:1406.4499 [hep-th]

IV. *The non-linear coupled spin 2 - spin 3 Cotton equation*
Hampus Linander and Bengt E.W. Nilsson
arXiv:1602.01682 [hep-th]

Contribution list

All projects have been carried out in close collaboration, unless otherwise stated the work is shared.

I. All calculations where performed independently by both authors. I also performed some verifications of the results by computer algebra methods. I contributed to the writing of the paper.

II. All calculations, discussions and choice of method was carried out jointly by both authors. Some of the calculations were verified by me using a computer tensor algebra system I started developing during this project. The paper was written by both authors.

III. I did the bulk of the calculations assisted by the tensor algebra system. I played an important role in the discussions and in choosing the methods of calculation. I wrote the majority of the paper.

IV. This work rests heavily on the tensor algebra system I developed during this and preceding papers, in which I carried out all the non-linear calculations. The paper was written by both authors and I played an important role in the discussions leading up to the results.

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Thank you Jennie for being a beacon of light in my life!

Göteborg, March 2016
Hampus Linander

Contents

1

Introduction

> *“ No fairer destiny could be allotted to any physical theory, than that it should of itself point out the way to the introduction of a more comprehensive theory, in which it lives on as a limiting case. ”*
>
> – Albert Einstein, *Relativity (translation) [1]*

In the search for a unified theory[1] of gravitation and quantum mechanics we have been led into the world of higher dimensions and symmetries. This thesis continues investigations of two enigmatic theories that seems to hold many keys to the understanding of our universe. The first is a unique theory in six dimensions and the second a three dimensional theory that both challange the currently available framework of model building in physics.

[1]The word *theory* should in the context of theoretical physics not be confused with the more everyday use of the word: it shares the meaning in that it is something that aims to explain a certain phenomena, but in contrast it is a very concrete mathematical model that enables exact calculations of the behaviour of its constituents.

In the early parts of the 20th century theoretical physics produced two wonderful theories that both completely changed the way we view the world around us: special relativity [2] and quantum mechanics [3]. They describe a strange reality where light travels at constant velocity independent of the reference frame and where things at a small scale become quantised and uncertain. Both have now been shown to describe our world extremely well [4]. It was therefore very annoying that quantum mechanics did not seem to be on friendly terms with special relativity. It took an enormous effort and the greater part of the 20th century to find the correct way of joining these two theories into what is now called quantum field theory. The efforts were not without reward because the result was the standard model of particle physics [5], a quantum field theory that together with the general theory of relativity describes almost everything[2] we see around us in terms of a handful of elementary particles and forces. Recently the last particle predicted by the standard model, the Higgs boson, was found independently by the ATLAS and CMS detectors at the LHC[3].

Einstein's theory of general relavitity describes gravity at length scales that are large in a certain very precise sense. It shows us how mass and energy curves space-time so that planets orbit their suns and light deflects when travelling past a cluster of galaxies. The theory predicts black holes, whose existence has long been infered by indirect measurements[4]. It also predicts gravitational waves and recently both have been observed together in spectacular fashion when LIGO[5] detected the signal from a black hole merger [8].

The standard model of particle physics and the general theory of relativity have shown remarkable aptitude in describing a wide range of phenomena but both have their limits. They are valid up to a certain energy scale where

[2]In terms of everyday phenomena such as a human being sitting in a chair on the planet earth this is certainly true, but to describe the singularity of a black hole or the origin of dark matter and dark energy there arises the need for new additions to our understanding.

[3]Large Hadron Collider, a particle accelerator at the CERN facilities on the France-Switzerland border near Geneva.

[4]See e.g. [6, 7]

[5]Laser Interferometer Gravitational-Wave Observatory in Washington and Louisiana, USA.

new physics needs to enter the picture, a scale that for gravity is much higher than for the standard model. At the time of writing the standard model has been shown to be valid, by particle collider experiments, up to energies of about[6] 1 TeV [9] but there are tentative signals of new physics at this scale [10, 11]. As for the theory of general relativity there are at this point no experimental evidence for any deviations, even in some of the most energetic processes available to us for observation [8, 12, 13]. Even so, there are no doubts as to its inability to describe certain phenomena. When it comes to details of a gravitational collapse inside the event horizon, or the very early events of our universe, it falls short. In these very energetic processes gravity is not well described by the classical theory of general relativity but rather is expected to have a quantum mechanical formulation. It is remarkable that general relativity has led us this far and so vividly pointed the way to new phenomena that seem to require radically new ways of thinking. The problem of quantum gravity is a difficult one, and to find a solution we are invited to broaden our horizons beyond our four-dimensional universe.

One should not despair at this uncomfortable leap into further dimensions and levels of abstraction. As Einstein said himself regarding special relativity in the quote, so too has now the general theory of relativity attained this fairest destiny. Even though a full answer to this question is still out of reach, there is a good candidate: *string theory* [14, 15].

String theory is higher dimensional in two ways. One in that it is formulated in more than four dimensions and another in that the fundamental objects are not zero-dimensional but rather one-dimensional: they are strings. Why is string theory interesting? It is a theory that describes both the fundamental particles and their interactions, and gives a quantum mechanical description of gravity. Why is string theory not the answer? Maybe it is, but even though much progress has been made in understanding the theory we still know too little about it to say for sure.

It is at the cross roads of string theory and quantum field theory that we

[6] In everyday quantities this can be stated as follows: The standard model correctly predicts the outcome of two protons colliding with the equivalent energy of a flying mosquito. Since a mosquito weighs roughly 10^{21} times as much as a proton this is quite a lot of energy for a single proton.

find the topic of Paper I – Paper III. The rather dull name of *(2,0) theory* does not convey the importance that it warrants. As will be covered in the next chapter it is a very special theory that enjoys a host of interesting properties, the most peculiar of which is that as of yet there is no framework where it can be explicitly defined. String theory and M-theory provides evidence for its existence yet quantum field theory cannot accomodate its formulation. This is the precarious situation that (2,0) theory finds itself in still today, about 20 years after its inception. It is the aim of this thesis to add a small piece of the puzzle that hopefully increases the understanding of this theory and in the end will result in its explicit formulation.

(2,0) theory can be construced as a certain limit of string theory, a feature that is thought to be shared by what is known as *higher spin theories*. This type of theory, of which the three dimensional one in Paper IV is an example, contain a large number of particles that generalize the ingredients of the standard model. String theory predicts a large number of new particles and in processes of very high energy these look just like the ones in described by these higher spin theories.

Both the six- and three dimensional theories can be related to string theory, but this is only a small piece of the puzzle. It turns out that they both are interesting by themselves and also, more suprisingly, that they are connected.

1.1 Outline

In chapter 2 an introduction to (2,0) theory, the main theme of the first part of the thesis, is given. First its origin in string theory and M-theory is reviewed. The theory is then described in terms of its symmetries and the problems of an explicit formulation are explained. The simpler non-interacting version of the theory is introduced with the field content of the tensor multiplet together with classical equations of motion.

Chapter 3 begins with an introduction to the concept of compactification. A simple example of a vector field in a circle geometry is worked through. Fibre bundles are then introduced to facilitate the generalization to circle fibrations. The last section in this chapter summarises Paper I where (2,0) theory on circle fibrations are investigated.

Chapter 4 concerns topological twisting, the topic of Paper II. Rigid supersymmetry on curved manifolds is used as a motivating problem and the technique is then introduced through a concrete example of $\mathcal{N} = 2$, $D = 4$ super Yang-Mills theory. This section ends with some comments on the general features of topologically twisted theories. The final sections gives an overview of Paper II and Paper III where (2,0) theory is topologically twisted and compactified to a four-dimensional theory.

Chapter 5 deals with the theory of higher spin, the topic of Paper IV. These types of theories deals with fundamental excitations of spin higher than two that turn out to have deep connections with string theory, nuclear physics and also has relevance for (2,0) theory. This chapter begins with an introduction to the intricacies of higher spin theories and motivations for why they are relevant. The last section gives a brief overview of the history leading up to consistent interacting theories of higher spin.

Chapter 6 describes the construction of a conformal higher spin theory in three dimensions which is the topic of Paper IV. The simpler theory of pure gravity is first introduced which is then rewritten as a gauge theory and finally extended to a higher spin theory through the higher spin algebra.

2

(2,0) theory: from strings to M-theory

The main subject of PAPER I – PAPER III is the six-dimensional superconformal theory known as (2,0) theory. It is the purpose of this chapter to try to give an overview of its place in theoretical physics and an introduction to the formulation of the free theory.

In the mid 90's there was a surge of activity in the field of string [16]. The ember for this explosion was the realization that the different versions of string theory are all related by dualities [17, 18] and that there is an underlying eleven-dimensional theory [17], now known as[1] M-theory [19, 24–29] with eleven-dimensional supergravity [30] as its low energy limit [17]. It was also around this time that the first evidence for a special six-dimensional theory was found as different limits of string theory and M-theory [31, 32].

Immidiately the difficulties, some of which will be reviewed here, with constructing an explicit formulation was recognised [33]. In the following years it

[1]There is no consensus on the meaning of the M in the name M-theory as proposed by Edward Witten [19], but suggestions include membrane, matrix and even mystery, magic or mother. "Membrane" can refer both to the interpretation of the branes as the fundamental objects of the theory but also eludes to the earlier investigations of the supermembrane of 11d supergravity [20, 21]. Matrix theory entered the picture [22, 23] as a possible formulation, hence the "Matrix".

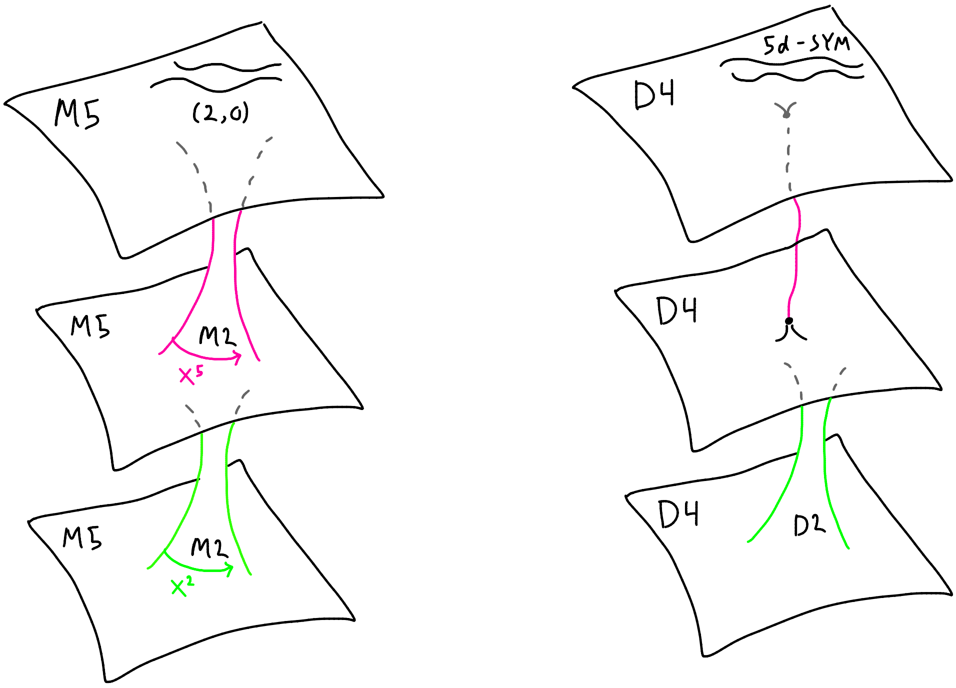
was realised that the low energy behaviour of the two[2] different branes of M-theory, the M2-brane [35] and M5-brane [36], are gouverned by superconformal theories in three and six dimensions [37]. The former share some of the intricacies of the six dimensional theory, but here there has been much progress with the advent of the BLG model [38–42] and ABJM theory [43], that seem to describe aspects of interacting M2-branes [34]. Multiple M5-branes of M-theory can interact by M2-branes ending on the fivebranes [32], as can be seen in the dual string theory picture [44] and reinforced by noting that the end of the M2-brane appears as 1-dimensional solitonic solutions to the low energy theory of the M5-brane [45]. These self-dual [45] strings seem to play an important part[3] in the puzzle and some progress has been made towards understanding their interactions [45–64]. Besides the string like solitonic solutions on the M5-brane there are also three-dimensional solitonic objects [36] that can be interpreted as intersecting M5-branes [65, 66].

An example of a configuration of M5- and M2-branes is presented in figure 2.1 on the next page, where a stack of three M5-branes is shown together with M2-branes stretching between them. Using the duality between M-theory and string theory this configuration corresponds in type IIA string theory to a stack of D4-branes with both D2-branes and fundamental strings, depending on if the M2-brane wraps the compact dimension or not. The low-energy limit of the M5-brane is the conjectured (2,0)-theory whereas the low-energy limit of the D4-brane is the more familiar five-dimensional supersymmetric Yang-Mills theory.

Since the D4-brane is obtained by compactification of the M5-brane on a circle, a natural conjecture is that five-dimensional supersymmetric Yang-Mills theory is obtained by compactifying (2,0)-theory on a circle. This is the topic of PAPER I where a non-interacting version of (2,0)-theory is compactified on circle fibrations. For some more progress in the description of different

[2] That there are two branes in M-theory can be seen from 11d supergravity where supersymmetry specifies the field content to consist of a metric G_{MN}, spin $\frac{3}{2}$ field Ψ_M^α and a three-form C_{MNP}. The three-form couples electrically to a two-dimensional object, the M2-brane, and magnetically to a five-dimensional object, the M5-brane [34].

[3] In the words of the discoverer of the Dp-branes: "Of all the new phases of gauge and string theories that have been discovered this is perhaps the most mysterious, and may be a key to understanding many other things." [15]



	1	2	3	4	5	6	7	8	9	10	11
M5	×	×	×	×	×						
M2					×	×					
M2		×	×								

Figure 2.1: *A stack of M5-branes in 11d M-theory is dual to a stack of D4 branes in 10d type IIA string theory. The theories are dual when one of the coordinates, x^5 in this example, is taken to be a circle of radius R where the type IIA picture on the right is reached in the limit $R \to 0$ [37].*

brane-configurations in M-theory see for [28, 67–69].

The general structure of the six dimensional theory was quickly recognised as a chiral[4] (2,0) supersymmetric conformal theory with no coupling constants [37, 70], but as of yet there is no local and covariant action. Instead, most of the modern understanding of the theory comes from string theory, gauge/gravity dualities [71] and more recently from numerical conformal field theory methods [72]. Another approach, used in Paper I – Paper III, has been to regard the low-energy limit of a single M5-brane: the non-interacting theory of one tensor multiplet. This simpler setup, as will be reviewed below, has well defined

[4]In the sense of a self-dual three-form in the spectrum.

classical equations of motion that can be analysed. It turns out that even the free theory and its solutions give remarkable insight into lower-dimensional theories [73, 74].

2.1 Shadows of (2,0): quantum field theory

> " *And my soul from out that shadow that lies floating on the floor. Shall be lifted — nevermore!* "
>
> – Edgar Allan Poe, *The Raven*

How symmetric can a theory be? It turns out that this question has a definite answer in the context of quantum field theories. The answer gives surprising restrictions on possible theories and at the very extreme end we find a special theory in six dimensions, *(2,0)-theory*. A non-interacting version can be explicitly defined in terms of a six-dimensional field theory. The difficulties, eluded to in the previous section, that a formulation of the interacting theory present might indicate that the free field theory is only a shadow, but a shadow non the less represents some features of its creator. It turns out that the non-interacting version has quite a lot to tell us.

Before constructing the free theory in terms of its equations of motion in section 2.1.3 some general results on space-time symmetry is reviewed in 2.1.2 and the role of symmetry in the following section.

2.1.1 Symmetries: prologue

To construct models of reality based on symmetry considerations has been a very fruitful idea. One prominent example is the standard model of particle physics, where the structure is defined solely in terms of its local gauge symmetries [5]. In a space-time based description of quantum fields this local symmetry is almost a necessity[5]. The name symmetry in this context is,

[5]This will be elaborated in more detail in the sections on higher spin.

however, slightly misleading and a more appropriate name might be redundancy. The physical states of the theory does not transform under local gauge transformations, it is only a symmetry of the mathematical framework that enables a covariant formulation. In light of the gauge/gravity dualities[6] [75, 76] this can even be said about general coordinate invariance. These considerations seem to imply that symmetry is really an emergent property, and not a fundamental guiding principle [18]. In fact, only Lorentz symmetry and its generalization to supersymmetry seem to be left standing[7].

Nevertheless, symmetry considerations has been, and continues to be, a powerful tool that also in the case of (2,0) theory constitutes a solid framework for defining some of its general features.

2.1.2 Restrictions on space-time symmetries

The space-time symmetry group of any physically viable theory must contain the Lorentz group to abide by the laws of special relativity. It can furthermore be invariant under scaling transformations as the theory of classical electromagnetism is. The smallest group containing both Lorentz transformations and scaling transformations is the conformal group which in addition to the aforementioned transformations also contain what is called special conformal transformations. By a theorem due to Coleman and Mandula [78], this is the largest possible space-time symmetry group for a consistent quantum field theory under some very natural conditions on the theory and on the form of the symmetry group. There can also be internal continuous symmetries of the theory that have to commute with the space-time group. It turns out however that there are some loopholes to this argument [79] that makes it possible to extend the spacetime symmetry of a theory. One such loophole is to regard symmetries that are not generated by ordinary numbers but rather by anticommuting numbers. This extends the possible symmetries to include what is called supersymmetry [80]. This is a symmetry that looks very peculiar, it exchanges fermions and bosons. If we combine the conformal group with the supersymmetry transformations we get what is called the superconformal

[6] This duality will be touched upon in chapter 5.

[7] And even here there are proposals to dispose of them as well [77].

group. Another loophole is an assumption of finitely many field. There will be reason to come back to this last assumption in the second part of this thesis, but let us first investigate the situation for the more manageable case of finitely many fields.

Armed with the knowledge of the possible symmetries a reasonable theory can possess, superconformal symmetry, we can now look for its representations. In the 70's Nahm [81] classified the superconformal algebras and showed among other things that they exist only in space-time dimensions less than or equal to six. Thus if we are looking for superconformal theories we need only look in dimension six and below. The existence of a superconformal algebra is a necessary condition but certainly not a sufficient one. Actually it was not at all clear that there should exist any well defined superconformal theories above space-time dimension four[8]. It was therefore a very interesting development when evidence for a theory in six dimensions was put forward [31].

2.1.3 Tensor multiplet

From the work of Nahm [81] we know that there is one possible representation of the (2,0) superconformal algebra called the tensor multiplet. This representation contains scalars, fermions and a three-form. Thus if we would like to try to write down a field theory description of the theory our field contents is fixed[9].

