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STATIC SOLUTIONS TO THE EINSTEIN–VLASOV SYSTEM WITH A NONVANISHING COSMOLOGICAL CONSTANT*

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Abstract. We construct spherically symmetric static solutions to the Einstein–Vlasov system with nonvanishing cosmological constant Λ. The results are divided as follows. For small $\Lambda > 0$ we show the existence of globally regular solutions which coincide with the Schwarzschild–deSitter solution in the exterior of the matter regions. For $\Lambda < 0$ we show via an energy estimate the existence of globally regular solutions which coincide with the Schwarzschild–anti-deSitter solution in the exterior vacuum region. We also construct solutions with a Schwarzschild singularity at the center regardless of the sign of Λ. For all solutions considered, the energy density and the pressure components have bounded support. Finally, we point out a straightforward method for obtaining a large class of global, nonvacuum spacetimes with topologies $\mathbb{R} \times S^3$ and $\mathbb{R} \times S^2 \times \mathbb{R}$ which arise from our solutions as a result of using the periodicity of the Schwarzschild–deSitter solution. A subclass of these solutions contains black holes of different masses.

Key words. Einstein equations, Einstein–Vlasov system, static solutions, Schwarzschild–deSitter, Schwarzschild–anti-deSitter, black holes

AMS subject classifications. 83C05, 83C20, 83C57

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1. Introduction.

Static solutions with Vlasov matter. In this work we consider matter described as a collisionless gas. In astrophysics this model is used to study galaxies and globular clusters where the stars are the particles of gas and where collisions between them are sufficiently rare as to be neglected. The particles interact due to the gravitational field which the particle ensemble creates collectively. Within the framework of general relativity the particle system is described by the Einstein–Vlasov system. The mathematical investigation of this system was initiated by Rein and Rendall in 1992 [24] in the context of the Cauchy problem and shortly thereafter the same authors provided the first study of static, spherically symmetric solutions to this system [23]. Since then, the Einstein–Vlasov system has been successfully studied in several contexts and many global results have been obtained during the past two decades. We refer to [2] for a review of these results, but let us mention in particular the recent monumental work on this system concerning the stability of the universe [25].

The purpose of the present work is to extend the class of static solutions of the Einstein–Vlasov system to the case with nonvanishing cosmological constant Λ. Several results on static and stationary solutions to this system have been obtained in the case when $\Lambda = 0$. The first result of this kind was provided in [23], where the authors construct spherically symmetric, isotropic, static solutions with compactly supported energy density and pressure. The solutions are asymptotically flat and thus

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serve as models for isolated, self-gravitating systems. Several generalizations of this result have since been obtained; in particular, solutions with nonisotropic pressure, and with a Schwarzschild singularity at the center, have been established [22, 20]. An approach by variational methods was developed by Wolansky [27]. The most difficult aspect of these proofs is showing that the matter has compact support. A neat and quite general method to treat this problem has recently been obtained by Ramming and Rein in [19]. However, this method does not apply to the situation we consider in this work. The cosmological constant changes the structure of the equations and this implies that inequality (1.23) in [19], on which this method is based, does not hold when $\Lambda > 0$. Hence, we rely on a different method in this work. The results discussed above all concern the spherically symmetric case. Let us point out that results beyond spherical symmetry have been established in the case of $\Lambda = 0$. The existence of stationary axially symmetric solutions to the Einstein–Vlasov system has recently been shown; see [9] and [10] for the nonrotating and the rotating case, respectively. In this context we also mention a result on static solutions for elastic matter which has been obtained without any symmetry assumption [1].

Static solutions with a nonvanishing cosmological constant. A specific class of solutions which concerns the Einstein equations with a nonvanishing cosmological constant Λ has not yet been discussed. The model solutions for the vacuum equations are the Schwarzschild–deSitter and Schwarzschild–anti-deSitter (Schwarzschild–AdS) solution for $\Lambda > 0$ and $\Lambda < 0$, respectively. Einstein's equations with nonvanishing Λ are of significant physical interest, where the case $\Lambda > 0$ applies to a universe with accelerated expansion [25], while the case $\Lambda < 0$ is relevant in the context of AdS-CFT correspondence [17]. Concerning the Einstein–Vlasov system, no existence results for the static Einstein equations with a nonvanishing cosmological constant are known. The aim of the present paper is to prove the existence of spherically symmetric static solutions to the Einstein–Vlasov system with a small positive or arbitrary negative cosmological constant. The solutions we construct are in general anisotropic. The results provided in this work are as follows.

Globally regular solutions for $0 < \Lambda \ll 1$. We construct globally regular static solutions for small $\Lambda > 0$.

The contribution of a positive cosmological constant in the main equation (2.24) drastically changes the behavior of the solution. The fundamental difference in the case of a vanishing cosmological constant is that for large radii the metric tends towards a cosmological horizon, which is however incompatible with any static ansatz for the distribution function that has so far been analyzed in the literature. It is unclear how to control the solution of (2.24) close to the cosmological horizon. To overcome this difficulty we show the existence of the solution up to a finite radius and then glue it to a Schwarzschild–deSitter exterior. For this to work, compactly supported matter quantities are necessary. However, all known methods for establishing this fail in the case of positive Λ. We construct a perturbation argument using a background solution with $\Lambda = 0$ and the Buchdahl inequality to overcome this problem and to obtain compactness.

In particular, we show that for small $\Lambda > 0$ the solutions we construct are close to the solutions corresponding to the $\Lambda = 0$ case for which the matter quantities have compact support, and in addition, the latter solutions obey a Buchdahl-type inequality. These facts imply that the support of the matter quantities can also be controlled in the case when $\Lambda > 0$. This method is introduced in the proof of Theorem 3.8, which is the core theorem of this paper. It yields a large class of globally regular solutions which coincide with a Schwarzschild–deSitter solution outside a compact set. Finally, the methods described above also apply to the case of solutions with singularities as described below.

Globally regular solutions for $\Lambda < 0$. The case of a negative cosmological constant is a priori simpler since the cosmological term has a good sign which yields a monotonically decreasing behavior of the lapse function. An energy argument following the general idea of [22] is used to establish global-in-r existence, yielding globally regular solutions for general $\Lambda < 0$. The result is given in Theorem 4.2.

Solutions with a Schwarzschild singularity for $0 < \Lambda \ll 1$. To construct solutions with singularities in the center, we start with the vacuum equations, which can be solved explicitly by the Schwarzschild–deSitter solution. This solution is considered up to a radius which allows us to continue the vacuum solution by a solution which at the same point satisfies an appropriate ansatz for the distribution function and eventually merges into a nonvacuum region. It is shown that the support of the matter quantities is compact and outside the matter region the solution can again be extended by a vacuum solution with the mass parameter corresponding to the interior mass of the black hole and matter. As in the nonsingular case these constructions only work for sufficiently small $\Lambda > 0$. The result is given in Theorem 5.5. These solutions can be interpreted as black holes surrounded by matter shells.

Solutions with a Schwarzschild singularity for $\Lambda < 0$. This point is similar to the case $\Lambda > 0$ with Schwarzschild singularities. In addition, a smallness condition for $|\Lambda|$ is also needed. The result is given in Theorem 5.9.

Solutions with topologies $\mathbb{R} \times S^3$ and $\mathbb{R} \times S^2 \times \mathbb{R}$. A significant generalization of the results with $\Lambda > 0$ is presented in the final section. The periodic structure of the Schwarzschild–deSitter space [16] allows us to consider solutions with regular massive centers and solutions with central black holes, and glue them to a periodic Schwarzschild–deSitter solution with a black hole region followed by another matter region—forming a spacetime with two nonvacuum ends and a black hole (or several) in between. The result is given in Theorem 6.1. These solutions with global nontrivial topology which arise only for positive Λ yield large classes of solutions with no counterpart in the $\Lambda = 0$ case.

All solutions constructed in this paper model isolated galaxies or configurations of isolated galaxies and black holes in an otherwise empty universe.

Outline of the paper. This paper is organized as follows. In section 2 we introduce the notation and give a short review on the static Einstein–Vlasov system in spherical symmetry. We discuss the anisotropic ansatz for the distribution function, variations of which are used in this work. A Buchdahl-type inequality, which applies to solutions of the Einstein–Vlasov system, is then briefly reviewed as it is used later in the existence proof for $\Lambda > 0$. The Einstein–Vlasov system in spherical symmetry with a specific ansatz for the distribution function reduces to an integro-differential equation given in (2.24). This equation lies at the heart of the analysis in the paper. In section 3 we prove the existence of globally regular solutions for small $\Lambda > 0$. The proof is divided into several steps, beginning with local-in-r existence in section 3.1, a continuation criterion in section 3.2, the existence up to sufficiently large radii to be able to reach the vacuum region section 3.3, and finally the proof of the existence theorem in 3.4. In section 4 the existence of globally regular solutions for arbitrary $\Lambda < 0$ is proven along with a result (see Theorem 4.2) which states the existence of such solutions outside a ball, which is eventually used to prove the existence of solutions with Schwarzschild singularities in the center. Section 5 begins with a generalization of the Buchdahl-type inequality, mentioned above, for solutions with Schwarzschild singularities. This result is useful for the construction of solutions of this kind when $\Lambda > 0$. These solutions are obtained in Theorem 5.5. Analogous solutions for the case of negative Λ are given in Theorem 5.9. Finally, section 6 discusses the globally nontrivial generalizations of the constructed solutions for $\Lambda > 0$.

2. Preliminaries.

2.1. Setup and notations. We consider the Einstein–Vlasov system with the cosmological constant $\Lambda \in \mathbb{R}$. For background information on this system and definition of coordinates we refer to [2]. Spatial indices are denoted by Latin letters, running from 1 to 3. For the spherically symmetric static Lorentzian metric g we use the standard ansatz

$$\mathrm{d}s^2 = -e^{2\mu(r)}\mathrm{d}t^2 + e^{2\lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2(\vartheta)\mathrm{d}\varphi^2. \tag{2.1}$$

In addition, assuming the matter distribution function f to be spherically symmetric and static one obtains the reduced system of equations

$$\frac{v^a}{\sqrt{1+|v|^2}}\frac{\partial f}{\partial x^a} - \sqrt{1+|v|^2}\mu'\frac{x^a}{r}\frac{\partial f}{\partial v^a} = 0, \tag{2.2}$$

$$e^{-2\lambda}(2r\lambda' - 1) + 1 - r^2\Lambda = 8\pi r^2\varrho, \tag{2.3}$$

$$e^{-2\lambda}(2r\mu' + 1) - 1 + r^2\Lambda = 8\pi r^2 p, \tag{2.4}$$

where $|v| = \sqrt{\delta_{ij}v^iv^j}$, $v_r = \frac{\delta_{ij}v^ix^j}{r}$ and the matter quantities read

$$\varrho = \int_{\mathbb{R}^3} f(x,v)\sqrt{1+|v|^2}\,\mathrm{d}v^1\mathrm{d}v^2\mathrm{d}v^3, \tag{2.5}$$

$$p = \int_{\mathbb{R}^3} \frac{f(x,v)}{\sqrt{1+|v|^2}}v_r^2\,\mathrm{d}v^1\mathrm{d}v^2\mathrm{d}v^3. \tag{2.6}$$

There is an additional Einstein equation

$$e^{-2\lambda}\left(\mu''\left(\mu + \frac{1}{r}\right)(\mu' - \lambda')\right) = 8\pi p_T, \tag{2.7}$$

where

$$p_T = \frac{1}{2}\int_{\mathbb{R}^3}\left|\frac{x\times v}{r}\right|^2 f(x,v)\frac{\mathrm{d}v}{\sqrt{1+|v|^2}}. \tag{2.8}$$

The quantity ϱ can be understood as energy density, p as radial pressure and p_T as tangential pressure. To ensure a regular center the following boundary condition is imposed:

$$\lambda(0) = 0. \tag{2.9}$$

This condition will be used in the first part of this work, but when we consider solutions with a Schwarzschild singularity at the center it will be dropped. A detailed derivation of the system (2.2)–(2.8) in the $\Lambda = 0$ case can be found in [24]. As shown in [26] a solution of the reduced system (2.2)–(2.6) also solves the full system. Considering the characteristic curves of the Vlasov equation (2.2) one can simplify the system of equations. Along these characteristic curves the quantities E and L, given by

$$E = e^{\mu(r)}\sqrt{1+|v|^2} =: e^{\mu(r)}\varepsilon \quad \text{and} \quad L = |x \times v|^2, \tag{2.10}$$

are conserved (see [23]). Therefore any ansatz for the matter distribution f of the form

$$f(x,v) = \Phi(E,L) \tag{2.11}$$

solves the Vlasov equation (2.2), and this equation drops out of the system of equations.

