THESIS FOR THE DEGREE OF LICENTIATE OF ENGINEERING

Automorphic string amplitudes

Henrik P. A. Gustafsson

Department of Fundamental Physics
CHALMERS UNIVERSITY OF TECHNOLOGY
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HENRIK P. A. GUSTAFSSON

Department of Fundamental Physics
Chalmers University of Technology
SE-412 96 Gothenburg
Sweden
Telephone +46 (0)31-772 1000

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To my family

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HENRIK P. A. GUSTAFSSON

Department of Fundamental Physics
Chalmers University of Technology

Abstract

This thesis explores the non-perturbative properties of higher derivative interactions appearing in the low-energy expansion of four-graviton scattering amplitudes in toroidal compactifications of type IIB string theory. We summarise the arguments for finding such higher derivative corrections in terms of automorphic forms using U-duality, supersymmetry and string perturbation theory. The perturbative and non-perturbative parts can then be studied from their Fourier expansions. To be able to compute such Fourier coefficients we use the adelic framework as an intermediate step which also gives a new perspective on the arithmetic content of the scattering amplitudes.

We give a review of known methods for computing certain classes of Fourier coefficients from the mathematical literature as presented in Paper I of the appended papers, and of our own work in Paper II towards computing some of the remaining coefficients of interest in string theory.

Keywords: string theory, automorphic forms, U-duality, non-perturbative effects, instantons, Eisenstein series, Whittaker vectors.

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List of publications

This thesis is based on the following appended papers listed in the order of intended reading, which is the order the projects were started, but not finished.

References to sections, equations or theorems of these papers will be denoted by the paper number followed by a dash and the reference number. For example, Paper I concludes with an outlook in chapter I-12.

Paper I

Eisenstein series and automorphic representations
Philipp Fleig, Henrik P. A. Gustafsson, Axel Kleinschmidt, Daniel Persson
`arXiv:1511.04265[math.NT]` (2015)
Submitted to Cambridge University Press

Paper II

Small automorphic representations and degenerate Whittaker vectors
Henrik P. A. Gustafsson, Axel Kleinschmidt, Daniel Persson
`arXiv:1412.5625[math.NT]` (2014)
Submitted to Journal of Number Theory

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Chapter 1

Introduction

Research in fundamental physics aims to describe the principles of nature that govern all scales in our universe and it is often by looking at the very small, or the very large scales that we may isolates or discern these principles. For example, by studying the collisions of elementary particles we have found a theory that describes the interactions between them – the standard model of particle physics, and by studying very distant objects we have found a theory that describes the evolution of our universe – the standard model of cosmology.

The cornerstones of these theories are different principles of symmetry. The standard model of particle physics is a quantum theory based on a gauge symmetry described by the groups $SU(3) \times SU(2) \times U(1)$ describing the strong, weak and electromagnetic forces. On the other hand, the standard model of cosmology is a classical theory of general relativity describing gravity and builds on the principle of coordinate invariance. In both these cases a lot of the structure is governed by symmetries.

When trying to unify the above theories into a quantum theory of gravity, using the usual prescriptions for quantising a classical theory, one runs into problems of non-renormalisable divergences and cannot compute meaningful physical observables. Therefore, a new approach to quantum gravity is needed with a completely different perspective.

String theory is a quantum theory with extended objects, such as strings and branes, instead of point particles as fundamental objects and naturally includes gravity making it a theory of quantum gravity. Using string theory we can then study the fundamental principles governing quantum gravity and a rich regime to explore is black holes where the effects from both gravity and quantum mechanics are strong. This leads us to the long term goal of this research project which is to understand the non-perturbative effects of quantum gravity through string theory such as instantons and black holes.

We study the fundamental symmetries of string theory called U-dualities and investigate their consequences for physical observables using the mathematical theory of automorphic forms, which are functions invariant under some discrete symmetry group. They play a crucial role in several areas of string theory [1,3,4,6,56,57,62,69]. We will, in this thesis, mainly focus on their part in scattering amplitudes and the low-energy effective action [15, 31–33, 40–42, 50, 55]. For their relevance in counting black hole degeneracies, statistical mechanics and pure mathematics, see, for example, [9, 20, 44, 51, 52, 60, 61] and chapter I-12 of Paper I.

Although many of these automorphic forms arising as coefficients in scattering amplitudes are known, they are typically expressed in terms of Eisenstein series as infinite sums over discrete subgroups and are hard to interpret. To extract physical information from them we study their Fourier expansions on different subgroups, but this becomes very hard when studying large symmetry groups corresponding to compactifying to lower dimensions. To be able to compute such Fourier coefficients we therefore lift our functions to the ring of adeles, which allows us to use very powerful adelic techniques to compute the resulting integrals, before restricting the final answer to a physical setting again.

For example, using symmetry arguments as described in chapter 2, one can obtain the automorphic forms describing all the perturbative and non-perturbative corrections to the first few derivative terms in the four-graviton scattering amplitude. Using the adelic framework makes it possible to find closed form expressions for some of the Fourier coefficients, such as the perturbative corrections, and Papers I & II give some important insight into how the remaining coefficients can be computed.

The perturbative terms agree with and extend what is known from string perturbation theory [31,50] whose corrections become increasingly hard to compute for higher orders, meaning world-sheets with larger genera. For example, complications arise from the problem of parametrising the moduli space of large genus surfaces and from computing superspace integrals [16]. As will be discussed below, it is also known that the perturbative terms do not give the complete picture requiring the inclusion of non-perturbative effects, which are not known how to take into account in general. The Fourier coefficients of automorphic forms giving rise to non-perturbative corrections agree with the expected behaviour based on single instanton computations in string theory [31] and gives us clues for how to formulate a non-perturbative theory.

Paper I gives an overview of the subject introducing the concepts and techniques required for the analysis of automorphic forms in the adelic framework. The aim of this paper is to give a pedagogical introduction, with explicit examples, to the subject for physicists and mathematicians alike, to highlight questions of interest to the string theory community, and to be a useful source of reference for the whole field.

Paper II was initiated during the work on Paper I. It applies some of the tools described in the mathematical literature addressed in Paper I, to do some explicit computations of Fourier coefficients on parabolic subgroups in terms of Whittaker vectors, the former of which are difficult to compute, while the latter are not.

This thesis is organised as follows. In chapter 2 we motivate the study of automorphic forms by exploring low-energy four-graviton scattering amplitudes in toroidally compactified type IIB string theory. From U-duality we get that the coefficient functions for the low-energy interactions are automorphic forms constrained by supersymmetry. We also discuss results from string perturbation theory and instanton calculations, all motivating the acquired form for the coefficient functions from which we then extract physical information by Fourier expansion.

Chapter 3 is dedicated to the theory of adelic automorphic forms on Lie groups G, where we, after a brief introduction of the adeles, in particular study adelic Eisenstein series and show how to recover the more familiar, real Eisenstein series. We define automorphic representations and different types of Fourier coefficients on G including so called Whittaker vectors.

It is not known how to compute a general Fourier coefficient, but in chapter 4, we summarise methods for computing Whittaker vectors from the mathematical literature using the adelic framework. Section 4.5 is devoted to our work from Paper II, where we develop methods, based on [24, 54], to obtain Fourier coefficients of interest in string theory for small automorphic representations in terms of known Whittaker vectors.

Chapter 2

Motivation from string theory

In this chapter we will discuss toroidal compactifications of the type IIB string theory and its low-energy effective theory: supergravity. We study their classical symmetries and, using the arguments of Hull–Townsend [47], find how those are restricted by quantum effects leading to the concept of U-dualities.

We then consider low-energy corrections to scattering amplitudes which are invariant under these dualities, and can be described by automorphic forms. From supersymmetry [40] and string perturbation theory [37] we obtain differential equations and weak coupling asymptotics for the amplitudes allowing us to completely determine all perturbative and non-perturbative corrections to the low-energy amplitudes in terms of Eisenstein series introduced below [31, 36].

2.1 Type IIB supergravity

The low-energy effective action for type IIB superstring theory in ten dimensions is the type IIB supergravity with $N = (2, 0)$ supersymmetry [39]. The particle content of this theory consists of two left-handed Majorana-Weyl gravitinos and two right-handed Majorana-Weyl dilatinos on the fermionic side. The bosonic particles, which will be the main focus of this thesis, are the NS-NS bosons: the metric $G_{\mu\nu}$, the two-form B_2 and the dilaton ϕ, and the R-R bosons: C_0, C_2 and C_4. We will denote the corresponding field strengths as $H_3 = dB_2$ and $F_{r+1} = dC_r$.

The bosonic part of the supergravity action can be split up into the terms

$$S_{\text{IIB}} = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}} \tag{2.1}$$

which are the Neveu-Schwarz, Ramond and Chern-Simons terms respectively. In the string frame [63]

$$\begin{aligned} S_{\text{NS}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G_S} e^{-2\phi} \Big(R_S + 4\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2} \left| H_3 \right|^2 \Big) \\ S_{\text{R}} &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G_S} \Big(\left| F_1 \right|^2 + \left| \tilde{F}_3 \right|^2 + \tfrac{1}{2} \left| \tilde{F}_5 \right|^2 \Big) \\ S_{\text{CS}} &= -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \, , \end{aligned} \tag{2.2}$$

where G_S and R_S are the space-time metric and Ricci scalar[1], and $\tilde{F}_3$ and $\tilde{F}_5$ are defined as

$$\begin{aligned}\tilde{F}_3 &= F_3 - C_0 \wedge H_3 \\ \tilde{F}_5 &= F_5 - \frac{1}{2} C_2 \wedge H_3 + \tfrac{1}{2} B_2 \wedge F_3\,,\end{aligned} \tag{2.3}$$

with the additional constraint that

$$* \tilde{F}_5 = \tilde{F}_5\,. \tag{2.4}$$

This action has an $SL(2,\mathbb{R})$ symmetry that becomes apparent when switching to the Einstein frame with

$$G_{\mu\nu} = (G_E)_{\mu\nu} := e^{-\phi/2}(G_S)_{\mu\nu} \tag{2.5}$$

and corresponding Ricci scalar R. Letting

$$\tau = C_0 + ie^{-\phi} \in \mathbb{H} \qquad G_3 = \tilde{F}_3 - ie^{-\phi} H_3 = F_3 - \tau H_3\,, \tag{2.6}$$

with $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ being the Poincaré upper half plane, we have that [5]

$$S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\Big(R - \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{2(\operatorname{Im}\tau)^2} - \frac{|G_3|^2}{2\operatorname{Im}\tau} - \frac{\big|\tilde{F}_5\big|^2}{4}\Big) + \frac{1}{8i\kappa_{10}^2}\int \frac{1}{\operatorname{Im}\tau} C_4 \wedge G_3 \wedge \overline{G}_3\,. \tag{2.7}$$

On this form, we see that S_{IIB} is invariant under a $SL(2,\mathbb{R})$ symmetry realised by

$$\begin{aligned}&\tau \to \gamma(\tau) = \frac{a\tau + b}{c\tau + d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \\ &\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \qquad G_{\mu\nu} \to G_{\mu\nu} \qquad \tilde{F}_5 \to \tilde{F}_5\,.\end{aligned} \tag{2.8}$$

In fact, the moduli space can be realised as the coset space $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ which is isomorphic to the Poincaré upper half plane $\mathbb{H}$ under the mapping

$$g \mapsto g(i) = \frac{mi + n}{pi + q} \qquad g = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in SL(2,\mathbb{R})\,. \tag{2.9}$$

A group element g in $SL(2,\mathbb{R})$ can be uniquely factorised using the Iwasawa decomposition and parametrised as

$$g = nak = \begin{pmatrix} 1 & \tau_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \tau_2^{1/2} & 0 \\ 0 & \tau_2^{-1/2} \end{pmatrix} k \qquad \tau_1, \tau_2 \in \mathbb{R},\ \tau_2 > 0,\ k \in SO(2,\mathbb{R}) \tag{2.10}$$

where n is unipotent and a is in the Cartan subgroup. This gives us

$$g(i) = \tau_1 + i\tau_2 = \tau\,, \tag{2.11}$$

[1] Subscript S for string frame.

independent of k. An element γ in $SL(2,\mathbb{R})$ acts on g by left-translation $g \mapsto \gamma g$. This also defines an action on $\tau = g(i)$ in $\mathbb{H}$ by

$$\gamma(\tau) = [\gamma g](i) = \frac{a\tau + b}{c\tau + d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), \tag{2.12}$$

which is exactly the $SL(2,\mathbb{R})$ symmetry acting on the moduli space as described above in (2.8).

This classical $SL(2,\mathbb{R})$ symmetry is not, however, a full symmetry of the ten-dimensional type IIB superstring theory, but is broken to a discrete subgroup which, at the largest, can be $SL(2,\mathbb{Z})$ [2,5,63]. This can be seen by considering a fundamental string carrying one unit charge of the B_2 field. Under (2.8), this is transformed to a string charged with d units of the B_2 field, which are quantised and must therefore be integer. The largest subgroup of $SL(2,\mathbb{R})$ with $d \in \mathbb{Z}$ is [5]

$$\left\{ \begin{pmatrix} a & \alpha b \\ c/\alpha & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\} \tag{2.13}$$

where $\alpha \in \mathbb{R}$ can be absorbed by rescaling C_2 leaving the discrete subgroup $SL(2,\mathbb{Z})$. It is conjectured that type IIB string theory is in fact invariant under this maximal allowed subgroup $SL(2,\mathbb{Z})$. Although this cannot be proven from first principles, it gives a consistent picture with the M-theory web of dualities [5].

The resulting string discussed above from the $SL(2,\mathbb{Z})$ transformation is a bound state of d fundamental strings and c D-strings called a (p,q)-string, here with $(p,q) = (d,c)$. A $(1,1)$-string, for example, can be imagined by first taking a fundamental string and a D-string aligned in the x^1-direction (which is a non-supersymmetric setup [63]). Then, to obtain a more energetically favoured configuration, the fundamental string would break and attach to the D-string with both charged endpoints. Between these points a B_2-flux along the D-string would appear. Both endpoints would then separate further and move to infinity, leaving only the D-string with B_2-flux forming a stable, supersymmetric bound state with the tension for the bound state being lower than the sum of the separate states. [63, 71].

When (p,q) are not coprime the system is only marginally stable against falling apart and it is believed that there are no bound states for these cases [63]. Note that the determinant condition for $SL(2,\mathbb{Z})$ gives us, together with Bézout's lemma, that (d,c) are coprime.