The tensor multiplet contains the fields summarised in table 2.1 where the bold face numbers indicate the dimensionality of the representations and a subscript c indicates that the spinor has positive chirality.

A few general comments on this field content is in order. Firstly the fermions Ψ are symplectic Majorana-Weyl, where the word symplectic stems from the fact that they transform[10] under $\mathrm{Spin}(5)_{\mathrm{R}} \cong \mathrm{Sp}(4)$. The observant reader will have noticed that the theory contains no gauge field in the usual

[8]There are now many examples of superconformal theories in dimension four and below, the most prominent one being $\mathcal{N}=4$ SYM in four dimensions.

[9]There is of course the possibility of having multiple tensor multiplets which would be the case for the general theory.

[10]This is in fact crucial for the Majorana reality condition which in six dimensions requires an interplay between the R-symmetry and complex conjugation.

Field	Spin(5,1)	$\times$	$\text{Spin}(5)_{\mathrm{R}}$
Φ	$\mathbf{1}$		$\mathbf{5}$
Ψ	$\mathbf{4}_{\mathrm{c}}$		$\mathbf{4}$
H	$\mathbf{10}$		$\mathbf{1}$

Table 2.1: *Field content of free (2,0) theory, the tensor multiplet.*

sense of a two-form field strength. Instead we have a three-form that, as can be seen from the dimensionality of its representation, must be self-dual. This is the source of the main mystery surrounding the theory. It is known that the theory is interacting and that it is classified by a choice of Lie group in the ADE-series, however the natural way to implement such an interaction would be through a gauge field which from just representation considerations is not present. Furthermore a dynamical theory of a self-dual three-form in six dimensions is notoriously difficult as we will shortly experience.

For the moment we can sidestep the problems of interactions and regard just the free theory. In this case the three-form poses no immediate conceptual difficulties, apart from the interesting features we will look closer at in a moment.

The next step in constructing a candidate theory would be to write down an action for the fields. In the case of the scalars and fermions the answer is essentially unique and is given by the standard expressions

$$S_\Phi = \int_{M_6} \mathrm{d}^6x\sqrt{-G}\left(\nabla_M\Phi\nabla^M\Phi + \frac{1}{5}R\Phi^2\right)$$

and

$$S_\Psi = \int_{M_6} \mathrm{d}^6x\sqrt{-G}\bar{\Psi}\Gamma^M\nabla_M\Psi.$$

Here R is the scalar curvature of the manifold M_6 and Γ^M are six-dimensional gamma matrices. The curvature term in the scalar action is required for the theory to be conformally invariant. These actions give rise to the local equations of motion

$$\nabla^2\Phi - \frac{1}{5}R\Phi = 0$$
$$\Gamma^M\nabla_M\Psi = 0.$$

When it comes to the three-form the situation is more precarious. The natural action for an n-form can be generalised from the action of electromagnetism $S_{\rm EM} = \int F\wedge\star F$, where F is the U(1) field strength. This form of the action carries over to the general case immediately and we are led to consider

$$S_H = \int_{M_6} H\wedge\star H.$$

Here we face a problem. Since H is self-dual we have that $\star H = H$ and substituting this into the action we find $H\wedge H = 0$. This is the second mystery, there is no known six-dimensional local and covariant action for a self-dual three-form. If we restrict attention to the equations of motion we are for the moment saved. The equation of motion that follow from the action for a general, non-self-dual, H is given by

$$\mathrm{d}\star H = 0.$$

This equation works perfectly well also for a self-dual field and reduces in this case to

$$\mathrm{d}H = 0,$$

the condition that H is a closed three-form. This means that a consistent set of equations for the three-form is

$$H = \star H$$
$$\mathrm{d}H = 0\,.$$

The theory should be invariant under the superconformal algebra and hence there should exist suitable supersymmetry variations transforming solutions to these equations of motion into each other. Indeed one finds that the transformations (2.1)-(2.3) below transforms solutions to the equation of motion into each other provided the supersymmetry parameter satisfies condition (2.4). A few words on notation is here warranted. Apart from the six-dimensional indices M,N,.. these expressions contain lower case Greek indices $\alpha, \beta, \cdots$ in the four-dimensional spinor representation of the $\mathrm{Spin}(5)_{\mathrm{R}}$ symmetry. The bilinear forms $M_{\alpha\beta}$ and $T^{\alpha\beta}$ lowers and raises indices in these representations. It is now convenient to regard the five-dimensional vector representation of $\mathrm{Spin}(5)_{\mathrm{R}}$ as the symmetric traceless tensor product of two spinor representations. This gives the space-time scalar Φ a representation as a bispinor, enabling a compact and computationally efficient form of the supersymmetry transformations.

$$\delta H_{MNP} = 3\nabla_{[M}\left(\overline{\Psi}_{\alpha}\Gamma_{NP]}\varepsilon^{\alpha}\right) \tag{2.1}$$

$$\delta \Phi^{\alpha\beta} = 2\overline{\Psi}^{[\alpha}\varepsilon^{\beta]} - \frac{1}{2}T^{\alpha\beta}\overline{\Psi}_{\gamma}\varepsilon^{\gamma} \tag{2.2}$$

$$\delta\Psi^{\alpha} = \frac{i}{12}H_{MNP}\Gamma^{MNP}\varepsilon^{\alpha} + 2iM_{\beta\gamma}\nabla_M\Phi^{\alpha\beta}\Gamma^M\varepsilon^{\gamma} + \frac{4i}{3}M_{\beta\gamma}\Phi^{\alpha\beta}\Gamma^M\nabla_M\varepsilon^{\gamma} \tag{2.3}$$

$$\nabla_M\varepsilon^{\alpha} - \frac{1}{6}\Gamma_M\Gamma^N\nabla_N\varepsilon^{\alpha} = 0 \tag{2.4}$$

Equation (2.4) is the conformal Killing spinor equation. For rigid supersymmetry the natural condition that comes to mind is for the parameter to be covariantly constant, this is however not a conformally invariant equation. The operator in (2.4) is, together with the Dirac operator, the only natural conformally invariant operators available [82]. On a manifold that admits two independent solutions to (2.4) the theory is maximally supersymmetric with 16 supercharges.

Conformal invariance of abelian[11] (2,0) theory manifests itself in the fact that the equations of motion and supersymmetry transformations depend only on the conformal class of the metric. This means that the theory is invariant under a change of the metric of the form

$$G \to e^{-2\sigma(x)}G,$$

where σ is a function on M_6.

At last we find ourselves with a starting point for explicit investigations, the tensor multiplet together with its classical equations of motion. This might

[11]The non-interacting version will sometimes be called the *abelian*. This is equivalent since all the fields transform in the adjoint representation.

seem like a poor substitute for the full interacting quantum theory but this is the only local covariant explicit formulation available at the moment. It seems reasonable that some general features of the theory should also be present in the free, classical version. In fact there are some quantities of the full quantum theory that can be calculated in terms of just the free classical theory, see [83] for such an example.

Before moving on to the description of PAPER I let us recall the counting of dimensionality of the representations in table 2.1. The scalar Φ is a vector under $\mathrm{Spin}(5)_{\mathrm{R}}$ and hence has five on-shell degrees of freedom. The three-form has a two-form potential with 15 independent components. Gauge invariance reduces this to 10 and the equations of motion removes four additional components[12]. The six remaining components are then finally reduced to three independent degrees of freedom by the self-duality condition.

The fermionic field Ψ is a chiral spinor in six dimensions, hence it transforms in a four-dimensional[13] representation of Spin(5,1). It also transforms in the four-dimensional spinor representation of $\mathrm{Spin}(5)_{\mathrm{R}}$, enabling the symplectic Majorana condition to be imposed. This condition halves the 16 complex components down to eight which the Dirac equation further reduces to four complex components, or eight real on-shell degrees of freedom.

Thus there are an equal number of bosonic and fermionic on-shell degrees of freedom (eight), a necessary condition for supersymmetry.

[12]Start with an unconstrained two-form potential B. Its field strength $H = \mathrm{d}B$ is invariant under $B \to B + \mathrm{d}\Lambda_1$ however Λ_1 is itself invariant under $\Lambda_1 \to \Lambda_1 + \mathrm{d}\Lambda_0$, thus there are five independent gauge parameters. Adopt light-cone coordinates and boost to a frame where $p^+ = 1$ and $p^{M\neq+} = 0$. The five independent gauge parameters can be used to impose for example $B_{-M} = 0$. The equation of motion then implies $B_{+M} = 0$, which in light of the gauge choice, imposes four further relations.

[13]A Dirac spinor in d dimensions has $2^{\lfloor \frac{d}{2} \rfloor}$ complex components which in six dimensions is 8, giving a chiral spinor 4 complex components.

3

Circle fibrations

The topic of Paper I is compactification of (2,0) theory on a circle fibration. It is the purpose of this chapter to introduce the concept of fibered spaces and how they are used in physics. There is a beautiful but rather large theoretical basis behind these methods which will not be covered, for an excellent exposition of these topics in the context of (2,0) theory see [84].

In section 3.1 the theory of compactification is reviewed with a simple example of a vector field on a circle. The generalization to circle fibrations and their geometry is covered in section 3.2 starting with a minimal introduction to fibre bundles. Finally the content of Paper I is described in section 3.3.

3.1 Compactification

Starting with a theory defined in D space-time dimensions there is a way to create a whole class of $d < D$-dimensional theories. This process is called compactification [85] and as the name implies it involves the use of compact manifolds. The concept dates back to the early 20th century when a unified theory of electromagnetism and gravity was sought. It was found that general relativity defined on a five-dimensional space-time, where one of the directions is periodic, describes gravity and electromagnetism in four dimensions [86].

The unifying theory of electromagnetism and gravity did not work out in the end but the concept of building lower dimensional theories from higher dimensional ones became a widely used method. The process of compactification can be readily described by an example, and usually the simplest possible candidate is the theory of a scalar field compactified on a circle. In the next section a slightly more involved example of a vector field compactified on a circle is described. The purpose of this is two-fold: first it provides an example where there arises new fields in the compactification of a different type than the original fields, secondly the specific example lies closer to the computations carried out in PAPER I and the reader may therefore find it elucidating to compare the results.

3.1.1 Theories on a circle

The canonical example of a compactification is when space-time is taken to be of the form[1]

$$M_D = \mathbb{R}_{D-1} \times \mathrm{S}_1 \,,$$

where S_1 denotes a circle. Let us investigate what happens to the theory of a single gauge field on this manifold. Given a one-form potential $A = A_M \mathrm{dx}^M$ and its field strength $F = \mathrm{d}A$, a Maxwell-like action is given by

$$S = \frac{1}{2\pi} \int_{M_D} F \wedge \star F \,, \tag{3.1}$$

where $\star$ denotes the Hodge dual[2]. The field strength F, and hence the action, is invariant under $A \to A + \mathrm{d}\Lambda$. Let us denote the index corresponding to the circle by φ and the $D-1$ other directions by lower case Greek letters. Let us also, by a slight abuse of notation, use φ to denote the coordinate on the circle. The fact that the circle is periodic enables us to make a Fourier expansion in the coordinate φ.

$$A = \sum_{n=-\infty}^{\infty} A_n(x_\mu) e^{in\varphi}$$

[1]For simplicity the space-time is taken to be flat but in general it can be curved, as will be the case for the later parts of this chapter.

[2]Here taken to be defined by $\star 1 = \mathrm{vol}(M)$.

Here $A_n = A^\star_{-n}$ are one-forms on M_D and can be split into the parts that lie in the direction of $\mathbb{R}_{D-1}$ and in the direction of the circle

$$A_n = A_n^{D-1} + A_n^\varphi \mathrm{d}\varphi.$$

Substituting this into the expansion and taking the exterior derivative, recalling that $\mathrm{d}\varphi \wedge \mathrm{d}\varphi = 0$, gives us three terms

$$\mathrm{d}A = \sum_{n=-\infty}^{\infty} \mathrm{d}A_n^{D-1} e^{in\varphi} + \sum_{n=-\infty}^{\infty} \mathrm{d}A_n^\varphi \wedge \mathrm{d}\varphi e^{in\varphi} + \sum_{n=-\infty}^{\infty} A_n^{D-1} ine^{in\varphi} \wedge \mathrm{d}\varphi\,.$$

Substituting this back into the action might not seem to give anything particularly nice but with a few observations an elegant answer emerges. The first observation is that the Hodge dual induces an inner product between forms and that the action in (3.1) is nothing but the inner product $\langle \mathrm{d}A, \mathrm{d}A\rangle$. In particular the basis 2-forms $\mathrm{dx}^M \wedge \mathrm{dx}^N$ are orthogonal with respect to this inner product which has as a consequence that

$$\begin{aligned}\langle \mathrm{d}A, \mathrm{d}A\rangle = &\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{in\varphi}e^{im\varphi} \\ &\cdot \Big[\left\langle \mathrm{d}A_n^{D-1},\, \mathrm{d}A_m^{D-1}\right\rangle - mn\left\langle A_n^{D-1}\wedge \mathrm{d}\varphi,\, A_m^{D-1}\wedge \mathrm{d}\varphi\right\rangle \\ &+ \left\langle \mathrm{d}A_n^\varphi \wedge \mathrm{d}\varphi,\, \mathrm{d}A_m^\varphi \wedge \mathrm{d}\varphi\right\rangle + im\left\langle \mathrm{d}A_n^\varphi \wedge \mathrm{d}\varphi,\, A_m^{D-1}\wedge \mathrm{d}\varphi\right\rangle \\ &+ in\left\langle \mathrm{d}A_n^\varphi \wedge \mathrm{d}\varphi,\, A_m^{D-1}\wedge \mathrm{d}\varphi\right\rangle \Big].\end{aligned}$$

The second observation is that $\int_{S_1} e^{im\varphi}e^{-in\varphi}\mathrm{d}\varphi = 2\pi\delta^{mn}$. Performing the integration over φ thus results in

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\langle \mathrm{d}A, \mathrm{d}A\rangle = &\sum_{n=-\infty}^{\infty}\Big[\left\langle \mathrm{d}A_n^{D-1}, \mathrm{d}A_{-n}^{D-1}\right\rangle + n^2\left\langle A_n^{D-1}, A_{-n}^{D-1}\right\rangle \\ &+ \langle \mathrm{d}A_n^\varphi, \mathrm{d}A_{-n}^\varphi\rangle - in\left\langle \mathrm{d}A_n^\varphi, A_{-n}^{D-1}\right\rangle + in\left\langle \mathrm{d}A_{-n}^\varphi, A_n^{D-1}\right\rangle\Big].\end{aligned}$$

By combining the second, third, fourth and fifth term for $n > 0$ gives the more transparent form

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\langle \mathrm{d}A, \mathrm{d}A\rangle = &\left\langle \mathrm{d}A_0^{D-1}, \mathrm{d}A_0^{D-1}\right\rangle + \langle \mathrm{d}A_0^\varphi, \mathrm{d}A_0^\varphi\rangle \\ &+ \sum_{n\neq 0}\Big[\left\langle \mathrm{d}A_n^{D-1}, \mathrm{d}A_{-n}^{D-1}\right\rangle + n^2\left\langle A_n^{D-1} - \frac{i}{n}\mathrm{d}A_n^\varphi, A_{-n}^{D-1} + \frac{i}{n}\mathrm{d}A_{-n}^\varphi\right\rangle\Big].\end{aligned}$$

The first line above describes a massive vector field A_0^{D-1} together with a massless scalar A_0^φ. In the second line, the mass term for the $A_{n\neq 0}^{D-1}$ vector fields has now been shifted by $\frac{i}{n}\mathrm{d}A_n^\varphi$. This shift can be cancelled by a gauge transformation $\delta A_n^{D-1} = \mathrm{d}\Lambda_n$, where the gauge parameter Λ is also Fourier expanded, resulting in a collection of massive $A_{n\neq 0}^{D-1}$ vector fields.

These are quite general features when compactifying a theory, the appearance of a tower of new massive fields as well as massless ones. It might seem distressful that there now emerges an infinite number of fields in the compactified theory. On the other hand, if the only interest is in what happens below a certain energy scale the increasingly heavy fields in the tower will not play a role. In the low energy limit only the massless fields remain, which as we have seen are finite in number.

3.2 Fibre bundles

In the previous section the basic features of compactification was worked out: a space-time with a compact direction gives rise to, in the low energy limit, a theory in one dimension lower containing new fields. This method is tremendously useful and has been used to derive a great deal of information about various theories, as well as constructing new ones [87]. A very prominent application is the compactifications of string theory and M-theory [88, 89]. There are some very natural generalizations to this scheme. Firstly the space-time be curved. This complicates the above discussion but does not change the basic results of compactification. Another interesting direction is to let the compact space vary in its geometry over the lower dimensional manifold. The concept of one space varying over another can be described in the language of *fibre bundles*.

Intuition for fibre bundles comes naturally with a simple example. Consider the “patch of grass” in figure 3.1. An idealised picture is a collection of one-dimensional fibres (grass) attached to a plane (ground). Thus the grass-covered ground can be thought of as a bundle of fibres.

More generally, the ingredients of a fibre bundle is a manifold M called the base and a manifold F called the fibre. The intuitive statement that a copy of

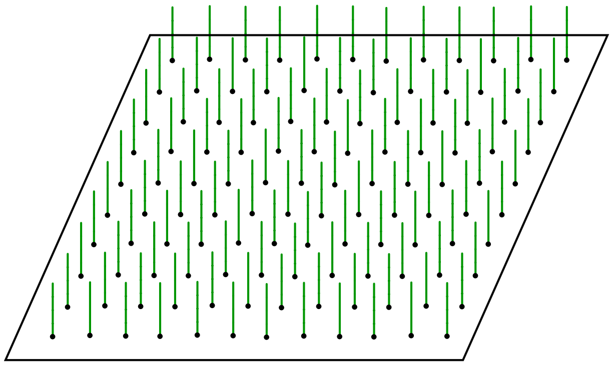


Figure 3.1: *Line segments fibered over a plane.*

the fibre F should be attached at every point of M can be made precise with the following definition.

A *fibre bundle* E over a manifold M with fibre F is a space that is locally diffeomorphic to the direct product of an open neighbourhood of M with F.

In other words if we take an open neighbourhood $V \subset E$ there exists an open neighbourhood $U \in M$ such that

$$V \cong U \times F.$$

This is illustrated schematically in figure 3.2, where over each point in the neighbourhood $U \subset M$ we find a copy of the fibre F which in this case is indicated by a line.