2.2. Relevant results. In the following we discuss the known results for the Einstein–Vlasov system with vanishing cosmological constant $\Lambda = 0$, which are relevant for the work presented in this paper. The existence of a unique solution $\mu(r)$, $\lambda(r)$ to given initial values $\mu(0) = \mu_0$ and $\lambda(0) = 0$ has been proved using the ansatz

$$f(x,v) = \Phi(E)[L - L_0]_+^{\ell}, \tag{2.12}$$

where $E > 0$, $L > 0$, $L_0 \geq 0$, $\ell > -1/2$, $\Phi \in L^\infty((0,\infty))$ for the matter distribution f [22]. Furthermore, it can be shown that the support of the matter quantities is contained in an interval $[0, R_0]$, $0 < R_0 < \infty$, if one takes an ansatz of the form

$$f(x,v) = \phi\left(1 - \frac{E}{E_0}\right) L^{\ell}, \tag{2.13}$$

where $\phi : \mathbb{R} \to [0,\infty)$ is measurable, $\phi(\eta) = 0$ for $\eta < 0$, and $\phi > 0$ a.e. on some interval $[0, \eta_1]$ with $\eta_1 > 0$ and E_0 is some prescribed cut-off energy [19]. Moreover, it is required that there exists $\gamma > -1$ such that for every compact set $K \subset \mathbb{R}$ there exists a constant $C > 0$ such that

$$\phi(\eta) \leq C\eta^\gamma, \quad \eta \in K. \tag{2.14}$$

In [21] this result is generalized to anisotropic matter distributions of the form

$$f(x,v) = \alpha[E_0 - E]_+^k [L - L_0]_+^{\ell}, \tag{2.15}$$

where $k \geq 0$, $\ell > -1/2$ fulfil the inequality $k < 3\ell + 7/2$ and $\alpha, E_0 > 0$, $L_0 \geq 0$. It is shown in [21] that for sufficiently small L_0 the support of f is contained in an interval $[R_i, R_0]$ where $0 \leq R_i < R_0 < \infty$ and $R_i > 0$ provided $L_0 > 0$.

By direct calculation one shows that the matter quantities fulfil the generalized Tolman–Oppenheimer–Volkov (TOV) equation

$$p'(r) = -\mu'(r)(p(r) + \varrho(r)) - \frac{2}{r}(p(r) - p_T(r)). \tag{2.16}$$

Another result which is relevant for the proof presented here is a generalized Buchdahl inequality [4], which is the content of the following lemma.

Lemma 2.1 (Theorem 1 in [4]). *Let $\lambda, \mu \in C^1([0,\infty))$ and let $\varrho, p, p_T \in C^0([0,\infty))$ be functions that satisfy the system of equations* (2.3)–(2.7) *and the condition* (2.9), *and such that $p + 2p_T \leq \varrho$. Then*

$$\sup_{r>0} \frac{2m(r)}{r} \leq \frac{8}{9}, \tag{2.17}$$

where

$$m(r) = 4\pi \int_0^r s^2 \varrho(s) \mathrm{d}s. \tag{2.18}$$

Remark 2.2. The inequality (2.17) holds for a more general class of functions; see [4]. Moreover, the inequality is sharp, and the solutions which saturate the inequality are infinitely thin shell solutions [4]. In [3] it is shown that there exist regular, arbitrarily thin shell solutions to the Einstein–Vlasov system such that the quantity $2m/r$ can be arbitrarily close to $8/9$. It should also be mentioned that Buchdahl-type inequalities have been obtained in the case of the nonvanishing cosmological constant [6, 7]. These results *assume* the existence of static solutions to the Einstein matter equations with a cosmological constant.

To prove the existence of solutions of the static Einstein–Vlasov system with nonvanishing Λ we make use of the results discussed above. To simplify calculations we define $y := \ln(E_0) - \mu$ as in [19] so that $e^\mu = E_0/e^y$. For the distribution function f we choose the ansatz[1]

$$\begin{aligned} f(x,v) = \Phi(E,L) = \alpha\phi\left(1 - \frac{E}{E_0}\right)[L - L_0]_+^\ell = \alpha\phi\left(1 - \varepsilon e^{-y}\right)[L - L_0]_+^\ell, \\ \phi(\eta) = [\eta]_+^k, \end{aligned} \tag{2.19}$$

where $k \geq 0$, $\ell \geq 0$ fulfil the inequality $k < 3\ell + 7/2$ and $\alpha, E_0 > 0$, $L_0 \geq 0$. For the construction of globally regular solutions L_0 has to be sufficiently small to ensure finite support of the matter quantities [21]. When considering solutions with a black hole at the center, there are positive lower bounds on L_0. The expressions for the matter quantities ϱ and p take the form

$$\varrho(r) = G_\phi(r, y(r)), \qquad p(r) = H_\phi(r, y(r)), \tag{2.20}$$

where

$$G_\phi(r,y) = c_\ell \alpha r^{2\ell} \int_{\sqrt{1+L_0/r^2}}^\infty \phi\left(1 - \varepsilon e^{-y}\right)\varepsilon^2\left(\varepsilon^2 - \left(1 + \frac{L_0}{r^2}\right)\right)^{\ell+\frac{1}{2}} \mathrm{d}\varepsilon, \tag{2.21}$$

$$H_\phi(r,y) = \frac{c_\ell \alpha}{2\ell+3} r^{2\ell} \int_{\sqrt{1+L_0/r^2}}^\infty \phi\left(1 - \varepsilon e^{-y}\right)\left(\varepsilon^2 - \left(1 + \frac{L_0}{r^2}\right)\right)^{\ell+\frac{3}{2}} \mathrm{d}\varepsilon, \tag{2.22}$$

given in [22]. The constant c_ℓ is given by

$$c_\ell = 2\pi \int_0^1 \frac{s^\ell}{\sqrt{1-s}} \mathrm{d}s. \tag{2.23}$$

LEMMA 2.3. *The functions $G_\phi(r,y)$ and $H_\phi(r,y)$ defined in (2.21) and (2.22), respectively, have the following properties.*

(i) *$G_\phi(r,y)$ and $H_\phi(r,y)$ are continuously differentiable in r and y.*

(ii) *The functions $G_\phi(r,y)$ and $H_\phi(r,y)$ and the partial derivatives $\partial_y G_\phi(r,y)$ and $\partial_y H_\phi(r,y)$ are increasing both in r and y.*

(iii) *There is vacuum, i.e., $f(r,\cdot) = p(r) = \varrho(r) = 0$ if $e^{-y(r)}\sqrt{1 + L_0/r^2} \geq 1$, in particular if $y(r) \leq 0$.*

[1] To be precise, any ϕ that is of the kind of ϕ in (2.13) would meet the assumptions of the following lemmas and theorems.

Proof. By performing a change of variables in the integrals in (2.21) and (2.22), the differentiability follows [22, Lemma 3.1]. The monotonicity can be seen directly from the structure of G_ϕ and H_ϕ. The last statement is obvious since $\phi(\eta) = 0$ if $\eta \leq 0$. ◻

2.3. Main equation. From the Einstein equations (2.3) and (2.4) one obtains the differential equation for y:

$$y'(r) = -\frac{4\pi}{1 - \frac{\Lambda r^2}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y(s))\mathrm{d}s} \times \left(r H_\phi(r, y(r)) - \frac{r\Lambda}{12\pi} + \frac{1}{r^2}\int_0^r s^2 G_\phi(s, y(s))\mathrm{d}s \right). \tag{2.24}$$

A solution to (2.24) yields a solution to the system (2.2)–(2.6). The aim in the following is to construct solutions to (2.24) which have compactly supported matter quantities. Outside the support of these quantities the metric should coincide with a Schwarzschild–deSitter solution. This gives rise to an appropriate boundary condition at the boundary of the support, $R_{0\Lambda}$, which is

$$\lim_{r \nearrow R_{0\Lambda}} E_0 e^{-y(r)} = \lim_{r \searrow R_{0\Lambda}} \left(1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}\right), \tag{2.25}$$

where $M = 4\pi \int_0^{R_{0\Lambda}} s^2 \rho(s) ds$. This is understood as a definition of E_0. We then express the function μ by $\mu(r) = \ln(E_0) - y(r)$. Furthermore, it should be mentioned that a solution to the system (2.2)–(2.6) provides a solution to all the Einstein equations. This is shown in [24], Theorem 2.1 in the case when $\Lambda = 0$. The proof is analogous in the case with nonvanishing Λ. The equation (2.24) is analyzed and solved in the remainder of this work.

3. Static, anisotropic globally regular solutions for $\Lambda > 0$. In this section we prove the existence of globally regular static solutions with small $\Lambda > 0$.

3.1. Local existence. The following local existence lemma corresponds to the first part of the proof of Theorem 2.2 in [23] for the case $\Lambda = 0$.

LEMMA 3.1. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form* (2.19) *and let G_ϕ, H_ϕ be defined by* (2.21) *and* (2.22)*, respectively. Then for every $y_0 \in \mathbb{R}$ and every $\Lambda > 0$ there is a $\delta > 0$ such that there exists a unique solution $y_\Lambda \in C^2([0, \delta])$ of* (2.24) *with initial value $y_\Lambda(0) = y_0$.*

Proof. The lemma can be shown using a contraction argument, as done in [23]. ◻

3.2. Continuation criterion. The solution y_Λ exists at least as long as the denominator of the right-hand side of (2.24) is strictly larger than zero. The following lemma formulates this assertion.

LEMMA 3.2. *Let $y_0 \in \mathbb{R}$ and let $R_c > 0$ be the largest radius such that the unique local C^2-solution y_Λ of* (2.24) *with $y_\Lambda(0) = y_0$ exists on the interval $[0, R_c)$. Then there exists $R_D \leq R_c$ such that*

$$\liminf_{r \to R_D} \left(1 - \frac{r^2\Lambda}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s\right) = 0. \tag{3.1}$$

Remark 3.3. Lemma 3.2 implies that the denominator on the right-hand side of (2.24) becomes arbitrarily small on $[0, R_c)$, i.e., the numerator has no singular behavior that would make the solution collapse as long as the denominator is larger than zero.

Remark 3.4. It is important to note that in contrast to the case of the vanishing cosmological constant, the numerator in (2.24) can be negative due to the Λ-term. This in turn implies that a zero of the denominator does not necessarily make the right-hand side of (2.24) singular as the numerator might also vanish at this particular point, possibly regularizing the full term. In this case, $R_D < R_c$.