This discrete $SL(2,\mathbb{Z})$ symmetry for type IIB string theory is called S-duality connecting a weakly coupled IIB theory with a strongly coupled IIB theory in the string coupling g_s. Indeed, with $a = d = 0$ and $b = -c = 1$ we get that $\tau \to \gamma(\tau) = -1/\tau$, which, for $\tau_1 = C_0 = 0$, amounts to $e^{-\phi} = \tau_2 \to 1/\tau_2 = e^{\phi}$ and we will see in section 2.3 that the string coupling is related to the constant mode of the dilaton as [70]

$$g_\text{s} = e^{\phi_0} \qquad \phi_0 = \lim_{X\to\infty} \phi\,, \tag{2.14}$$

where the limit is taken with respect to the space-time coordinates X.

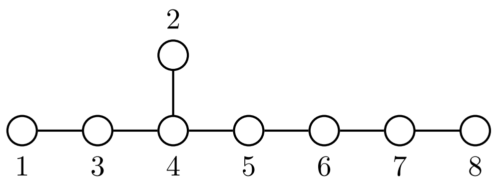

Figure 2.1: Dynkin diagram of E_8 with the Bourbaki convention for node labels. The groups of table 2.1 can be obtained by adding nodes in the order of the labeling.

2.2 Toroidal compactifications

When compactifying the ten-dimensional type IIB string theory on a circle with radius r we get a larger moduli space which includes r, and an additional duality: T-duality, under which the IIB theory on r transforms to a IIA theory on a circle of radius α'/r. The total, unified, U-duality group is then the combination of the S-duality and T-duality described by $SL(2,\mathbb{Z}) \times \mathbb{Z}_2$.

Upon further compactification on a torus T^d to $D = 10 - d$ dimensions, we obtain a larger T-duality group $O(d, d; \mathbb{Z})$ together with the S-duality group $SL(2,\mathbb{Z})$, but the full discrete symmetry group of the theory is conjectured to be larger than the product $SL(2,\mathbb{Z}) \times O(d, d; \mathbb{Z})$ in general [47].

To find the maximal possible U-duality group for a given dimension D, let us consider the classical symmetries of the corresponding low-energy supergravity actions denoted by $G(\mathbb{R})$ in table 2.1. In figure 2.1 we see the corresponding Dynkin diagrams which can be obtained by adding nodes in the order of the Bourbaki labeling. The moduli space of each theory is given by the coset $G(\mathbb{R})/K(\mathbb{R})$ where $K(\mathbb{R})$ is the maximal compact subgroup of (the split real) $G(\mathbb{R})$, which, at each further compactification gets enlarged by the introduction of new scalars. The symmetry group for a higher dimensional theory is contained in that of a lower dimensional one. The moduli spaces $G(\mathbb{R})/K(\mathbb{R})$ in lower dimensions are not, in general, equipped with a complex structure as was the case for $SL(2,\mathbb{R})/SO(2,\mathbb{R}) \cong \mathbb{H}$.

To describe the moduli space one can use the Iwasawa decomposition for a general connected semisimple real Lie group $G(\mathbb{R})$

$$G(\mathbb{R}) = B(\mathbb{R})K(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R})K(\mathbb{R}) \tag{2.15}$$

where B is the Borel subgroup, K is the maximal compact subgroup, N the subgroup of unipotent elements in B and A the Cartan subgroup. The precise constructions of these subgroups will be described when discussing parabolic subgroups in section 2.7, but for $G(\mathbb{R}) = SL(n,\mathbb{R})$ we have that B consists of upper triangular matrices, N of upper triangular matrices with ones on the diagonal, A of the diagonal matrices with positive entries, and K consists of the orthogonal matrices.

In four dimensions $G(\mathbb{R}) = E_7(\mathbb{R})$ and the supergravity theory contains 28 vector bosons A^I_μ, $I = 1, \ldots, 28$ with field strengths $F^I_{\mu\nu}$. Taking certain linear combinations of F and $*F$ one can form a dual field strength G_I which combines with F_I to a 56-dimensional vector of field strengths $\mathcal{F}$ in the vector representation of E_7

Table 2.1: Classical symmetry groups $G(\mathbb{R})$ with maximal compact subgroups K and corresponding U-duality groups $G(\mathbb{Z})$ when compactifying on T^d to $D = 10 - d$ dimensions. From [13, 47] summarised in [33]. The split real forms $E_{n(n)}$ are here denoted E_n for brevity.

D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

transforming as [47]

$$\mathcal{F} = \begin{pmatrix} F^I \\ G_I \end{pmatrix} \to \Lambda \begin{pmatrix} F^I \\ G_I \end{pmatrix} \qquad \Lambda \in E_7(\mathbb{R})\,. \tag{2.16}$$

The corresponding charges p^I and q_I are the magnetic and Noether electric charges respectively, and satisfy the Dirac-Schwinger-Zwanziger quantisation condition for two dyons $\mathcal{Q} = (p^I, q_I)^T$ and $\mathcal{Q}' = (p'^I, q'_I)^T$

$$\mathcal{Q}^T \Omega \mathcal{Q}' = p^I q'_I - p'^I q_I = n \in \mathbb{Z}\,, \tag{2.17}$$

where

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{2.18}$$

This is invariant under $\mathcal{Q} \to \Lambda' \mathcal{Q}$ with $\Lambda' \in Sp(56,\mathbb{R}) \supset E_7(\mathbb{R})$, but assuming that all types of p^I and q_I charges exists, the only Λ' that preserve the charge lattice are those in $Sp(56,\mathbb{Z})$ [47]. Thus, the full duality group of the four-dimensional theory, taking quantum effects into account, can at most be

$$E_7(\mathbb{R}) \cap Sp(56,\mathbb{Z}) \tag{2.19}$$

which also coincides with the Chevalley subgroup $E_7(\mathbb{Z})$ [68], which is defined as the discrete subgroup of $E_7(\mathbb{R})$ that stabilises the integer lattice spanned by the Chevalley generators under the adjoint action [12]. It is conjectured that this is *exactly* the U-duality group in four dimensions [47].

We note that the S-duality and T-duality groups in four dimensions are strictly contained in $E_7(\mathbb{Z})$

$$SL(2,\mathbb{Z}) \times O(6,6;\mathbb{Z}) \subset E_7(\mathbb{Z})\,. \tag{2.20}$$

As stated above and depicted in figure 2.1 the symmetry group $G_{D'}$ for dimensions $D' < D$ contains the symmetry group G_D which especially means that G_D for $D > 4$

can be embedded in the symmetry group E_7 in four dimensions. This makes it natural to conjecture that the duality groups for $D > 4$ are [47]

$$G_D(\mathbb{Z}) = G_D(\mathbb{R}) \cap E_7(\mathbb{Z}) \,, \tag{2.21}$$

the results of which are shown in table 2.1. For dimensions lower than four, which are also shown in 2.1, see [47]. We note that the respective S-duality groups and T-duality groups are contained in the U-duality groups $G(\mathbb{Z})$.

All physical observables, such as the free energy and scattering amplitudes, are functions that are invariant under this discrete symmetry.

2.3 Four-graviton scattering

The study of scattering amplitudes in string theory can be made by expanding in several different parameters. One parameter is $\alpha' = \ell_s^2$ where ℓ_s is the characteristic length of a string. Expanding in small α' amounts to a derivative expansion of the background fields in the two-dimensional world-sheet theory or taking the low-energy limit of the corresponding momenta. Another parameter is the string coupling g_s specifying how easily strings split and merge.

Computing string scattering amplitudes in the path integral formulation leads to an integration over world-sheet geometries S with different topologies. The constant mode ϕ_0 of the dilaton gives a topological factor in the string partition function of the form $\exp(-\phi_0\chi(S))$ where $\chi(S)$ is the Euler characteristic of the surface S, thus giving the expansion parameter: the string coupling $g_s = \exp(\phi_0)$. Since, for a surface with genus g and no boundaries, $\chi(S) = 2(1-g)$ we then obtain an expansion in different genera of the world-sheet in the case of closed strings as pictured in figure 2.2.

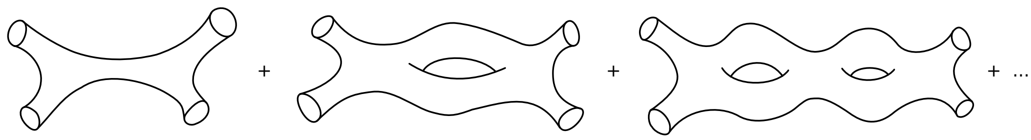

Figure 2.2: Genus expansion in string scattering amplitudes where each topology is weighted by the string coupling to the power of the negative Euler characteristic.

In this thesis we are interested in the scattering of four gravitons which have been computed at tree-level and one-loop in g_s for all α' in [37, 39, 43]. Both share the same kinematic structure, which factorises into a left- and right-moving part each similar to the kinematic factor for open strings [43]. Let us therefore study the kinematic structure for open strings.

In the operator formalism, this structure comes from the trace of fermionic zero modes S_0^a of the string where a is a target space spinor index in the transverse spin(8) group after light-cone gauge fixing. The massless ground states are attached to the $\mathbf{8_v} \oplus \mathbf{8_c}$ representation of spin(8) spanned by eight bosonic states $|i\rangle$ and eight fermionic states $|\dot{a}\rangle$. The S_0-trace of an operator A is then

$$\mathrm{tr}_{S_0}(A) = \sum_i \langle i|A|i\rangle - \sum_{\dot{a}} \langle \dot{a}|A|\dot{a}\rangle \,. \tag{2.22}$$

When calculating loop amplitudes, one needs to compute the trace of an even number of S_0 operators. Let $R_0^{ij} = \frac{1}{4}\gamma^{ij}_{ab} S_0^a S_0^b$ which is the only independent tensor that can be made of two S_0's allowing us to write any bilinear in S_0 in terms of products of R_0's. The first non-vanishing trace of R_0 operators is [37]

$$\begin{aligned} t^{ijklmnpq} &:= \mathrm{tr}_{S_0}(R_0^{ij} R_0^{kl} R_0^{mn} R_0^{pq}) \\ &= -\frac{1}{2}\epsilon^{ijklmnpq} - \frac{1}{2}\Big((\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})(\delta^{mp}\delta^{nq} - \delta^{mq}\delta^{np}) \\ &\qquad + (\delta^{km}\delta^{ln} - \delta^{kn}\delta^{lm})(\delta^{pi}\delta^{qj} - \delta^{pj}\delta^{qi}) \\ &\qquad + (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})(\delta^{kp}\delta^{lq} - \delta^{kq}\delta^{lp}) \Big) \\ &\quad + \frac{1}{2}\Big(\delta^{jk}\delta^{lm}\delta^{np}\delta^{qi} + \delta^{jm}\delta^{nk}\delta^{lp}\delta^{qi} + \delta^{jm}\delta^{np}\delta^{qk}\delta^{li} + \\ &\qquad \text{antisymmetrisation on each pair of indices} \Big) \end{aligned} \tag{2.23}$$

Letting $k_1, \ldots, k_4$ and $\zeta_1, \ldots, \zeta_4$ be the momenta and polarisations vectors for the four external vector states for the open string in the light-cone gauge, the kinematic factor takes the form [37]

$$K_{\text{open}} := K_{ijkl}(k_1, \ldots, k_4)\zeta_1^i \zeta_2^j \zeta_3^k \zeta_4^l := t_{ijklmnpq} k_1^i \zeta_1^j k_2^k \zeta_2^l k_3^m \zeta_3^n k_4^p \zeta_4^q \tag{2.24}$$

Returning to the closed string, we have that for IIB the massless ground states are attached to the representation $(\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c})$ giving a kinematic factor[2] [37]

$$K_{\text{cl}} := K_{ijkl}(\tfrac{k_1}{2}, \ldots, \tfrac{k_4}{2}) K_{mnpq}(\tfrac{k_1}{2}, \ldots, \tfrac{k_4}{2}) \zeta_1^{im} \zeta_2^{jn} \zeta_3^{kp} \zeta_4^{lq} \tag{2.25}$$

with $\zeta_1, \ldots, \zeta_4$ now being the polarisation tensors for the external gravitons.

Remark 2.1. In type IIA the massless ground states are attached to $(\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s})$ where $\mathbf{8_s}$ is the (undotted) spinor representation, giving two different K's differing in a few signs but not affecting the S-matrix [37, 43].

In covariant notation the kinematic structure can be expressed as follows with [63]

$$\begin{aligned} K_{\text{open}} &= \frac{1}{8}\Big(4M^1_{\mu\nu} M^2_{\nu\sigma} M^3_{\sigma\rho} M^4_{\rho\mu} - M^1_{\mu\nu} M^2_{\nu\mu} M^3_{\sigma\rho} M^4_{\rho\sigma} \Big) + \\ &\qquad (1234 \to 1342) + (1234 \to 1423) \\ &= t_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} k_1^\mu \zeta_1^\nu k_2^\rho \zeta_2^\sigma k_3^\alpha \zeta_3^\beta k_4^\gamma \zeta_4^\delta \,, \end{aligned} \tag{2.26}$$

where $M^i_{\mu\nu} = k^i_\mu \zeta^i_\nu - k^i_\nu \zeta^i_\mu$.

[2] In Paper I, this factor is denoted by $\mathcal{R}^4$, the reason for which is explained below.

The ten-dimensional scattering amplitude of four gravitons to one-loop order in g_s is then [15, 35, 37, 39, 43]

$$\begin{aligned}\mathcal{A}^{(4)} &= g_\mathrm{s}^{-2} K_\mathrm{cl}\Big(\mathcal{A}^{(4)}_\mathrm{tree} + g_\mathrm{s}^2 \mathcal{A}^{(4)}_\text{one-loop}\Big)\\ \mathcal{A}^{(4)}_\mathrm{tree} &= \frac{1}{stu}\frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}\\ \mathcal{A}^{(4)}_\text{one-loop} &= 2\pi \int_{\mathscr{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2}\mathcal{B}_1(s,t,u;\tau)\end{aligned} \tag{2.27}$$

where s, t and u are the dimensionless Mandelstam variables

$$s = -\frac{\alpha'}{4}(k_1+k_2)^2 \qquad t = -\frac{\alpha'}{4}(k_1+k_3)^2 \qquad u = -\frac{\alpha'}{4}(k_1+k_4)^2\,, \tag{2.28}$$

and $\mathcal{B}_1$ a modular invariant function defined in [15] with $\mathscr{F}$ being the fundamental domain of the modular group. The tree-level correction is also computed in chapter I-2 using the path-integral formalism.

Remark 2.2. Upon compactification the integral over the fundamental region in the one-loop correction receives an extra modular factor in (2.27) while the tree level remains unchanged [39].