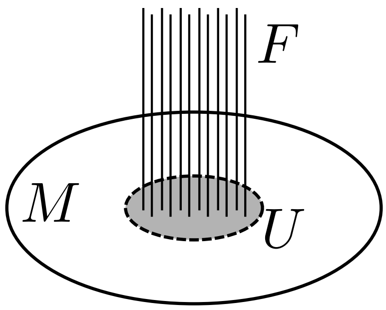


Figure 3.2: *A fibre bundle. Over each point in a neighbourhood U we find a copy of the fibre F. Note that the figure only shows a subset of the fibers, in reality there is a dense cover of fibers over all points in U.*

3.2.1 Circle fibration

In the special case when the fibre F is a circle, the resulting bundle is called a circle bundle. This means that at each point in the base M we find a circle. In this work a manifold that can be described as a circle bundle will be called a circle fibration. In PAPER I the six-dimensional space-time is a circle fibration over a five-dimensional base. The manifolds involved can be summarised with the diagram in figure 3.3 below. Here the manifold M_6 is a circle fibration with fibre S_1. The arrow from S_1 to M_6 indicates the embedding of the fibre in M_6. The base of the fibration is M_5 and the arrow from M_6 to M_5 indicates that to each point in M_6 there is an associated point in the base given by a projection π.

$$\begin{array}{ccccc} \mathrm{S}_1 & \longrightarrow & M_6 & \overset{\text{locally}}{\cong} & M_5 \times \mathrm{S}_1 \\ & & \downarrow \pi & & \\ & & M_5 & & \end{array}$$

Figure 3.3: *A diagrammatic description of a circle fibration M_6.*

3.2.2 Geometry of circle fibrations

So far, the only focus has been the smooth structure of the manifolds in question. This section will give a brief overview of how the metric information about a manifold is represented in the special situation of a circle fibration.

On a general six-dimensional Lorentzian manifold M_6 there is a semi-definite metric tensor G_{MN}. If M_6 is a circle fibration, locally

$$M_6|_V \cong M_5|_U \times S_1.$$

I.e. an open neighbourhood of M_6 is isometric to a product of an open neighbourhood in M_5 and a circle. The metric on the product space on the right is not necessarily a product metric, i.e. it will in general not have a block diagonal structure.

Let φ be the coordinate in the direction of the circle, and as before we also let φ be the value of the index for this direction. The metric G_{MN} will then have the structure

$$G = G_{\mu\nu}\mathrm{dx}^\mu\mathrm{dx}^\nu + 2G_{\mu\varphi}\mathrm{dx}^\mu\mathrm{d}\varphi + G_{\varphi\varphi}\mathrm{d}\varphi\mathrm{d}\varphi\,. \tag{3.2}$$

There are more convenient parametrizations than the above. A better way to keep track of the reparametrization invariance in six dimensions is

$$G = g_{\mu\nu}\mathrm{dx}^\mu\mathrm{dx}^\nu + r^2\left(\mathrm{d}\varphi + \theta_\mu\mathrm{dx}^\mu\right)^2 .$$

This is just a renaming of the components in (3.2). It keeps track of reparametrization invariance since

$$\varphi \to \varphi + \lambda(x^\mu) \qquad \Leftrightarrow \qquad \theta_\mu \to \theta_\mu + \partial_\mu\lambda.$$

The vector θ transforms as a U(1) gauge field under reparametrizations and the fact that the six-dimensional theory is reparametrization invariant means in essence that the compactified theory can only depend on the gauge invariant, non-dynamical, field strength

$$\mathcal{F}_{\mu\nu} = \partial_\mu\theta_\nu - \partial_\nu\theta_\mu\,.$$

The scalar r also naturally corresponds to the radius of the fibre which can be seen if we take $\theta_\mu = 0$ thereby making the metric block diagonal.

3.3 (2,0) theory on circle fibrations

In Paper I, (2,0) theory is compactified on a space-time that is a circle fibration over a five-dimensional base manifold. Already in the original work [31] proposing the existence of (2,0) theory some aspects of its compactification on $\mathbb{R}^5 \times S_1$ were discussed. Compactification arguments also played an essential role in deducing its existence through the web of string theory dualities. By various considerations one can make it very plausible that the theory compactified on a circle will give rise to five-dimensional super Yang-Mills theory [90]. The lack of an explicit construction of the theory of course means that there are no proofs for such claims but only indications. Lately there have been efforts to show that (2,0) theory can perhaps be completely described by a five-dimensional theory [91, 92].

The purpose of Paper I is to continue these efforts in the more general setup that circle fibrations provide.

3.3.1 Overview of paper I

The plan is summarised in figure 3.4 below. The starting point is to regard one free tensor multiplet of (2,0) theory on a circle fibration. A low energy limit is taken and we find a five-dimensional supersymmetric theory. This is illustrated in figure 3.5 below which shows an M5-brane[3] where one direction is compactified and the surviving zero-mode on the resulting D4-brane in the low-energy limit. This theory is shown to have a unique extension to an interacting theory which we derive. That the extension is unique makes it plausible that this is the low energy limit of the interacting (2,0) theory in six dimensions.

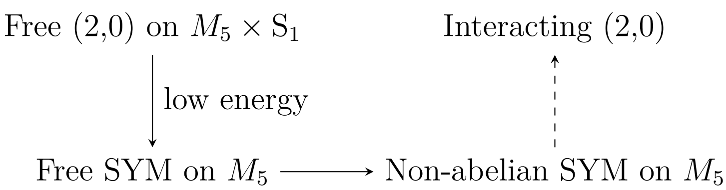


Figure 3.4: *Method of* Paper I. *Starting from the abelian theory in six dimensions we take the low energy limit on a circle fibration and extend to an interacting theory.*

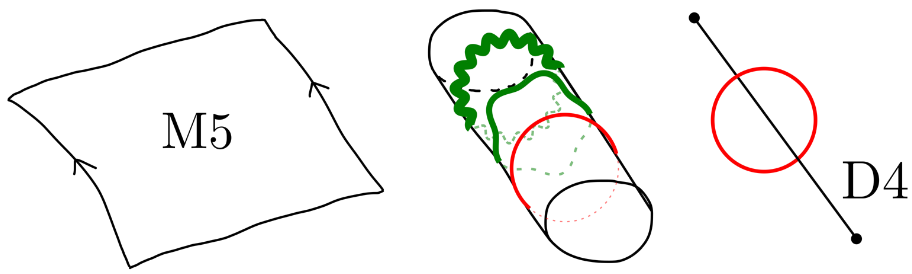


Figure 3.5: *When one of the directions along the M5-brane (left) is compactified (center) the possible modes in this direction are quantized according to their Fourier expansion, starting with the constant zero-mode (red) and continuing with an infinite tower of increasingly massive excitations (green). In the low-energy limit only the zero-mode, which has no dependence on the circle coordinate, survives (right).*

[3]The M5-brane should not be confused with the base manifold M_5 of the circle fibration.

3.3.2 Abelian compactification

Let M_6 be a circle fibration over M_5 as described in section 3.2.2. Recall that the fibration metric is given by

$$G = g_{\mu\nu}\mathrm{dx}^\mu\mathrm{dx}^\nu + r^2\left(\mathrm{d}\varphi + \theta_\mu\mathrm{dx}^\mu\right)^2 . \tag{3.3}$$

It is now a straight forward calculation to show that the scalar equation of motion[4]

$$\hat{\nabla}_M\hat{\nabla}^M\Phi - \frac{1}{5}\hat{R}\Phi = 0\,,$$

reduce in five dimensions to an equation that can be integrated to the action

$$S_\phi = \int \mathrm{d}^5x\sqrt{-g}\left(-\frac{1}{r}\nabla_\mu\phi_{\alpha\beta}\nabla^\mu\phi^{\alpha\beta} - \frac{1}{5}R\phi_{\alpha\beta}\phi^{\alpha\beta} + K(g,r,\theta)\phi_{\alpha\beta}\phi^{\alpha\beta}\right)\,, \tag{3.4}$$

where

$$K(g,r,\theta) = \frac{1}{r^3}\nabla_\mu r\nabla^\mu r - \frac{3}{5}\frac{1}{r^2}\nabla_\mu\nabla^\mu r + \frac{1}{20}r\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$$

contains the geometric information about the circle fibration, i.e. the fibration radius r and the non-dynamical[5] field strength $\mathcal{F}_{\mu\nu} = \partial_\mu\theta_\nu - \partial_\nu\theta_\mu$. Note that in the general case the geometric data r and $\mathcal{F}$ are functions on M_5 and so can vary over the manifold.

For the fermions the computation is also in principle straight forward but is complicated by the somewhat more heavy machinery of spinors in curved backgrounds. In Paper I we provide a self contained description of this process. The result is that the fermion equation of motion

$$\Gamma^M\hat{\nabla}_M\Psi = 0\,,$$

reduce in five dimensions to an equation that integrates to an action of the form

$$S_\psi = \int \mathrm{d}^5x\sqrt{-g}\left(\frac{1}{r}i\bar{\psi}\gamma^\mu\nabla_\mu\psi - \frac{1}{8}\mathcal{F}_{\mu\nu}\bar{\psi}\gamma^{\mu\nu}\psi\right)\,. \tag{3.5}$$

[4]Here the six-dimensional covariant derivative and scalar curvature are indicated with a hat to distinguish them from their five-dimensional counterparts.

[5]The field strength $\mathcal{F}_{\mu\nu}$, together with r, constitutes the geometric information in the fibration which from a five-dimensional perspective can be regarded as background fields.

Both (3.4) and (3.5) have the appearance of five-dimensional super Yang-Mills but with additional geometric terms stemming from the fibration.

Now the self-dual three-form enters the picture, and the story gets more interesting. Let us delve a bit deeper into the calculations to elucidate some of the features of its compactification. On the fibration geometry the three-form H can be written as

$$H = E + F \wedge \mathrm{d}\varphi\,, \tag{3.6}$$

with E a three-form on M_5 and F a two-form on M_5. A self-dual three-form in six dimensions has 10 independent components, which is the same number as for a three-form *or* a two-form in five dimensions. One would therefore expect that the components of F and E are identified and this is precisely what happens. Writing out the self-duality condition $H = \star H$ results in

$$E = -\frac{1}{r} \star F + \theta \wedge F. \tag{3.7}$$

Thus, in the end we have a two-form field strength in five dimensions, precisely what is needed for a standard gauge theory[6].

It is now immediate that the equation of motion $\mathrm{d}H = 0$ implies first that $\mathrm{d}F = 0$ from (3.6) and also that $\mathrm{d}E = 0$, which with the identification in (3.7) gives an equation of motion for F that can be integrated to the action

$$S_F = \int_{M_5} \left(-\frac{1}{r} F \wedge \star F + \theta \wedge F \wedge F \right) .$$

At this point a feature of the compactification of (2,0) theory emerges that is very unusual but which has been known from its inception. Even though the above theory is non-interacting, the form of the coupling constant from the factors in the first term can be anticipated. It would seem that we have a coupling constant $\sqrt{r}$ which is the inverse of what would be expected from a standard dimensional reduction where we integrate out the circle and pick up a factor of r in the numerator, giving rise to a coupling constant $\frac{1}{\sqrt{r}}$. In the context of compactification on circle fibrations this comes about very naturally from the geometry of the fibration.

[6]This can be compared to the example in section 3.1.1 where the potential gives rise to two new fields that are independent.

In a similar fashion to the reduction of the equation of motions, the five-dimensional supersymmetry transformations can be derived as

$$\delta\phi^{\alpha\beta} = 2\overline{\psi}^{[\alpha}\varepsilon^{\beta]} - \frac{1}{2}T^{\alpha\beta}\bar{\psi}_{\gamma}\varepsilon^{\gamma}\,,$$

$$\begin{aligned}\delta F_{\mu\nu} = &-2i\nabla_{[\mu}\overline{\psi}_{\alpha}\gamma_{\nu]}\varepsilon^{\alpha} + i\frac{1}{r}\nabla^{\rho}r\overline{\psi}_{\alpha}\gamma_{\mu\nu\rho}\varepsilon^{\alpha} - 2i\frac{1}{r}\nabla_{[\mu}r\overline{\psi}_{\alpha}\gamma_{\nu]}\varepsilon^{\alpha} \\ &+ r\mathcal{F}_{\mu\nu}\overline{\psi}_{\alpha}\varepsilon^{\alpha} + \frac{3}{2}r\mathcal{F}_{[\mu}{}^{\rho}\overline{\psi}_{\alpha}\gamma_{\nu]\rho}\varepsilon^{\alpha} - \frac{1}{4}r\mathcal{F}^{\rho\sigma}\overline{\psi}_{\alpha}\gamma_{\mu\nu\rho\sigma}\varepsilon^{\alpha}\end{aligned}$$

and

$$\begin{aligned}\delta\psi^{\alpha} = &\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{\alpha} + 2iM_{\beta\gamma}\nabla_{\mu}\phi^{\alpha\beta}\gamma^{\mu}\varepsilon^{\gamma} \\ &+ 2i\frac{1}{r}M_{\beta\gamma}\phi^{\alpha\beta}\nabla_{\mu}r\gamma^{\mu}\varepsilon^{\gamma} - rM_{\beta\gamma}\phi^{\alpha\beta}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{\gamma}\,.\end{aligned}$$

The five-dimensional theory is invariant under these transformations provided the supersymmetry parameter satisfies the reduced version of the conformal Killing spinor equation

$$\nabla_{\mu}\varepsilon^{\alpha} = \frac{1}{2}\frac{1}{r}\nabla^{\nu}r\gamma_{\mu}\gamma_{\nu}\varepsilon^{\alpha} + \frac{i}{8}r\mathcal{F}^{\rho\sigma}\gamma_{\mu}\gamma_{\rho\sigma}\varepsilon^{\alpha} + \frac{i}{4}r\mathcal{F}_{\mu}{}^{\nu}\gamma_{\nu}\varepsilon^{\alpha}.$$

3.3.3 Interacting generalization

In the second part of Paper I we extend the abelian theory in five dimensions to include interactions. This process is strongly constrained by the symmetries present. It turns out that there seems to be only one possible interacting extension.

Let us list the symmetries of the five-dimensional theory. Apart from five-dimensional Lorentz symmetry and the R-symmetry there is, if the background geometry permits, maximal supersymmetry. The introduction of the length scale r has broken the conformal symmetry but there is still a remnant of it. To see this, note that the theory in six dimensions only depends on the conformal class of the metric. This means that we end up with the same theory in five dimensions if we instead regard the metric

$$G' = e^{-2\sigma}G\,,$$

where σ is a smooth function on M_5. In terms of the fibration geometry this means that the theory is invariant under

$$
\begin{aligned}
g_{\mu\nu} &\to e^{-2\sigma} g_{\mu\nu}\,, \\
r &\to e^{-\sigma} r\,,
\end{aligned}
$$

which can be deduced from the form of the metric in (3.3).

Any modifications to the theory must respect these symmetries. The plan is now straight forward, promote F to be the field strength of a connection of a non-abelian gauge group. Let the scalars and fermions transform in the adjoint representation of this gauge group. We then proceed to promote all the covariant derivatives to gauge covariant derivatives. At this point the theory is gauge invariant but no longer supersymmetric. To continue we begin by satisfying the condition that in the case of a trivial fibration the theory should reduce to supersymmetric Yang-Mills. To this end we add the standard Yukawa and ϕ^4 terms.

$$
S = S_\phi + S_\psi + S_F + \int_{M_5} \mathrm{d}^5x\sqrt{-g}\Big(2\frac{1}{r}f^{abc}\phi_a\bar{\psi}_b\psi_c + \frac{1}{r}f^{ab}{}_e f^{cde}\phi_a\phi_b\phi_c\phi_d\Big)\,.
$$

To preserve the supersymmetry of the action we also modify the supersymmetry variations according to

$$
\delta\psi_a = \cdots + 2f_a{}^{bc}\phi_b\phi_c\epsilon\,.
$$

With these modifications the theory reduces to five-dimensional super Yang-Mills in the case of a trivial fibration with product metric.

The main result of Paper I is that these modifications also in the case of a general fibration geometry constitutes a supersymmetric theory. We also argue that the above modifications constitute the only possible extension to the abelian theory respecting all the symmetries present.

4

Topological twisting

This chapter concerns the main technique used in Paper II and Paper III, topological twisting. Briefly, it is a method to create topological field theories out of supersymmetric theories. From another perspective it can be viewed as a method to create supersymmetric theories on general curved manifolds from theories on flat manifolds. Topological twisting was introduced to theoretical physics as a technique in the 80's [93], and has since been used to derive many striking results [94–98].

Section 4.1 reviews the basic requirements for supersymmetry on curved backgrounds that can be satisfied by performing a topological twist, reviewed by an example in section 4.2. The topological twisting of (2,0) theory is covered in section 4.3 where the contents of Paper II and Paper III i presented.

4.1 Supersymmetry on curved manifolds

Let us start from the perspective of trying to create a globally supersymmetric theory on a curved manifold. A supersymmetry transformation is parametrised by a constant spinor ε. On a flat manifold there is no ambiguity in what we mean by a constant parameter, it simply has no space-time dependence. However when the theory lives on a curved manifold the situation becomes more tricky. The proper generalization to being constant is to be covariantly constant. So to have a good parameter for supersymmetry we need to find a covariantly constant spinor. This is in general impossible and imposes severe constraints on the geometry of the manifold as can be easily seen. Suppose we have a spinor ε that is covariantly constant:

$$D_\mu \epsilon = 0.$$

The above condition trivially implies $[D_\mu, D_\nu]\,\varepsilon = 0$ and using the fact that the covariant derivatives commutes to the Riemann tensor we find

$$R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\epsilon = 0.$$

This is an integrability condition for the curvature on the manifold which in general is not satisfied.

4.2 The twist

Topological twisting solves this problem in a very elegant fashion using the tools of group theory. The idea is to replace the space-time group by a new one, combining the space-time symmetries with the R-symmetry. The spinor representation of the original theory will now be reducible and will, under certain circumstances, contain a part that does not transform at all under the new space-time group. This means in particular that this part of the spinor also transforms trivially under the new space-time holonomy group and thereby can be considered as a rigid supersymmetry parameter.

4.2.1 Example: $\mathcal{N} = 2$, $D = 4$ SYM

The details of the technique is best explained by an example. A very instructive example is that of the original paper introducing the concept, namely the twisting of $N = 2$ super Yang-Mills theory on four-dimensional Euclidean space-time. [93].

This theory has an SU(2) R-symmetry (denoted below by $\mathrm{SU}(2)_\mathrm{R}$) and the space-time symmetry group is Spin(4). The space-time group is isomorphic to $\mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)_\mathrm{r}$, where the two groups are often referred to as the left and right part (here indicated by the subscripts). The supersymmetry parameters

ε are chiral spinors transforming in the two-dimensional representation of the R-symmetry group $SU(2)_R$. Let us look at one of these parameters, the one transforming under $SU(2)_r$. Its transformation properties are summarised as.

$$\begin{array}{c|ccccc} & SU(2)_l & \times & SU(2)_r & \times & SU(2)_R \\ \hline \varepsilon \in & \mathbf{1} & & \mathbf{2} & & \mathbf{2}\,. \end{array}$$

Let us now define a new SU(2). For concreteness let us denote the generators of $SU(2)_r$ by $\{T_l^i\}$ and the generators of $SU(2)_R$ by $\{T_R^i\}$ with $i \in 1,2,3$. We define a new set of generators T^i_{twist} generating a new SU(2) as follows.

$$T^i_{\text{twist}} = T^i_r + T^i_R$$

The new group is thus what is called the diagonal of $SU(2)_r \times SU(2)_R$, we rotate in both factors at the same time.