Proof. Assume

$$1 - \frac{r^2\Lambda}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s > 0 \tag{3.2}$$

for all $r \in [0, R_c)$. Otherwise $R_D < R_c$ (with R_D characterized as above) occurs due to the continuity of y_Λ and G_ϕ and the lemma follows. Assume now that the assertion of the lemma does not hold, i.e., there is a constant $a > 0$ such that

$$1 - \frac{r^2\Lambda}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s \geq a \tag{3.3}$$

for all $r \in [0, R_c)$. First we show that this implies the existence of a $C > 0$ such that for all $r \in [0, R_c)$ we have $|y'_\Lambda(r)| \leq C$. Therefore we consider

$$|y'_\Lambda(r)| \leq \frac{4\pi}{a}\left(rH_\phi(r, y_\Lambda(r)) + \frac{r\Lambda}{12\pi} + \frac{1}{r^2}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s\right). \tag{3.4}$$

Here we have used that H_ϕ and G_ϕ are positive. It is obvious that the second term, $\frac{r\Lambda}{12\pi}$, is bounded on the interval $[0, R_c)$. We show that the right-hand side of (3.4) is uniformly bounded on this interval. Assume the opposite,

$$\limsup_{r\to R_c} H_\phi(r, y_\Lambda(r)) = \infty \quad \text{or} \quad \limsup_{r\to R_c}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s = \infty. \tag{3.5}$$

The second possibility implies $\limsup_{r\to R_c} G_\phi(r, y_\Lambda(r)) = \infty$. On the interval $[0, R_c)$ we have the upper bounds $H_\phi(r, y_\Lambda(r)) \leq H_\phi(R_c, y_\Lambda(r))$ and $G_\phi(r, y_\Lambda(r)) \leq G_\phi(R_c, y_\Lambda(r))$, see (ii) of Lemma 2.3. And since $H_\phi(r, y)$ and $G_\phi(r, y)$ are increasing functions in y (see Lemma 2.3) this in turn implies

$$\limsup_{r\to R_c} y_\Lambda(r) = \infty. \tag{3.6}$$

It follows that for all $\varepsilon > 0$ sufficiently small there exists $r \in (R_c - \varepsilon, R_c)$ such that $y'_\Lambda(r) > 0$, which on the other hand implies

$$rH_\phi(r, y_\Lambda(r)) + \frac{1}{r^2}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s < \frac{r\Lambda}{12\pi}, \tag{3.7}$$

by (2.24) for y'_Λ. This contradicts the assumption that either $H_\phi(r, y_\Lambda(r))$ or the integral $\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s$ diverges as the right-hand side of (3.7) is bounded. Thus $|y'_\Lambda(r)|$ is uniformly bounded on $[0, R_c)$.

In the remainder of this proof it is shown that the solution can be continued beyond R_c in order to yield the desired contradiction. To achieve this, methods similar to those in the proof of Lemma 3.1 will be used. Consider a radius $r_1 \in (0, R_c)$ and let $\delta > 0$. Define $y_1 = y_\Lambda(r_1)$ and the interval I_δ by $I_\delta = [r_1, r_1 + \delta]$.

Consider the operator

$$(T_1 u)(r) = y_1 + \int_{r_1}^{r} \frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{8\pi}{s}\int_0^s \sigma^2 G_\phi(\sigma, u_y(\sigma))\mathrm{d}\sigma} \times \left(sH_\phi(s, u(s)) - \frac{s\Lambda}{12\pi} + \frac{1}{s^2}\int_0^s \sigma^2 G_\phi(\sigma, u_y(\sigma))\mathrm{d}\sigma \right)\mathrm{d}s, \tag{3.8}$$

where

$$u_y(r) := \begin{cases} y_\Lambda(r); & r \in [0, r_1] \\ u(r); & r \in (r_1, r_1+\delta] \end{cases}, \tag{3.9}$$

acting on the set

$$M_1 = \Big\{ u : I_\delta \to \mathbb{R} \mid u(r_1) = y_1, y_1 - 1 \le u(r) \le y_1 + 1, \frac{r^2\Lambda}{3} + \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, u_y(s))\mathrm{d}s \le c < 1, r \in I_\delta \Big\}. \tag{3.10}$$

Using (3.3) and $|y'_\Lambda(r)| < C$ on $[0, R_c)$ for $C > 0$ one can prove that T_1 acts as a contraction on M_1 for δ sufficiently small. In virtue of Banach's fixed point theorem the operator T_1 has a fixed point $w \in M_1$ such that $(w)_y$ defined by (3.9) solves (2.24) on the interval $(0, r_1 + \delta)$. Note that δ is independent of the choice of r_1 due to the uniform bound on $|y'_\Lambda(r)|$. Thus by choosing $r_1 \in (0, R_c)$ sufficiently close to R_c the solution extends beyond $r = R_c$. But this contradicts the definition of R_c and the lemma follows. ∎

3.3. Existence beyond the nonvacuum region.

PROPOSITION 3.5. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form (2.19) and let y be the unique global C^1-solution of (2.24) in the case $\Lambda = 0$ where $y(0) = y_0 > 0$ [22]. As proved in [22], f has bounded spatial support $[0, R_0)$ where $y(R_0) = 0$ defines R_0 uniquely. Let y_Λ be the unique C^2-solution of (2.24) with $\Lambda > 0$ and $y_\Lambda(0) = y(0)$, that according to Lemma 3.1 exists at least on an interval $[0, \delta]$ for a certain $\delta > 0$, and let f_Λ be the distribution function corresponding to y_Λ.*

Then y_Λ exists at least on $[0, R_0 + \Delta R]$ and the spatial support of f_Λ is bounded by some $R_{0\Lambda} < R_0 + \Delta R$ if Λ and $\Delta R > 0$ are chosen such that

$$0 < \Lambda < \min\left\{ \frac{|y(R_0 + \Delta R)|}{C_y(R_0 + \Delta R)}, \frac{\frac{1}{18}}{C_v(R_0 + \Delta R)} \right\} \tag{3.11}$$

holds. The constants $C_y(r)$ defined in (3.25) and $C_v(r)$ defined in (3.23) are determined by the background solution y.

Remark 3.6. Note that the upper bound for Λ in (3.11) is strictly larger than zero since $|y(R_0 + \Delta R)| > 0$. This holds because the globally existing background solution y is strictly monotone and we have $y(R_0) = 0$ by definition of R_0.

Before we present the proof of this proposition we state a lemma containing a crucial but also lengthy estimate.

LEMMA 3.7. *Let G_ϕ and H_ϕ rather be as given by (2.21) and (2.22), respectively. Furthermore, let $R > 0$ and let $y, y_\Lambda : [0, R] \to \mathbb{R}$ be functions solving (2.24) with vanishing and positive cosmological constant Λ, respectively, common initial value $y_0 = y(0) = y_\Lambda(0)$, and such that the conditions*

$$1 - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y(s))\mathrm{d}s \ge \frac{1}{9}, \qquad 1 - \frac{r^2\Lambda}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y(s))\mathrm{d}s \ge \frac{1}{18},$$

and $|y_\Lambda(r) - y(r)| \leq |y(R)|$ *hold for all* $r \in [0, R]$*. Then we have the estimate*

$$|G_\phi(r, y_\Lambda(r)) - G_\phi(r, y(r))| + |H_\phi(r, y_\Lambda(r)) - H(r, y(r))| \leq \Lambda C_{gh}(r) \tag{3.12}$$

for an increasing function $C_{gh}(r)$*.*

Proof. Since

$$\begin{aligned} &|G_\phi(r, y_\Lambda(r)) - G_\phi(r, y(r))| + |H_\phi(r, y_\Lambda(r)) - H_\phi(r, y(r))| \\ &\quad \leq \left(\sup_{u \in [y_\Lambda(r), y(r)]} |\partial_u G_\phi(r, u)| + \sup_{u \in [y_\Lambda(r), y(r)]} |\partial_u H_\phi(r, u)| \right) |y_\Lambda(r) - y(r)| \end{aligned} \tag{3.13}$$

we calculate

$$\begin{aligned} |y_\Lambda(r) - y(r)| &\leq \int_0^r |y'(s) - y'_\Lambda(s)| \mathrm{d}s \\ &\leq \int_0^r \Bigg[\underbrace{\frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{2m_\Lambda(s)}{s}}}_{\leq 72\pi} \\ &\quad \times \Bigg(\left| -\frac{s\Lambda}{12\pi} \right| + s|H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))| \\ &\quad + \underbrace{\frac{1}{s^2} \int_0^s \sigma^2 |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| \mathrm{d}\sigma}_{I_1} \Bigg) \\ &\quad + \left(sH_\phi(s, y(s)) + \frac{1}{s^2} \int_0^s \sigma^2 G_\phi(\sigma, y(\sigma)) \mathrm{d}\sigma \right) \\ &\quad \times \underbrace{\left(\frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{2m_\Lambda(s)}{s}} - \frac{4\pi}{1 - \frac{2m(s)}{s}} \right)}_{I_2} \Bigg] \mathrm{d}s. \end{aligned}$$

We estimate I_1 and I_2 separately:

$$\begin{aligned} I_1 &= \int_0^r \frac{1}{s^2} \int_0^s \sigma^2 |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| \mathrm{d}\sigma \mathrm{d}s \\ &\leq \int_0^r \int_0^r |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| \mathrm{d}\sigma \mathrm{d}s \\ &\leq r \int_0^r |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| \mathrm{d}\sigma, \\ I_2 &= \frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{2m_\Lambda(s)}{s}} - \frac{4\pi}{1 - \frac{2m(s)}{s}} \\ &\leq 4\pi \cdot 18 \cdot 9 \cdot \left(\frac{s^2\Lambda}{3} + \frac{8\pi}{s} \int_0^s \sigma^2 |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| \mathrm{d}\sigma \right) \\ &\leq 648\pi \left(\frac{s^2\Lambda}{3} + 8\pi s \int_0^s |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| \mathrm{d}\sigma \right). \end{aligned}$$

So, using that y is decreasing we have

$$
\begin{aligned}
&|y_\Lambda(r) - y(r)| \\
&\leq \Lambda \int_0^r \left(6s + 216\pi s^3 \left(H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0)\right)\right) \mathrm{d}s \\
&\quad + 72\pi r \int_0^r |H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))| \mathrm{d}s \\
&\quad + \left(72\pi r + 5184\pi^2 \frac{r^3}{3} \left(H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0)\right)\right) \\
&\quad \times \int_0^r |G_\phi(s, y_\Lambda(s)) - G_\phi(s, y(s))| \mathrm{d}s \\
&\leq \Lambda \left(3r^2 + 54\pi r^4 \left(H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0)\right)\right) \\
&\quad + \left(72\pi r + 1728\pi^2 r^3 \left(H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0)\right)\right) \\
&\quad \times \int_0^r (|H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))| + |G_\phi(s, y_\Lambda(s)) - G_\phi(s, y(s))|) \mathrm{d}s \\
&\leq \Lambda C_1(r) + C_2(r) \int_0^r (|H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))| + |G_\phi(s, y_\Lambda(s)) - G_\phi(s, y(s))|) \mathrm{d}s.
\end{aligned}
$$

The derivatives with respect to y of $G_\phi(r, y)$ and $H_\phi(r, y)$ are strictly increasing both in r and y (see Lemma 2.3). And since $|y_\Lambda(r) - y(r)| \leq |y(R)|$ we can write

$$
\begin{aligned}
&\left(\sup_{u \in [y_\Lambda(r), y(r)]} |\partial_u G_\phi(r, u)| + \sup_{u \in [y_\Lambda(r), y(r)]} |\partial_u H_\phi(r, u)|\right) \\
&\qquad \leq \left|\partial_u G_\phi(\tilde{r}^*, u)|_{y_0 + |y(R)|}\right| + \left|\partial_u H_\phi(\tilde{r}^*, u)|_{y_0 + |y(R)|}\right| =: C_3.
\end{aligned}
$$

So we have obtained that (3.13) is of the form

$$
\begin{aligned}
&|H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))| + |G_\phi(s, y_\Lambda(s)) - G_\phi(s, y(s))| \\
&\leq C_4(r)\Lambda + C_5(r) \int_0^r (|H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))| \\
&\quad + |G_\phi(s, y_\Lambda(s)) - G_\phi(s, y(s))|) \mathrm{d}s
\end{aligned} \tag{3.14}
$$

Note that $C_4(r)$ is strictly increasing. Grönwall's inequality yields

$$
\begin{aligned}
&(|G_\phi(s, y_\Lambda(s)) - G_\phi(s, y(s))| + |H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))|) \\
&\leq C_4(r) e^{\int_0^r C_5(r) \mathrm{d}s} = C_4(r) \Lambda e^{r C_5(r)} =: C_{gh}(r)\Lambda.
\end{aligned} \tag{3.15}
$$

Note that $C_{gh}(r)$ is increasing when r is increasing. This ends the proof of the lemma. ∎

With this lemma at hand we now turn to the proof of Proposition 3.5.