Expanding in α', that is, taking the low-energy limit, the four-graviton scattering amplitude (2.27) subject to momentum conservation, $s+t+u=0$, becomes [15]

$$\begin{aligned}\mathcal{A}^{(4)}_\mathrm{tree} &= \frac{3}{\sigma_3} + 2\zeta(3) + \zeta(5)\sigma_2 + \frac{2}{3}\zeta(3)^2\sigma_3 + \mathcal{O}((\alpha')^4)\\ \mathcal{A}^{(4)}_\text{one-loop} &= 4\zeta(2) + \frac{4}{3}\zeta(2)\zeta(3)\sigma_3 + \mathcal{O}((\alpha')^4) + \text{non-analytic terms}\end{aligned} \tag{2.29}$$

where

$$\sigma_2 = s^2+t^2+u^2 \qquad \sigma_3 = s^3+t^3+u^3\,. \tag{2.30}$$

and the higher order terms are polynomials in σ_2 and σ_3.

K_cl is the linearised form of [15, 30, 35]

$$R^4 \coloneqq t^{\mu_1\rho_1\ldots\mu_4\rho_4} t^{\nu_1\sigma_1\ldots\nu_4\sigma_4} R_{\mu_1\rho_1\nu_1\sigma_1}\cdots R_{\mu_4\rho_4\nu_4\sigma_4} \tag{2.31}$$

which appears as the first α'-correction to the Einstein-Hilbert term in the low-energy effective action. Terms of order $\sigma_2^p\sigma_3^q$ with weight $w \coloneqq 2p+3q$ in (2.29) give higher-derivative contributions denoted by $D^{2w}R^4$ [15]. See [32] for an exact form.

The σ_3^{-1}-term of $\mathcal{A}^{(4)}_\mathrm{tree}$ in (2.29) comes from the classical Einstein-Hilbert term [38]. The remaining terms give the following low-energy effective Lagrangian in the string frame

$$\begin{aligned}\mathcal{L}_S \propto R_S &+ (\alpha')^3\Big(2\zeta(3) + 4\zeta(2)g_\mathrm{s}^2 + \ldots\Big)R_S^4 + (\alpha')^5\Big(\zeta(5)+\ldots\Big)D^4R_S^4 +\\ &(\alpha')^6\Big(\frac{2}{3}\zeta(3)^2 + \frac{4}{3}\zeta(2)\zeta(3)g_\mathrm{s}^2 + \ldots\Big)D^6R_S^4 + \mathcal{O}((\alpha')^7)\end{aligned} \tag{2.32}$$

where the dots correspond to higher genus corrections. In the Einstein frame, which is the frame that makes the connection to automorphic forms[3], we instead have

$$\mathcal{L} \propto R + (\alpha')^3\Big(2\zeta(3)g_s^{-3/2} + 4\zeta(2)g_s^{1/2} + \dots\Big)R^4 + (\alpha')^5\Big(\zeta(5)g_s^{-5/2} + \dots\Big)D^4R^4 + (\alpha')^6\Big(\frac{2}{3}\zeta(3)^2g_s^{-3} + \frac{4}{3}\zeta(2)\zeta(3)g_s^{-1} + \dots\Big)D^6R^4 + \mathcal{O}((\alpha')^7) \quad (2.33)$$

omitting the extra dilaton terms arising from a Weyl transformation of the Riemann tensor.

The parenthesis in front of the different R^4 terms in (2.33) are functions $\mathcal{E}_{(p,q)}$ on the moduli space $\mathcal{M}_{10} = SL(2,\mathbb{R})/SO(2,\mathbb{R}) \cong \mathbb{H}$ parametrised by the axion-dilaton $\tau = \chi + ig_s^{-1}$ with

$$\mathcal{L} \propto R + (\alpha')^3\mathcal{E}_{(0,0)}(\tau)R^4 + (\alpha')^5\mathcal{E}_{(1,0)}(\tau)D^4R^4 + (\alpha')^6\mathcal{E}_{(0,1)}(\tau)D^6R^4 + \mathcal{O}((\alpha')^7)\,. \quad (2.34)$$

Since the theory is invariant under $SL(2,\mathbb{Z})$ transformations acting on τ by

$$\tau \to \gamma(\tau) = \frac{a\tau + b}{c\tau + d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \quad (2.35)$$

each function $\mathcal{E}_{(p,q)}(\tau)$ needs to be invariant under this action as well.

Remark 2.3. Similarly, when compactifying on a T^d torus to $D = 10 - d$ dimensions we obtain the same structure with coefficient functions $\mathcal{E}^{(D)}_{(p,q)}$ on the moduli space $\mathcal{M}_D = G(\mathbb{R})/K(\mathbb{R})$ invariant under $G(\mathbb{Z})$.

$$\mathcal{L} \propto R + (\alpha')^3\mathcal{E}^{(D)}_{(0,0)}(g)R^4 + (\alpha')^5\mathcal{E}^{(D)}_{(1,0)}(g)D^4R^4 + (\alpha')^6\mathcal{E}^{(D)}_{(0,1)}(g)D^6R^4 + \mathcal{O}((\alpha')^7)\,. \quad (2.36)$$

We will continue to denote $\mathcal{E}^{(10)}_{(p,q)}$ as $\mathcal{E}_{(p,q)}$ for brevity.

In the next section, we will show that the coefficient functions $\mathcal{E}^{(D)}_{(p,q)}$ satisfy certain differential equations. Together with the weak coupling conditions found by string perturbation theory above, and the invariance under $G(\mathbb{Z})$, this means that the functions $\mathcal{E}^{(D)}_{(p,q)}$ are so called automorphic forms. We will here give a definition of automorphic forms for groups over the reals, but in chapter 3 we will make a more refined and detailed definition for groups over the adeles.

Automorphic forms are functions $f(g)$ on a Lie group $G(\mathbb{R})$ taking values in $\mathbb{C}$ that satisfy the following conditions

(A) *Automorphic invariance*: $f(\gamma g) = f(g)$ for all γ in $G(\mathbb{Z})$

(B) *Differential equations*: f is an eigenfunction to certain differential operators

(C) *Growth condition*: f should grow at most as a polynomial.

[3]Note that it was in the Einstein frame that the classical $SL(2,\mathbb{R})$ symmetry of type IIB supergravity became apparent.

Each statement will be made exact in chapter 3.

We will often restrict ourselves to so called *spherical* automorphic forms, which for the maximal compact subgroup $K(\mathbb{R}) \subset G(\mathbb{R})$, satisfies

$$f(gk) = f(g) \qquad \text{for all } k \in K(\mathbb{R})\,. \tag{2.37}$$

that is, f is a function on $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$. The main example of an automorphic form is an Eisenstein series which we will study below.

2.4 Supersymmetry constraints

Besides being constrained by U-duality, the coefficient functions in the effective action are also constrained by the continuous symmetries of the theory. In this section we will obtain a differential eigenvalue equation for $\mathcal{E}_{(0,0)}$ in (2.34) using constraints from supersymmetry. So far, we have only studied the four-graviton correction to the low-energy effective action, but there are many other interactions entering the effective action at the same order in α' as the R^4 term, all related by supersymmetry [30]. In order find the constraint on $\mathcal{E}_{(0,0)}$ we have to consider these additional terms as well which we will do using a superspace approach.

In four dimensions a chiral superfield Φ satisfies the schematical linear equation

$$D^*\Phi = 0 \tag{2.38}$$

where D^* is the anti-holomorphic supercovariant derivative. In ten dimensions, with θ^a, $a = 1, \ldots, 16$ being a complex Grassmannian coordinate transforming as a Weyl spinor under $SO(9,1)$, a similar holomorphicity constraint for a superfield Φ in ten dimensions can be imposed in flat space, but not in curved space where the integrability condition $\{D^*_a, D^*_b\} = 0$ is not satisfied in general [14].

It is, however, possible to consider such a field in linearised[4] supersymmetry with the rigid generators [30]

$$Q_a = \frac{\partial}{\partial\theta^a} \qquad\qquad Q^*_a = -\frac{\partial}{\partial\theta^{*a}} + 2i(\overline{\theta}\gamma^\mu)_a\partial_\mu \tag{2.39}$$

anticommuting with the corresponding covariant derivatives

$$D_a = \frac{\partial}{\partial\theta^a} + 2i(\gamma^\mu\theta^*)_a\partial_\mu \qquad\qquad D^*_a = -\frac{\partial}{\partial\theta^{*a}}\,. \tag{2.40}$$

After imposing (2.38) and the constraints

$$D^4\Phi = D^{*4}\Phi^*\,, \tag{2.41}$$

the on-shell superfield Φ describes all fields of the IIB supergravity [14, 46] and the linearised interactions can be obtained by integrating a superpotential F of Φ over

[4] In terms of fluctuations around a flat background with constant complex scalar $\tau = C_0 + ig_s^{-1}$.

the sixteen components of θ leading to all the possible interactions at this order in α' [30, 40]

$$\mathcal{L}_{(\alpha')^3} = f_{12}(\tau)\lambda^{16} + f_{11}(\tau)\hat{G}\lambda^{14} + \ldots + f_0(\tau)R^4 + \ldots + f_{-12}(\tau)\lambda^{*16}\,. \tag{2.42}$$

Here $f_w(\tau)$ are modular forms with holomorphic weight w and anti-holomorphic weight $-w$ with $f_0 = \mathcal{E}_{(0,0)}$ from above, λ is the dilatino, and $\hat{G}$ includes the R-R and NS-NS 2-forms, the gravitino and the dilatino. The terms λ^{16}, $\hat{G}\lambda^{14}$ and λ^{*16} are defined in [40]. A modular form h with holomorphic weight w and anti-holomorphic weight $\hat{w}$ transforms as

$$h(\tau) \to h(\gamma(\tau)) = (c\tau + d)^w (c\overline{\tau} + d)^{\hat{w}} h(\tau) \qquad \tau \to \gamma(\tau) = \frac{a\tau + b}{c\tau + d}\,. \tag{2.43}$$

From the fact that the different f_w terms, in the linear approximation, are related by the Taylor expansion of F, one obtains that [30]

$$f_{12}(\tau) = D^{12} f_0(\tau) = D_{11} D_{10} \ldots D_0 f_0(\tau) \tag{2.44}$$

where D_w is the refined modular covariant derivative

$$D_w = i\Big(\tau_2 \frac{\partial}{\partial\tau} - i\frac{w}{2}\Big) \tag{2.45}$$

which maps a modular form of holomorphic weight w into a weight $w+1$ modular form (with corresponding changes to the antiholomorphic weight). Finding an eigenvalue equation for f_{12} then gives us the sought for condition for f_0.

This will be achieved by studying the supersymmetry transformation of the action S order by order in α' using the Noether method. Expanding the supersymmetry transformation on an arbitrary field Ψ as

$$\delta\Psi = (\delta^{(0)} + \alpha'\delta^{(1)} + (\alpha')^2\delta^{(2)} \ldots)\Psi \tag{2.46}$$

acting on the effective action

$$S = S^{(0)} + (\alpha')^3 S^{(3)} + (\alpha')^4 S^{(4)} + \ldots \tag{2.47}$$

we obtain constraints such as

$$\delta^{(0)} S^{(0)} = 0 \qquad \delta^{(0)} S^{(3)} + \delta^{(3)} S^{(0)} = 0 \qquad \ldots \tag{2.48}$$

for each order in α', where $\delta^{(0)}$ corresponds to the supersymmetry transformation of the classical theory described by $S^{(0)}$ and $S^{(3)}$ contains the interaction terms of (2.42).

The reason for studying f_{12} instead of f_0 is that f_{12} together with f_{11} mix only with each other and no other terms under $\delta^{(0)}$, while a general transformation can become quite complicated [30]. From the order $(\alpha')^3$ condition in (2.48) we get the following two factors of two independent terms that have to vanish [30]

$$\begin{gathered} D_{11} f_{11} + \frac{4}{3\cdot 144} f_{12} = 0 \\ \overline{D}_{-12} f_{12} + 3\cdot 144\cdot \frac{15}{2} f_{11} + 15g = 0 \end{gathered} \tag{2.49}$$

where g is another modular form yet to be decided and

$$\overline{D}_w = -i\Big(\tau_2\frac{\partial}{\partial\overline{\tau}} + i\frac{w}{2}\Big) \tag{2.50}$$

which transforms a modular form of weight w to a weight $w-1$ modular form.

By also requiring that the algebra closes for the field λ^* defined in [30] at order $(\alpha')^3$, i.e. $\Big(\delta^{(0)}_{\epsilon_1}\delta^{(3)}_{\epsilon_2^*} - \delta^{(3)}_{\epsilon_2^*}\delta^{(0)}_{\epsilon_1}\Big)\lambda^* = 0$, and comparing with the equation of motion for λ^* one finds that [30]

$$32 D_{11} g = f_{12}\,. \tag{2.51}$$

Together with (2.49), this gives us that [40]

$$\Delta_{(-),12} f_{12} = \Big(-132 + \frac{3}{4}\Big) f_{12} \tag{2.52}$$

where

$$\Delta_{(-),w} \coloneqq 4D_{w-1}\overline{D}_{-w} = 4\tau_2^2\frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}} - 2iw\tau_2\Big(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\overline{\tau}}\Big) - w(w-1) \tag{2.53}$$

is the Laplace operator acting on weight w modular forms. Inserting this into (2.44), gives us for $f_0 = \mathcal{E}_{(0,0)}$ that

$$(\Delta - \tfrac{3}{4})\mathcal{E}_{(0,0)}(\tau) = 0 \tag{2.54}$$

where $\Delta = 4\tau_2^2\frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}}$ is the usual Laplace operator on the upper half plane. In this way one can obtain eigenequations for all interaction coefficients f_{2w} at the order of the R^4 term as shown in [30, 40].

The method presented above was extended for the corresponding terms at order of D^4R^4 as well in [67] where it was shown that the $\mathcal{E}_{(1,0)}$ coefficient satisfies

$$(\Delta - \tfrac{15}{4})\mathcal{E}_{(1,0)}(\tau) = 0\,. \tag{2.55}$$

Remark 2.4. The D^6R^4 term is different from the other two terms studied above. It does not satisfy an eigenfunction equation to the Laplace operator but includes an extra inhomogeneous term [42]

$$(\Delta - 12)\mathcal{E}_{(0,1)}(\tau) = -\Big(\mathcal{E}_{(0,0)}(\tau)\Big)^2\,. \tag{2.56}$$

In lower dimensions we have similar eigenfunction equations. We will only state the results here and refer the reader to I-2 for more details. The constraints in lower dimensions from supersymmetry are [36] (see (I-2.12))

$$\begin{aligned}
R^4:&\quad \Big(\Delta_{G/K} - \tfrac{3(11-D)(D-8)}{D-2}\Big)\,\mathcal{E}^{(D)}_{(0,0)}(g) = 6\pi\delta_{D,8}\\
D^4R^4:&\quad \Big(\Delta_{G/K} - \tfrac{5(12-D)(D-7)}{D-2}\Big)\,\mathcal{E}^{(D)}_{(1,0)}(g) = 40\zeta(2)\delta_{D,7}\\
D^6R^4:&\quad \Big(\Delta_{G/K} - \tfrac{6(14-D)(D-6)}{D-2}\Big)\,\mathcal{E}^{(D)}_{(0,1)}(g) = 40\zeta(3)\delta_{D,6} - \Big(\mathcal{E}^{(D)}_{(0,0)}(g)\Big)^2\,,
\end{aligned} \tag{2.57}$$

where $\Delta_{G/K}$ is the Laplace operator on $G(\mathbb{R})/K(\mathbb{R})$.