How will ϵ transform under this new group? What we have been doing simply amounts to taking the tensor product of the two representations and the answer is that

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}.$$

The representation $(\mathbf{1},\mathbf{2},\mathbf{2})$ under $SU(2)_l \times SU(2)_r \times SU(2)_R$ therefore splits into $(\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$ under $SU(2)_l \times SU(2)_{\text{twist}}$.

$$\begin{array}{ccc} SU(2)_l \times SU(2)_r \times SU(2)_R & \xrightarrow{\text{twist}} & SU(2)_l \times SU(2)_{\text{twist}} \\ (\mathbf{1},\mathbf{2},\mathbf{2}) & & (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \end{array}$$

Note that the first term is a singlet under both factors. Thus we find that from the original 8 supercharges we have constructed one scalar supercharge under the new Lorentz group. Let this scalar supercharge be called simply Q.

$$Q \in (\mathbf{1},\mathbf{1})$$

This supercharge have many interesting properties. From the supersymmetry algebra

$$\left\{Q^\alpha, Q^\beta\right\} = (\gamma_\mu)^{\alpha\beta} P^\mu\,,$$

we see that after the twisting we have a Lorentz scalar in the left hand side but there is no Lorentz scalar operator available for the right hand side so we must have

$$Q^2 = 0.$$

Here we have an operator that squares to zero. This invokes a strong urge to immediately look for Q-closed and Q-exact quantities and examine what their cohomology looks like. In words the cohomology of this operator means that we are looking at supersymmetric quantities but we don't care if they differ by a quantity that is the supersymmetry transformation of something else.

A first observation is that the expectation value of a Q-exact operator vanishes.

$$\begin{aligned} \langle \delta_Q \mathcal{O} \rangle &= \int \mathcal{D}\Phi \; (\delta_Q \mathcal{O}) \; e^{-S[\Phi]} \\ &= \int \mathcal{D}\Phi \, \delta_Q \left(\mathcal{O} \, e^{-S[\Phi]} \right) \\ &= \int \mathcal{D}\Phi \left(\mathcal{O} \left[\Phi + \delta\Phi \right] e^{-S[\Phi+\delta\Phi]} - \mathcal{O} e^{-S[\Phi]} \right) \\ &= 0 \end{aligned} \tag{4.1}$$

In the first step we use that the action is supersymmetric. The last step assumes that the path integral measure is supersymmetric so that the total supersymmetry variation can be absorbed by a change of variables in field space, $\Phi' = \Phi + \delta\Phi$ with $\mathcal{D}\Phi' = \mathcal{D}\Phi$.

It also turns out that in this theory the stress tensor is a Q-exact quantity.

$$T^{\mu\nu} = \delta_Q \lambda^{\mu\nu}$$

This has a very interesting consequence. Lets regard the expectation value of a supersymmetric and metric independent operator O and lets see how it behaves under a metric perturbation.

$$\delta_g \langle \mathcal{O} \rangle = \int \mathcal{D}\Phi \mathcal{O} \delta_g e^{-S[\Phi]}$$

By the definition of the stress tensor and the fact that it is Q-exact it follows that

$$\begin{aligned} \delta_g \langle \mathcal{O} \rangle &= \int \mathcal{D}\Phi \mathcal{O} \delta g_{\mu\nu} T^{\mu\nu} e^{-S[\Phi]} \\ &= \int \mathcal{D}\Phi \mathcal{O} \delta g_{\mu\nu} \delta_Q \lambda^{\mu\nu} e^{-S[\Phi]} \\ &= \left\langle \delta_Q \left(\delta g_{\mu\nu} \lambda^{\mu\nu} \right) \right\rangle \\ &= 0 \, . \end{aligned}$$

In the last step the result in (4.1) is used, i.e. that the expectation value of a Q-exact operator vanishes. The upshot of all this is that if attention is restricted to Q-cohomology then the theory is in fact topological.

4.3 Twisting (2,0)

The existence of the interacting (2,0) theory in six dimensions has, as we have seen in the previous chapters, provided an explanation of many properties of lower dimensional theories. One beautiful example of this is the construction due to Gaiotto [99]. Here a whole class of four-dimensional supersymmetric gauge theories[1] are constructed by compactifying (2,0) theory on a Riemann surface with possible defects. Their common origin in the six-dimensional theory induces a web of dualities between these theories, a kind of S-duality. Closely related to this construction is a recent conjecture that there is a correspondence between four-dimensional $N = 2$ gauge theories and two-dimensional conformal field theories. This conjecture[2] is referred to as the AGT[3] correspondence [102] and states, among other things, that correlation functions in Liouville theory can be computed by the Nekrasov [103] partition function of a four-dimensional $N = 2$ super Yang-Mills theory.

One very natural explanation of this was put forward in [104] and also indicated in [105]. The observation is that if we could somehow compactify (2,0) theory on the four-dimensional manifold instead we would end up with a conformal theory on the Riemann surface. The idea would then be to look for quantities that are protected under both compactifications and this would then hopefully explain the correspondence.

Again the fact that there is no explicit formulation of (2,0) theory means that these ideas rest on the assumption of the existence of the theory. Furthermore in [104] it is assumed that when the compactification on the Riemann surface is performed and the holonomy of the four manifold is twisted, the result is a topological field theory on the four manifold. In Paper II we investigate these claims by explicit calculations in the setting of the abelian theory

[1] Usually referred to as class $\mathcal{S}$ [100, 101].

[2] There is as of yet no formal proof of this correspondence.

[3] Alday, Gaiotto and Tachikawa [102].

version of (2,0) theory.

One of the main differences between our treatment and that of [104] is that we work in Lorentzian signature. There are certain conceptual difficulties when formulating the theory in Euclidean signature. One of them is that the Hodge dual does not square to one but rather to minus one which implies that we cannot regard a real self-dual three-form but rather a complex three-form. The situation for the spinors is also different in the two signatures. To be as explicit as possible and to avoid any pitfalls with the choice of Euclidean signature we choose to carry out our investigation in Lorentzian signature.

4.3.1 Overview of paper II and III

The setup used in both PAPER II and PAPER III is summarised in figure 4.1. The starting point is to regard abelian (2,0) theory on a Lorentzian manifold $M_{1,5} = C \times M_4$, where C is a compact two manifold. We choose to work in Lorentzian signature to avoid the problems associated to formulating (2,0) theory in Euclidean signature. This choice of signature is not ideal from the perspective of twisting as will be shown shortly, however we try to stay as close as possible to the proposed construction in Euclidean signature and see where it leads us.

Abelian (2,0) theory on $M_{1,5} = C \times M_4$

↓

Twist M_4 holonomy

↓

Compactify on C

Figure 4.1: *Setup for* PAPER II *and* PAPER III. *Abelian (2,0) theory is twisted and compactified on a Lorentzian manifold of the form $C \times M_4$.*

In PAPER II the details of the twisted theory is worked out and the flat theory on M_4 is shown to possess a Q-exact stress tensor. The extension to a formulation on curved manifolds is investigated where problems arise when trying to naively covariantize the theory. PAPER III resolves these issues and constructs a compact[4], off-shell formulation in terms of the Q-cohomology.

4.3.2 Lorentzian twist

In chapter 2 we saw that (2,0) theory has an R-symmetry group Spin(5), in this section indicated by a subscript R. Thus on a Lorentzian manifold $M_{1,5}$ the bosonic part of the symmetry group of the theory is given by

$$\mathrm{Spin}(1{,}5) \times \mathrm{Spin}(5)_{\mathrm{R}}.$$

We now take the six manifold to be of the form

$$M_{1,5} = C \times M_4,$$

where C is a compact two manifold with Minkowski signature and M_4 is a four manifold. The space-time symmetry group is broken into two parts and we now have

$$\underbrace{\mathrm{Spin}(1{,}1)}_{C} \times \underbrace{\mathrm{Spin}(4)}_{M_4} \times \mathrm{Spin}(5)_{\mathrm{R}}.$$

It should here be pointed out that the space-time symmetry group contains a non-compact part when working in Minkowski signature. Normally to be able to find a scalar supercharge the subgroup of the holonomy group that the untwisted supercharge transforms under needs to be twisted. This means that the corresponding subgroup of the Lorentz group must be embeddable into the R-symmetry group of the theory. Since it is not possible to embed a non-compact group into a compact group we will only perform a partial twist corresponding to the relevant parts of the holonomy on M_4.

Now we make a few observations regarding the structure of the symmetry groups. The Lorentz group on M_4 is Spin(4) which is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$. As before we let a subscript l and r denote the left and the right factor respectively. For the R-symmetry group we have that $\mathrm{Spin}(3) \times \mathrm{Spin}(2) \subset \mathrm{Spin}(5)_R$. Using that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ we thus have that

$$\mathrm{SU}(2)_{\mathrm{l}} \times \mathrm{SU}(2)_{\mathrm{r}} \times \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_{\mathrm{R}} \subset \mathrm{Spin}(4) \times \mathrm{Spin}(5)_{\mathrm{R}}. \tag{4.2}$$

[4] In the notational sense.

The supersymmetry parameter ε is a symplectic Majorana-Weyl spinor of negative chirality. In terms of the Lorentz group on M_4 it therefore[5] transforms as $(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$ under $\mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)_\mathrm{r}$. It transforms in the $\mathbf{4}$ of $\mathrm{Spin}(5)_\mathrm{R}$ which under the subgroup $\mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{R}$ transforms as $\mathbf{2}^{\frac{1}{2}} \oplus \mathbf{2}^{-\frac{1}{2}}$.

Combining this information we have that the supersymmetry parameter transforms under the subgroups in (4.2) as

$$\varepsilon \in (\mathbf{1},\mathbf{2},\mathbf{2})^{\pm\frac{1}{2}} \oplus (\mathbf{2},\mathbf{1},\mathbf{2})^{\pm\frac{1}{2}}.$$

From the results in section 4.2.1 we now see what needs to be done to find a scalar supercharge. There are two possibilities that are equivalent. We choose to twist $\mathrm{SU}(2)_\mathrm{r}$ and $\mathrm{SU}(2)_\mathrm{R}$. Let $\mathrm{SU}(2)' = \mathrm{SU}(2)_\mathrm{r} \times \mathrm{SU}(2)_\mathrm{R}$, then the first term in 4.3.2 will be twisted according to table 4.1.

$\mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)_\mathrm{r} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{R}$	$\xrightarrow{\text{twist}}$	$\mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)' \times \mathrm{U}(1)_\mathrm{R}$
$(\mathbf{1},\mathbf{2},\mathbf{2})^{\pm\frac{1}{2}}$		$(\mathbf{1},\mathbf{1})^{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})^{\pm\frac{1}{2}}$

Table 4.1: *Twisting of the supersymmetry parameter representation.*

After twisting there are two scalar components on M_4 with positive and negative $\mathrm{U}(1)_\mathrm{R}$ charge. There is now a definite choice of which supercharge to use. To see this we need to consider the transformation properties under $\mathrm{Spin}(1,1)_\mathrm{C}$. Let $\pm$ indicate the two one-dimensional representations of $\mathrm{Spin}(1,1) \cong \mathbb{R}$. Here we let the $+$ correspond to the representation that in Euclidean signature would have positive charge under $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. Then the full representation of the part of ε in table 4.1 is given by

$$^{+}(\mathbf{1},\mathbf{2},\mathbf{2})^{\pm\frac{1}{2}} \quad \xrightarrow{\text{twist}} \quad ^{+}(\mathbf{1},\mathbf{1})^{\pm\frac{1}{2}} \oplus {}^{+}(\mathbf{1},\mathbf{3})^{\pm\frac{1}{2}}. \tag{4.3}$$

The supersymmetry parameter is a spinor of negative chirality which fixes the choice of $\mathrm{Spin}(1,1)$ representation to be $+$ in (4.3).

[5] The spinor representation $\mathbf{4}$ of negative chirality decomposes under $\mathrm{Spin}(1,1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ as $(\mathbf{2},\mathbf{1})_- \oplus (\mathbf{1},\mathbf{2})_+$ where $\pm$ indicates the eigenvalue under Γ^{01}, for details see Paper II.

$M_6 \xrightarrow{\text{twist}}$	M_4
Φ	$E_{\mu\nu}, \bar{\sigma}, \sigma$
Ψ	$\psi_\mu, \tilde{\psi}_\mu, \chi_{\mu\nu}, \tilde{\chi}_{\mu\nu}, \eta, \tilde{\eta}$
H	$F^-_{\mu\nu}, F^+_{\mu\nu},\ A_\mu$

Table 4.2: *Field content of the twisted theory in four dimensions stemming from the fields in six dimensions.*

If the twisting was carried out in Euclidean signature then we would also have to twist away the dependence on Spin(1,1) by taking the diagonal embedding of $\mathrm{U}(1)'$ in $\mathrm{Spin}(2)_C \times \mathrm{Spin}(2)_\mathrm{R}$. The twisted U(1) charge is then simply the sum of the individual charges. It is then clear that the component of the M_4 scalar representation in (4.3) that would have zero charge under $\mathrm{U}(1)'$ is the one with negative $\mathrm{U}(1)_\mathrm{R}$ charge.

Therefore we choose to regard the M_4 scalar supercharge in the representation $(\mathbf{1},\mathbf{1})^{-\frac{1}{2}}$ when performing the calculations in Paper II.

4.3.3 Twisted tensor multiplet

The fields of the tensor multiplet gives rise to a number of fields when the twisting is performed. With similar arguments as in the previous section one arrives at the field content in (4.2). Here $E_{\mu\nu}$ is a real self-dual two form, σ a complex scalar, $\{\psi,\ \tilde{\psi}\}$ fermionic one forms, $\{\chi,\ \tilde{\chi}\}$ fermionic self-dual two forms, $\{\eta,\ \tilde{\eta}\}$ fermionic scalars, $\{F^-,\ F^+\}$ anti self-dual and self-dual real two forms and finally A_μ a real one-form.

In Paper II the derivation of the twisted field content is performed in detail. Let us here only confirm the counting. The space-time scalar Φ transforms in a five-dimensional representation corresponding to the three components of the self-dual two-form and the two components in the complex scalar. The fermionic fields, counting from left to right, contain $4+4+3+3+1+1=16$ real components corresponding to the 16 real components of the symplectic Majorana-Weyl spinor Ψ. Similarly the fields arising from the self-dual three-form sums to 10 real components.

The six-dimensional equations of motion and supersymmetry transformations give rise to corresponding equations and transformations for the twisted fields. This is also derived in detail in Paper II from the explicit relations between the twisted and untwisted fields of the tensor multiplet.

It turns out that the twisted theory splits into two sectors, one containing $\{E_{\mu\nu}, \tilde{\psi}_\mu, A_\mu, \chi_{\mu\nu}, \eta\}$ and the other containing $\{F_{\mu\nu}, \tilde{\chi}_{\mu\nu}, \psi_\mu, \sigma, \bar{\sigma}, \tilde{\eta}\}$.The latter sector is equivalent on-shell to Donaldson-Witten theory, the unique twist of $N = 2$ super Yang-Mills, whereas the former is related to the Vafa-Witten twist of $N = 4$ super Yang-Mills.

4.3.4 Stress tensor

After a compactification on C the equations for the twisted fields will be purely four-dimensional. We then proceed to determine the stress tensor for the theory defined on a flat manifold. This is done in two steps. First an ansatz for the stress tensor for the fields arising from Φ and Ψ is made from the metric variation of the action that does exists for these fields. For the fields arising from H an ansatz is made starting from the stress tensor for a general three-form. It is found that this ansatz needs to be modified for the stress tensor to be supersymmetric. The modified stress tensor is then shown to be Q-exact with

$$T^{\mu\nu} = \delta_Q \lambda^{\mu\nu},$$

where

$$\begin{aligned}\lambda^{\mu\nu} = \frac{1}{2}\Big(&\sqrt{2}i\psi^{(\mu}\partial^{\nu)}\sigma + \tilde{\psi}^{(\mu}\partial^{\rho}E^{\nu)}{}_{\rho} + \partial_\rho\tilde{\psi}^{(\mu}E^{\nu)}{}_{\rho} - \partial^{(\mu}\tilde{\psi}^{\rho}E^{\nu)}{}_{\rho}\\ &+ i\tilde{\psi}^{(\mu}A^{\nu)} - \frac{i}{2}\tilde{\chi}^{(\mu}{}_{\rho}F^{-\nu)\rho} - \frac{i}{\sqrt{2}}g^{\mu\nu}\psi_\rho\partial^\rho\sigma - \frac{1}{2}g^{\mu\nu}\tilde{\psi}_\rho\partial_\sigma E_{\rho\sigma} - \frac{i}{2}g^{\mu\nu}\tilde{\psi}_\rho A^\rho\Big).\end{aligned}$$

Here Q denotes the scalar supercharge transforming as $(\mathbf{1},\mathbf{1})^{-\frac{1}{2}}$ under the twisted Lorentz group, described in the previous section.

The question then arises as to how this expression carries over to the case of a curved manifold. In Paper II this extension is investigated with the result that a naive covariantization meets some problems pertaining to the bosonic self-dual fields. The conclusion was that it seemed difficult to construct necessary curvature corrections to allow the stress tensor to be both conserved and Q-exact.

4.4 Off-shell structure and resolution

It turns out that the issues with a conserved covariant stress tensor can be solved by certain additions to the equations of motion for the self dual two-form $E_{\mu\nu}$. In PAPER II it was shown that the obstruction to a conserved stress tensor consisted of certain curvature terms of the form

$$\mathrm{D}^{\mu}T_{\mu\nu} = -\frac{1}{4}\mathrm{D}_{\tau}R_{\rho\kappa}E^{\tau\kappa}E_{\nu}{}^{\rho} + \frac{1}{8}\mathrm{D}_{\nu}R_{\mu\kappa\rho\tau}E^{\tau\kappa}E^{\mu\rho}.$$

This might seem surprising but since the stress tensor in this twisted theory is not derived from an action, but rather from the equations of motion, conservation is not guaranteed. In fact, the procedure to find the correct stress tensor in PAPER II consisted of using an ansatz obtained from the scalar action that was then modified to ensure supersymmetry. A natural guess would then be to look for suitable modifications to the equations of motion for the self-dual two form to ensure also conservation of the stress tensor. From the six dimensional perspective one very natural addition in the curved case is that of the conformally coupled scalar. For the six dimensional scalar action to be conformally invariant an addition of the form $R\Phi^2$ is needed, which would imply a correction to the equations of motion for $E_{\mu\nu}$ of the form

$$\mathrm{D}^2 E_{\mu\nu} = aRE_{\mu\nu}. \tag{4.4}$$

In PAPER II it is shown that this kind of addition is not enough.