Proof. We define

$$
m(r) = 4\pi \int_0^r s^2 \varrho(s) \mathrm{d}s, \qquad m_\Lambda(r) = 4\pi \int_0^r s^2 \varrho_\Lambda(s) \mathrm{d}s, \tag{3.16}
$$

$$
v(r) = 1 - \frac{2m(r)}{r}, \qquad v_\Lambda(r) = 1 - \frac{r^2 \Lambda}{3} - \frac{2m_\Lambda(r)}{r}. \tag{3.17}
$$

Consider the continuous function v_Λ. Note that $v_\Lambda(0) = 1$. We define

$$r^* := \inf\{r \in [0, R_c) \,|\, v_\Lambda(r) = 1/18\}, \tag{3.18}$$

i.e., r^* is the smallest radius where $v_\Lambda(r) = \frac{1}{18}$. Lemma 3.2 assures that $r^* < R_c$, if R_c is finite, otherwise r^* is clearly finite due to the form of v_Λ, i.e., r^* is well defined. Note that $v_\Lambda(r)$ is the quantity in Lemma 3.2. In addition, we define

$$\tilde{r} := \inf\{r \in [0, R_c] \,|\, |y_\Lambda(r) - y(r)| > |y(R_0 + \Delta R)|\}. \tag{3.19}$$

The right-hand side of this inequality is given by the background solution y, which exists globally. Note that $|y(R_0 + \Delta R)| > 0$ since y is strictly monotone, and $y(0) = y_\Lambda(0) = y_0$, so $0 < \tilde{r}$ by continuity of y and y_Λ. Let

$$\tilde{r}^* := \min\{r^*, \tilde{r}\}. \tag{3.20}$$

Choosing Λ s.t. (3.11) holds, we will show that $\tilde{r}^* > R_0 + \Delta R$. We assume the opposite, $\tilde{r}^* \leq R_0 + \Delta R$, and consider the sum $|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)|$ on the interval $[0, \tilde{r}^*]$. By the mean value theorem we have

$$\begin{aligned}&|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)| \\ &\quad = \left(\left| \partial_y G_\phi(r, y) \big|_{u_1} \right| + \left| \partial_y H_\phi(r, y) \big|_{u_2} \right| \right) |y_\Lambda(r) - y(r)|\end{aligned} \tag{3.21}$$

where $u_1, u_2 \in [y(r), y_\Lambda(r)]$ are chosen appropriately. For $r \leq \tilde{r}^*$ the assumptions of Lemma 3.7 are clearly satisfied. So for $r \leq \tilde{r}^*$ we have the estimate

$$|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)| \leq \Lambda C_{gh}(\tilde{r}^*), \tag{3.22}$$

where C_{gh} is defined in (3.15). Note that $C_{gh}(r)$ is increasing in r. Still on $[0, \tilde{r}^*]$ we compute

$$\begin{aligned}|v(r) - v_\Lambda(r)| &\leq \frac{r^2 \Lambda}{3} + \frac{2}{r} |m_\Lambda(r) - m(r)| = \frac{r^2 \Lambda}{3} + \frac{8\pi}{r} \int_0^r s^2 |\varrho_\Lambda(s) - \varrho(s)| \mathrm{d}s \\ &\leq \left(\frac{(\tilde{r}^*)^2}{3} + \frac{8\pi}{3} (\tilde{r}^*)^2 C_{gh}(\tilde{r}^*) \right) \Lambda =: C_v(\tilde{r}^*) \Lambda\end{aligned} \tag{3.23}$$

Since we have $v(r) \geq \frac{1}{9}$ (Buchdahl inequality, see Lemma 2.1) and $\Lambda < \frac{1/18}{C_v(R_0 + \Delta R)}$ by the choice of Λ we can conclude

$$v_\Lambda(r) \geq v(r) - \Lambda C_v(\tilde{r}^*) > \frac{1}{9} - \frac{1/18}{C_v(R_0 + \Delta R)} C_v(\tilde{r}^*) \geq \frac{1}{18} \tag{3.24}$$

on $[0, \tilde{r}^*]$ since $C_v(\tilde{r}^*) < C_v(R_0 + \Delta R)$ because $C_v(r)$ is increasing and $\tilde{r}^* \leq R_0 + \Delta R$ by assumption.

We also consider the distance between y and y_Λ on $[0, \tilde{r}^*]$. Following the procedure depicted in the proof of Lemma 3.7 one obtains

$$\begin{aligned}|y_\Lambda(r) - y(r)| &\leq \Lambda \left(3r^2 + 29\pi r^4 \left(H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0) \right) \right) \\ &\quad + 72\pi \left(r + 24\pi r^2 \left(H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0) \right) \right) \\ &\quad \times \int_0^r C_{gh}(s) \Lambda \mathrm{d}s \\ &=: C_y(r) \Lambda \leq C_y(\tilde{r}^*) \Lambda.\end{aligned} \tag{3.25}$$

Since $C_y(\tilde{r}^*) \leq C_y(R_0 + \Delta R)$ and $\Lambda < \frac{|y(R_0+\Delta R)|}{C_y(R_0+\Delta R)}$ on $[0, \tilde{r}^*]$ by assumption, the relation

$$|y_\Lambda(r) - y(r)| < |y(R_0 + \Delta R)| \tag{3.26}$$

already holds. Equations (3.24) and (3.26) state that $v_\Lambda(\tilde{r}^*) > \frac{1}{18}$ and $|y_\Lambda(\tilde{r}^*) - y(\tilde{r}^*)| < |y(R_0 + \Delta R)|$, respectively, on the interval $[0, \tilde{r}^*]$, which is a contradiction of the definition of $\tilde{r}^*$. Thus we have $\tilde{r}^* > R_0 + \Delta R$ as desired.

Note that due to the assumption (3.11) $C_y(r)$ is uniformly bounded on the interval $[0, R_0+\Delta R]$, so the inequality (3.26) in fact holds for all $r \leq \min\{\tilde{r}^*, R_0+\Delta R\}$. Since $\tilde{r}^* > R_0 + \Delta R$ we can conclude the following. We have shown that y_Λ exists at least on $[0, R_0 + \Delta R]$ as the continuation criterion applies and from (3.26) we already know that $y_\Lambda(R_0 + \Delta R) < 0$. Since y_Λ is continuous it has at least one zero in the interval $(R_0, R_0 + \Delta R)$. In particular there exists an interval $(R_{0\Lambda}, R_0 + \Delta R)$ where y_Λ is strictly smaller than zero. $R_{0\Lambda}$ is the largest zero of y_Λ in $(R_0, R_0 + \Delta R)$. So, the spatial support of f_Λ is contained in the interval $[0, R_{0\Lambda})$ and this implies the assertion. ∎

3.4. Global regular solutions for $\Lambda > 0$. In the last two sections we have seen that for suitably chosen Λ there exists a unique solution y_Λ to (2.24) on the interval $[0, R_0 + \Delta R]$ for some $\Delta R > 0$. This solution uniquely induces a solution μ_Λ, λ_Λ of (2.3), (2.4) on $[0, R_0 + \Delta R]$ whose distribution function f_Λ is of bounded support in space. By gluing a Schwarzschild–deSitter metric to this solution one can construct a global static solution to the Einstein–Vlasov system in the following sense. The solutions possess a cosmological horizon $r_C > R_0 + \Delta R$ determined by the constant Λ and the total mass of the matter M, being the largest zero of $1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}$. Different coordinates are used for the regions $\{r < r_C\}$ and $\{r > r_C\}$. μ_Λ and λ_Λ will be shown to extend to the region $\{r < r_C\}$. In section 6 we will discuss in more detail how these solutions can be extended beyond the comsological horizon, yielding global solutions.

THEOREM 3.8. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form* (2.19) *and let G_ϕ and H_ϕ be given by* (2.21) *and* (2.22)*, respectively. Then, for every initial value $\mu_0 < 0$ there exists a constant $C = C(\mu_0, \phi) > 0$ such that for every $0 < \Lambda < C$ there exists a unique global solution $\mu_\Lambda, \lambda_\Lambda \in C^2([0, r_C))$ of the static, spherically symmetric Einstein–Vlasov system* (2.2)–(2.6) *with $\mu_\Lambda(0) = \mu_0$, and $\lambda_\Lambda(0) = 0$ such that the spatial support of the distribution function is bounded. This solution coincides with the Schwarzschild–deSitter metric in the vacuum region.*

Proof. According to Lemma 3.1 there exists a C^2-solution y_Λ of (2.24) on a small interval $[0, \delta]$. In the proof of Proposition 3.5 we saw that this solution can be extended at least up to $r = R_0 + \Delta R$ for any ΔR if one chooses Λ small enough. Beyond the support of ϱ_Λ and p_Λ, i.e., for $r \in [R_{0\Lambda}, R_0 + \Delta R]$, (2.24) takes the form

$$y_\Lambda'(r) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}\right) \tag{3.27}$$

where $M = m_\Lambda(R_{0,\Lambda})$. This equation is solved by the (shifted) Schwarzschild–deSitter metric, whose corresponding y-coefficient y_S is given by

$$y_S(r) = -\frac{1}{2}\ln\left(1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}\right) - \ln\left(e^{-\lambda(R_{0\Lambda})}\right) \tag{3.28}$$

for $r \in [R_{0\Lambda}, r_C)$, where r_C is defined as the largest zero of $1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}$. The shift has been chosen such that y_Λ can be extended by y_S as a C^2-solution of (2.24) on

$[0, r_C)$, fulfilling the boundary condition (2.25), using a modified ansatz for the matter distribution f_Λ. Namely, for $r > R_0 + \Delta R$ we drop the original ansatz Φ for f_Λ and continue f_Λ by the constant zero function, i.e.,

$$f_\Lambda(x, v) = \begin{cases} \alpha\,[1 - \varepsilon e^{-y}]_+^k\,[L - L_0]_+^\ell, & r \in [0, R_0 + \Delta R] \\ 0, & r \in (R_0 + \Delta R, r_C) \end{cases}. \tag{3.29}$$

Note, that f_Λ is not losing any regularity due to the gluing procedure. Via $\mu_\Lambda = \ln(E_0) - y_\Lambda$ and

$$e^{-2\lambda_\Lambda} = 1 - \frac{r^2\Lambda}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s \tag{3.30}$$

one can construct a local solution $\mu_\Lambda, \lambda_\Lambda \in C^2([0, R_c))$ of (2.3), (2.4), where $R_c > R_0 + \Delta R$. This solution fulfils the boundary conditions $\lambda_\Lambda(0) = 0$, $\mu_\Lambda(0) = \ln(E_0) - y_0$, $\lambda'_\Lambda(0) = \mu'_\Lambda(0) = 0$. We now see that $E_0 = e^{\mu(R_{0\Lambda})}$ and continue μ_Λ and λ_Λ with the Schwarzschild–deSitter coefficients μ_S, λ_S given by

$$e^{2\mu_S} = e^{-2\lambda_S} = 1 - \frac{r^2\Lambda}{3} - \frac{2M}{r} \tag{3.31}$$

in a continuous way beyond $R_0 + \Delta R$. From (3.27) we deduce that the derivatives of μ_Λ and μ_S can also be glued together in a continuous way. The functions μ_Λ, λ_Λ, and f_Λ solve the Einstein–Vlasov system (2.2)–(2.4) globally. ∎

Remark 3.9. In the isotropic case, i.e., $L_0 = \ell = 0$ in the ansatz (2.19) for the distribution function f, the matter quantities ϱ and p are monotonically decreasing. This implies that their support in space is a ball. In the anisotropic case, however, so-called shell solutions occur [8]. The support of such matter shells is in general not connected.