2.5 Instantons and non-perturbative effects

In 1994, Polchinski [64] found that introducing boundaries to the string world-sheet (corresponding to open string endpoints) that satisfy Dirichlet boundary conditions leads to non-perturbative corrections to scattering amplitudes on the form e^{-1/g_s}. Even before that though, Shenker [66] argued in 1990, by studying the large order behaviour of string perturbation theory, that non-perturbative effects had to contribute on the form e^{-1/g_s} instead of the form e^{-1/g_s^2} which is typical for non-perturbative effects in field theory.

In the computations of the scattering amplitudes in section 2.3, only string world-sheets without boundaries were included as illustrated in figure 2.2, but when summing over topologies, we should also include Riemann surfaces with an arbitrary number of both holes (boundaries) and handles. A boundary may have Neumann boundary conditions in $p+1$ space-time directions and Dirichlet boundary conditions in $9-p$ directions. In [64], Polchinski argued that these boundaries are attached to extended *physical* objects called Dp-branes where p is the spatial dimension of the brane corresponding to the Neumann conditions of the string endpoint. The addition of such a physical object, which can have several boundaries attached to it, gives different combinatorial properties to the scattering amplitudes compared to treating each boundary independently. This way of counting leads to the cancellation of divergences between world-sheets with boundaries [29, 64].

D-instantons are D(-1)-branes, which are localised to a single point in both space and time, and were found to give the sought for e^{-1/g_s} corrections to scattering amplitudes [64].

Following [29, 64], when computing a scattering amplitude $\mathcal{A}$ with D-instanton effects, we thus need to sum over the number of D-instantons n at positions y_i in space-time with $i = 1, \ldots, n$

$$\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n \tag{2.58}$$

where $\mathcal{A}_0$ is the usual scattering of closed strings without boundaries considered in section 2.3. For each instanton i we also need to sum over the number of world-sheet boundaries b_i that are attached to it, and integrate over the instanton position y_i to recover translational invariance. As we will shortly see, this integration also imposes momentum conservation of the external states. We then have that

$$\mathcal{A}_n = \prod_{i=1}^{n} \bigg(\int d^{10} y_i \sum_{b_i=0}^{\infty} \bigg) f_{b_1,\ldots,b_n}(y_1, \ldots, y_n) \tag{2.59}$$

where $f_{b_1,\ldots b_n}(y_1, \ldots, y_n)$ includes the usual summation over world-sheets with arbitrarily many handles and now including disconnected world-sheets[5], all with appropriate symmetry factors for exchanging D-instantons, disconnected components of the world-sheet and the boundaries of those components. The scattering amplitude

[5]Following the combinatoric arguments from ordinary field theory, only world-sheet components which are connected to the external states as seen from target space (i.e. that have boundaries attached to a D-instanton shared by a world-sheet with vertex operator insertions) contribute to scattering amplitudes [29].

factorises over these disconnected components and we may consider the components without vertex operator insertions (punctures) separately.

Each world-sheet component S is weighted by the factor $g_\mathrm{s}^{-\chi(S)} = g_\mathrm{s}^{2g-2+b}$ where $\chi(S)$ is the Euler characteristic of the surface S with genus g and b boundaries. The leading order contributions for a single D-instanton are shown in figure 2.3.

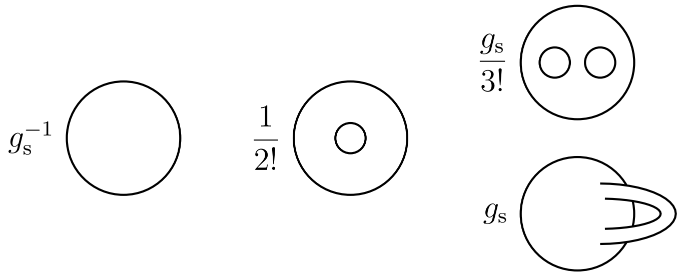

Figure 2.3: Lowest order puncture-free world-sheet components with weights and boundary symmetry factors where each boundary is attached to the same D-instanton. First we have the disk with one boundary and no handles followed by the annulus, and lastly a disk with two holes and a disk with one handle.

Let us now consider the first few corrections in the single D-instanton $n = 1$ case. When summing over world-sheet topologies we should sum over the number of disks, annuli, etc. of figure 2.3 together with their weights, boundary symmetry factors and with a symmetry factor for exchanging identical world-sheet components — we get a gas of different types of components all with boundaries at the same point y.

Denoting the amplitude factor for a single disk as $\langle \bigcirc \rangle$ and the factor for a single annulus as $\langle \circledcirc \rangle$ we then get an exponentiation [29]

$$\begin{aligned} \mathcal{A}_1 &= \int d^{10}y \sum_{d_1=0}^{\infty} \frac{1}{d_1!}\Big(g_\mathrm{s}^{-1}\langle \bigcirc \rangle\Big)^{d_1} \sum_{d_2=0}^{\infty} \frac{1}{d_2!}\Big(\frac{1}{2!}\langle \circledcirc \rangle\Big)^{d_2} \cdots \mathcal{A}_1^{(\text{vertex ops})} \\ &= \int d^{10}y \, \exp\Big(g_\mathrm{s}^{-1}\langle \bigcirc \rangle + \frac{1}{2!}\langle \circledcirc \rangle + \ldots\Big) \mathcal{A}_1^{(\text{vertex ops})} \end{aligned} \tag{2.60}$$

where the dots signify higher order world-sheet components in the string coupling. From this we see that with a negative $\langle \bigcirc \rangle$ we get the anticipated non-perturbative corrections on the form e^{-1/g_s}.

The leading contribution to $\mathcal{A}_1^{(\text{vertex ops})}$ for the four-graviton scattering amplitude in the single D-instanton background comes from four disks with one vertex operator insertion on each. This correction was computed in [31] for a single unit charged D-instanton as

$$\mathcal{A}_1 = C e^{2\pi i \tau} \int d^{10}y \, e^{iy\cdot(k_1+k_2+k_3+k_4)} K_\mathrm{cl} \,, \tag{2.61}$$

where the integral over y imposes momentum conservation. Together with (2.33), this gives us a coefficient function $\mathcal{E}_{(0,0)}$ on the form

$$\mathcal{E}_{(0,0)}(\tau) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \ldots + C e^{2\pi i \tau} + \ldots \ . \tag{2.62}$$

For the perturbative amplitudes in section 2.3, the K_cl factor was obtained by tracing over sixteen fermionic zero modes, or by integrating over sixteen Grassmann coordinates in (2.42), which is half of type IIB superspace [42]. Here, in the stringy D-instanton picture, it is obtained by integrating over fermionic zero modes from open string vertex operators attached to the world-sheet boundaries. Each of the four disks with graviton external states have four boundary open string vertex operators attached to them.

The factor $e^{2\pi i\tau} = e^{2\pi i\chi - 2\pi/g_s}$ can be motivated semi-classically from supergravity. As illustrated in figure 2.4 we can approximate the brane-string-interaction by a non-trivial background in supergravity. In the left illustration the brane emits and absorbs closed strings or gravitons and, as seen above, summing over all possible world-sheet boundaries attached to the brane leads to an exponentiation. In the classical theory, this exponentiation can be seen as a deformation $e^{-\delta S}$ of the action coming from the expansion of field fluctuations around a *non-trivial* classical solution to the equations of motion. Schematically, for a field theory with field φ with non-trivial classical solution φ_0 and quantum fluctuations $\tilde{\varphi}$

$$\varphi = \varphi_0 + \tilde{\varphi} \qquad S[\varphi] = S[\varphi_0] + \frac{1}{2}\frac{\delta^2 S}{\delta\varphi\delta\varphi}\bigg|_{\varphi=\varphi_0} \tilde{\varphi}^2 + \dots\,, \tag{2.63}$$

where the linear term vanishes since φ_0 satisfies the equations of motion.

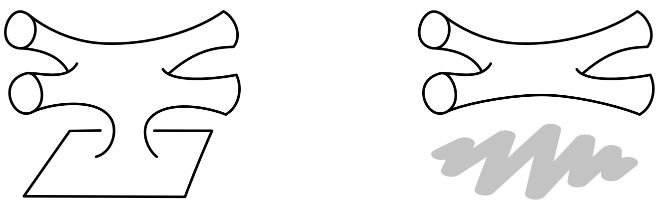

Figure 2.4: Strings interacting with a brane (left) approximated by a non-trivial supergravity background (right).

The supergravity solution which describes a single D-instanton is [29]

$$G_{\mu\nu} = \delta_{\mu\nu} \qquad e^{\phi} = g_s + \frac{c}{r} \qquad \chi = \chi_\infty \pm (e^{-\phi} - g_s^{-1}) \tag{2.64}$$

where $G_{\mu\nu}$ is the space-time metric and c is a positive constant related to the Noether charge Q under translations of χ (whose sign depends on the choice of sign in χ above). The remaining fields are trivial. The charge is defined as a nine-dimensional surface integral around the space-time origin and is conserved under surface deformations as long as we do not cross $r = 0$. In this sense the instanton is localised at the origin.

By inserting our solution into (2.2) we obtain the instanton action corresponding to $S[\varphi_0]$ in (2.63) as

$$S^{(N)}_{\text{inst}} = -i\,|Q|\,\tau = -2\pi i\,|N|\,\tau \tag{2.65}$$

where Q has been quantised by the Dirac-Nepomechie-Teitelboim argument [31] as $Q = 2\pi N$ with N integer. For $N = 1$ we obtain the factor $e^{-S_{\text{inst}}} = e^{2\pi i\tau}$ of (2.61).

Remark 2.5. The supergravity solution breaks sixteen supercharges, meaning that it is a 1/2 BPS state, and gives sixteen fermionic zero modes for the instanton from which one obtains the kinematic factor $K_{\text{cl}} = \mathcal{R}^4$ after integration [31]. The higher derivative corrections D^4R^4 and D^6R^4 require more fermionic zero modes and in lower dimensions they are related to instantons many of which are dimensionally reduced 1/4 and 1/8 BPS black hole states respectively [31, 33].

2.6 Eisenstein series in ten-dimensional scattering amplitudes

Let us summarise the results we have found for higher order corrections to the ten-dimensional type IIB supergravity theory from four-graviton scattering amplitudes in string theory. More details can be found in chapter I-2. The corrections were structured in orders of α' with coefficients that were functions on the moduli space $G(\mathbb{R})/K(\mathbb{R})$ invariant under $G(\mathbb{Z})$. That is, they satisfy the automorphicity condition (A) for automorphic forms as defined at the end of section 2.3.

In (2.33) we found the first perturbative corrections to the coefficient functions as

$$\begin{aligned}
\mathcal{E}_{(0,0)}(\tau) &= 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \dots \\
\mathcal{E}_{(1,0)}(\tau) &= \zeta(5)\tau_2^{5/2} + \dots \\
\mathcal{E}_{(0,1)}(\tau) &= \frac{2}{3}\zeta(3)^2\tau_2^3 + \frac{4}{3}\zeta(2)\zeta(3)\tau_2 + \dots
\end{aligned} \tag{2.66}$$

with $\tau_2 = g_s^{-1}$. This means that that these coefficient functions satisfies the growth condition (C) for automorphic forms.

We also showed in (2.54) and (2.55) that $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$ are eigenfunctions to the Laplace operator on $\mathbb{H}$ with eigenvalues 3/4 and 15/4 respectively, which means that they also satisfy condition (B). Thus, they satisfy all the conditions for being an automorphic form.

Remark 2.6. The D^6R^4 coefficient $\mathcal{E}_{(0,1)}$ is not an eigenfunction to the Laplace operator as discussed in remark 2.4, and therefore not an automorphic form in a strict sense. This term is discussed further in section I-12.1.2.

2.6.1 Eisenstein series on $SL(2,\mathbb{R})$

We will now show that $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$ are in fact Eisenstein series, but first we need some definitions and to study the properties of Eisenstein series. There are several ways to define them, but we will use a definition that is more easily generalisable to larger groups $G(\mathbb{R})$.

An Eisenstein series on $SL(2,\mathbb{R})$ is defined by a multiplicative characters $\chi : B(\mathbb{Z})\backslash B(\mathbb{R}) \to \mathbb{C}^\times$, where B is the Borel subgroup as defined in (2.15), determined by its restriction on A. That is,

$$\chi(b'b) = \chi(b) \qquad \text{and} \qquad \chi(na) = \chi(a) \tag{2.67}$$

for $b, b' \in B(\mathbb{R})$, $n \in N(\mathbb{R})$ and $a \in A(\mathbb{R})$. It is trivially extended to all of $G(\mathbb{R})$ by $\chi(nak) = \chi(na) = \chi(a)$ for $k \in K(\mathbb{R})$. Mapping to the upper half plane, this means that χ is a function of $\tau_2 = \mathrm{Im}(\tau)$. Such characters are parametrised by a complex number s as

$$\chi(\tau) = \mathrm{Im}(\tau)^s \,. \tag{2.68}$$

Then, the corresponding Eisenstein series is defined by

$$E(s;\tau) = \sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \mathrm{Im}(\gamma(\tau))^s = \sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \frac{\tau_2^s}{|c\tau+d|^{2s}} \tag{2.69}$$

We note that the sum in (2.69) makes the Eisenstein series manifestly invariant under $G(\mathbb{Z}) = SL(2,\mathbb{Z})$ fulfilling condition (A) in the definition of an automorphic form. The $B(\mathbb{Z})\backslash SL(2,\mathbb{Z})$ coset is needed since χ is trivial under $B(\mathbb{Z})$. To see how this expression relates to Eisenstein series defined as a lattice sum over coprime integers c and d see section I-1.1.

Additionally, $\chi(\tau)$ is an eigenfunction to the Laplace-Beltrami operator on the Poincaré upper half plane $\mathbb{H}$

$$\Delta\chi(\tau) = s(s-1)\chi(\tau) \qquad \Delta = 4\tau_2^2 \frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar\tau} = \tau_2^2\left(\frac{\partial^2}{\partial\tau_1^2} + \frac{\partial^2}{\partial\tau_2^2}\right), \tag{2.70}$$

and since Δ is an $SL(2,\mathbb{R})$ invariant operator, i.e., if $(\gamma\cdot f) = f(\gamma(\tau))$ then $\Delta(\gamma\cdot f) = \gamma\cdot\Delta f$ for some function f and for all γ in $SL(2,\mathbb{R})$, we have that

$$\Delta E(s;\tau) = s(s-1)E(s;\tau)\,. \tag{2.71}$$

This eigenfunction condition is exactly that of (B) in the definition of an automorphic form.