Fortunately, it turns out that there is another possible curvature correction. Since $E_{\mu\nu}$ is self-dual any new term must also respect this symmetry. Besides the term in (4.4), the only other possibility is

$$(P^{+})_{\mu\nu}{}^{\tau\sigma}R_{\tau\sigma}{}^{\rho\lambda}E_{\rho\lambda},$$

where P^{+} is the projection on the self-dual part. In PAPER III it is shown that an addition of this form, together with the term in (4.4) makes the stress tensor conserved.

4.4.1 Q-cohomology formulation

In the example of twisted $N = 2$ super Yang-Mills, the stress tensor is Q-exact. One way to show this is by noting that the action is itself Q-exact with

$L = \{Q, V\}$. The quantity V is Grassmann odd and can be thought of as a supersymmetric potential of the action. This implies that a metric variation of the action takes the form

$$\begin{aligned}\delta_g \int_{M_4} \sqrt{g}\, L &= \int \sqrt{g} \left(\frac{1}{2} \mathrm{Tr}(\delta g)\{Q, V\} + \delta_g \{Q, V\} \right) \\ &= \int \sqrt{g}\, \delta g_{\mu\nu} \left\{ Q, \frac{1}{2} \mathrm{Tr}(\delta g) V + \frac{\delta V}{\delta g_{\mu\nu}} \right\}.\end{aligned}$$

The last step gives a manifestly Q-exact expression for the stress energy tensor.

In Paper III the same structure is shown to also exist for both sectors of the twisted theory on M_4. The result is a very compact formulation of the theory in terms of the quantity V given by

$$V = -\left(\frac{1}{2}(iA_\mu + h_\mu) - \mathrm{D}_\nu E_\mu{}^\nu \right) \tilde{\psi}^\mu - \frac{1}{8} \left(F_{\mu\nu} - \frac{1}{2} B_{\mu\nu} \right) \tilde{\chi}^{\mu\nu} + \frac{i}{\sqrt{2}} \psi^\mu \mathrm{D}_\mu \sigma,$$

together with the supersymmetry variations

$$\begin{aligned}\delta E_{\mu\nu} &= i\chi_{\mu\nu} v\,, \\ \delta \tilde{\psi}_\nu &= ivA_\nu - v\mathrm{D}_\mu E_\nu{}^\mu\,, \\ \delta A_\mu &= \mathrm{D}_\mu \eta\,,\end{aligned} \qquad \begin{aligned}\delta F_{\mu\nu} &= -4\mathrm{D}_{[\mu} \psi_{\nu]} v\,, \\ \delta \tilde{\chi}_{\mu\nu} &= 2ivB_{\mu\nu}\,, \\ \delta \psi_\nu &= -vi\sqrt{2}\mathrm{D}_\nu \bar{\sigma}\,, \\ \delta \sigma &= \sqrt{2} \tilde{\eta} v\,,\end{aligned}$$

where the variations of the remaining fields are zero. Notice that since the supersymmetry variation of the field $B_{\mu\nu}$ vanishes the Lagrangian is just $L_B = 2iB_{\mu\nu}B^{\mu\nu}$ which implies that $B_{\mu\nu} = 0$ on-shell. This auxiliary field makes the off-shell formulation of the theory in terms of the potential V possible by ensuring that $Q^2 = 0$, which otherwise is only valid on-shell.

The full relationship derived in Paper III between the quantities V, L, $\lambda^{\mu\nu}$ and $T^{\mu\nu}$ can be summarised in the commuting diagram in figure 4.2 on the next page.

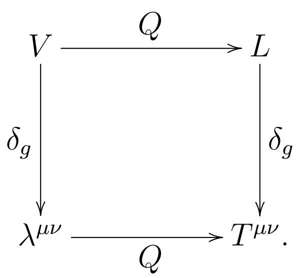


Figure 4.2: *The relationship between the quantities V, $\lambda^{\mu\nu}$ that generate the Lagrangian and the stress tensor. Metric variations are denoted by δ_g and supersymmetry transformations with Q.*

5

Higher spin

In the standard model of particle physics there are fundamental particles of spin 0 (Higgs), $\frac{1}{2}$ (Leptons and quarks) and 1 (Vector bosons). An outstanding problem is to include gravity at the quantum level, although it is clear that the force is mediated by a spin 2 boson: the graviton. Losely speaking any theory that contains fundamental constituents of spin greater than 2 is said to be a theory of higher spin. These kinds of theories have a long history stretching back all the way to the inception of quantum field theory where they were at first discarded as inconsistent. They have seen periods of popularity as it was later realised that consistent theories could be formulated and today higher spin is a very active field of research with connections to string theory, nuclear physics and holography. This chapter serves as a selective introduction to the parts relevant for this thesis.

Theories of higher spin have many features that makes them an interesting topic to study. As is common in the world of fundamental theories of physics, there are many possible ways to arrive at a their doorstep. One very natural place to start in the context of this thesis is with the quest for symmetry. Higher spin theories are indeed very symmetric, a point that will be elaborated on in section 6.2, but before we dive into these more abstract considerations I would like to start by giving an overview of how higher spin theory fit into larger picture of physics and make contact with some of its history. There are also tantalising connections to the topic of the first half of this thesis, (2,0) theory, to be elaborated on in section 5.2.2. As a first taste of the intricacies that these theories present, why not begin by investigating what the general principles of physics tell us about their properties?

5.1 Go, no-go: An existential crisis

Within the framework of quantum field theory there are strong restrictions for any interacting theory of matter that abide by the rules of special relativity and other relevant symmetry properties. These restrictions can be summarised in so called "no-go" theorems, previously considered in section 2.1.2, where one makes natural assumptions for a physical theory and then see what limits these entail. In the case of higher spin there are a number of such "no-go" theorems and one might at first glance come to the conclusion that such theories cannot exist in any useful form relevant for physics. However, under closer scrutiny, it turns out that the assumptions leave some possibilities open. To appreciate the situation it is instructive to take a look at one of these theorems and then see how it can be avoided.

Let's start with a very compact argument due to Weinberg [106][1]. This particular example is also a very nice demonstration of how simple principles can be used to draw striking conclusions about the structure of a theory. Consider a quantum field theory in four space-time dimensions. Let it contain some massive scalar particle ϕ (for example the Higgs particle). Now lets first regard its minimal coupling to a new massless particle of spin 1. Minimal coupling implies here the introduction of the simplest possible[2] interaction term in the Lagrangian for the spin 1 particle, commonly denoted A_μ, which for a scalar particle takes the form[3]

$$eA^\mu \phi \partial_\mu \phi \,,$$

where e is the charge of ϕ. The astute reader well versed in the wonders of QFT can now without effort take the Fourier transform of this expression and

[1] The exposition here will be hand-waving at best, for a detailed derivation see [107].

[2] Note that minimal coupling is enough to describe all of the standard model.

[3] We are looking for a scalar containing A_μ which can only be constructed by introducing a derivative. Alternatively this vertex follows from the Klein-Gordon action $D^\mu \phi D_\mu \phi$ with the gauge-covariantised derivative $D_\mu \phi = (\partial_\mu + eA_\mu)\,\phi$.

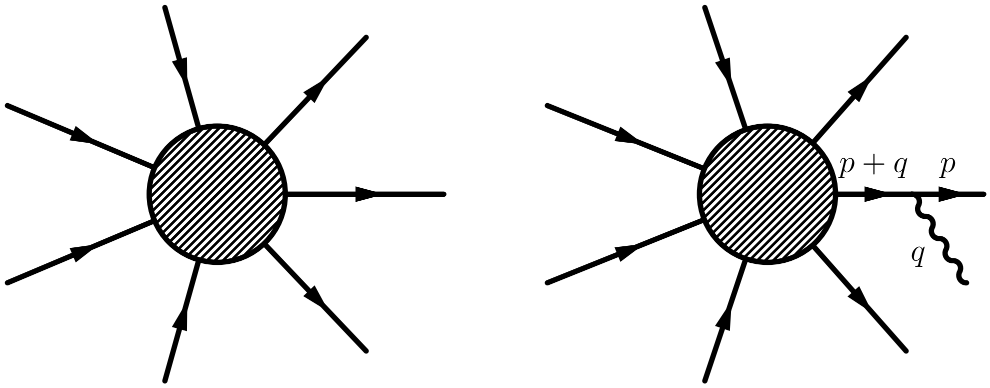


Figure 5.1: *General n-particle process without and with the emission of an additional massless particle.*

conclude that the corresponding vertex will carry a factor of the momentum of the scalar particle. In figure 5.1 a general process is depicted where the blob represents some form of interaction of the original theory (tree level or loop).

In the right-hand diagram the same process is supplemented by one of the legs emitting the new massless particle with momenta q. Momentum conservation then specifies the new propagator to carry momentum $p+q$. If the original process involving n particles was given by the matrix element M_n then the new process will multiply this with an additional factor for the new vertex and propagator[4] roughly of the form

$$M'^{\mu}_{n+1} = M_n \sum_i \frac{e\eta_i \left(p_i^\mu + q^\mu\right)}{(p_i+q)^2 - m^2} = \sum_i M_n \frac{e\eta_i \left(p_i^\mu + q^\mu\right)}{2p_i \cdot q},$$

where the sum is over all the external particles, η_i is ± 1 for initial/final state and in the last step it is used that $p^2 = m^2$ and $q^2 = 0$. For very small q the leading contribution is

$$M'^{\mu}_{n+1} \approx M_n \sum_i \frac{e\eta_i p_i^\mu}{2p_i \cdot q} \tag{5.1}$$

The amplitude corresponding to this process is given by contracting this matrix element with the polarization vector for the massless particle. For the spin 1

[4]The denominator corresponds to the massive propagator and the numerator stems from the vertex.

particle considered here the polarization vector is given by ϵ_μ.

$$\mathcal{A} = \epsilon_\mu M'^{\mu}_{n+1}$$

Recall now that the covariant theory of a massless spin 1 particle, formulated in terms of a vector potential A_μ, enjoys a gauge symmetry where it is invariant under $A_\mu \longrightarrow A_\mu + \partial_\mu \alpha(x)$. In terms of the polarization vector this transformation reads

$$\epsilon_\mu \rightarrow \epsilon_\mu + \alpha q_\mu \, .$$

Hence, if the amplitude is to be gauge invariant it is demanded that

$$q_\mu M'^{\mu}_{n+1} = 0 \, .$$

Looking at (5.1), this implies that

$$\sum_i e\eta_i = 0 \, , \tag{5.2}$$

i.e. the process conserves the total charge. If this process is repeated for a massless spin 2 particle the matrix element (5.1) receives an additional factor of momentum in the numerator[5] and so the analogue of (5.2) becomes

$$\sum_i e\eta_i p_i^\mu = 0 \, ,$$

but since the total momentum is conserved (also by Lorentz invariance) this implies that all the charges have to be equal, i.e. all particles must couple in the same way to a massless spin 2 particle. This is the quantum version of the equivalence principle: gravity acts universally on all matter.

Now what happens if we consider adding a massless particle of spin 3? Already for spin 2 the resulting constraint is very tight! Again the numerator in (5.1) receives another momentum factor and the final relation reads

$$\sum_i e\eta_i p_i^\mu p_i^\nu = 0.$$

[5]This can be seen easily by considering the corresponding coupling term in the Lagrangian which now must contain an additional derivative so as to create a scalar out of the spin 2 field with two indices.

This relation simply cannot be satisfied for arbitrary momenta unless $e = 0$, i.e. if massless spin 3 particle does not interact at all.

A quick glance at this might result in statements such as “Interacting massless higher spin particles are forbidden”. Looking carefully at the assumptions a more correct conclusion is that *minimally coupled massless higher spin particles in Minkowski space are forbidden*.

However, the first statement is not entirely wrong. If we take it to mean that interacting massless higher spin particles are forbidden in the low energy limit it is actually perfectly correct! By the theorem above such particles would have to be non-minimally coupled which results in interaction terms that are irrelevant in the Wilsonian sense and hence will vanish in the low energy limit [108]. Weinberg’s arguments tells us that we should not expect to observe massless higher spin particles in our everyday life, which we confirm empirically: there are no observed long range forces mediated by a higher spin carrier.

Another way of succinctly rewording the above conclusion is that it is very difficult to construct a theory for a massless particle of higher spin that removes the unphysical polarizations. In the case of lower spin this problem is what leads us to gauge invariance: The covariant theory of a photon is formulated in terms of a four-vector A_μ that a priori propagate four degrees of freedom. This is only brought down to the two physical polarizations after we construct the theory to be gauge invariant. The arguments above, that rests firmly on the principle of gauge invariance, shows that for higher spin this is not possible in the minimal setting[6].

5.2 Why higher spin?

5.2.1 Massless higher spin, nuclear physics and string theory

So why are massless higher spin particles interesting? Certainly there are *massive* higher spin particles observed in nature. The spectrum of particles in nuclear physics contain a proliferation of these. Here they arise as com-

[6] In the sense of the previous paragraphs

posite particles of the more fundamental quarks for which we do have a good description, quantum chromodynamics. As is by now well understood, a good microscopic description might not be a good tool to understand the more macroscopic physics. For example we do not use quantum electrodynamics to calculate how many light bulbs a certain fuse can support, we use the simpler Maxwell theory of electromagnetism which for this purpose is very accurate. In the case of hadronic resonances or nuclei interacting at low energy one is led to look for an effective theory valid for calculation in energy regimes relevant for nuclear processes. Here one looks for local Lagrangian for the effective degrees of freedom. Even for massive higher spin this poses big difficulties where minimal coupling to electromagnetism implies superluminous propagation [109]. Thus higher spin theory is interesting also from a nuclear physics perspective.

The concept of high energy limits presents, at least for the author, the most compelling reasons to study higher spin. Not in the context of nuclear physics but in the search for fundamental theories of physics. In the quest for a quantum description of gravity, one of the most promising directions is string theory. The spectrum of string theory contains, apart from the building blocks needed for the standard model and quantum gravity, an infinite number of massive higher spin states[7]. These states have mass squared proportional to the string tension $\frac{1}{\alpha'}$ that for a given spin s satisfies [15]

$$M^2 > C\frac{s}{\alpha'},$$

where C is a constant. Thus string theory contains higher spin states but they are massive so that they will be invisible at low enough energy. However in the limit where $\alpha' \to \infty$ they become massless. So also in the case of string theory there is a limit that is related to an interacting theory of massless higher spin states [110]. In fact this relation has today been explored to some extent and many interesting results are now known [111–116].

[7]The connection to nuclear physics is not by chance, string theory arose from the attempts to create a local interacting theory of massive higher spin objects observed in nuclear physics.

5.2.2 Holography

Recently a new door opened up into the toolbox of mathematical physics: the AdS/CFT correspondence [75], for a modern review see [117]. This duality between theories in AdS spacetime and conformal field theory on its boundary has had profound implications ranging from a better understanding of string theory [75, 118], strongly coupled systems such as the quark gluon plasma [119], black holes [120] and recently also higher spin theory [121].

The connection to higher spin theory can be loosely derived starting from the considerations in the previous section. In the standard incarnation of the duality the theories of interest are type IIB string theory on $\mathrm{AdS}_5 \times S^5$ with N units of flux on S^5 and $\mathcal{N} = 4$ supersymmetric $\mathrm{U}(N)$ Yang-Mills theory. The parameters of the string theory are the string coupling g_s and the tension $\frac{1}{\alpha'}$ whereas the Yang-Mills theory has the coupling constant g_{YM}. In its original formulation [75] the duality states that these theories are equivalent when their parameters (up to numerical factors) are related as

$$\begin{aligned} g_{\mathrm{YM}}^2 N &= R^4 \frac{1}{\alpha'^2}, \\ g_s &= g_{\mathrm{YM}}^2, \end{aligned} \tag{5.3}$$

in a certain limit: the large N limit is taken while keeping $g_{\mathrm{YM}}^2 N$ fixed and finally the t'Hooft coupling $g_{\mathrm{YM}}^2 N$ is taken to infinity as well.

On the string theory side the first limit is, as can be seen above, the small string coupling limit where tree level string theory becomes a good approximation. The second one corresponds to a small string length limit, i.e. a point-like limit where supergravity is a good description. From the gauge theory perspective the first limit is the planar limit with an infinite number of colors. Here the effective gauge coupling is not g_{YM} but rather $g_{\mathrm{YM}}\sqrt{N}$, so that the first limit is the planar limit for fixed effective gauge coupling[8]. The second limit is then clearly the strong coupling limit.

When the dusts settles after this bonanza of limits and identifications the result is that the strongly coupled Yang-Mills theory is dual to the weakly coupled string theory in the low energy limit, i.e. supergravity. In terms of

[8]Note in particular that the first limit implies $g_{\mathrm{YM}} \to 0$, this does *not* imply that the gauge theory becomes weakly coupled.

computational complexity this is a relation between the difficult strongly coupled Yang-Mills theory and the easy weakly coupled supergravity theory. This easy-hard duality is the heart of what makes the AdS/CFT correspondence so fantastic.

This original form of the duality is, as we see above, a very specific limit both in energy and parameters. There is by now evidence [76] that the duality is much more far reaching where the most optimistic conjecture is that it is valid generally, i.e. for arbitrary parameters.

In the context of higher spin one looks at the above relations and thinks about a different limit. As was shown in the previous section the tensionless limit of string theory should correspond to a theory of massless higher spin. In the relation (5.3) the right hand side tends to zero as $\frac{1}{\alpha'} \to 0$ with R large and fixed. On the other side this mean that now the t'Hooft coupling tends to zero so that the Yang-Mills theory becomes free. If one is to trust this implication and if the zero tension limit of string theory really is a higher spin theory then a strange conclusion is made: Higher spin theory on AdS_d is dual to a free conformal field theory in $d-1$ dimensions. At first this might seem unreasonable, how can an interacting theory be equivalent to a free theory? Or even worse, does this imply that higher spin theory is trivial in some sense? The fears of the latter question fortunately are not materialised and in fact the answer turns out to be very interesting. Actually the concept of interacting theories expressed in terms of free theories is not new, bosonization in two dimensions is one such example. Instead this fact becomes a virtue for the higher spin theory where the free conformal theory can be used to calculate properties of the higher spin theory [121, 122].

Recently the duality for higher spin has been partly verified and in fact it has at this point become one of the corner stone examples of these types of dualities. One reason for this is that it turns out that in particular instances of the duality there is a match between a free $\mathrm{O}(N)$ CFT and higher spin theory where the coupling is proportional to $\frac{1}{\sqrt{N}}$. This means that in the large N limit there is a correspondence between a free higher spin theory and a "free" CFT in the sense that the correlation functions factorise. Thus this is an example of a weak-weak duality which is an excellent chance to be able to show the correspondence rigorously where explicit calculations can be carried out on both sides [121, 122].