4. Static, anisotropic, globally regular solutions for $\Lambda < 0$.

4.1. Local existence. In this section an existence lemma for $\Lambda < 0$ is stated for small radii. This lemma corresponds to the first part of the proof of Theorem 2.2 in [23] for the case $\Lambda = 0$.

LEMMA 4.1. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form* (2.19) *and let G_ϕ, H_ϕ be defined by* (2.21) *and* (2.22)*, respectively. Then for every $y_0 \in \mathbb{R}$ and every $\Lambda < 0$ there exists a $\delta > 0$ such that there exists a unique solution $y_\Lambda \in C^2([0, \delta])$ of* (2.24) *with initial value $y_\Lambda(0) = y_0$.*

Proof. The proof works in an exact analogous way as in the case $\Lambda > 0$. ∎

4.2. Globally regular solutions for $\Lambda < 0$. For negative cosmological constants the global existence of solutions can be proved in an analogous way as done in [22] for the case $\Lambda = 0$. After establishing the local existence of solutions analogous to the $\Lambda > 0$ case, we show that the metric components stay bounded for all $r \in \mathbb{R}_+$ with an energy estimate. This will yield the global existence of solutions of the Einstein–Vlasov system with a negative cosmological constant. In the next step we show, by virtue of a suitable choice of ansatz for the matter distribution f, that the matter quantities ϱ and p are of bounded support.

In the following theorem the existence on spatial intervals of the form $\mathbb{R}_+ \setminus [0, r_0)$ for $r_0 > 0$ is included for the purpose of applying the same theorem to the construction of static spacetimes with Schwarzschild singularities in the center (see section 5.2).

The solutions of interest here are those in which the radius variable takes values in all of $\mathbb{R}_+$.

THEOREM 4.2. *Let* $\Lambda < 0$ *and let* $\Phi : \mathbb{R}^2 \to [0, \infty)$ *be of the form* (2.19) *and let* G_ϕ *and* H_ϕ *be defined by* (2.21) *and* (2.22). *Then for every* $r_0 \geq 0$ *and* $\mu_0, \lambda_0 \in \mathbb{R}$ *there exists a unique solution* $\lambda_\Lambda, \mu_\Lambda \in C^1([r_0, \infty))$ *of the Einstein–Vlasov system* (2.2)–(2.6) *with* $\mu_\Lambda(r_0) = \mu_0$ *and* $\lambda_\Lambda(r_0) = \lambda_0$. *One has* $\lambda_0 = 0$ *if* $r_0 = 0$.

Proof. We use an energy argument similar to [22]. Let $y_\Lambda \in C^2([r_0, r_0 + \delta])$ be the local solution of (2.24) with $y_\Lambda(r_0) = \ln(E_0)e^{-\mu_0}$. If $r_0 = 0$ the existence of this local solution is established by Lemma 4.1 and in the case $r_0 > 0$ the existence of a local solution follows directly from the regularity of the right-hand sides of (2.3) and (2.4). Let $[r_0, R_c)$ be the maximal interval of existence of this solution. By $\mu_\Lambda = \ln(E_0) - y_\Lambda$ and

$$e^{-2\lambda_\Lambda} = 1 - \frac{\Lambda}{3}\left(r^2 - \frac{r_0^3}{r}\right) - \frac{2}{r}\left(\frac{r_0}{2}\left(1 - e^{-2\lambda_0}\right) + 4\pi \int_{r_0}^{r} s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s\right) \tag{4.1}$$

one constructs a local solution $\mu_\Lambda, \lambda_\Lambda \in C^2([r_0, R_c])$ of (2.3) and (2.4). We define

$$w_\Lambda(r) = -\frac{\Lambda}{12\pi} + \frac{1}{r^3}\left(-\frac{r_0^3\Lambda}{24\pi} + \frac{r_0}{8\pi}\left(1 - e^{-2\lambda_0}\right) + \int_{r_0}^{r} s^2 \varrho_\Lambda(s)\mathrm{d}s\right). \tag{4.2}$$

The Einstein equation (2.3) implies

$$\mu_\Lambda'(r) = 4\pi r e^{2\lambda_\Lambda(r)}\left(p_\Lambda(r) + w_\Lambda(r)\right). \tag{4.3}$$

By adding (2.3) and (2.4) we have

$$(\mu_\Lambda'(r) + \lambda_\Lambda'(r)) = 4\pi r e^{2\lambda_\Lambda(r)}(p_\Lambda(r) + \varrho_\Lambda(r)). \tag{4.4}$$

We assume $R_c < \infty$ and consider the quantity $e^{\mu_\Lambda + \lambda_\Lambda}(p_\Lambda + w_\Lambda)$ on the interval $\left[\frac{R_c}{2}, R_c\right)$. On this interval, in particular away from the origin, a differential inequality will be established that will allow us to deduce that both μ_Λ and λ_Λ are bounded on $\left[\frac{R_c}{2}, R_c\right)$. Using the TOV equation (2.16) we obtain for $r \in \left[\frac{R_c}{2}, R_c\right)$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left(e^{\mu_\Lambda + \lambda_\Lambda}(p_\Lambda + w_\Lambda)\right) &= e^{\mu_\Lambda + \lambda_\Lambda}\left(-\frac{2p_\Lambda}{r} - \frac{3w_\Lambda}{r} - \frac{\Lambda}{4\pi r} + \frac{2p_{T\Lambda}}{r} + \frac{\varrho_\Lambda}{r}\right) \\ &\leq C_1 e^{\mu_\Lambda + \lambda_\Lambda} = \underbrace{\frac{C_1}{p_\Lambda + w_\Lambda}}_{=:C_2}(p_\Lambda + w_\Lambda)e^{\mu_\Lambda + \lambda_\Lambda}. \end{aligned} \tag{4.5}$$

In the course of this estimate we have used that $\frac{\Lambda}{4\pi r}$, $p_{T\Lambda}(r)/r$ and $\varrho_\Lambda(r)/r$ stay bounded for $r \in \left[\frac{R_c}{2}, R_c\right)$. The constant C_2 is bounded since $w_\Lambda(r) > 0$ for negative Λ. It follows

$$\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(e^{\mu_\Lambda + \lambda_\Lambda}(p_\Lambda + w_\Lambda)\right) \leq C_2 \quad \Rightarrow \quad \lambda_\Lambda + \mu_\Lambda < \infty. \tag{4.6}$$

Equation (4.3) implies that $\mu_\Lambda'(r) \geq 0$ and therefore $\mu_\Lambda(r) \geq \mu_0$. We also have

$$e^{-2\lambda_\Lambda} \leq 1 + \frac{r^2|\Lambda|}{3} \leq \frac{3 + R_c^2|\Lambda|}{3} < \infty. \tag{4.7}$$

This in turn implies $\lambda_\Lambda > -\infty$ and we deduce from (4.6) that both μ_Λ and λ_Λ are bounded on $\left[\frac{R_c}{2}, R_c\right)$. This allows us to continue μ_Λ and λ_Λ as C^2-solutions of the Einstein equations beyond R_c, which contradicts its definition. So $R_c = \infty$. ◻

We prove in the following theorem that the distribution function in the previous theorem is compactly supported.

THEOREM 4.3. *Let $\Phi : \mathbb{R}^2 \to [0,\infty)$ be of the form* (2.19)*, let $\mu_0 \in \mathbb{R}$ and $r_0 \geq 0$, and let $\lambda_\Lambda, \mu_\Lambda \in C^1([r_0,\infty))$, $f(x,v) = \Phi(E,L)$ be the unique global-in-r solution of the Einstein–Vlasov system* (2.2)*–*(2.6) *with a negative cosmological constant, where $\mu_\Lambda(0) = \mu_0$ such that $y_0 = \ln(E_0)e^{-\mu_0} > 0$. Then there exists $R_0 \in (r_0,\infty)$ such that the spatial support of f_Λ is contained in the interval $[r_0, R_0)$.*

Proof. Due to part (iii) of Lemma 2.3, we have vacuum if $y_\Lambda(r) \leq 0$. By assumption we have $y_\Lambda(0) > 0$. In the following we show that $\lim_{r\to\infty} y_\Lambda(r) < 0$. Since y_Λ is continuous and monotonically decreasing, this implies that y_Λ possesses a single zero R_0 and the support of the matter quantities ϱ_Λ and p_Λ is contained in $[0, R_0)$.

We define $y_{\mathrm{vac},\Lambda}$ by

$$y_{\mathrm{vac},\Lambda} = y_0 - \frac{1}{2}\ln\left(1 - \frac{r^2\Lambda}{3}\right). \tag{4.8}$$

So we have

$$y'_{\mathrm{vac},\Lambda}(r) = -\frac{4\pi}{1-\frac{\Lambda r^2}{3}}\left(-\frac{r\Lambda}{12\pi}\right) \tag{4.9}$$

and $y_{\mathrm{vac},\Lambda}(0) = y_\Lambda(0) = y_0$. Furthermore, since $y'_\Lambda(r) < y'_{\mathrm{vac},\Lambda}(r)$ which can be seen immediately by means of (2.24), we have

$$y_\Lambda(r) \leq y_{\mathrm{vac},\Lambda}(r) = y_0 - \frac{1}{2}\ln\left(1 + \frac{r^2|\Lambda|}{3}\right) \overset{r\to\infty}{\longrightarrow} -\infty < 0 \tag{4.10}$$

and the theorem follows. ◻

Remark 4.4. The solution coincides with the Schwarzschild–AdS solution for $r \geq R_0$ if the continuity condition

$$\mu_\Lambda(R_0) = \ln(E_0) - y_\Lambda(R_0) = \frac{1}{2}\ln\left(1 - \frac{R_0^2\Lambda}{3} - \frac{2M}{R_0}\right) \tag{4.11}$$

is fulfilled, where $M = 4\pi\int_0^{R_0} s^2\varrho_\Lambda(s)\mathrm{d}s$. So, if y_0 is given, the corresponding value of E_0 in the ansatz Φ for the matter distribution f can be read off.

5. Solutions with a Schwarzschild singularity at the center. In this section we construct spherically symmetric, static solutions of the Einstein–Vlasov system with a nonvanishing cosmological constant that contain a Schwarzschild singularity at the center. We consider both cases with a positive and a negative cosmological constant. The construction for the case $\Lambda > 0$ makes use of the corresponding solutions with vanishing Λ. In the following we will call this solution, where $\Lambda = 0$, a *background solution*. The global existence of the background solution is proved in [22]. The matter quantities belonging to this background solution are of finite support.