They also satisfy the growth condition (C), which is most easily seen by studying their Fourier expansion. The Eisenstein series is invariant under shifts $\tau \to \tau + 1$ given by $g \to \gamma g$ with

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G(\mathbb{Z})\,. \tag{2.72}$$

We can then Fourier expand it as

$$E(s;\tau) = \sum_{m\in\mathbb{Z}} a_m(\tau_2) e^{2\pi i m\tau_1} = C(\tau_2) + \sum_{m\neq 0} a_m(\tau_2) e^{2\pi i m\tau_1} \tag{2.73}$$

where $C(\tau_2)$ is referred to as the constant term (with respect to τ_1).

As is shown in appendix I-B exploiting the lattice form of the Eisenstein series or, more generally, in chapter I-7 using the framework of adeles, the Fourier expansion is

$$E(s;\tau) = \tau_2^s + \frac{\xi(2s-1)}{\xi(2s)}\tau_2^{1-s} + \frac{2\tau_2^{1/2}}{\xi(2s)} \sum_{m\neq 0} |m|^{s-1/2}\,\sigma_{1-2s}(m) K_{s-1/2}(2\pi\,|m|\,\tau_2) e^{2\pi i m\tau_1} \tag{2.74}$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, K_s is the modified Bessel function of the second kind, and

$$\sigma_{1-2s}(m) = \sum_{d\,|\,m} d^{1-2s} \tag{2.75}$$

is a sum over all positive divisors d of m. In the weak coupling limit corresponding to $\tau_2 \to \infty$ we see that $E(s,\tau)$ grows polynomially as

$$E(s,\tau) \sim \tau_2^s + \frac{\xi(2s-1)}{\xi(2s)}\tau_2^{1-s} + \ldots \qquad \tau_2 \to \infty\,. \tag{2.76}$$

Thus, the Eisenstein series $E(s,\tau)$ on $SL(2,\mathbb{R})$ are automorphic forms with eigenvalues $s(s-1)$ to the Laplace-Beltrami operator and comparing with the eigenvalues of $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$ in (2.54) and (2.55) respectively, and the low coupling limit (2.76) with (2.66) we see that the following ansätze satisfy all the required conditions

$$\begin{aligned} \mathcal{E}_{(0,0)}(\tau) &= 2\zeta(3)E(3/2,\tau) \\ \mathcal{E}_{(1,0)}(\tau) &= \zeta(5)E(5/2,\tau) \end{aligned} \tag{2.77}$$

and it was shown in [59] for $\mathcal{E}_{(0,0)}$ that there are no additional automorphic forms which can be added to this without changing the asymptotic behaviour (2.66).

2.6.2 Extracting physical information

From the Fourier expansion of $\mathcal{E}_{(0,0)}$ we can extract a lot of physical information. When expanding the Bessel functions for the non-zero modes in (2.74) in the weak coupling limit $\tau_2 \to \infty$ (keeping the exponential suppression) we obtain

$$\mathcal{E}_{(0,0)}(\tau) \sim 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + 2\pi \sum_{N\neq 0} \sqrt{|N|}\sigma_{-2}(N)e^{-S_{\text{inst}}^{(N)}(\tau)}\Big[1 + \mathcal{O}(\tau_2^{-1})\Big]\,, \tag{2.78}$$

As expected, the non-perturbative terms come with a factor of $e^{-S_{\text{inst}}^{(N)}}$ with the instanton action (2.65)

$$S_{\text{inst}}^{(N)}(\tau) = -2\pi i\,|N|\,\tau \tag{2.79}$$

for an instanton with charge N. The omitted corrections $\mathcal{O}(\tau_2^{-1})$ in the non-perturbative terms come from higher genus instanton scattering amplitudes. Comparing with (2.62) we see that (2.78) captures both the perturbative and non-perturbative behaviour.

The expansion tells us that there are no further perturbative corrections beyond one loop and, interpreting the factor $\sqrt{|N|}\sigma_{-2}(N)$ as the instanton measure counting the number of instantons states with charge N we see that we have a degeneracy equal to the number of ways N can be factorised into two integers. In the type IIA picture, these two integers are the winding number and charge of a T-dual D-particle wrapping the circle [41].

2.7 Lower dimensions and larger groups

As shown in (2.36), the coefficient functions $\mathcal{E}^{(D)}_{(p,q)}$ in the low-energy effective action when compactifying on a torus are functions on the moduli space $G(\mathbb{R})/K(\mathbb{R})$ in table 2.1 invariant under the U-duality group $G(\mathbb{Z})$. Due to supersymmetry, the lower order terms $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ satisfy the eigenfunction equations (2.57) disregarding the in-homogeneous contributions for $D = 8$ and $D = 7$. Furthermore, it is possible to obtain their weak coupling asymptotics using string perturbation theory in lower dimensions.

Similar to ten dimensions, it is then possible to find the exact forms of $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ in terms of Eisenstein series, but now on the larger group $G(\mathbb{R})$.

Such Eisenstein series can be defined not only with respect to the Borel subgroup B but also with respect to all parabolic subgroups $P \supseteq B$. It is also possible to Fourier expand automorphic forms on larger groups G in different periodic variables corresponding to different choices of unipotent subgroups $U \subseteq N$. Such a unipotent subgroup is often constructed as part of another parabolic subgroup P. Before defining Eisenstein series on larger groups we will therefore study parabolic subgroups which will also give a precise definition of the Borel subgroup as the minimal parabolic subgroup together with the Iwasawa decomposition discussed in section 2.2.

Let $\mathfrak{g}$ be the Lie algebra of G with Cartan subalgebra $\mathfrak{h}$, roots Δ, positive roots Δ_+ and simple roots Π. As seen in section I-4.1.3 a parabolic subgroup P is specified by a choice of simple roots $\Sigma \subseteq \Pi$ generating a root system $\langle\Sigma\rangle$. The roots defining the Lie algebra $\mathfrak{p}$ of P are then $\Delta(\mathfrak{p}) = \Delta_+ \cup \langle\Sigma\rangle$ with

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{p})} \mathfrak{g}_\alpha \,, \tag{2.80}$$

where

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h, g] = \alpha(h) \quad \forall h \in \mathfrak{h}\} \tag{2.81}$$

for a root α.

The minimal parabolic subgroup corresponding to an empty subset Σ is the Borel subgroup B constructed from all positive roots.

The parabolic subgroup P can be decomposed into a semi-simple Levi subgroup L and a unipotent subgroup U as $P = UL$ defined by

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\langle\Sigma\rangle} \mathfrak{g}_\alpha \qquad \mathfrak{u} = \bigoplus_{\alpha\in\Delta(\mathfrak{u})} \mathfrak{g}_\alpha \qquad \Delta(\mathfrak{u}) = \Delta_+\backslash(\Delta_+ \cap \langle\Sigma\rangle)\,. \tag{2.82}$$

For the minimal parabolic we have the corresponding decomposition $B = NA$ where $U = N$ is a unipotent subgroup of B and $L = A$ is the Cartan subgroup. These form the Iwasawa decomposition of G which uniquely factorises an element into $G = BK = NAK$, where K is the maximal compact subgroup introduced above. Note that for a general parabolic subgroup P, the decomposition $G = PK = ULK$ is not unique.

Another special class of parabolic subgroups are those that include all simple roots in Σ except one α_j, called maximal parabolic subgroups denoted by P_{α_j}. These will be the main parabolic subgroups of interest besides the minimal one.

A parabolic Eisenstein series is given by a multiplicative character $\chi : P(\mathbb{Z})\backslash P(\mathbb{R}) \to \mathbb{C}^\times$ which is determined by its restriction on L, that is $\chi(ul) = \chi(l)$, and trivially extended to all of G by $\chi(ulk) = \chi(l)$. Then, similar to (2.69), the Eisenstein series is defined as the sum over images

$$E_P(\chi; g) = \sum_{\gamma \in P(\mathbb{Z})\backslash G(\mathbb{Z})} \chi(\gamma g) . \tag{2.83}$$

The characters χ can also be parametrised by weights $\lambda \in \mathfrak{h}^*$ as described in section I-5.3 similar to the complex number s which parametrised the characters on B of $SL(2, \mathbb{R})$. We will use the character χ and the weight λ interchangeably.

We note that the sum makes the Eisenstein series inherently invariant under $G(\mathbb{Z})$ fulfilling the condition (A). They also satisfy the eigenfunction condition (B), which is shown in appendix I-C for the Laplace operator. Since automorphic forms are invariant under discrete translations in $G(\mathbb{Z})$, they may be Fourier expanded as will be discussed in chapter 3 and is also one of the main topics of Paper I. When studying their Fourier coefficients one sees that also the growth condition (C) is satisfied. For brevity we will continue to denote the Eisenstein series $E_B(\chi, g)$ corresponding to the minimal parabolic $P = B$ as simply $E(\chi, g)$.

For dimensions $D = 5$, 4 and 3 corresponding to $G = E_6$, E_7 and E_8 respectively, the coefficient functions are conjectured to be the following solutions to (2.57) and the lower dimensional versions of (2.66) in terms of maximal parabolic Eisenstein series [36]

$$\begin{aligned} R^4 : \quad & \mathcal{E}^{(D)}_{(0,0)}(g) = 2\zeta(3) E_P(\lambda_{s=3/2}; g) \\ D^4 R^4 : \quad & \mathcal{E}^{(D)}_{(1,0)}(g) = \zeta(5) E_P(\lambda_{s=5/2}; g) \end{aligned} \tag{2.84}$$

where $P = P_{\alpha_1}$ is given by the choice of simple roots $\Sigma = \Pi\backslash\{\alpha_1\}$ of G and $\lambda_s = 2s\Lambda_P - \rho$ with Λ_P being the fundamental weight orthogonal to the Levi subgroup L of P and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha$ the Weyl vector.

To extract physical information from these Eisenstein series we compute their Fourier coefficients just as in the $SL(2, \mathbb{R})$ case. But for larger groups G we have more periodic variables to expand in and shifts of two different variables need not commute meaning that we have to compute non-abelian Fourier coefficients which we will define in the next chapter.

We also have a choice of unipotent subgroup – a choice of variables – to expand in corresponding to studying different limits of the moduli space, giving insight to different kinds of perturbative and non-perturbative effects. There are three typical limits that one would like to study in string theory in $D = 10 - d$ dimensions corresponding to choosing unipotent subgroups U of three different maximal parabolic subgroups P [33].

1. **Decompactification limit:** $P_{\alpha_{d+1}}$
   The limit where one of the compactified dimensions becomes large, which amounts to studying instantons coming from wrapped, higher dimensional black-holes. The maximal parabolic subgroup is obtained by removing the last root in the Dynkin diagram in figure 2.1. Note that, in this limit, the Levi subgroup of P contains the higher dimensional symmetry group G_{D+1}.

2. **String perturbation limit:** P_{α_1}
   The limit of weak string coupling $g_{\mathrm{s}} \to 0$ as was studied for $SL(2,\mathbb{R})$, which amounts to studying D-instantons and NS5-instantons.

3. **M-theory limit:** P_{α_2}
   The limit where the M-theory torus becomes large and the eleven-dimensional supergravity becomes a good approximation. This limit studies the wrapping of M2- and M5-branes.

The vast amount of physical information contained in these limits strongly motivates the study of automorphic forms on larger groups and their Fourier expansions with respect to different parabolic subgroups. In chapters 3 and 4 we will discuss the computation of such Fourier coefficients on larger groups G. This is, in general, a very complicated task, especially when without the comfort of a complex structure of the underlying space or a lattice formulation of the Eisenstein series as for $G = SL(2)$.

However, using the adelic framework, we will see that the Fourier coefficients factorise over primes $p < \infty$ together with a factor from the so called real part denoted by $p = \infty$, dividing the task into more manageable pieces. In terms of the $SL(2)$-expansion in (2.78), these factors correspond to the arithmetic information of the instanton measure for the former, and the Bessel function (with the instanton action) for the latter.

In chapter 4 we will present closed form expressions for the perturbative terms and the arithmetic factors $p < \infty$ for the non-perturbative terms for general groups G. Corresponding closed form expressions for the remaining real parts $p = \infty$ are not known in the literature, and have to be computed case by case. In Paper I we do this for $SL(2)$ in chapter I-7 and for $SL(3)$ in section I-9.6. Besides computing Fourier-like integrals, the real parts can also be studied from the differential equations (2.57), but the numerical coefficients, the arithmetic parts, are not specified by these. Concretely, for $SL(2)$, inserting $f(\tau) = \sum_m c_m(\tau_2)\exp(2\pi i m \tau_1)$ with unknown coefficients into the eigenvalue equation $\Delta f(\tau) = s(s-1)f(\tau)$ gives that

$$f(\tau) = A(s)\tau_2^s + B(s)\tau_2^{1-s} + \tau_2^{1/2} \sum_{m\neq 0} a_m(s) K_{s-1/2}(2\pi\,|m|\,\tau_2) e^{2\pi i m \tau_1}\,, \tag{2.85}$$

with the arithmetic factors $A(s)$, $B(s)$ and $a_n(s)$ unknown.

Chapter 3

Automorphic forms

In section 2.3 we gave an introduction to automorphic forms defined on a real group $G(\mathbb{R})$. To be able to compute Fourier coefficients, we will lift the automorphic forms to the adeles introduced below and, in particular, discuss adelic Eisenstein series and show how they are reduced to the real case.

We will then discuss automorphic representations, which are of great importance in Paper II, followed by a detailed definition of what we mean by a Fourier coefficient on a group $G(\mathbb{A})$. In chapter 4 we will present different methods for computing such Fourier coefficients.

For references on adelic automorphic forms (besides Paper I) we recommend the books [22, 48] where the framework was first developed. For some good introductions, see [10, 21, 27, 28].