The connection to the first part of this thesis is made when one looks at another example of the AdS/CFT correspondence in light of higher spin. Here one starts with the conjectures correspondence between M-theory on $\text{AdS}_7 \times S_4$ and the A_N series of (2,0) theory in six dimensions [75]. If the same type of tensionless limit is taken as before there emerges a duality between a higher spin theory in AdS_7 and free (2,0) theory [123]! What this really means is a bit puzzling since here there is no coupling constant which means that the A_N theory for large N is inherently interacting. At least the small N limit would in some sense correspond to the abelian theory and thus would presumably be related to a strongly interacting higher spin theory.

5.3 The road to interactions

> “*The difficulty of this problem is illustrated by the fact that the most immediate method of taking into account the effect of the electromagnetic field, proposed by Dirac (1936), leads to inconsistent equations as soon as the spin is grater than 1.*”
>
> – Markus Fierz & Wolfgang Pauli, *1939*

The first considerations of higher spin particles was initiated by Dirac [124] when investigating relativistic field equations in 1936. Two years later a systematic study by Fierz and Pauli [125] concluded that a set of symmetric traceless fields $\Phi^{\mu_1,\cdots,\mu_n}$ in four dimensions governed by the equations

$$\begin{aligned} \partial_\mu \partial^\mu \Phi^{\mu_1,\cdots,\mu_n} &= m^2 \Phi^{\mu_1,\cdots,\mu_n} \\ \partial^\nu \Phi_{\nu\mu_2\cdots\mu_n} &= 0, \end{aligned}$$

describe consistently free particles of spin n and identified the problems of a simple gauge procedure to introduce interactions. More than this they identified that these equations could be reproduced through a Lagrangian formulation with auxiliary fields and took the first steps towards interactions by noting that seemingly consistent couplings with an external field could be accommodated in this formulation. The auxiliary fields turn out to be necessary to enforce proper transversality conditions on the propagating fields.

These ideas was furthered by Fronsdal [126] and Chang [127] and resulted in a Lagrangian formulation for massive higher spin fields of arbitrary spin in 1974 by Singh and Hagen [128]. Here it was noted that the transversality condition takes the form of successive identities

$$\partial^{\mu_1} \cdots \partial^{\mu_\lambda} \Phi_{\mu_1 \dots \mu_s} = 0 \,,$$

for $\lambda \leq s$ where s is the spin.

At this point in time there is however a growing number of negative results starting with the soft photon argument (see section 5.1) of Weinberg in 1964 and including the Aragone-Deser [129] calculations of the presence of non-physical modes in the coupling of a spin $\frac{5}{2}$ field to gravity. These two results in combination was particularly devastating since gravity, according to Weinberg, has to couple universally. Furthermore the problems persisted also for the massive case where Velo and Zwanziger [130] showed that there is superluminous propagation when interactions are present.

It was another twenty years until finally a consistent interacting higher spin theory in four dimensions was constructed by Vasiliev and Fradkin [131–133]. They achieved this by noting that the previous no-go results all concerned flat space and set out to construct a theory in the other maximally symmetric space-times, de Sitter and anti-de Sitter space. The result is an interacting theory where the interaction terms comes dressed with an infinite sum of higher derivative terms. This means that the interaction is in some ways non-local but are controlled by the fact that they come for higher powers of the inverse AdS radius. This is also the heuristic reason for why there is no limit of these theories in flat space, it is singular. There is by now a large literature on this subject, see [134] for a recent review.

Even before the details of the Vasiliev-type[9] theories was worked out it

[9] Vasiliev continued the program of investigating these higher spin theories and they have since been referred to as "Vasiliev theory" in the literature.

was noted that many of the no-go theorems can also be evaded by going down to three dimensions [135, 136]. For example the Aragone-Deser obstructions come with the Weyl tensor which vanish in three dimensions. These three dimensional theories provide another angle on the problem of higher spin theory and it is in this direction that the work in this thesis builds.

6

Conformal higher spin in 3d

Higher spin in general spacetime dimension is a very difficult problem that only recently was shown to be consistent as an interacting theory [133, 134].

It turns out that for three dimensions the situation is slightly better. Here there exists formulations not only of interacting higher spin of Vasiliev type but also constructions based on Chern-Simons type theories. In particular there are theories of higher spin where the infinite tower of higher spin can be truncated consistently so that only spins up to a finite number is present. This is very different from the situation in for example four dimensions where there are no such truncated theories.

6.1 3d gravity

Quantum gravity in four space-time dimensions is hard, this is one of the current outstanding problems. It is hard for a number of reasons. Perhaps the most basic one being that the Einstein-Hilbert action, that gives rise to the Einstein field equations, is perturbatively non-renormalisable [137]. In three dimensions, the situation is drastically different [138].

It is often the case that things get easier as one moves down in space-time dimension. One can hope that by investigating a certain theory in lower dimensions it is still possible to draw some conclusions on its behavior in higher ones. Though what often happens is that the lower dimensional theory becomes either too simple or different so that in the end it is hard to do just that.

In the case of gravity there is a huge simplification going from four to three dimensions, no propagating degrees of freedom remain. Remarkably, despite this, there are still many interesting phenomena persisting such as black holes [139, 140]. The most interesting feature is that there has been much progress towards a quantized theory [141]. That a theory of quantum gravity seems to exists with an explicit formulation in three dimensions makes it a very interesting toy model to study. Early work include the investigation of the classical theory coupled to matter [142] where the geometry from point like sources results in conical structure. The non-perturbative quantization was carried out in [143] using a Chern-Simons formulation. More recent progress seems to point to some subtleties in the exact definition of the quantum configurations relevant for the path integral [141, 144–146][1].

The feature that makes three dimensional gravity tractable is that it is closely related to a simpler topological model called Chern-Simons theory. This also provides the starting point for an extension to a theory of higher spin considered in Paper IV. As a warm up to the more complicated setup in the higher spin theory the next section will provide an introduction to the gauge formulation of gravity in three dimensions.

6.1.1 Gravity as a gauge theory

The starting point for three dimensional quantum gravity, as well as for the coming investigations, is the fact that Einstein gravity can be formulated as a Chern-Simons gauge theory [148].

Let M be an oriented Lorentz three-manifold and A an iso(2,1) connection. Consider the Chern-Simons [149] action

$$S = \int_M \mathrm{tr}\left(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A\right),$$

where tr is an invariant bilinear form[2]. There is a lot to be said about this

[1] For a recent review see [147].

[2] That such a bilinear form exists [138] for iso(2,1) is another special circumstance in three dimensions that is not true in general for iso($d-1$,1)

functional, as a starting point in context of mathematical physics the reader is referred to [94]. At this point, however, let us take it at face value and see explicitly what it implies. Let T^a be a basis for the Lie algebra. The connection can then be written as $A = A_a T^a$, so that the first term above takes the form

$$\mathrm{tr}\left(T^a T^b A_a \wedge \mathrm{d}A_b\right) = \mathrm{tr}\left(T^a T^b\right) A_a \wedge \mathrm{d}A_b.$$

Choose a particular set of generators of iso(2,1) and write the connection as

$$A = e_a P^a + \omega_a M^a\,, \tag{6.1}$$

where e_a, $\omega_a \in \Lambda^1(M)$ and P^a, M^a satisfy the commutation relations

$$\left[P^a, P^b\right] = 0 \tag{6.2}$$
$$\left[M^a, P^b\right] = \epsilon^{ab}{}_c P^c$$
$$\left[M^a, M^b\right] = \epsilon^{ab}{}_c M^c. \tag{6.3}$$

Here P^a are the three generators for translations and M^a the three Lorentz transformations. This form of the algebra looks very simple but might be slightly confusing for someone working in higher dimensions. In four (and higher) dimensions a rotation can be specified by two directions spanning a plane in which the rotation takes place. In three dimensions, on the other hand, these two directions are uniquely specified by the direction that is not rotated. The relation between these two perspectives is facilitated by the totally antisymmetric epsilon tensor so that the more familiar form is recovered as

$$M^{ab} = \epsilon^{ab}{}_c M^c.$$

Returning now to the Chern-Simons action for the iso(2,1) connection, what equations of motion does it imply? A variation of the connection gives[3]

$$\begin{aligned}\delta S &= \int_M \delta\mathrm{tr}\left(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A\right)\\ &= 2\int_M \mathrm{tr}\Big(\left(\mathrm{d}A + A \wedge A\right) \wedge \delta A\Big),\end{aligned} \tag{6.4}$$

[3] In this calculation one uses the linearity of the trace, the product rule for the variation followed by an appropriate partial integration and repeated use of the cyclic property of the trace.

where one finds the gauge covariant field strength $F = \mathrm{d}A + A \wedge A$ and concludes that the stationary point of the action functional is given by solutions to

$$F = 0\,,$$

i.e. flat connections. What does this imply for the specific parametrization of A in (6.1)?

$$\begin{aligned}
\mathrm{d}A + A \wedge A &= \mathrm{d}e_a P^a + \mathrm{d}\omega_a M^a + (e_a P^a + \omega_a M^a) \wedge \left(e_b P^b + \omega_b M^b\right) \\
&= \mathrm{d}e_a P^a + \mathrm{d}\omega_a M^a + \\
&\quad \tfrac{1}{2}\left[P^a, P^b\right] e_a \wedge e_b + \left[P^a, M^b\right] e_a \wedge \omega_b + \tfrac{1}{2}\left[M^a, M^b\right] \omega_a \wedge \omega_b \\
&= \mathrm{d}e_a P^a + \mathrm{d}\omega_a M^a + \epsilon^{ab}{}_c e_a \wedge \omega_b P^c + \tfrac{1}{2}\epsilon^{ab}{}_c \omega_a \wedge \omega_b M^c = 0
\end{aligned}$$

Since the generators are algebraically independent the above relation imply

$$\mathrm{d}e_a + \epsilon^{bc}{}_a \omega_b \wedge e_c = 0 \tag{6.5}$$

$$\mathrm{d}\omega_a + \tfrac{1}{2}\epsilon^{bc}{}_a \omega_b \wedge \omega_c = 0\,. \tag{6.6}$$

These equations are precisely the zero torsion condition and the vanishing of the curvature two-form, if the fields e_a and ω_a are interpreted as the frame field and spin connection respectively. In fact, the zero curvature equation turns out to be exactly the Einstein equation in three dimensions.

Let us press on a bit further to make this plausible. From the first equation we identify a Lorentz covariant exterior derivative as

$$\mathrm{D}V_a = \mathrm{d}V_a + \epsilon_a{}^{bc} \omega_b \wedge V_c\,,$$

for a form $V \in \mathrm{iso}(2{,}1) \otimes \Lambda^1(M)$. Note that it is covariant under iso(2,1) since (6.5) corresponds to a component of the field strength. Let us see what the curvature for this covariant derivative is by computing

$$\begin{aligned}
\mathrm{DD}V_a &= \mathrm{D}\left(dV_a + \epsilon_a{}^{bc}\omega_b \wedge V_c\right) \\
&= \epsilon_a{}^{bc}\omega_b \wedge \mathrm{d}V_c + \epsilon_a{}^{bc}\mathrm{d}\left(\omega_b \wedge V_c\right) + \epsilon_a{}^{de}\omega_d \wedge \left(\epsilon_e{}^{bc}\omega_b \wedge V_c\right) \\
&= \epsilon_a{}^{bc}\mathrm{d}\omega_b \wedge V_c + \epsilon_a{}^{de}\epsilon_d{}^{bc}\omega_e \wedge \omega_b \wedge V_c\,,
\end{aligned}$$

from which the curvature can be identified as

$$\tilde{R}_{ac} = \epsilon_{ac}{}^{b}\mathrm{d}\omega_b + \omega_a \wedge \omega_c\,.$$

Note that it is a two-form and that it is antisymmetric in its gauge indices. Dualizing this once results in

$$\begin{aligned} R_f &= \epsilon_f{}^{ac} \tilde{R}_{ac} \\ &= -2\mathrm{d}\omega_f + \epsilon_f{}^{ac} \omega_a \wedge \omega_c \,, \end{aligned}$$

which is exactly two times (6.6). Thus the covariant derivative[4] from (6.5) gives rise to a curvature tensor that is consistent with (6.6).

Note now that the vanishing of the Riemann tensor implied in (6.6) really is an equation for the Ricci tensor. This follows from the fact that in three dimensions the Riemann tensor can be expressed in terms of the Ricci tensor, there are no further independent components. This can be seen in many different ways, the most explicit being the fact that in three dimensions there is an identity[5]

$$R_{\mu\nu}{}^{\rho\sigma} = 4\delta_{[\mu}{}^{[\rho} R_{\nu]}{}^{\sigma]} - \delta_{[\mu}{}^{\rho} \delta_{\nu]}{}^{\sigma} R \,,$$

expressing the Riemann tensor in terms of the Ricci tensor. Hence if the latter vanishes then so does the former. This means that vacuum solutions to the Einstein equation in three dimensions are flat spacetimes or, in the presence of a cosmological constant, spacetimes of constant curvature. In four dimensions the Riemann tensor do have additional independent components and can thus be non-trivial in vacuum. It turns out that spacetime is curved outside an energy-mass source, whereas in three dimensions this is no longer the case.

To reiterate, the Chern-Simons action for the gauge group ISO(2,1) gives rise to equations of motion that are equivalent to the Einstein equations in three dimensions. This is encouraging, however it must be remembered that the spacetime symmetry group ISO(2,1) enters here as the gauge group. This means that translations and Lorentz transformations are gauge transformations. For this to make sense it must be verified that they give rise to the standard transformation rules for the dreibein and spin connection.

[4]If we interpret (6.5) as the vanishing of torsion for a dreibein e_a.

[5]This can be easily derived using the fact that in three dimensions two antisymmetric indices is equivalent to one using the epsilon tensor. Rewriting the two antisymmetric pairs of the Riemann tensor this way and then using the epsilon identities immediately gives the sought relation.

The Chern-Simons action is invariant under[6] gauge transformations

$$\delta A = \mathrm{d}\Lambda + [A, \Lambda] \ ,$$

where Λ is a Lie-algebra valued function. To see this one can first note that F satisfies the Bianchi identity[7]

$$\mathrm{d}F + A \wedge F - F \wedge A = 0 \, ,$$

which is immediate from the form of F. Now the result in (6.4) can be used with the particular form of δA from above[8]

$$\begin{aligned} \delta_{\text{gauge}} S &= \int_M \mathrm{tr}\Big(F \wedge (\mathrm{d}\Lambda + [A,\Lambda]) \Big) \\ &= \int_M \mathrm{tr}\Big(- \mathrm{d}F\Lambda + F \wedge [A,\Lambda] \Big) \\ &= \int_M \mathrm{tr}\Big(A \wedge F\Lambda - F \wedge \Lambda A \Big) \, , \end{aligned}$$

where the first step is an integration by parts and the second uses the Bianchi identity. This expression vanishes since the second term is an even permutation of the first[9]. Thus the Chern-Simons action is invariant under infinitesimal gauge transformations.

How do these transformations act in the case at hand? The gauge parameter Λ can written as[10]

$$\Lambda = \rho_a P^a + \tau_a M^a \, . \tag{6.7}$$

One then quickly calculates using the commutation relations (6.2)-(6.3) that

$$\begin{aligned} \delta A &= \mathrm{d}\Lambda + \Big[e_b P^b + \omega_b M^b, \Lambda \Big] \\ &= \mathrm{d}\rho_a P^a + \mathrm{d}\tau_a M^a - \epsilon^{ab}{}_c \tau_a e_b P^c + \epsilon^{ba}{}_c \rho_a \omega_b P^c + \epsilon^{ba}{}_c \tau_a \omega_b M^c \, . \end{aligned}$$

---

[6]A more complete analysis shows that it is invariant under gauge transformations connected to the identity.

[7]Which is the statement that F is covariantly constant.

[8]Note that we want to show that the action is gauge invariant without the use of the equation of motion, i.e. we use the form of δS derived previously which is valid in general but we are not using that $F = 0$.

[9]The trace is cyclic and the wedge product between A and F is even since F is a two-form.

[10]For the convenience of the reader the same notation as in [143] is used.

From which we read off the variations

$$\delta e_a = \mathrm{d}\rho_a - \epsilon^{bc}{}_a \tau_b e_c - \epsilon^{bc}{}_a \rho_b \omega_c \,,$$
$$\delta \omega_a = \mathrm{d}\tau_a - \epsilon^{bc}{}_a \tau_b \omega_c \,.$$

Take a look at how the parameter τ_a enters. From (6.7) it is clear that it should parametrize the Lorentz transformations. Without doing the detailed computation we can observe that the second term in δe_a corresponds exactly to a local Lorentz transformation $\tau_b (T^b)_a{}^c e_c$ with $(T^b)_a{}^c = \epsilon^b{}_c{}^a$ (which precisely corresponds to the antisymmetric generators of $so(2,1)$), so that the two boosts are T^2, T^3 and the spatial rotation is T^1.

Looking at $\delta\omega_a$, it contains the correct vector transformation but also the inhomogeneous term $\mathrm{d}\tau_a$ needed to be compatible with the solution of ω_a in terms of e_a from the zero torsion equation[11] (6.5).

The translations are a bit more subtle for a very interesting reason. In standard gravity they should correspond to local diffeomorphisms, however here we already have a notion of diffeomorphisms in that the Chern-Simons theory lives on a background geometry. If we were to find that these two notions of local diffeomorphisms generate different transformations on the fields then the interpretation as standard Einstein gravity would fall apart. They turn out to be equivalent up to terms that vanish using the equations of motion [138]. This means that at least classically the theory corresponds exactly to Einstein gravity.

This has profound consequences for three dimensional gravity. The first consequence is that this formulation makes very explicit the fact that the theory is topological [138]. This is clear already from the Einstein field equations in light of the fact that there are no more independent component in the Riemann tensor apart from the Ricci tensor. Thus there is no room for any propagating degrees of freedom in the metric. Here however this is built into the formulation where the Chern-Simons action makes no reference to any metric on the base manifold at all. Since there is a well defined quantization of Chern-Simons theory this also means that the gravity theory can be quantized resulting in a consistent quantum theory of gravity in three dimensions [150].

[11] I.e. the solution of ω_a contains derivatives on e_a which after a local Lorentz transformation then generates terms of the form $\mathrm{d}\tau_a$.

Since the theory is topological one might think that it is trivial in some sense but it turns out that it is still a very rich system where many of the phenomena from four dimensions show up. Coupled to matter in the form of point particles it turns out that even though the vacuum is flat it curls up the space into cones so that the dynamics of particles do feel the presence of each other [142]. For a negative cosmological constant there are even black holes [139], which together with the fact that there is an explicit quantum description makes it possible to study the quantum behavior of black holes directly.