5.1. Matter shells immersed in Schwarzschild–deSitter spacetime. The construction of the solution with $\Lambda > 0$ can be outlined as follows. In the vacuum case, i.e., when the right-hand sides of the Einstein equations (2.3) and (2.4) are zero, the solutions are given by

$$e^{2\mu(r)} = 1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}, \quad e^{2\lambda(r)} = \left(1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}\right)^{-1}, \quad r > r_{B\Lambda} \tag{5.1}$$

where $r_{B\Lambda}$ is defined to be the black hole event horizon, i.e., the smallest positive zero of $1-r^2\Lambda/3-2M_0/r$. If one chooses L_0 and M_0 appropriately and Λ sufficiently small the following configuration is at hand. For small $r > r_{B\Lambda}$ one sets $f(x,v) \equiv 0$ and the metric is given by the Schwarzschild-deSitter solution. Thus one has the coefficients (5.1). Increasing the radius r one reaches an interval $[r_{-\Lambda}, r_{+\Lambda}]$ where also an ansatz $f(x,v) = \Phi(E,L)$ of the form (2.19) yields vacuum, i.e., $G_\phi(r,y(r)) = H_\phi(r,y(r)) = 0$. In this interval it is possible to glue the Schwarzschild–deSitter solution (5.1) to a nonvacuum solution, solving the Einstein–Vlasov system. It will be shown that the matter quantities ϱ_Λ and p_Λ of this solution have finite support. Beyond the support of the matter quantities the solution will be continued again by the Schwarzschild-deSitter solution.

For the negative cosmological constant, globally defined solutions can also be constructed. As for the case above, the black hole is surrounded by a vacuum shell which is itself surrounded by a shell containing matter. In the outer region, we again have vacuum.

Before we consider the system with $\Lambda \neq 0$ we establish a generalized Buchdahl-type inequality for solutions of the Einstein equations with a Schwarzschild singularity at the center. This inequality is relevant for the proof of the existence of solutions of the Einstein–Vlasov system with $\Lambda > 0$.

LEMMA 5.1. *Let $\lambda, \mu \in C^1([0,\infty))$ and let $\varrho, p, p_T \in C^0([0,\infty))$ be functions that satisfy the system of equations* (2.3)–(2.7) *with a Schwarzschild singularity with mass parameter $M_0 > 0$ at the center, and such that $p + 2p_T \leq \varrho$. Then the inequality*

$$\frac{2(M_0 + m(r))}{r} \leq \frac{8}{9} \tag{5.2}$$

holds for all $r \in \left[\frac{9}{4}M_0, \infty\right)$ where $m(r)$ is given by

$$m(r) = 4\pi \int_{2M_0}^{r} s^2 \varrho(s) \mathrm{d}s. \tag{5.3}$$

Proof. For the proof of the lemma we apply techniques that are already used in [18] to prove the Buchdahl inequality for globally regular solutions without a Schwarzschild singularity. Only the steps that differ from the proof of [18], Theorem 4.1, or [5, Theorem 1] for the charged case, will be described in detail.

By integrating the Einstein equation (2.3) over the interval $\left(\frac{9M_0}{4}, r\right)$ we obtain

$$e^{-2\lambda} = 1 - \frac{9M_0}{4r}\left(1 - e^{-2\lambda_0}\right) - \frac{8\pi}{r}\int_{\frac{9M_0}{4}}^{r} s^2 \varrho(s) \mathrm{d}s, \tag{5.4}$$

where $\lambda_0 = \lambda\left(\frac{9M_0}{4}\right)$. Since we have vacuum on $\left(2M_0, \frac{9M_0}{4}\right)$ on this interval the metric is given by the Schwarzschild metric and one can compute λ_0 explicitly. One finds that

$$e^{-2\lambda} = 1 - \frac{2(M_0 + m(r))}{r}. \tag{5.5}$$

We plug this into the other Einstein equation (2.4) and obtain the differential equation

$$\mu'(r) = \frac{1}{1 - \frac{2(M_0+m(r))}{r}} \left(4\pi r p + \frac{M_0 + m(r)}{r^2} \right). \tag{5.6}$$

We now introduce the variables

$$x = \frac{2(M_0 + m(r))}{r}, \quad y = 8\pi r^2 p(r). \tag{5.7}$$

Note that $x < 1$ and $y \geq 0$. The first inequality must hold true since otherwise the metric function λ will not stay bounded. Next we let $\beta = 2\ln(r)$ and consider the curve $\left(x\left(e^{\beta/2}\right), y\left(e^{\beta/2}\right)\right)$ parameterized by β in $[0,1) \times [0,\infty)$. In the following a dot denotes the derivative with respect to β. Using the Einstein equations and the generalized TOV equation (2.16) one checks that x and y satisfy the equations

$$8\pi r^2 \varrho = 2\dot{x} + x, \tag{5.8}$$

$$8\pi r^2 p = y, \tag{5.9}$$

$$8\pi r^2 p_T = \frac{x+y}{2(1-x)}\dot{x} + \dot{y} + \frac{(x+y)^2}{4(1-x)}. \tag{5.10}$$

By virtue of these equations (5.8)–(5.10) the condition $p + 2p_T \leq \varrho$ can be written in the form

$$(3x - 2 + y)\dot{x} + 2(1-x)\dot{y} \leq -\frac{\alpha(x,y)}{2}, \quad \alpha = 3x^2 - 2x + y^2 + 2y. \tag{5.11}$$

From now on the proof is analogous to the proof of [5], Theorem 1 for the charged case. One defines the quantity

$$w(x,y) = \frac{(3(1-x) + 1 + y)^2}{1-x} \tag{5.12}$$

and shows that since $0 \leq x < 1$ and $y \leq 0$ this quantity is bounded by 16 along the curve (x, y) with an optimization procedure. The inequality $w \leq 16$ is already equivalent to

$$\frac{2(M_0 + m(r))}{r} \leq \frac{8}{9} \tag{5.13}$$

for all $r \in \left[\frac{9M_0}{4}, \infty\right)$ and the proof is complete. ∎

Remark 5.2. In the case when $M_0 = 0$ it is known that the inequality is sharp; see [4] and [18]. For the purpose of this work the bound (5.2) is sufficient and we have not tried to show sharpness.

In the course of the proof of Theorem 5.5 we will need a continuation criterion for the solution of the Einstein equations, namely the following statement.

LEMMA 5.3. *Let $\Lambda > 0$, $\mu_0 \in \mathbb{R}$ and $M_0, r_0 > 0$. Let G_ϕ and H_ϕ be defined by (2.21) and (2.22). Then the equation*

$$\begin{aligned} \mu'_\Lambda = {} & \frac{1}{1 - \frac{\Lambda}{3}\left(r^2 - \frac{r_0^3}{r}\right) - \frac{2}{r}\left(M_0 + 4\pi\int_{r_0}^r s^2 G_\phi(s, \mu_\Lambda(s))\mathrm{d}s\right)} \\ & \times \left(4\pi r H_\phi(s, \mu_\Lambda(s)) - \Lambda\left(\frac{r}{3} + \frac{r_0^3}{6r^2}\right)\right. \\ & \left. + \frac{1}{r^2}\left(M_0 + 4\pi\int_{r_0}^r s^2 G_\phi(s, \mu_\Lambda(s))\mathrm{d}s\right)\right) \end{aligned} \tag{5.14}$$

has a unique local C^2-solution μ_Λ with $\mu(r_0) = \mu_0$, with maximal interval of existence $[r_0, R_c)$, $R_c > 0$. Moreover, there exists $R_D \le R_c$ such that

$$\liminf_{r \to R_D} \left(1 - \frac{\Lambda}{3}\left(r^2 - \frac{r_0^3}{r}\right) - \frac{2}{r}\left(M_0 + 4\pi \int_{r_0}^{r} s^2 G_\phi(s, \mu_\Lambda(s))\mathrm{d}s\right)\right) = 0. \tag{5.15}$$

Proof. The local existence of a C^2-solution of (5.14) follows from the regularity of the right-hand side. Basically, one has the situation of Lemma 3.2, i.e., the case with a regular center and $\Lambda > 0$, except for the fact that there are additional terms containing r_0 and M_0. On a finite interval $[r_0, R_c)$, however, these terms are bounded and well behaved, i.e., the proof can be carried out in an analogous way. □

Remark 5.4. Lemma 5.3 implies that if the denominator of the right-hand side of (5.14) is strictly larger than zero on an interval $[r_0, r)$, then μ_Λ can be extended beyond r as a solution of (5.14).

The following theorem states the existence of solutions for $\Lambda > 0$ with a Schwarz schild singularity at the center.

THEOREM 5.5. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form (2.19) with $E_0 = 1$, let $L_0, M_0 \ge 0$ such that $L_0 > 16M_0^2$, and let G_ϕ and H_ϕ be given by (2.21) and (2.22), respectively. Then there exists a unique solution $\mu_\Lambda, \lambda_\Lambda \in C^2((r_{B\Lambda}, r_C))$ of the Einstein–Vlasov system (2.2)–(2.6) for $\Lambda > 0$ sufficiently small. The spatial support of the distribution function f_Λ is contained in a shell $\{r_{+\Lambda} < r < R_{0\Lambda}\}$. In the complement of this shell the solution of the Einstein equations is given by the Schwarzschild–deSitter metric.*

Remark 5.6. In the course of the proof one will come across the fact that in one of the vacuum regions, either $r \le r_{+\Lambda}$ or $r \ge R_{0\Lambda}$, the component μ_{vac} given by $e^{2\mu_{\text{vac}}} = 1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}$ of the Schwarzschild–deSitter metric will be shifted by a constant. But this shift is just a reparameterization of the time t [22]. Thus the shell of Vlasov matter causes a redshift.

Proof. In the first part of the proof we consider the black hole region and show that the chosen parameters lead to the configuration depicted in Figure 1. We, then make use of the existence of a background solution and construct the desired solution μ_Λ.

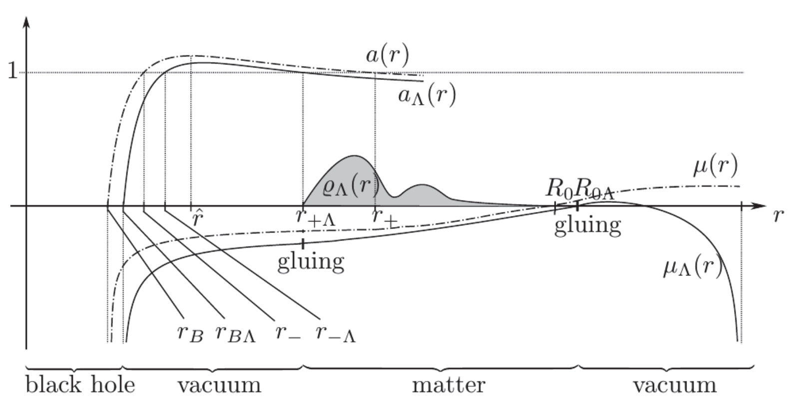

FIG. 1. *Qualitative sketch of a black hole configuration surrounded by a shell of matter.*

We define the functions

$$a(r) = \sqrt{1 - \frac{2M_0}{r}}\sqrt{1 + \frac{L_0}{r^2}}, \tag{5.16}$$

$$a_\Lambda(r) = \sqrt{1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}}\sqrt{1 + \frac{L_0}{r^2}}. \tag{5.17}$$

Moreover, we define r_- and r_+ to be the first and second radii where $a(r) = 1$, respectively, and $r_B := 2M_0$ to be the event horizon of the black hole. Since $L_0 > 16M_0^2$ we have $r_B < r_- < r_+$ [22]. Note also that $r_+ > 4M_0 > \frac{18}{5}M_0$.