3.1 Adelisation and Eisenstein series

For a fixed prime p the p-adic number system is an extension of the rational numbers, but instead of using the ordinary archimedean norm, which gives the real numbers as an extension, one uses the following non-archimedean norm. For a rational number $q = p_1^{k_1} \cdots p_r^{k_r}$ with primes p_j and integers k_j the p_i-adic norm is defined as

$$|q|_{p_i} = p_i^{-k_i} \,. \tag{3.1}$$

For an arbitrary prime p not included in $p_1, \ldots p_r$ the corresponding k, called the p-adic valuation of q, is zero giving $|q|_p = 1$. If $q = 0$ the norm is defined to be $|q|_p = 0$ for all p. That it is non-archimedean means that it satisfies a stronger triangle inequality

$$|x + y|_p \leq \max(|x|_p \,, |y|_p) \,. \tag{3.2}$$

Taking equivalence classes of Cauchy sequences from $\mathbb{Q}$ with respect to this norm we obtain the p-adic numbers $\mathbb{Q}_p$. The p-adic integers are defined as

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\} \,, \tag{3.3}$$

which includes the integers $\mathbb{Z}$, and we can now define the adeles to include all these extensions of $\mathbb{Q}$ as the restricted product

$$\mathbb{A} = \mathbb{R} \times \prod_{p<\infty}{}' \mathbb{Q}_p \tag{3.4}$$

where the restriction, denoted by the prime in the product, includes only elements $x = (x_\infty; x_2, x_3, x_5, \dots)$ with $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ where all but a finite x_p are in $\mathbb{Z}_p$. It is often convenient to include $\mathbb{R}$ as $\mathbb{Q}_\infty$. An object or result that only treats one prime (or place) including ∞, is called *local*, while an adelic object or result is called *global*.

The rational numbers can be diagonally embedded into the adeles

$$\mathbb{Q} \ni q \mapsto (q; q, q, \dots) \in \mathbb{A} \tag{3.5}$$

and in proposition I-3.24 we show that $\mathbb{Q}$ is discrete in $\mathbb{A}$. This is a large simplification compared to studying $\mathbb{Z}$ in $\mathbb{R}$ since $\mathbb{Z}$ is only a ring while $\mathbb{Q}$ is a field. The adelic norm is defined as

$$|x|_\mathbb{A} = |x_\infty|_\infty \prod_{p<\infty} |x_p|_p \qquad x \in \mathbb{A} \tag{3.6}$$

and, using the fundamental theorem of arithmetic, $|q|_\mathbb{A} = 1$ for a rational number q diagonally embedded in $\mathbb{A}$. We will, from here on, suppress the subscript $\mathbb{A}$ of the adelic norm.

The adelisation of a function $f_\mathbb{R} : \mathbb{R} \to \mathbb{C}$ is a function $f_\mathbb{A} : \mathbb{A} \to \mathbb{C}$ taking $x = (x_\infty; x_2, x_3, \dots) \in \mathbb{A}$, with $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$, to $f_\mathbb{A}(x)$ such that

$$f_\mathbb{A}\Big((x_\infty; 1, 1, \dots)\Big) = f_\mathbb{R}(x_\infty)\,. \tag{3.7}$$

As discussed in section I-3.6 multiplicative characters on $\mathbb{Q}\backslash\mathbb{A}$ are parametrised by rational numbers m and factorise as follows

$$\psi^{(m)}(x) = \psi^{(m)}_\infty(x_\infty) \prod_{p<\infty} \psi^{(m)}_p(x_p) = e^{2\pi i m x_\infty} \prod_{p<\infty} e^{-2\pi i [m x_p]_p} \tag{3.8}$$

where the p-adic fractional part $[\,\cdot\,]_p$ is defined in section I-3.3.

The notion of p-adic groups is discussed in chapter I-4.2. For real Lie groups we had the unique Iwasawa decomposition $G(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R})K(\mathbb{R})$. Due to the definition of a p-adic integer (3.3) the maximal compact subgroup for a p-adic Lie group is $K_p = G(\mathbb{Z}_p)$ and we instead get the non-unique p-adic Iwasawa decomposition $G(\mathbb{Q}_p) = N(\mathbb{Q}_p)A(\mathbb{Q}_p)K_p$. Note that $G(\mathbb{Z}) \subset G(\mathbb{Z}_p) = K_p$.

An adelic Lie group is defined as the product

$$G(\mathbb{A}) = G(\mathbb{R}) \times G_f \qquad G_f = \prod_{p<\infty}{}' G(\mathbb{Q}_p) \tag{3.9}$$

restricting to elements $(g_\infty; g_2, g_3, \dots)$ with all but finitely many $g_p \in G(\mathbb{Z}_p)$. The adelic version of the compact subgroup is

$$K_\mathbb{A} = K(\mathbb{R}) \times K_f \qquad K_f = \prod_{p<\infty} K_p \tag{3.10}$$

with Iwasawa decomposition

$$G(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})K_\mathbb{A}\,. \tag{3.11}$$

Since $\mathbb{Q}$ is discrete in $\mathbb{A}$ we also have that $G(\mathbb{Q})$ is discrete in $G(\mathbb{A})$.

Adelic automorphic forms are functions $\varphi(g)$ on a Lie group $G(\mathbb{A})$ taking values in $\mathbb{C}$ that satisfy the following conditions (a more detailed definition can be found in chapter I-5)

(A') *Automorphic invariance*: $\varphi(\gamma g) = \varphi(g)$ for all γ in some discrete subgroup $\Gamma \subset G$

(B') *Differential equations*: φ is an eigenfunction to all G-invariant differential operators

(C') *Growth condition*: for any norm $\|\cdot\|$ on $G(\mathbb{A})$, $\varphi(g)$ has at most polynomial growth in $\|g\|$.

We will in this thesis consider only $\Gamma = G(\mathbb{Q})$ and spherical φ, that is, φ is a function on $G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\mathbb{A}}$.

The strong approximation theorem I-4.5 states that

$$G(\mathbb{Z})\backslash G(\mathbb{R}) \cong G(\mathbb{Q})\backslash G(\mathbb{A})/K_f\,. \tag{3.12}$$

This means that we can lift automorphic forms on $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$ to automorphic forms on $G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\mathbb{A}}$.

Let us now explicitly construct Eisenstein series on $G(\mathbb{A})$ and see that they are the adelisation of the real Eisenstein series of (2.83) in the meaning of (3.7). This construction does not explicitly use the strong approximation theorem, but the theorem does tell us that this is the only extension of the Eisenstein series to $G(\mathbb{A})$ [19].

The Eisenstein series in (2.83) were constructed as the sum over images of a character $\chi : P(\mathbb{Z})\backslash P(\mathbb{R}) \to \mathbb{C}^\times$. The corresponding object to consider in the adelic framework is a character $\chi : P(\mathbb{Q})\backslash P(\mathbb{A}) \to \mathbb{C}^\times$. It is trivially extended to $G(\mathbb{A})$ by $\chi(pk) = \chi(p)$ for $k \in K_{\mathbb{A}}$ and factorises as $\chi(g) = \chi_\infty(g_\infty)\prod_{p<\infty}\chi_p(g_p)$. In particular, this means that $\chi_p(k_p) = 1$ for $k_p \in K_p$.

An adelic Eisenstein series is then obtained by the sum over images

$$E_{\mathbb{A}}(\chi; g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \chi(\gamma g) \tag{3.13}$$

and satisfies all the conditions for an adelic automorphic form.

We will now show that this is an adelisation of real Eisenstein series in the case of $SL(2,\mathbb{R})$. In example I-4.9 we show that $B(\mathbb{Z})\backslash SL(2,\mathbb{Z}) \to B(\mathbb{Q})\backslash SL(2,\mathbb{Q})$ given by $B(\mathbb{Z})\gamma \mapsto B(\mathbb{Q})\gamma$ is an isomorphism which means that each $B(\mathbb{Q})g$ coset for $g \in SL(2,\mathbb{Q})$ has a representative in $SL(2,\mathbb{Z})$. Choosing such representatives we obtain

$$E_{\mathbb{A}}(\chi; g) = \sum_{\gamma \in B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} \chi(\gamma g) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \chi(\gamma g)\,. \tag{3.14}$$

Restricting to $g = (g_\infty; \mathbb{1}, \mathbb{1}, \ldots)$ we see that

$$\chi(\gamma g) = \chi_\infty(\gamma g_\infty) \prod_{p<\infty} \chi_p(\gamma) = \chi_\infty(\gamma g_\infty) \tag{3.15}$$

since $\gamma \in SL(2,\mathbb{Z}) \subset SL(2,\mathbb{Z}_p) = K_p$. Hence

$$E_{\mathbb{A}}\Big(\chi; (g_\infty, \mathbb{1}, \mathbb{1}, \ldots)\Big) = E_{\mathbb{R}}(\chi_\infty; g_\infty) \tag{3.16}$$

From here on, we will, in the context of automorphic forms and Eisenstein series, mean the adelic versions defined here and suppress the subscript $\mathbb{A}$ in $E_{\mathbb{A}}$.

For a discussion of non-spherical automorphic forms and how holomorphic modular forms fit in this description see sections I-5.1.3 and I-5.5.

3.2 Automorphic representations

Let us denote the space of automorphic forms on $G(\mathbb{A})$ invariant under $G(\mathbb{Q})$ as $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. We will now investigate how the group $G(\mathbb{A})$ acts on an automorphic form $\varphi \in \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ by the right-translation action π

$$[\pi(h)\varphi](g) = \varphi(gh) \qquad h, g \in G(\mathbb{A})\,. \tag{3.17}$$

Before we do that though we need to discuss a subtlety in the definition of an automorphic form φ that we have glossed over in this summary. The rigorous definition I-5.6 includes an additional K-finiteness condition, that is

$$\dim_{\mathbb{C}}\Big(\operatorname{span}\{\varphi(gk) \mid k \in K_{\mathbb{A}}\}\Big) \leq \infty\,. \tag{3.18}$$

In our discussion we have mainly treated spherical automorphic forms which are invariant under $K_{\mathbb{A}}$ and are thus automatically K-finite. However, the condition (3.18) is not invariant under right-translations of $G(\mathbb{A})$ meaning that ρ takes us outside the space $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ [10, 26].

Instead, we consider the actions

$$\begin{aligned} [\pi_f(h_f)\varphi](g) &= \varphi(g(\mathbb{1}; h_f)) && h_f \in G_f \\ [\pi_{K(\mathbb{R})}(k_\infty)\varphi](g) &= \varphi(g(k_\infty; \mathbb{1})) && k_\infty \in K(\mathbb{R}) \\ [\pi_{\mathfrak{g}}(X)\varphi](g) &= \tfrac{d}{dt}\varphi(ge^{tX})\big|_{t=0} && X \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \end{aligned} \tag{3.19}$$

where $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ is the universal enveloping algebra of $\mathfrak{g}(\mathbb{C})$, and by $(a; b)$ we mean $(a; b_2, b_3, b_5, \ldots) \in G(\mathbb{A})$ for $a \in G(\mathbb{R})$ and $b \in G_f$. All these actions preserve the space $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. An automorphic representation π of $G(\mathbb{A})$ is then an irreducible constituent in the decomposition of $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ under the simultaneous actions of (3.19). The automorphic representation π factorises over primes $\pi = \prod_{p\leq\infty} \pi_p$ where π_p for $p < \infty$ furnishes a representation of $G(\mathbb{Q}_p)$.

To each representation π we can assign a notion of size: the functional dimension $\operatorname{GKdim}(\pi)$, called the Gelfand-Kirillov dimension, which is defined as the smallest number of variables that are required to realise all the functions in π. In Paper II we are especially interested in small representations to which the automorphic forms for the R^4 and D^4R^4 interactions in chapter 2 are attached.

Indeed, the Eisenstein series $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ of (2.84) are attached to the minimal and next-to-minimal representations with respect to the Gelfand-Kirillov dimension.

As discussed in sections 4.5 and I-6.4.2, each automorphic representation π can be associated to a nilpotent orbit $\mathcal{O}_\pi$ of a nilpotent element $X \in \mathfrak{g}(\mathbb{C})$

$$\mathcal{O} = \{gXg^{-1} \mid g \in G(\mathbb{C})\}\,. \tag{3.20}$$

For an introduction to nilpotent orbits see section II-2.2. In particular, the minimal and next-to-minimal representations, π_{min} and π_{ntm}, are associated to the minimal and next-to-minimal orbits, $\mathcal{O}_{\text{min}}$ and $\mathcal{O}_{\text{ntm}}$, respectively. The latter are defined by the partial ordering of nilpotent orbits defined in section II-2.2. This connection to orbits will be useful when discussing Fourier coefficients below and is central to Paper II.

3.3 Fourier coefficients of automorphic forms

Before considering Fourier coefficients of adelic automorphic forms, let us first introduce some concepts using the example of spherical Eisenstein series $E(s;\tau)$ on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$. The expansion was summarised in (2.74). There we used the isomorphism $SL(2,\mathbb{Z})/SO(2,\mathbb{R}) \cong \mathbb{H}$ to expand $E(s;\tau)$ in shifts of $\tau \to \tau + 1$, but we will now use a group-theoretical approach which can be generalised to the other moduli spaces.

From (2.10) we obtained the Iwasawa decomposition of $SL(2,\mathbb{R}) = NAK$ with

$$N = \left\{ n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}. \tag{3.21}$$

and we noted that shifts $\tau \to \tau + 1$ was equivalent to translating $g \to ng$ with $x = 1$. The different Fourier modes $e^{2\pi i m \tau_1}$ of (2.73) are here described by unitary multiplicative characters $\psi^{(m)} : N(\mathbb{Z})\backslash N(\mathbb{R}) \to U(1)$

$$\psi^{(m)}(n) = e^{2\pi i m x} \tag{3.22}$$

We then have that

$$\begin{aligned} E(s;\tau) &= \sum_{m\in\mathbb{Z}} a_m(\tau_2) e^{2\pi i m \tau_1} = \sum_{m\in\mathbb{Z}} \int_0^1 dx\, E(s; x + i\tau_2) e^{-2\pi i m (x - \tau_1)} \\ &= \sum_{m\in\mathbb{Z}} \int_0^1 dx\, E(s; \tau + x) e^{-2\pi i m x} \end{aligned} \tag{3.23}$$

where, in the last step, we used the variables substitution $x \to x + \tau_1$ and that $E(\tau)$ is periodic. This form may be translated to group-theoretical terms as

$$\begin{aligned} E(s;\tau) &= \sum_{m\in\mathbb{Z}} \quad \int_0^1 dx \quad E(s;\tau + x) \quad e^{-2\pi i m x} \\ E(s;g) &= \sum_{\psi} \quad \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} dn \quad E(s; ng) \quad \overline{\psi(n)} \end{aligned} \tag{3.24}$$

where we sum over different characters ψ parametrised by m and dn is the Haar measure on $N(\mathbb{R})$ normalised such that $\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} dn = 1$.