6.2 Conformal gravity and higher spin algebra

Using the gauge theory formulation of gravity reviewed in section 6.1 we are now in a position to introduce the relevant theory for Paper IV. It builds on the same foundation as the Chern-Simons gauge theory but starts with the gauge group SO(3,2) instead, being the conformal group in three dimensions. By using a particular presentation of the corresponding algebra the extension to higher spin become almost immediate.

The conformal version is interesting for many reasons. First, in the quest for the most symmetric theories this is the natural starting point. Furthermore the conformal theory becomes relevant when doing holography. It turns out that in general, a higher spin theory in AdS_4 of Vasiliev type is dual to a three dimensional conformal higher spin theory [151][12].

There are even indications that holography of higher spin theories might be more rich than expected. Here the boundary dual seems to be a special case of a more general duality to a conformal higher spin theory on a generic three-manifold embedded in AdS_4 [151, 152].

Recently conformal higher spin theories have played a role in continuing the program of verifying the AdS/CFT duality and provided some striking new results in this direction [153].

[12]This statement seems to be in tension with the hand waving arguments in section 5.2.2 but it turns out that certain truncated versions of the Vasiliev theory is dual to free bosonic and fermionic vector models [151].

Before introducing the construction relevant for PAPER IV the gauge description of gravity will first be extended to the conformal algebra in the following section. Following this the higher spin algebra is introduced in 6.2.2 together with some comments on the necessary computational tools in section 6.2.3.

6.2.1 Conformal gravity

The conformal group in three dimensions consists of the generators P^a and M^a of ISO(2,1) together with dilation D and special conformal transformations K^a. It is isomorphic to SO(3,2). The additional commutators are given by

$$\begin{aligned} [D, P^a] &= P^a \\ [D, K^a] &= -K^a \\ [P^a, K^a] &= -2\epsilon^{ab}{}_c M^c - 2\eta^{ab} D \,. \end{aligned}$$

It is now possible to carry out the same program[13] as in section 6.1.1 and arrive at a theory for conformal gravity instead of Einstein gravity [154].

Another peculiarity of three dimensions relevant here is that the Weyl tensor, which is conformally invariant, vanishes[14] and so the equation of motion resulting from this system is not related to the Weyl tensor but rather to the Cotton tensor [154]. In the standard notation the resulting equation reads

$$\epsilon^{\mu\nu\rho} D_\mu \left(R_{\nu\sigma} - \frac{1}{4} g_{\nu\sigma} R \right) = 0 \,, \tag{6.8}$$

and it can be shown that solutions are conformally flat spacetimes [155].

In the case of conformal gravity in the gauge formulation some new phenomena shows up. To see this first note that the gauge field now has two new fields corresponding to the new generators,

$$A = e_a P^a + \omega_a M^a + bD + f_a K^a \,,$$

[13]In a certain sense the conformal case is more straight forward since the group SO(3,2) is semi-simple, thereby having a unique invariant bilinear form. See [154] for details.

[14]This is a restatement of the fact that the Riemann tensor can be expressed in terms of the Ricci tensor.

together with new gauge parameters corresponding to the dilation and special conformal transformations,

$$\Lambda = \rho_a P^a + \tau_a M^a + \gamma D + \sigma_a K^a \,.$$

Also here the question presents itself as to whether this extension corresponds to what would normally be called conformal gravity [156, 157]. It turns out that here as well the theories are equivalent at the classical level. The way to show this is to use the gauge parameter σ_a to gauge away the field b [154]. The resulting equations of motion then first relate f_a to e_a, in fact f_a is nothing but the Schouten tensor [154],

$$f_\mu{}^a = \frac{1}{2} e^{\nu a} \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) \,.$$

The remaining equation is then exactly (6.8). Note that here there seemed to be more fields, but in the end the larger gauge symmetry makes it possible to go back to a formulation where again the vielbein is the only remaining independent field. This is something that will manifest itself when generalizing to higher spins.

6.2.2 Poisson algebra

Now we will introduce a specific kind of presentation of the so(3,2) algebra with the goal of extending conformal gravity to higher spin. The first parts of this section is not strictly necessary if one is willing to take the final result of the Poisson algebra formulation at face value, but for the interested reader we provide a bottom up derivation.

First note that the three dimensional spin group in Lorentz signature is isomorphic to SL(2,ℝ) [158]. One possible choice for the generators are[15]

$$T_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad T_{12} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} , \quad T_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ,$$

[15]In the standard presentation of sl(2,1) the generator T_{12} is multiplied by a factor 2, which slightly alters the commutation relations. The version used here is convenient for our purposes.

which satisfy the commutation relations

$$[T_{11}, T_{22}] = -2T_{12}$$
$$[T_{11}, T_{12}] = T_{11}$$
$$[T_{22}, T_{12}] = -T_{22} \,.$$

Introducing the two dimensional epsilon symbol $\epsilon^{\alpha\beta}$ with $\epsilon^{12} = 1$, these generator can be written in the compact form

$$(T_{\alpha\beta})^{\gamma}{}_{\delta} = \delta^{\gamma}_{(\alpha}\epsilon_{\beta)\delta} \,. \tag{6.9}$$

Now let q^α and p_α be two real SL(2,$\mathbb{R}$) spinors, where indices are raised and lowered with the antisymmetric $\epsilon_{\alpha\beta}$. Looking at them for the moment as coordinates and conjugate momenta we can contract them with the relevant indices of (6.9) to obtain a generating function

$$T_{\alpha\beta} = p_{(\alpha}q_{\beta)} \,,$$

that satisfies the same commutation relations under the Poisson bracket

$$\big\{f, g\big\}_{\mathrm{PB}} = \frac{\partial f}{\partial q^\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial g}{\partial q^\alpha}\frac{\partial f}{\partial p_\alpha} \,.$$

Let us do one example very explicitly,

$$\begin{aligned}
\big\{T_{11}, T_{22}\big\}_{\mathrm{PB}} &= \big\{p_1q_1, p_2q_2\big\}_{\mathrm{PB}} \\
&= \frac{\partial(p_1q_1)}{\partial q^2}\frac{\partial(p_2q_2)}{\partial p_2} - \frac{\partial(p_2q_2)}{\partial q^1}\frac{\partial(p_1q_1)}{\partial p_1} \\
&= \epsilon^{21}p_1q_2 - \epsilon^{12}p_2q_1 \qquad (6.10)\\
&= -2p_{(1}q_{2)} \\
&= -2T_{12} \,,
\end{aligned}$$

where the epsilon tensors in (6.10) arise when lowering the index on the partial derivative and the vanishing terms in the sum over the index α have been dropped.

Let us at this point change the notation slightly. Since we will always be working with only p's and q's let us calculate

$$\big\{q^\alpha, p_\beta\big\}_{\mathrm{PB}} = \delta^\alpha_\beta \,,$$

and then define

$$[q^\alpha, p_\beta] = \delta^\alpha_\beta\,.$$

Note now that since the Poisson bracket satisfies the usual Leibniz's rule of the commutator, all the necessary information is contained in the above relation. Thus in what follows we will use the commutator with the above definition.

Pressing on, we can use the three dimensional gamma matrices γ^0,γ^1 and γ^2 to get a vector expression for the generators[16]

$$M^a = -\tfrac{1}{2}(\gamma^a)_\alpha{}^\beta q^\alpha p_\beta\,,$$

where the commutation relation now takes the form

$$\Big[M^a, M^b\Big] = \epsilon^{ab}{}_c M^c\,.$$

This now finally makes contact with the standard form (in three dimensions) of the commutation relations for the Lorentz algebra so(2,1). At this point one might rightly wonder what the virtue of all this is. The first nice realization is that the remaining generators of so(3,2) are found by simply including the missing bilinears in p_α and q^α,

$$D = -\frac{1}{2} q\cdot p, \quad P^a = -\frac{1}{2}\left(\gamma^a\right)_{\alpha\beta} q^\alpha q^\beta, \quad K^a = -\frac{1}{2}\left(\gamma^a\right)^{\alpha\beta} p_\alpha p_\beta\,.$$

A more interesting upshot of this formulation is that it is now very natural to ask what happens if we include higher order polynomials in q^α and p^α. The first thing to note is that if we include any generator with more factors then we must include all generators with more factors all the way up to infinity. To see this note that the Poisson bracket removes one q^α and one p^α by the differentiation. Let $G(n_q + n_p)$ denote a generator with the indicated sum of the number of p's and q's. Then we have schematically that

$$\Big[G(n_1), G(n_2)\Big] = G\Big(n_1 + n_2 - 2\Big)\,.$$

This implies that the $n_q + n_p = 2$, i.e. so(3,2) generators close under the commutator as we have seen. It also implies that if $n_1 = 2$ but $n_2 > 2$ then

$$\Big[G(2), G(n_2)\Big] = G(n_2)\,,$$

[16]The normalization introduced in this step aims to follow the one in [159].

so that the so(3,2) generators also preserve the sum $n_q + n_p$ of higher order generators. If however both $n_q > 2$ and $n_p > 2$ then they commute to a generator of strictly higher order, which implies that we must include all higher order generators to close the algebra.

What is the representation structure of these higher order generators? It turns out that on the sl(2,$\mathbb{R}$) side things are easy. Start by finding the irreducible parts[17] of the simple bilinear $T^{\alpha\beta} = q^\alpha p^\beta$, by writing it as a sum of the symmetric and antisymmetric part

$$T^{\alpha\beta} = q^{(\alpha}p^{\beta)} + q^{[\alpha}p^{\beta]} \,.$$

Since the antisymmetric tensor product in two dimensions contains just one independent component this term is proportional to $\epsilon^{\alpha\beta}$,

$$T^{\alpha\beta} = q^{(\alpha}p^{\beta)} - \tfrac{1}{2}\epsilon^{\alpha\beta}q^\gamma p_\gamma \,, \tag{6.11}$$

which can be verified by contracting with $\epsilon_{\alpha\beta}$. Notice that when doing this contraction the first term vanishes by symmetry, i.e. it is automatically traceless[18]. The conclusion, to be elaborated on in the next section, is that the irreducible tensors in q^α and p^α are totally symmetric products with possible factors of $q \cdot p = q^\alpha p_\alpha$ together with the two dimensional epsilon tensor.

Let us now introduce the following definition,

$$G(n_q,n_p,c)^{\alpha_1 \ldots \alpha_{n_q+n_p-2c}} = C(n_q,n_p,c)q^{(\alpha_1} \cdots q^{\alpha_{n_q-c}}p^{\alpha_{n_q-c+1}} \cdots p^{\alpha_{n_p+n_q-2c})}(q\cdot p)^c \,,$$

with $c \leq \frac{n_q+n_p}{2}$, where n_q and n_p denotes the total number of p's and q's and where the number of contracted pairs are specified by c. The constant C is a normalization that is unimportant for the discussion here, see Paper IV for details. Note that the number of free indices uniquely specifies c so that sometimes when they are specified the c will be dropped.

Since we will only be working with even spins the generators are most conveniently presented with vector indices,

[17] This is a standard technique, see for example [160].

[18] Recall that $\epsilon^{\alpha\beta}$ is the metric in the spinor representation.

$$G(n_q,n_p,c)^{a_1\dots a_N} = C(n_q,n_p,c)\times$$
$$(\gamma^{a_1})_{\alpha_1\alpha_2}\cdots(\gamma^{a_N})_{\alpha_{2N-1}\alpha_{2N}}q^{(\alpha_1}\cdots q^{\alpha_{n_q-c}}p^{\alpha_{n_q-c+1}}\cdots p^{\alpha_{2N})}(q\cdot p)^c\,. \quad (6.12)$$

To reiterate, the irreducible generators are in one to one correspondence with the triples $\{n_q,n_p,c\}$. The commutation relations can now be calculated by using the fundamental relation $[q^\alpha,p_\beta]=\delta^\alpha_\beta$ followed by a decomposition into the irreducible parts[19].

As an example let us rederive the commutation relation for two so(3,2) generators P^a and M^a.

$$\begin{aligned}\left[P^a,M^b\right] &= \tfrac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)_\gamma{}^\tau\left[q^\alpha q^\beta,q^\gamma p_\tau\right]\\ &= \tfrac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)_\gamma{}^\tau q^\gamma\left(\left[q^\alpha,p_\tau\right]q^\beta+q^\alpha\left[q^\beta,p_\tau\right]\right)\\ &= \tfrac{1}{4}(\gamma^a)_{\tau\beta}(\gamma^b)_\gamma{}^\tau q^\gamma q^\beta+\tfrac{1}{4}(\gamma^a)_{\alpha\tau}(\gamma^b)_\gamma{}^\tau q^\gamma q^\alpha\\ &= \tfrac{1}{2}(\gamma^a)_{\tau\beta}(\gamma^b)_\gamma{}^\tau q^\beta q^\gamma\\ &= \tfrac{1}{2}\epsilon^{ab}{}_c(\gamma^c)_{\beta\gamma}q^\beta q^\gamma\\ &= \epsilon^{ab}{}_cP^c\,,\end{aligned} \quad (6.13)$$

where the identity $(\gamma^{[a})_{\alpha\delta}(\gamma^{b]})_\beta{}^\delta=\epsilon^{ab}{}_c(\gamma^c)_{\alpha\beta}$ has been used.

A generator with more vector indices, i.e. more q's and p's, will in general carry higher values of spin. To see this, first note that a generator of the form in (6.12) is automatically irreducible in its vector indices. It is symmetric because of the symmetrization on the q's and p's and it is traceless because of the Fierz identity

$$(\gamma^a)_{\alpha\beta}(\gamma_a)_{\gamma\delta}=\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}+\epsilon_{\beta\gamma}\epsilon_{\alpha\delta}\,.$$

Together, this means that a generator $G^{a_1\dots a_j}$ is totally symmetric and traceless, i.e. it carries spin j.

It is now possible to calculate the commutation relations for the higher spin algebra using the same techniques as in the example calculation (6.13). This can be done by hand for the lower spin levels however they quickly become

[19]This is most easily performed by doing the decomposition for the sl(2,1) parts of the expression which then implies irreducibility also for the vector expression.

tedious and prone to errors. The next section describes how this calculation was instead implemented in a computer algebra system.

6.2.3 sl(2,ℝ) tensor product decomposition

Here the more general decomposition of sl(2,ℝ) tensors, that generalize the simple example (6.11), will be presented.

Let $S^{\alpha_1\ldots\alpha_r}$ be a totally symmetric (i.e. irreducible) sl(2,ℝ) tensor. The decomposition of the tensor product with an sl(2,ℝ) vector V^β is then given by[20]

$$V^\beta S^{\alpha_1\ldots\alpha_r} = V^{(\beta} S^{\alpha_1\ldots\alpha_r)} + \frac{r}{r+1}\epsilon^{\beta(\alpha_1} S^{\alpha_2\ldots\alpha_r)\gamma} V_\gamma\,.$$

Let us confirm the counting: The tensor $S^{\alpha_1\ldots\alpha_r}$ sits in the symmetric tensor product of rank r which for this two dimensional representation is[21] $\binom{r+1}{1} = r+1$ dimensional. The tensor product above is therefore $2(r+1)$ dimensional which decomposes into the two symmetric rank $r+1$ and $r-1$ representations.

Using the above relation one can recursively decompose a general product of tensors into its irreducible parts. This is doable by hand for smaller representations but it quickly becomes tedious and prone to errors. The decomposition is however easily implemented in a computer algebra system. For the calculation of the algebra structure constants the above decomposition was implemented as a Mathematica program.

6.3 Paper IV: 3d conformal higher spin

In Paper IV an interacting theory of higher spin is constructed using the Chern-Simons based formulation with the higher spin gauge algebra described in section 6.2.2. This theory was first considered in [161] where the first analysis of the spin content was carried out at the linear level. Paper IV builds

[20] This formula can be easily derived by doing an ansatz for the two possible irreducible tensors that can be constructed with the symmetries of the left hand side and then taking an appropriate trace or symmetrization to isolate the terms. Thus it is a simple generalization from the two-index version in (6.11).

[21] This formula can be established for a general d-dimensional representation by realizing that the independent components are in one to one correspondence with the number of ways to place $d-1$ separators between the r indices.

on previous work [159, 162] and the main new addition is the analysis of the coupled spin 2 - spin 3 system at the non-linear level, something that has previously been unfeasible due to the size of the resulting calculations. This obstacle is tackled by performing the tensor algebra and related calculations in a computer algebra system developed for this purpose. This system consists of an implementation of symbolic tensor algebra routines in Mathematica together with an interface to the index canonicalization package xPerm [163].

6.3.1 Higher spin gauge theory

Let A be a spacetime one-form taking values now in the infinite dimensional higher spin algebra defined in 6.2.2. It can be written as a formal sum

$$A = \sum_{s=2}^{\infty} A_s \,,$$

where each A_s contains generators with $n_q + n_p = 2(s-1)$ that turns out to correspond to spin up to s. For example the spin 2 and 3 parts are given by[22]

$$\begin{aligned} A_2 &= e_a \underset{(2,0)}{P^a} + \omega_a \underset{(1,1)}{M^a} + b \underset{(1,1)}{D} + f_a \underset{(0,2)}{K^a} \\ A_3 &= e_{ab} \underset{(4,0)}{P^{ab}} \\ &\quad + \tilde{e}_{ab} \underset{(3,1)}{\tilde{P}^{ab}} + \tilde{e}_a \underset{(3,1)}{\tilde{P}^a} \\ &\quad + \tilde{\omega}_{ab} \underset{(2,2)}{\tilde{M}^{ab}} + \tilde{\omega}_a \underset{(2,2)}{\tilde{M}^a} + \tilde{b} \underset{(2,2)}{\tilde{D}} \\ &\quad + \tilde{f}_{ab} \underset{(1,3)}{\tilde{K}^{ab}} + \tilde{f}_a \underset{(1,3)}{\tilde{K}^a} \\ &\quad + f_{ab} \underset{(0,4)}{K^{ab}} \,. \end{aligned}$$

Here the naming convention for the spin 3 generators and fields follow that of spin 2 and in fact they are quite appropriate as will be shown. The q and p content is indicated by the underset (n_q,n_p). Note that for a given value of s the irreducible content might be of lower spin, this happens precisely when the generator contains contracted pairs $q \cdot p$.

[22]The generators are here given a specific name for convenience so that for example $G(3,1,1)^a = \tilde{P}^a$.

The spacetime spin s here is different from the spin j considered previously in the context of the algebra. This is because the gauge potential is a one-form, i.e. it carries a spacetime index, so that the spacetime spin contents of ${A_\mu}^{a_1 \cdots a_j}$ contains spin up to $s = j + 1$.

To clarify the above expansion, let us look at how the spin 2 part, i.e. the so(3,2) generators, fit in this picture. For $s = 2$, there are three possible combinations of (n_q,n_p) given by (2,0), (1,1) and (0,2). The first and the last one contain a single irreducible part since there are no possible contractions of $q^\alpha q^\beta$ and $p^\alpha p^\beta$. For the middle one there are two irreducible parts, $q^{(\alpha} p^{\beta)}$ and $q \cdot p$, corresponding to $G(1{,}1{,}0)$ and $G(1{,}1{,}1)$ that we have given the name M^a and D.