Since $9M_0^2\Lambda < 1$ by assumption (Λ is chosen to be small), there exists a black hole horizon $r_{B\Lambda}$ of the Schwarzschild–deSitter metric with parameters M_0 and Λ. It can be calculated explicitly by

$$r_{B\Lambda} = -\frac{2}{\sqrt{\Lambda}}\cos\left(\frac{1}{3}\arccos\left(-3M_0\sqrt{\Lambda}\right) + \frac{\pi}{3}\right). \tag{5.18}$$

Note that $r_B < r_{B\Lambda}$. We construct an upper bound to $r_{B\Lambda}$. Set $v(r) = 1 - \frac{2M_0}{r}$.

$$\begin{aligned} v(r_{B\Lambda}) &= \int_{r_B}^{r_{B\Lambda}} v'(s)\mathrm{d}s + \underbrace{v(r_B)}_{=0} \\ &\geq \int_{r_B}^{r_{B\Lambda}} \left(\inf_{s\in[r_B, r_{B\Lambda}]} v'(s)\right)\mathrm{d}s = (r_{B\Lambda} - r_B)v'(r_{B\Lambda}) \\ &\Rightarrow \quad r_{B\Lambda} \leq r_B + \frac{v(r_{B\Lambda})}{v'(r_{B\Lambda})}. \end{aligned} \tag{5.19}$$

A short calculation yields $v(r_{B\Lambda}) = \frac{r_{B\Lambda}^2\Lambda}{3}$ and $v'(r_{B\Lambda}) = \frac{2M_0}{r_{B\Lambda}^2}$. One also checks by explicit calculation that $\frac{\mathrm{d}r_{B\Lambda}}{\mathrm{d}\Lambda} > 0$. So the distance

$$r_{B\Lambda} - r_B \leq \frac{r_{B\Lambda}^4\Lambda}{6M_0} \tag{5.20}$$

between the two horizons can be made arbitrarily small if Λ is chosen to be sufficiently small. In particular we need Λ to be small enough to assure that $r_{B\Lambda} < r_-$.

Next we define $r_{-\Lambda}$ and $r_{+\Lambda}$ to be the first and second radii where $a_\Lambda(r) = 1$. Note that $a(r) > a_\Lambda(r)$ for all $r \in (r_{B\Lambda}, r_C)$, where r_C is the cosmological horizon of the vacuum solution, and therefore the second positive zero of $1 - r^2\Lambda/3 - 2M_0/r$. Between r_- and r_+ the function $a(r)$ has a unique maximum at $r = \hat{r}$, given by

$$\hat{r} = \frac{L_0 - \sqrt{L_0^2 - 12M_0^2L_0}}{2M_0}. \tag{5.21}$$

We consider the distance between $a^2(r)$ and $a_\Lambda^2(r)$ at this radius $\hat{r}$:

$$|a^2(\hat{r}) - a_\Lambda^2(\hat{r})| = \Lambda\frac{\hat{r}^2 + L_0}{3}. \tag{5.22}$$

Choosing Λ sufficiently small one can attain $|a^2(\hat{r}) - a_\Lambda^2(\hat{r})| < a^2(\hat{r}) - 1$. This implies that $a_\Lambda(r) - 1$ has exactly two zeros in the interval (r_-, r_+). This in turn yields the desired configuration

$$2M_0 = r_B < r_{B\Lambda} < r_- < r_{-\Lambda} < \hat{r} < r_{+\Lambda} < r_+. \tag{5.23}$$

In the vacuum region $[r_{-\Lambda}, r_{+\Lambda}]$ the function $a_\Lambda(r)$ coincides with the expression $e^{-y_\Lambda(r)}\sqrt{1+\frac{L_0}{r^2}}$. Part (iii) of Lemma 2.3 therefore implies that for $r \in [r_{-\Lambda}, r_{+\Lambda}]$ the ansatz Φ for the distribution function f also yields $\varrho_\Lambda(r) = G_\phi(r, y_\Lambda(r)) = 0$ and $p_\Lambda(r) = H_\phi(r, y_\Lambda(r)) = 0$. So at $r = r_{+\Lambda}$ one can continue f by the ansatz Φ in a continuous way and for $r \geq r_{+\Lambda}$ the Einstein equations lead to the differential equation

$$\mu'_\Lambda = \frac{1}{1-\frac{\Lambda}{3}\left(r^2-\frac{r_{+\Lambda}^3}{r}\right)-\frac{2}{r}\left(\frac{r_{+\Lambda}}{2}\left(1-e^{-2\lambda_0}\right)+4\pi\int_{r_{+\Lambda}}^r s^2\varrho_\Lambda(s)\mathrm{d}s\right)} \times\left(4\pi r p_\Lambda-\Lambda\left(\frac{r}{3}+\frac{r_{+\Lambda}^3}{6r^2}\right)+\frac{r_{+\Lambda}}{2r^2}\left(1-e^{-2\lambda_0}\right) + \frac{4\pi}{r^2}\int_{r_{+\Lambda}}^r s^2\varrho_\Lambda(s)\mathrm{d}s\right) \tag{5.24}$$

where $\lambda_0 = \lambda(r_{+\Lambda})$.

There exists a background solution $\mu \in C^2((2M_0, \infty))$ to the Einstein equations with $\Lambda = 0$ [22]. For $r \in (2M_0, r_{+\Lambda}]$ this solution is given by the Schwarzschild metric and for $r > r_{+\Lambda}$ as a solution of (5.24) with $\Lambda = 0$. The background solution is continuous at $r_{+\Lambda}$ if

$$\frac{r_{+\Lambda}}{2}\left(1-e^{-2\lambda_0}\right) = M_0. \tag{5.25}$$

Furthermore, the background solution μ has the property that there exists $R_0 > 0$ such that $\mu(R_0) = 0$, which implies that the support of matter quantities ϱ and p is contained in the interval (r_+, R_0) [22]. In the remainder of the proof we show that using properties of this background solution μ one obtains a global solution μ_Λ of (5.24). We set

$$\mu_{0\Lambda} = \frac{1}{2}\ln\left(1-\frac{r_{+\Lambda}^2\Lambda}{3}-\frac{2M_0}{r_{+\Lambda}}\right), \tag{5.26}$$

$$\mu_0 = \mu(r_{+\Lambda}) = \frac{1}{2}\ln\left(1-\frac{2M_0}{r_{+\Lambda}}\right). \tag{5.27}$$

In the following we seek a solution μ_Λ of (5.24) on an interval beginning at $r = r_{+\Lambda}$ with the initial value $\mu_{0\Lambda}$ as given in (5.26) that we can glue to the vacuum solution on $(r_{B\Lambda}, r_{+\Lambda}]$. Note that $\mu_{0\Lambda} < 0$. Since there are no issues with an irregular center the local existence of μ_Λ on an interval $(r_{+\Lambda}, r_{+\Lambda}+\delta]$, $\delta > 0$ follows from the regularity of the right-hand side of (5.24). So let $(2M_0, R_c)$ be the maximum interval of the existence of μ_Λ. We define

$$v_{M_0}(r) = 1-\frac{2}{r}\left(M_0+4\pi\int_{r_{+\Lambda}}^r s^2\varrho(s)\mathrm{d}s\right), \tag{5.28}$$

$$v_{M_0\Lambda}(r) = 1-\frac{\Lambda}{3}\left(r^2-\frac{r_{+\Lambda}^3}{r}\right)-\frac{2}{r}\left(M_0+4\pi\int_{r_{+\Lambda}}^r s^2\varrho_\Lambda(s)\mathrm{d}s\right) \tag{5.29}$$

as the denominator of the right-hand side of (5.24). We set

$$\Delta v_0 := \frac{1}{18}v_{M_0\Lambda}(r_{+\Lambda}) = \frac{1-\frac{2M_0}{r_{+\Lambda}}}{18} \leq \frac{1}{18}, \tag{5.30}$$

define the radii

$$\begin{aligned} r^* &= \inf\{r \in (r_{+\Lambda}, R_c) \mid v_{M_0\Lambda}(r) = \Delta v_0\}, \\ \tilde{r} &= \sup\{r \in (r_{+\Lambda}, R_c) \mid |\mu_\Lambda(r) - \mu(r)| \leq \mu(R_0 + \Delta R)\}, \end{aligned} \tag{5.31}$$

and set $\tilde{r}^* := \min\{\tilde{r}, r^*\}$. Note that $\mu(R_0 + \Delta R) > 0$ since $\mu(R_0) = 0$ and μ is strictly increasing. We assume that $r \leq \tilde{r}^*$ and calculate $|\mu(r) - \mu_\Lambda(r)|$. To make the calculations more convenient, we extend ϱ and p on $[0, 2M_0]$ as constant zero such that integrals of ϱ and p over (r_+, r) can be replaced by integrals over $(0, r)$. First we calculate

$$|\mu_0 - \mu_{0\Lambda}| = \frac{1}{2} \ln \left[1 + \frac{r_{+\Lambda}^2 \Lambda}{3} \left(1 - \frac{r_{+\Lambda}^2 \Lambda}{3} - \frac{2M_0}{r_{+\Lambda}}\right)^{-1}\right] =: C_{0\Lambda}(r). \tag{5.32}$$

We write
(5.33)

$$\begin{aligned} |\mu(r) - \mu_\Lambda(r)| \leq \int_{r_{+\Lambda}}^r \frac{1}{v_{M_0\Lambda}(s)} \Bigg[& 4\pi s |p_\Lambda(s) - p(s)| - \Lambda \left(\frac{s}{3} + \frac{r_{+\Lambda}^3}{s^2}\right) \\ & + \frac{4\pi}{s^2} \int_0^s \sigma^2 |\varrho_\Lambda(\sigma) - \varrho(\sigma)| \mathrm{d}\sigma \Bigg] \mathrm{d}s \\ & + \int_{r_{+\Lambda}}^r \left(4\pi s p(s) + \frac{4\pi}{s^2} \int_0^s \sigma^2 \varrho(\sigma) \mathrm{d}\sigma\right) \left|\frac{1}{v_{M_0\Lambda}(s)} - \frac{1}{v_{M_0}(s)}\right| \mathrm{d}s \\ & + C_{0\Lambda}(r) \end{aligned}$$

We would like to apply the generalized Buchdahl inequality (Lemma 5.1) to the background solution μ on the interval $[r_{+\Lambda}, \infty)$. We have that $r_{+\Lambda} > \hat{r} \geq 3M_0 > 9/4M_0$. The crucial condition is the existence of a vacuum region on $\left(2M_0, \frac{9}{4}M_0\right]$. However, this is ensured by virtue of the assumption $L_0 > 16M_0^2$ which implies $r_+ > 4M_0$. So the difference $|\mu(r) - \mu_\Lambda(r)|$ can be further simplified and estimated. Using similar estimates as in the proof of Lemma 3.7 we obtain an inequality of the form

$$|\mu(r) - \mu_\Lambda(r)| \leq C_\Lambda(r) + C(r) \int_0^r \left(|p(s) - p_\Lambda(s)| + |\varrho(s) - \varrho_\Lambda(s)|\right) \mathrm{d}s \tag{5.34}$$

where $C(r)$ is increasing in r, $C_\Lambda(r)$ is increasing both in Λ and r and we have $C_\Lambda(r) = 0$ if $\Lambda = 0$. Note that the constants are fully determined by M_0, L_0, ϕ and μ.