For an automorphic form φ, the functions

$$W_N(\varphi,\psi;g) = \int\limits_{N(\mathbb{Z})\backslash N(\mathbb{R})} dn\, \varphi(ng)\overline{\psi(n)} \tag{3.25}$$

have the following property

$$\begin{aligned} W_N(\varphi,\psi;nak) &= \int\limits_{N(\mathbb{Z})\backslash N(\mathbb{R})} dn'\, \varphi(n'nak)\overline{\psi(n')} = \int\limits_{N(\mathbb{Z})\backslash N(\mathbb{R})} dn''\, \varphi(n''ak)\overline{\psi(n''n^{-1})} \\ &= \psi(n)W_N(\varphi,\psi;ak) \end{aligned} \tag{3.26}$$

and for spherical φ

$$W_N(\varphi,\psi;nak) = \psi(n)W_N(\varphi,\psi;a)\,. \tag{3.27}$$

Such functions are called Whittaker vectors and are examples of Fourier coefficients prominent in both Paper I and Paper II. In chapter I-6.1 we compute them for $SL(2,\mathbb{R})$ using the adelic framework, and for general groups in chapter I-9. Note that it is only for abelian N that

$$\varphi(g) = \sum_{\psi} W_N(\varphi,\psi;g)\,. \tag{3.28}$$

Let us now generalise this discussion to include larger groups and use the adelic framework introduced above. Since an adelic automorphic form φ is a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$ and $\mathbb{Q}$ sits discretely in $\mathbb{A}$ we can Fourier expand φ in some periodic variables n_i. There are several choices of expansions depending on which periodic variables we would like to consider equivalent to choosing a unipotent subgroup U. We will here mainly consider unipotent subgroups that are part of a parabolic subgroup P.

Fourier coefficients on this unipotent subgroup are then specified by a unitary multiplicative character $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$. Similar to above, these characters factorise into p-adic characters defined in section I-3.3, and real characters expressed as ordinary exponential functions. If $u \in U$ is given by $u = \prod_\alpha \exp(u_\alpha E_\alpha)$ with $\alpha \in \Delta(\mathfrak{u})$, E_α the corresponding (positive) Chevalley generator, and $u_\alpha \in \mathbb{A}$ then ψ_U might be parametrised by so called charges $m_\alpha \in \mathbb{Q}$ as

$$\begin{aligned} \psi_U(u) &= \psi_{U,\infty}(u_\infty) \prod_{p<\infty} \psi_{U,p}(u_p) \\ &= \exp\Big(2\pi i \sum_{\alpha\in\Delta(\mathfrak{u})} m_\alpha u_{\alpha,\infty}\Big) \prod_{p<\infty} \exp\Big(-2\pi i \sum_{\alpha\in\Delta(\mathfrak{u})} \big[m_\alpha u_{\alpha,p}\big]_p\Big)\,. \end{aligned} \tag{3.29}$$

Characters with all $m_\alpha \neq 0$ are called generic with the special case of all $m_\alpha = 1$ being called unramified. If at least one $m_\alpha = 0$ then the character is called degenerate. When we would like to specify the charges explicitly we denote the character as $\psi_U^{(m_\alpha, m_\beta, \ldots)}$. A Fourier coefficient is then defined as

$$F_U(\varphi,\psi_U;g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}\, du\,. \tag{3.30}$$

We have that

$$F_U(\varphi, \psi_U; u'g) = \psi_U(u')F_U(\varphi, \psi_U; g) \tag{3.31}$$

and since ψ_U is multiplicative this tells us that F_U can only capture the abelianisation $[U, U]\backslash U$ of U where the commutator subgroup is defined as

$$[U, U] = \{u_1 u_2 u_1^{-1} u_2^{-1} \mid u_1, u_2 \in U\}\,. \tag{3.32}$$

Let $U^{(1)} = U$ and $U^{(n+1)} = [U^{(n)}, U^{(n)}]$. A complete Fourier expansion on U requires that we recursively include all commutator subgroups

$$\varphi(g) = F_U(\varphi, 1; g) + \sum_{\psi_{U^{(1)}} \neq 1} F_{U^{(1)}}(\varphi, \psi_{U^{(1)}}; g) + \sum_{\psi_{U^{(2)}} \neq 1} F_{U^{(2)}}(\varphi, \psi_{U^{(2)}}; g) + \ldots \tag{3.33}$$

where $F_U(\varphi, 1; g)$ is called the constant term on U. The Fourier coefficients on $U^{(1)}$ are called abelian, while the Fourier coefficients on $U^{(n)}$ for $n > 1$ are called non-abelian.

As discussed in section 2.7, the Borel subgroup is the minimal parabolic subgroup. Its unipotent subgroup U, denoted by N, is defined by $\mathfrak{n} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$. The corresponding Fourier coefficients are called Whittaker vectors

$$W_N(\varphi, \psi_N; g) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_N(n)}\, dn\,, \tag{3.34}$$

as discussed above. Similar to (3.28) we have for a spherical automorphic form that

$$W_N(\varphi, \psi_N; nak) = \psi(n)W_N(\varphi, \psi_N; a) \tag{3.35}$$

We will now show that when restricting to $g = nak = (g_\infty; \mathbb{1}, \mathbb{1}, \ldots)$ the Whittaker vector $W_N(\varphi, \psi_N^{(m_1, m_2, \ldots)}; g)$, with character parametrised by rational charges m_i, is non-vanishing only for integer charges. Then, we will show that these Whittaker vectors exactly recover the real Whittaker vectors of (3.25). For brevity, we will restrict to the case of $SL(2, \mathbb{A})$, but the arguments for general groups are exactly the same. Let $\hat{n} = (\mathbb{1}; \hat{n}_2, \hat{n}_3, \ldots) \in N(\mathbb{A})$ with $\hat{n}_p \in N(\mathbb{Z}_p) \subset K_p$ parametrised by

$$\hat{n} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad x \in \mathbb{A}\,. \tag{3.36}$$

We then also have that $\hat{n} \in K_\mathbb{A}$ and

$$\begin{aligned} W_N(\varphi, \psi_N; a) = W_N(\varphi, \psi_N; a\hat{n}) &= W_N(\varphi, \psi_N; a\hat{n}a^{-1}a) \\ &= \psi_N(a\hat{n}a^{-1})W_N(\varphi, \psi_N; a) \end{aligned} \tag{3.37}$$

which means that $\psi_N(a\hat{n}a^{-1}) = 1$ for $W_N(\varphi, \psi_N; a)$ to be non-vanishing. Restricting to the $SL(2, \mathbb{A})$ case with character $\psi_N^{(m)}$ parametrised by a charge $m \in \mathbb{Q}$ and inserting $a = (a_\infty; \mathbb{1}, \mathbb{1}, \ldots)$ we then get that

$$1 = \psi_N^{(m)}(a\hat{n}a^{-1}) = \psi_{N,\infty}(\mathbb{1}) \prod_{p<\infty} \psi_{N,p}(\hat{n}_p) = \exp\Big(-2\pi i \sum_{p<\infty} [mx_p]_p\Big) \tag{3.38}$$

Using proposition I-3.13 we then have that $m \in \mathbb{Z}$. Thus,

$$\begin{aligned}\varphi\Big((g_\infty; \mathbb{1}, \mathbb{1}, \ldots)\Big) &= \sum_{m\in\mathbb{Z}} W_N\Big(\varphi, \psi_N^{(m)}; (g_\infty; \mathbb{1}, \ldots)\Big) \\ &= \sum_{m\in\mathbb{Z}} W_N\Big(\varphi, \psi_N^{(m)}; (a_\infty; \mathbb{1}, \ldots)\Big)\psi_{N,\infty}(n_\infty)\end{aligned} \tag{3.39}$$

and by orthogonality arguments we then have that $W_N\Big(\varphi, \psi_N^{(m)}; (a_\infty; \mathbb{1}, \ldots)\Big)$ are the same Whittaker vectors as the ones defined on $SL(2, \mathbb{R})$ in (3.25).

Chapter 4

Main results

In this chapter we will summarise the main results of Paper I and Paper II for computing adelic Fourier coefficients and, in particular, Whittaker vectors. Sections 4.1 to 4.4 are based on Paper I reviewing the existing literature, while section 4.5 summarises our own work in Paper II based on the methods of [24, 54].

Section 4.1 starts with some preliminary steps and explains the different cases we need to study in sections 4.2 and 4.3. We will use the notation

$$W_N(\chi, \psi_N; g) = W_N(E(\chi; \cdot), \psi_N; g) \tag{4.1}$$

and we will sometimes suppress the subscript in the character ψ_N or drop the character all together. Recall that, for spherical automorphic forms, W_N is determined by its values on the Cartan subgroup, and that a character χ can also be specified by a weight $\lambda \in \mathfrak{h}^*$. We will use the short-hand notation

$$\chi(a) = \left|a^{\lambda+\rho}\right| \qquad\qquad \rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha\,, \tag{4.2}$$

for this relation as explained in section I-5.3.1, where ρ is the Weyl vector.

For applications of many of the results below in the case of $SL(3)$ see section I-9.6, and examples I-10.29 and I-10.30.

4.1 Preliminary steps

Using the Bruhat decomposition [7]

$$G(\mathbb{Q}) = \bigcup_{w\in\mathcal{W}} B(\mathbb{Q})wB(\mathbb{Q}) \tag{4.3}$$

where $\mathcal{W}$ is the Weyl group of $G(\mathbb{R})$, we will now rewrite the Whittaker vectors of Eisenstein series in a way that factorises their computation over primes p. Let

$$N^{(w)}(\mathbb{A}) = \prod_{\substack{\alpha>0\\ w\alpha<0}} N_\alpha(\mathbb{A}) \tag{4.4}$$

where the product is over positive roots that become negative after reflection with w, and $N_\alpha(\mathbb{A}) = \{\exp(n_\alpha E_\alpha) \mid n_\alpha \in \mathbb{A}\}$. Also, for a character ψ on N, let

$$\mathcal{C}_\psi = \{w \in \mathcal{W} \mid w\alpha < 0 \text{ for all } \alpha \in \operatorname{supp}(\psi)\} \tag{4.5}$$

where $\operatorname{supp}(\psi) = \{\alpha \in \Pi \mid m_\alpha \neq 0\}$.

Then, it is shown in sections I-8.2 and I-9.1 that a Whittaker vector of a spherical Eisenstein series can be expressed as

$$W_N(\chi; a) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(\gamma n a)\overline{\psi(n)}\, dn = \sum_{w \in \mathcal{C}_\psi} F_w(\chi; a)\,. \tag{4.6}$$

where

$$F_w(\chi; a) = \int\limits_{N^{(w)}(\mathbb{A})} \chi(wna)\overline{\psi(n)}\, dn \tag{4.7}$$

Each of the terms F_w factorises as follows

$$F_w(\chi; a) = \prod_{p \leq \infty} F_{w,p}(\chi_p; a_p) \qquad F_{w,p}(\chi_p; a_p) = \int\limits_{N^{(w)}(\mathbb{Q}_p)} \chi_p(wna_p)\psi_p(n)\, dn \tag{4.8}$$

where we include $p = \infty$ with $\mathbb{Q}_\infty = \mathbb{R}$. We will now compute this in the separate cases (defined in section 3.3)

$\psi = 1$	The Langlands constant term formula
ψ unramified	}
ψ generic	} The Casselman-Shalika formula
ψ degenerate	}

4.2 The Langlands constant term formula

Based on the factorisation above, we would now like to compute the constant term

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, na)\, dn \tag{4.9}$$

corresponding to $\psi = 1$. Since $\operatorname{supp}(\psi) = \emptyset$ we have that $\mathcal{C}_\psi = \mathcal{W}$. The resulting integrals can be computed by induction over primitive Weyl reflections $w = w_1 w_2 \ldots w_l$. At each step one is left with an integral that can be computed using adelic methods. The resulting constant term is given by the following theorem proved in chapter I-8 where $\langle \cdot \mid \cdot \rangle$ is the Killing form on $\mathfrak{g}$.

Theorem I-8.1. (Langlands' constant term formula [53]). *The constant term of $E(\lambda, g)$ with respect to N is given by:*

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\lambda, ng)\, dn = \sum_{w \in \mathcal{W}} \left| a^{w\lambda + \rho} \right| M(w, \lambda)\,. \tag{4.10}$$

where

$$M(w, \lambda) = \prod_{\alpha > 0 \,|\, w\alpha < 0} \frac{\xi(\langle \lambda | \alpha \rangle)}{\xi(1 + \langle \lambda | \alpha \rangle)}\,. \tag{4.11}$$

For constant terms along different parabolic subgroups, see section I-8.9.

4.3 The Casselman-Shalika formula

Following chapter I-9, we will now consider the non-constant Whittaker vectors on N. For generic characters, $\operatorname{supp}(\psi) = \Pi$ and only the longest Weyl word reflects all simple roots into negative roots, giving $\mathcal{C}_\psi = \{w_{\text{long}}\}$.

The Casselman-Shalika formula will enable us to compute the finite $p < \infty$ part of $F_{w_{\text{long}}}(\chi; a)$. There is currently no general closed-form expression for the archimedean $p = \infty$ part, but it is computed for $SL(2,\mathbb{A})$ in chapter I-7 and for $SL(3,\mathbb{A})$ in section I-9.6.

We will first consider the unramified case where all $m_\alpha = 1$, and this case will then be used to calculate generic Whittaker vectors. Finally, degenerate Whittaker vectors will be computed using the known expressions for generic Whittaker vectors on smaller subgroups.

4.3.1 Unramified Whittaker vectors

Let $\hat{\psi}$ be the unramified character on N with $m_\alpha = 1$ for all $\alpha \in \Pi$. We have that $\mathcal{C}_{\hat{\psi}} = \{w_{\text{long}}\}$ giving

$$W_N(\chi, \hat{\psi}; a) = F_{w_{\text{long}}}(\chi, \hat{\psi}; a) = F_{w_{\text{long}},\infty}(\chi_\infty, \hat{\psi}_\infty; a_\infty) \prod_{p<\infty} F_{w_{\text{long}},p}(\chi_p, \hat{\psi}_p; a_p) \quad (4.12)$$

with

$$F_{w_{\text{long}},p}(\chi_p, \hat{\psi}_p; a_p) = \int\limits_{N(\mathbb{Q}_p)} \chi_p(w_{\text{long}} n a_p) \overline{\hat{\psi}_p(n)}\, dn \quad (4.13)$$

where we have used that $N^{(w_{\text{long}})} = N$. The Casselman-Shalika formula, then gives us these $F_{w_{\text{long}},p}$ for $p < \infty$ as follows.