At the next level $s = 3$ there are more solutions to $n_q + n_p = 2(s-1) = 4$. They are (4,0), (3,1), (2,2), (1,3) and (0,4). These will be called the spin 3 generators.

At this point it might be pertinent to point out again that although this part of A is referred to as the spin 3 sector, it contains fields of lower spin content such as $\tilde{e}_a$. The reason that this convention makes sense is that, as will be shown, the lower spin fields will be completely solved for in terms of fields of spin 3 (or gauged away) so that the only remaining fields will be of spin 3. In fact all fields will be shown to be related to e_{ab}, the generalization of the spin 2 frame field.

The gauge parameter Λ now also becomes a formal sum over the infinite set of generators,

$$\Lambda = \sum_{s=2}^{\infty} \Lambda_s \,,$$

which for the spin 3 sector takes the form

$$\Lambda_3 = \underset{(4,0)}{\Lambda_{ab}} P^{ab} + \underset{(3,1)}{\Lambda_{ab}} \tilde{P}^{ab} + \underset{(3,1)}{\Lambda_a} \tilde{P}^a + \underset{(2,2)}{\Lambda_{ab}} \tilde{M}^{ab} + \underset{(2,2)}{\Lambda_a} \tilde{M}^a + \underset{(2,2)}{\Lambda} \tilde{D} + \underset{(1,3)}{\Lambda_{ab}} \tilde{K}^{ab} + \underset{(1,3)}{\Lambda_a} \tilde{K}^a + \underset{(0,4)}{\Lambda_{ab}} K^{ab} \,.$$

Using the gauge transformations above there are many possible gauge choices. It turns out that from the structure of the transformation laws stemming from δA there is a natural choice where the calculations simplify somewhat, see Paper IV for details, given by

$$\begin{aligned} \tilde{e}_\mu{}^a &= 0\,, \\ \tilde{\omega}_\mu{}^a &= e_\mu{}^a \hat{\omega}\,, \\ \tilde{f}_\mu{}^a &= \epsilon_\mu{}^{ab} \hat{f}_b + e_\mu{}^a \hat{f}\,. \end{aligned} \tag{6.14}$$

The equations of motion stemming from the Chern-Simons action now naturally generalize to the higher spin setting. Just as the resulting equation $F = 0$ splits into separate equations for each generator in section 6.1.1, here it splits into an infinite number of equations consisting of a growing number of equations at each spin level.

For spin 3 this results in a system that is summarized in table 6.1. Here the second column notation $f_1(f_2)$ indicates that field f_1 is solved for in terms of f_2. Note also that this system is written in the gauge (6.14), which results in that some equations of motion become constraints. There is a general pattern here where the equation at level (n_q,n_p) is used to solve for fields one level down, i.e. $(n_q - 1,n_p + 1)$. The last equation can not be used to solve for any fields and becomes the dynamical equation for the remaining spin 3 field e_{ab}. In analogy with spin 2 this equation is called the Cotton equation.

The pattern above can be seen purely from algebra consideration as follows. A given (n_q, n_p, c) component of $F = \mathrm{d}A + A \wedge A$ will contains terms that are proportional to $G(n_q,n_p,c)$. Since the first term in F does not contain any commutator it will give rise to $\mathrm{d}A(n_q,n_p,c)$ while the second term gives contributions whenever there are generators $G_1(m_1,n_1,c_1)$ and $G_2(m_2,n_2,c_2)$ such that the commutator $[G_1, G_2]$ contains $G(n_q,n_p,c)$.

From the fundamental relation $[q^\alpha, p_\beta] = \delta^\alpha_\beta$ this is possible when

$$m_1 + m_2 - 1 = n_q \tag{6.15}$$

$$n_1 + n_2 - 1 = n_p, \tag{6.16}$$

since the commutator removes one q and one p in every resulting term. Take now $G_1^a = P^a$, i.e. $m_1 = 2$, $n_1 = 0$, $c_1 = 0$. The solution for G_2 is then $m_2 = n_q - 1$ and $n_2 = n_p + 1$.

Taking a step back, this means that there will be a term in the equation of motion $F(n_q,n_p,c) = 0$ proportional to the product of e_a and $A(n_q - 1,n_p + 1)$. Under the assumption of an invertible frame field e_a this means that

$F(n_{\mathrm{q}},n_{\mathrm{p}}) = 0$	Solution
$F^{ab}(4,0) = 0$	$\tilde{e}_\mu^{ab}\left(e_\mu^{ab}\right)$
$F^{ab}(3,1) = 0$	$\tilde{\omega}_\mu^{ab}\left(\tilde{e}_\mu^{ab}\right)$
$F^{a}(3,1) = 0$	b_μ, $\hat{\omega}$
$F^{ab}(2,2) = 0$	$\tilde{f}_\mu^{ab}\left(\tilde{\omega}_\mu^{ab}\right)$
$F^{a}(2,2) = 0$	$\hat{f}_a$, $\hat{f}$
$F(2,2) = 0$	
$F^{ab}(1,3) = 0$	$f^{ab}\left(\tilde{f}^{ab}\right)$
$F^{a}(1,3) = 0$	
$F^{ab}(0,4) = 0$	Cotton equation.

Table 6.1: *Spin 3 equations of motion and the solution cascade.*

$F(n_q,n_p) = 0$ gives a solution of $A(n_q - 1,n_p + 1)$ in terms of $\mathrm{d}A(n_q,n_p)$ plus other terms not involving a derivative.

The above gives an indication that this structure is possible but relies on specific commutators to be non-zero which might not be the case. It can in fact be shown that the commutators relevant for the above results are always non-zero.

Another important feature is that when G_1 is stepped down to lower q and higher p the relation (6.16) implies that $n_2 \leq n_p + 1$. Thus there will only be terms containing fields $A(m_2,n_2)$ with $n_2 \leq n_p + 1$ which in turn means that the solution for $A(n_q - 1,n_p + 1)$ above will be in terms of fields with lower p content.

Together this implies that there is a solution cascade that in the spin 3 case looks like

$$(0,4) \to (1,3) \to (2,2) \to (1,3) \to (4,0)\,, \tag{6.17}$$

as indicated in table 6.1. This should be read from left to right where each successive field content is solved for in terms of the previous.

6.3.2 Spin 3 Cotton equation

Recall that the component $F(1{,}3)$ was used to solve for $\underset{(0,4)}{f_{ab}}$ which means that the last equation $F(0{,}4)$ remains as a differential equation for the remaining field e_{ab}. This equation is of order 5 since there are 4 field substitutions in the cascade (6.17) and one from the equation $F(0{,}4) = 0$. Generalizing this, it is easy to see that for spin s the last equation will be of order $2(s-1)+1 = 2s-1$.

Furthermore the equation at level $(n_q{,}n_p)$ will contain, apart from $\mathrm{d}A(n_q{,}n_p)$ and $A(n_q - 1{,}n_p + 1)$, terms containing $A(n_q + 1{,}n_p - 1)$ stemming from the commutator[23] with $f_a K^a$. The last type of term, where $m_1 = 1$ and $n_1 = 1$ corresponds to the commutator with $\omega_a M^a$ and will according to the counting above contain exactly $A(n_q{,}n_p)$. It is comforting, but of course a necessity of the construction, that this term always exactly corresponds to the correct spin connection contraction for a Lorentz covariant derivative together with $\mathrm{d}A(n_q{,}n_p)$ of the schematic form

$$D = d + \omega\,.$$

At this point the possible terms are exhausted which means that the $(n_q{,}n_p)$ component of F will have the form

$$F(n_q{,}n_p) = \mathrm{D}A(n_q{,}n_p) + \underset{(2,0)}{e}A(n_q - 1{,}n_p + 1) + \underset{(0,2)}{f}A(n_q + 1{,}n_p - 1)\,. \quad (6.18)$$

Of course there are many possible tensor contractions that will be realized so that the actual expression is more complicated, however this general form will always be adhered to. Here it is seen explicitly that the second term in (6.18) makes it plausible that there will be a solution for $A(n_q + 1{,}n_p - 1)$.

Returning to the last equation, $F(0{,}4) = 0$, the general form (6.18) stipulates that the second term will then not be present (since for example $n_q - 1 = -1$). How does the equation look? It turns out that already at the spin 3 level this equation in terms of e_{ab} contains on the order of 10^3 terms. This might seem excessive given that there is only one field but recall that it is of fifth

[23]This corresponds to the solution of (6.15)-(6.16) when $m_1 = 0$ and $n_1 = 2$.

order and couples to the spin 2 Schouten tensor so in general it contains terms of the form

$$\begin{gathered} \mathrm{D}^5 \underset{(4,0)}{e}\,, \quad \underset{(0,2)}{f}\,\mathrm{D}^3 \underset{(4,0)}{e}\,, \quad \underset{(0,2)}{f^2}\,\mathrm{D} \underset{(4,0)}{e} \\ \mathrm{D}^2 \underset{(0,2)}{f}\,\mathrm{D} \underset{(4,0)}{e}\,, \quad \mathrm{D} \underset{(0,2)}{f}\,\mathrm{D} \underset{(4,0)}{e}\,, \quad \underset{(0,2)}{f}\,\mathrm{D} \underset{(0,2)}{f} \underset{(4,0)}{e}\,. \end{gathered} \tag{6.19}$$

Since the equation $F(0,4)_{\mu\nu}{}^{ab} = 0$ carries two antisymmetric (spacetime form indices) and two symmetric traceless indices (algebra indices), there will also be trace terms where the above terms are supplemented by the flat metric η^{ab}. The above terms are of course tensors, which then have a multitude of possible index contractions. The slightly more rigorous way of estimating the number of terms is to look at the representation contents. The first term in (6.19) is the product

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes (\mathbf{7} + \mathbf{5} + \mathbf{3})\,, \tag{6.20}$$

where the five $\mathbf{3}$'s correspond to the covariant derivatives and the field

$$e_\mu{}^{ab} \in \mathbf{3} \otimes \mathbf{5} = \mathbf{7} \oplus \mathbf{5} \oplus \mathbf{3}\,.$$

After some calculations[24] one finds that product (6.20) is

$$40(\mathbf{1}) + 105(\mathbf{3}) + 135(\mathbf{5}) + 125(\mathbf{7}) + 90(\mathbf{9}) + 50(\mathbf{11}) + 21(\mathbf{13}) + 6(\mathbf{15}) + \mathbf{17}\,. \tag{6.21}$$

The Cotton equation transforms in $\mathbf{3} \otimes \mathbf{5}$ (two antisymmetric indices and two symmetric traceless) so naively the possible terms are of the form $\mathbf{7} + \mathbf{5} + \mathbf{3}$ however one of the results of PAPER IV is that both the $\mathbf{5}$ and $\mathbf{3}$ vanish so that it transforms only in the $\mathbf{7}$ dimensional representation. The decomposition in (6.21) then stipulates around 125 terms of this type. Similar considerations for the other types of terms brings this number up to the order of 10^3 terms, in rough agreement with the explicit expressions obtained through the computer algebra.

6.4 Computer algebra

The work in PAPER IV would not have been possible without the help of computers. This help comes in the form of a tensor algebra system developed in

[24] Or more conveniently by using a Lie algebra system such as LieART [164]

parallel with the investigations of paper. It is implemented in Mathematica which provides a solid foundation for a variety of mathematical problems but where a general symbolic tensor algebra system is lacking. This is in part because this is a very complex task with varying requirements depending on the particular situation. There are however a number of symbolic tensor manipulation packages available for a number of different platforms such as (in no particular order) Mathematica (xAct), Maple (GRTensorII) and Maxima (itensor) together with the more low level libraries for Python (Sympy), C++ (Redberry) as well as stand alone applications (Cadabra). Many of these could have served equally well as a foundation for the calculations instead of developing new code, however the specific needs of this project made a custom solution with an interface to xPerm [163] necessary. xPerm is as the name suggests a core algorithm in xAct which provides efficient canonicalization of tensor expressions. The problem of canonicalization is quite general and xPerm provides a very solid implementation that has become the standard for many computer algebra systems.

6.4.1 Canonicalization of tensor expressions

One of the main problems of computer based tensor calculation is that of canonicalization. Essentially this is the question of whether two tensor expressions are equivalent. When working with small rank tensors in expressions with only a few tensor products this is seldom a problem. As an example lets look at an expression containing a symmetric tensor S_{ab} and a general tensor T_{ab},

$$S_{ab}T^{bc}S_{cd} - S_{ca}S^{b}{}_{d}T^{c}{}_{b}\,.$$

After staring at this for a while, renaming some dummy indices, raising and lowering indices, and using the symmetry of S_{ab} one concludes that this expression is in fact zero. Can this procedure of comparing two terms be formulated as an algorithm? For smaller expressions such as this one it is often enough to lexicographically sort the factors, rename and raise/lower dummy indices and sort the indices using the symmetries. However when the expression is larger and there are more complicated symmetries involved, this simple procedure becomes ambiguous.

One way to solve this problem is to have an algorithm that maps a tensor expression to a certain *canonical* form. This should be canonical in the sense that all equivalent expressions should be mapped to the same form. An instructive way to regard this problem is to see the various operations such as renaming dummy indices and symmetry permutations as transformations in a "space of expressions". Equivalent expressions will then lie on the same orbit of all the allowed operations and the problem is to find a canonical representative of these orbits.

This is in essence what is solved with the Butler-Portugal algorithm [165] for tensor canonicalization. An efficient implementation of this algorithm in C is provided in the package xPerm [163] that has been interfaced through Mathematica in this work.

6.4.2 Verification

Since the computer algebra system has been developed in parallel during this work there has been much effort to validate the results. One of the major confirmations of the systems validity comes from the solution of the constraints stemming from the particular gauge choice. The resulting system is overdetermined and provide many non-trivial checks of the consistency of the computer algebra computations. Another check comes from the fact that the remaining gauge transformations after a partial gauge fixing can be evaluated either before or after substituting the cascade fields. On a more basic level, the linear truncation has been verified to agree with calculations by hand for the full spin 3 system.

7

Conclusions and outlook

The results of Paper I, among which is that the compactification of free (2,0) theory on circle fibrations seems to have a unique extension to an interacting supersymmetric theory, is another step towards the ultimate goal of an explicit construction of the interacting (2,0) theory in six dimensions. In light of its strong connections to M-theory and string theory this continues to be an important challange for the future. There have been many interesting developments regarding the compactification of (2,0) theory in the last few years. In direct relation to Paper I there have been some recent devlopments regarding circle fibrations in the context of (2,0) theory [98, 166–168]. The relationship to five-dimensional super Yang-Mills continues to be an active area of research with many recent results [169–173]. In [174] the co-author of Paper I, using the results of Paper I, investigated a particular example of a singular circle fibration and solved the equations of motion for the gauge field. It was shown that the theory couples to additional degrees of freedom living on the singularity in the form of a Wess-Zumino-Witten model.

Another very interesting direction related to the compactifications in Paper I-Paper III has been the construction of a large class of four-dimensional theories through compactification of (2,0) theory on a Riemann surface [99]. Through their common origin in six dimension there is a whole web of dualities between these theories extending S-duality of $N = 4$ SYM in four dimensions. Similar to this is the compactification on three-dimensional manifolds, also resulting in a number of dualities [175–177]. Investigating these dualities in terms of the tensor multiplet using similar techniques as in Paper I is also a possible future direction.

In Paper II-Paper III the twisting, and subsequent compactification, of (2,0) theory on spacetimes of the form $C \times M_4$ was carried out. The result is a compact formulation of the four-dimensional topological theory that makes use of the Q-cohomology structure.

This geometric setup is particularly interesting since it relates to the conjectured AGT correspondance [102]. It would be very interesting to continue in this direction to see if it is possible for the free tensor multiplet to say anything about the correspondance. One possible way forward would be to include higher excitations from the two-dimensional manifold so that an explicit calculation of observables on both sides would be possible.

The topological twisting of (2,0) theory also continues to be a useful tool with recent results including [101, 178, 179] and [180] which also has strong connections to Paper I. Another very interesting development that lies outside the scope of this thesis is the conformal bootstrap of (2,0) theory, which has provided many new concrete results using completely different methods [72].

In Paper IV the focus is shifted towards theories of higher spin. Using a tensor algebra system developed for these purposes the non-linear Cotton equation for the coupled spin 2 - spin 3 system has been analyzed in the context of conformal higher spin in three dimensions in the Poisson algebra formulation. To reach this point the full non-linear spin 3 system was solved in the frame field formulation.

By gauging the analogue of Lorentz transformations for the spin 3 frame field the resulting Cotton equation is shown to correspond at the linear level to previous results in three dimensional higher spin. Here these solutions arise naturally from the Chern-Simons based formulation, paving the way for a more fundamental description of the conformal higher spin system.

At the non-linear level the equations and solutions are complicated, on the order of 10^3 terms for the spin 3 system, making manual computation very difficult. Despite this complexity, several key symmetry properties of the non-linear Cotton equation has been verified using the computer algebra system.

To make contact with the next spin level the linearised analysis has been extended to spin 4 and makes it plausible that a similar non-linear analysis can be carried out in the future.

The explicit and highly constrained structure of the theory makes it an excellent target for future investigations of higher spin systems. Some immediate questions consists of a more compact description of the spin 3 system and an extension to higher spin levels. One possible direction here is to solve the system before gauging, thereby hopefully making the process more amenable to a more algorithmic solution suitable for the computer algebra system. Another direction is to include higher order contributions from the higher spin star-product algebra.

A more long term goal is the coupling of the pure higher spin system to matter which facilitates solutions connecting to the other known higher spin theories in three dimensions. This could give some new insight into the structure of higher spin theories on AdS background and possibly shed some light on the structure of the higher order vertices currently being investigated using the HS/CFT duality [121].

In all these investigations, both for (2,0) theory and higher spin, computer aided calculations have been an indispensible tool. In particular for Paper IV, the results would not have been feasible to obtain by hand.

7.1 Final remarks

In light of their position on the frontier of high energy physics, the importance of (2,0) theory and higher spin cannot be overstated. A proper understanding of the former will be crucial in the quest for M-theory, the mysterious theory that seems to underlie all the different formulations of string theory. Also higher spin seems to hold many important keys to unlock the secrets of the high energy limit of string theory. Furthermore, they have provided important evidence for holographic dualities and seems to indicate an even richer structure than previously imagined. Through holographic duality (2,0) theory and higher spin finally meet. Although the details of this encounter is out of reach for the moment, it is a tantalising future prospect.

It is clear that (2,0) theory and higher spin continue to provide very challenging and important questions for the future. By twisting, turning and spinning, some aspects of their elusive dance has been uncovered in this thesis.

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