Theorem I-9.1. (The Casselman-Shalika formula [11]). *The local factors with $p < \infty$ of the unramified Whittaker vector are given by*

$$F_{w_{long},p}(\chi_p, \hat{\psi}_p a_p) = \frac{1}{\zeta(\lambda)} \sum_{w \in \mathcal{W}} \epsilon(w\lambda) \left| a^{w\lambda+\rho} \right|_p \quad (4.14)$$

where

$$\rho = \frac{1}{2} \sum_{\alpha>0} \alpha \qquad \zeta(\lambda) = \prod_{\alpha>0} \frac{1}{1 - p^{-(\langle\lambda|\alpha\rangle+1)}} \qquad \epsilon(\lambda) = \prod_{\alpha>0} \frac{1}{1 - p^{\langle\lambda|\alpha\rangle}}\,. \quad (4.15)$$

The proof is given in section I-9.3 and consists of finding a functional equation for $F_{w_{\text{long}}}$ under Weyl transformations of λ, construct a Weyl invariant function out of the Whittaker vector which is then expressed as a sum over Weyl images. This is the sum that appears in the right hand side of (4.14). Lastly, one of the terms in this sum is evaluated, giving all the other terms.

4.3.2 Generic Whittaker vectors

In section I-9.4 we show that a character $\psi_N^{(m_\alpha)}$ with charges m_α can be expressed as a twisting of the unramified character $\hat{\psi}$ as

$$\psi_N^{(m_\alpha)}(n) = \hat{\psi}^{\tilde{a}}(n) := \hat{\psi}(\tilde{a}n\tilde{a}^{-1})$$
$$\tilde{a} = \exp\Big(\sum_{\alpha\in\Pi} \log(v_\alpha) H_\alpha\Big) \qquad v_{\alpha_j} = \prod_{i=1}^{r} (m_{\alpha_i})^{(A^{-1})_{ij}} \tag{4.16}$$

where r is the rank of the group, H_α the Cartan element dual to the root α and A^{-1} the inverse of the Cartan matrix $A_{ij} = \alpha_j(H_{\alpha_i})$.

As stated above, we have for generic characters that $C_\psi = \{w_{\text{long}}\}$. This allows us to compute the generic Whittaker vector as

$$\begin{aligned} W_N(\chi, \psi_N^{(m_\alpha)}; a) = F_{w_{\text{long}}}(\chi, \psi_N^{(m_\alpha)}; a) &= \int_{N^{(w_{\text{long}})}(\mathbb{A})} \chi(w_{\text{long}} n a) \overline{\psi_N^{(m_\alpha)}(n)}\, dn \\ &= \int_{N^{(w_{\text{long}})}(\mathbb{A})} \chi(w_{\text{long}} n a) \overline{\hat{\psi}(\tilde{a}n\tilde{a}^{-1})}\, dn \end{aligned} \tag{4.17}$$

Now, we want to make the variable substitution $n' = \tilde{a}n\tilde{a}^{-1}$ under which $dn = \chi(w_{\text{long}}\tilde{a}w_{\text{long}}^{-1})|\tilde{a}^{-(w_{\text{long}}\lambda+\rho)}|\, dn'$. We then make use of the multiplicativity of χ on $B \supset A$ to write

$$\chi(w_{\text{long}}\tilde{a}w_{\text{long}}^{-1})\chi(w_{\text{long}}na) = \chi(w_{\text{long}}\tilde{a}w_{\text{long}}^{-1})\chi(w_{\text{long}}\tilde{a}^{-1}n'\tilde{a}a) = \chi(w_{\text{long}}n'\tilde{a}a)\,. \tag{4.18}$$

Hence,

$$\begin{aligned} W_N(\chi, \psi_N^{(m_\alpha)}; a) &= |\tilde{a}^{-(w_{\text{long}}\lambda+\rho)}| \int_{N^{(w_{\text{long}})}(\mathbb{A})} \chi(w_{\text{long}} n'\tilde{a} a) \overline{\hat{\psi}(n')}\, dn' \\ &= |\tilde{a}^{-(w_{\text{long}}\lambda+\rho)}| W_N(\chi, \hat{\psi}; \tilde{a}a)\,, \end{aligned} \tag{4.19}$$

whose local $p < \infty$ factors are given by the Casselman-Shalika formula above.

4.3.3 Degenerate Whittaker vectors

Degenerate Whittaker vectors have characters without support on all simple roots, meaning that we obtain a larger $\mathcal{C}_\psi$ and more terms in the sum (4.6).

Let ψ_N be a degenerate character with $\text{supp}(\psi_N) = \Pi' \subset \Pi$ with associated subgroup $G'(\mathbb{A}) \subset G(\mathbb{A})$ and Weyl group $\mathcal{W}'$ with longest Weyl word w'_{long}. The Weyl words in $\mathcal{C}_\psi$ can then be parametrised by certain, carefully chosen representatives $w_c w'_{\text{long}}$ of $\mathcal{W}/\mathcal{W}'$ with w_c satisfying $w_c\alpha > 0$ for all $\alpha \in \Pi'$, as shown in section I-9.5

The Whittaker vector then becomes

$$W_N(\chi, \psi_N; a) = \sum_{w_c w'_{\text{long}} \in \mathcal{W}/\mathcal{W}'} F_{w_c w'_{\text{long}}}(\chi, \psi_N; a) \tag{4.20}$$

It is then possible to split the integration over $N^{(w_c w'_{\text{long}})}$ in $F_{w_c w'_{\text{long}}}$ into two factors: one which becomes a generic Whittaker vector on the subgroup $G'(\mathbb{A})$ and another containing the factor $M(w_c^{-1}, \lambda)$ of (4.11) in the Langlands constant term formula. Note that since the character ψ_N has support only on $\text{supp}(\psi_N) = \Pi'$ it can also be viewed as a character $\psi'_{N'}$ on N' of $G'(\mathbb{A})$.

The resulting Whittaker vector is given in the following theorem proved in section I-9.5 from [18, 45].

Theorem I-9.5. *The Whittaker vector with degenerate character ψ_N is given by*

$$W_N(\chi, \psi_N; a) = \sum_{w_c w'_{long} \in \mathcal{W}/\mathcal{W}'} \left| a^{(w_c w'_{long})^{-1}\lambda + \rho} \right| M(w_c^{-1}, \lambda) W'_{N'}(\lambda', \psi'_{N'}; \mathbb{1}) \qquad (4.21)$$

where $W'_{N'}$ is a generic Whittaker vector on $G'(\mathbb{A})$ with weight λ' given by the orthogonal projection of $w_c^{-1}\lambda$ on the weight space of $G'(\mathbb{A})$.

4.4 The method of Piatetski-Shapiro and Shalika

We saw above that a degenerate Whittaker vector can be expressed in terms of Whittaker vectors on a smaller group. The method of Piatetski-Shapiro and Shalika is another example taking advantage of this technique for the case of $G = GL(n, \mathbb{R})$. Generalisations for other groups are discussed in [54].

The method first considers the Fourier expansion along the maximal parabolic subgroup P with unipotent subgroup along the first row of upper triangular matrices N in $GL(n)$.

$$P = \begin{pmatrix} * & U \\ 0 & GL(n-1) \end{pmatrix} \qquad (4.22)$$

As will be discussed in the next section, these Fourier coefficients can all be expressed as γ-translates of the Fourier coefficients charged only on α_1, where γ is in the Levi subgroup $GL(n-1)$ of P. These Fourier coefficients are themselves automorphic forms on $GL(n-1)$ and we may repeat the process by expanding along the first row of upper triangular matrices in $GL(n-1)$. At the end we obtain [54, 58, 65][6]

$$\varphi(g) = \sum_{m_{\alpha_1}, \dots m_{\alpha_{n-1}} \in \mathbb{Z}} \; \sum_{\gamma \in N(n-1,\mathbb{Z}) \backslash GL(n-1,\mathbb{Z})} W_\psi(\gamma g) \qquad (4.23)$$

[6] We have here, for simplicity, neglected the constant term at each iteration which amounts to assuming that φ is a cusp form.

4.5 Character variety orbits and wavefront sets

We now turn to general Fourier coefficients that we want to compute in terms of Whittaker vectors (which were determined above) as discussed in Paper II.

Let P be a parabolic subgroup of G with unipotent subgroup U and Levi subgroup L. A Fourier coefficient on U was defined in (3.30) as

$$F_U(\varphi, \psi_U; g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}\, du\,. \tag{4.24}$$

For an element $\gamma \in L(\mathbb{Q})$ we have, using the automorphic invariance (A) of φ, that

$$\begin{aligned} F_U(\varphi, \psi_U; \gamma g) &= \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(u\gamma g)\overline{\psi_U(u)}\, du = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(\gamma^{-1}u\gamma g)\overline{\psi_U(u)}\, du \\ &= \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(u'g)\overline{\psi_U(\gamma u'\gamma^{-1})}\, du' = F_U(\varphi, \psi_U^\gamma; g) \end{aligned} \tag{4.25}$$

where we have used the fact that L leaves U invariant under conjugation, and $du' = du$ for $u' = \gamma^{-1}u\gamma$ with $\gamma \in L(\mathbb{Q})$ and where $\psi_U^\gamma(u) \coloneqq \psi_U(\gamma u \gamma^{-1})$ is a similar twisting to the one introduced above. This means that if we know one Fourier coefficient in the $L(\mathbb{Q})$-orbit of ψ_U, called a character variety orbit, we may compute the rest by $L(\mathbb{Q})$-translations of the argument. As discussed in section I-6.4.1, one may parametrise the charges of ψ_U as an element ω in the dual space $\mathfrak{u}^*$ on which $L(\mathbb{Q})$ acts by conjugation. This means that we may describe the character variety orbits using the theory of nilpotent orbits[7].

We will now discuss how these character variety orbits are related to the orbit $\mathcal{O}_\pi$ associated to the automorphic representation π of φ. Disregarding a few subtleties discussed in section I-6.4.2, the wavefront set $\mathrm{WF}(\pi)$ of an automorphic representation π can be defined as follows. Let φ be an automorphic form in the automorphic representation π. If the Fourier coefficients of φ in the character variety orbit $L(\mathbb{Q})\omega$ are non-vanishing, then $G(\mathbb{C})\omega \subset \mathrm{WF}(\pi)$ and otherwise not.

It has been shown in [8, 49] that the wavefront set is the Zariski closure of a single nilpotent orbit $\mathcal{O}_\pi$, that is, the union of all orbits $\mathcal{O} \leq \mathcal{O}_\pi$. This is the nilpotent orbit introduced in section 3.2 and corresponds to the minimal and next-to-minimal orbits for π_{min} and π_{ntm} respectively. Concretely, this means that small automorphic representations have few non-vanishing Fourier coefficients.

In Paper II, we explicitly, using the methods of [23–25], construct so called orbit Fourier coefficients $F_\mathcal{O}$ of an automorphic form φ in π for each orbit in $G = SL(3)$, and then $SL(4)$, whose character variety orbit intersects $\mathcal{O}$. This means that, by design, $F_\mathcal{O}$ vanishes unless $\mathcal{O} \leq \mathcal{O}_\pi$. We then prove the following Theorem I of Paper II.

[7] We will identify adjoint and coadjoint orbits using the non-degenerate Killing form.

Theorem II-I. *Let $E(\chi, g)$ be an Eisenstein series on the group $G(\mathbb{A}) = SL(3, \mathbb{A})$ or $G(\mathbb{A}) = SL(4, \mathbb{A})$ in the automorphic representation π. Then $E(\chi, g)$ can be expanded as*

$$E(\chi, g) = \sum_{\mathcal{O}} \mathcal{F}_{\mathcal{O}}(\chi, g) \tag{4.26}$$

where the sum is over all nilpotent orbits $\mathcal{O}$ of G and each $\mathcal{F}_{\mathcal{O}}$ is a sum of integrals over translated orbit Fourier coefficients $F_{\mathcal{O}}$ that is non-vanishing only for $\mathcal{O} \leq \mathcal{O}_\pi$.

In this way we can examine the contributions to $E(\chi, g)$ for each representation separately. We find explicit expressions for $F_{\mathcal{O}_{\text{min}}}$ in terms of Whittaker vectors and obtain the following theorem for $SL(3)$ and $SL(4)$.

Theorem II-II. *In the minimal representation, the only non-vanishing orbit Fourier coefficients are the trivial and $F_{\mathcal{O}_{min}}$ the latter of which can be expanded in terms of translated maximally degenerate Whittaker vectors.*

This means that φ, and all of its Fourier coefficients, is completely determined by the constant term and maximally degenerate Whittaker vectors similar to the theorem of [54] for E_6 and E_7. For the next-to-minimal representation we continued the expansion for $SL(4)$.

Theorem II-III. *In the next-to-minimal representation for $SL(4)$, the only non-vanishing orbit Fourier coefficients are the trivial, $F_{\mathcal{O}_{min}}$ and $F_{\mathcal{O}_{ntm}}$ all of which can be expanded in terms of translated Whittaker vectors that are either trivial, maximally degenerate or charged on two commuting roots.*

In general, Fourier coefficients of φ pick up certain terms in the expansion (4.26) and the Whittaker vectors therein. But, we found, for $SL(3)$ and $SL(4)$, that Fourier coefficients on maximal parabolic subgroups reduce to only a single translated maximally degenerate Whittaker vector in the minimal representation. A self-contained example of this for $SL(3)$ can be found in example I-10.30.

Lastly, we test if maximal parabolic Fourier coefficients on E_6 on E_7 and E_8 also simplify to single translated Whittaker vectors in the minimal representation by comparing with known so called spherical local vectors for both $p < \infty$ and $p = \infty$ as explained in more detail in sections II-5 and I-10.4.4.

4.6 Outlook

We are currently working on generalising the results of Paper II for E_6, E_7 and E_8 corresponding to $D = 5$, 4 and 3 respectively. In particular, we are interested in the maximal parabolic Fourier coefficient in small representations, which we have found evidence for to simplify to translated Whittaker vectors. If proven true, such a statement would allow for the computations of instanton effects in the three different limits discussed in section 2.7.

Another interesting development is the compactification to dimensions lower than three, leading to Kac-Moody symmetry groups E_9, E_{10} and E_{11} as discussed in section I-12.7. Eisenstein series on such groups would, in general, have infinitely many terms in its constant Fourier-mode, but string theory predicts a finite number of perturbative corrections. For a certain choice of Eisenstein series though, corresponding to the $s = 3/2$ and $s = 5/2$ in chapter 2, for example, the number of terms collapses to only a few [17, 18]. This means that, although the definition of a small automorphic representation used in section 3.2 cannot be applied to infinite-dimensional groups, there is a similar mechanism restricting the number of non-vanishing Fourier coefficients.

Lastly, we have briefly remarked on the D^6R^4 coefficient $\mathcal{E}_{(0,1)}$, which, as seen in, (2.57) satisfies an inhomogeneous Laplace equation and does not satisfy the definition of an automorphic form as defined above. This is also discussed in section I-12.1.2. In [34], Green, Miller and Vanhove found the solution for $\mathcal{E}_{(0,1)}$ in terms of a sum over images, similar to the definition of an Eisenstein series (2.69), but not of a character χ. It seems that string theory requires an extended framework of automorphic forms, the development of which will positively bring new exciting insights to both physics and mathematics.

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