By virtue of the mean value theorem, the sum $|p_\Lambda - p| + |\varrho_\Lambda - \varrho|$ can be estimated as

$$|p_\Lambda(r) - p(r)| + |\varrho_\Lambda(r) - \varrho(r)| \leq C \cdot |\mu_\Lambda(r) - \mu(r)|, \tag{5.35}$$

where the constant C is determined by the derivatives of G_ϕ and H_ϕ. A Grönwall argument yields $|\mu_\Lambda(r) - \mu(r)| \leq C_{\mu\Lambda}(r)$ implying $|\varrho_\Lambda(r) - \varrho(r)| \leq C_{g\Lambda}(r)$ with certain constants $C_{g\Lambda}$ and $C_{\mu\Lambda}$.

One can choose Λ small enough such that for all $r \in (r_{+\Lambda}, R_0 + \Delta R]$ we have

$$|\mu_\Lambda(r) - \mu(r)| < \mu(R_0 + \Delta R). \tag{5.36}$$

Moreover, we consider the difference

$$|v_{M_0}(r) - v_{M_0\Lambda}(r)| \leq \frac{\Lambda}{3} \left|r^2 - \frac{r_{+\Lambda}^3}{r}\right| + \frac{8\pi r^2}{3} C_{g\Lambda}(r). \tag{5.37}$$

Lemma 5.1 implies $v_{M_0}(r) \geq \frac{1}{9}$ for all $r \in (r_{+\Lambda}, \infty)$. Choosing Λ sufficiently small, such that for all $r \in (r_{+\Lambda}, R_0 + \Delta R]$ we have $|v_{M_0}(r) - v_{M_0\Lambda}(r)| \leq \frac{1}{18}$, one obtains $v_{M_0\Lambda} \geq \frac{1}{18}$ on $(r_{+\Lambda}, R_0 + \Delta R]$.

In all, we have deduced that $\tilde{r}^* \geq R_0 + \Delta R$ if Λ is chosen sufficiently small. This implies that μ_Λ exists at least on $[0, R_0 + \Delta R]$ by Lemma 5.3 and also that $\mu_\Lambda(R_0 + \Delta R) > 0$. From the latter property one deduces that there exists a radius $R_{0\Lambda} > R_0$ such that for all $r \in [R_{0\Lambda}, R_0 + \Delta R]$ we have $\varrho_\Lambda(r) = p_\Lambda(r) = 0$. On this interval, we can glue an appropriately shifted Schwarzschild-deSitter metric to μ_Λ. This yields the desired solution defined on $(r_{B\Lambda}, r_C)$. ∎

Remark 5.7. To see that the solutions constructed in Theorem 5.5 are nonvacuum, one checks that for $r \geq r_{+\Lambda}$ one has

$$\frac{\mathrm{d}}{\mathrm{d}r} a_\Lambda(r) < 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}r^2} a_\Lambda(r) \leq 0. \tag{5.38}$$

Since $a_\Lambda(r)$ corresponds to $e^{-y_\Lambda(r)}$, this implies that for some $r > r_{+\Lambda}$ the quantity $e^{-y_\Lambda(r)}\sqrt{1 + \frac{L_0}{r^2}} < 1$ which in turn implies by part (iii) of Lemma 2.3 that $\varrho_\Lambda(r), p_\Lambda(r) > 0$ for some $r > r_{+\Lambda}$.

Remark 5.8. In contrast to the metric without a singularity at the center, the metric with a Schwarzschild singularity does not coincide with the nonshifted Schwarzschild–deSitter solution for $r > R_{0\Lambda}$. This can be seen as follows. We have

$$\mu'_\Lambda(r) \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r} \ln\left(1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}\right). \tag{5.39}$$

Certainly, the mass parameter M of the vacuum solution, which is glued on in the outer region, is larger than M_0. This implies

$$1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r} > 1 - \frac{r^2\Lambda}{3} - \frac{2M}{r} \tag{5.40}$$

for all $r \in (r_{B\Lambda}, r_C)$. So there is no ansatz Φ for the matter distribution that yields a metric component μ_Λ that connects the two vacuum solutions without any shift. By suitable choice of Φ and E_0, however, one can determine whether the inner or the outer Schwarzschild–deSitter metric is shifted. For the maximal C^2-extension of the metric constructed in Theorem 5.5 we will need the solution to coincide with the nonshifted Schwarzschild–deSitter metric for $r > R_{0\Lambda}$.

5.2. Matter shells immersed in Schwarzschild–AdS spacetimes. We construct solutions of the Einstein–Vlasov system with a Schwarzschild singularity at the center for the case $\Lambda < 0$. The result is given in the following theorem.

THEOREM 5.9. *Let* $\Phi : \mathbb{R}^2 \to [0, \infty)$ *be of the form* (2.19), *let* $L_0, M_0 \geq 0$ *such that* $L_0 < 16M_0^2$, *and let* G_ϕ *and* H_ϕ *be given by* (2.21) *and* (2.22), *respectively. Then there exists a unique solution* $\mu_\Lambda, \lambda_\Lambda \in C^2((r_{B\Lambda}, \infty))$ *of the Einstein–Vlasov system* (2.2)–(2.6) *for* $\Lambda < 0$ *and* $|\Lambda|$ *sufficiently small. The spatial support of the distribution function* f_Λ *is contained in a shell,* $\{r_{+\Lambda} < r < R_{0\Lambda}\}$. *In the complement of this shell, the solution of the Einstein equations is given by the Schwarzschild–AdS metric.*

Proof. The theorem can be proved by applying the same ideas as in the proof of Theorem 5.5. ∎

6. Solutions on $\mathbb{R} \times S^3$ and $\mathbb{R} \times S^2 \times \mathbb{R}$. In sections 3.4 and 5.1 we constructed spherically symmetric static solutions of the Einstein–Vlasov system with small positive cosmological constant Λ. For small radii the Λ-term plays only a minor role. This was crucial for the method of the proof. However, the global structure of the constructed spacetime is substantially different when $\Lambda > 0$ and shows interesting properties. In particular, it allows for solutions with different global topologies.

The following theorem gives a class of new solutions to the nonvacuum field equations with nontrivial global topology. These solutions are constructed from pieces consisting of the solutions constructed in Theorems 3.8 and 5.5.

Theorem 6.1. *Let $\Lambda > 0$ be sufficiently small and let $\mathscr{M}_1 = \mathbb{R} \times S^3$ and $\mathscr{M}_2 = \mathbb{R} \times S^2 \times \mathbb{R}$. The following types of static metrics solving the Einstein–Vlasov system exist on these topologies.*

(i) *There is a class of static metrics on $\mathscr{M}_1$, which is characterized in Figure* 2. *In regions* I *and* IV *a metric in this class coincides with two a priori different solutions of the type constructed in Theorem* 3.8 *with identical total mass, but with possibly different matter distributions and radii of the support of the matter quantities R_1 and R_2 and regular centers. The metric in regions* II *and* III *is vacuum.*

(ii) *There is a class of static metrics on $\mathscr{M}_1$, which is characterized in Figure* 3. *A metric in this class consists of two regular centers with finitely extended matter distribution around each of the centers of equal mass, but with possibly different matter distributions and radii R_1, R_2 of the type constructed in Theorem* 3.8. *These two regions are connected by a chain of black holes of identical masses (the diagram shows the minimal configuration with one black hole).*

(iii) *There is a class of metrics on $\mathscr{M}_2$, which is characterized in Figure* 4. *The spacetime consists of an infinite sequence of black holes, each surrounded by matter shells of possibly different radii and positions. In regions* IV, VII, X, *and* XIII *these solutions coincide with those constructed in Theorem* 5.5. *The necessary conditions on the masses are $M_\varrho^{A_1} = M_\varrho^{A_2}$, $M_\varrho^{B_1} = M_\varrho^{B_2}$ and $M_0^A + M_\varrho^{A_2} = M_\varrho^{B_1} + M_0^B$, where M_0^i, $i = A, B$, denote the mass parameter of the black holes and $M_\varrho^{i_j}$, $i = A, B$, $j = 1, 2$ denote the quasilocal mass of the matter shells defined in* (6.11).

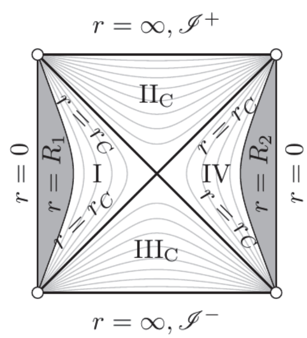

Fig. 2. *Penrose diagram of the maximal C^2-extension of a metric constructed as a spherically symmetric solution of the Einstein–Vlasov system. Region* I *corresponds to the region $0 < r < r_C$. The metric is extended in an analogous way to the standard extension of the deSitter metric. The gray lines are surfaces of constant r.*

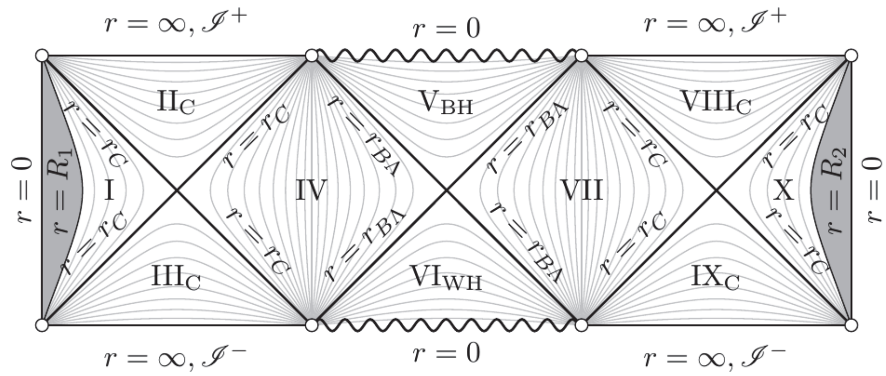

FIG. 3. *Penrose diagram of the maximal C^2-extension of a metric constructed as a spherically symmetric solution of the Einstein–Vlasov system. Region* I *corresponds to the region $0 < r < r_C$. In this region matter (represented by the shaded area) is present and the metric is regular. This metric is extended with the Schwarzschild–deSitter metric that leads to a periodic solution. The periodic course stops when a matter region appears again preventing the metric from being singular at $r = 0$. The gray lines are surfaces of constant r.*

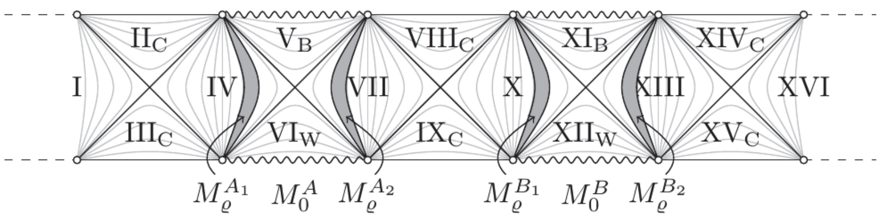

FIG. 4. *Penrose diagram of the maximal C^2-extension of a metric constructed as a spherically symmetric solution of the Einstein–Vlasov system. The solution coincides with the Schwarzschild–deSitter spacetime in the vacuum regions and the black holes are surrounded by shells of Vlasov matter (gray shaded domains). Notably, the black holes do not necessarily have the same mass. The gray lines are surfaces of constant r.*

Remark 6.2.

(a) The black hole masses in the third class of solutions in the previous theorem can be pairwise different. Only the total mass of the black hole and matter shell have to agree pairwise; see the condition in (iii) above.

(b) Combinations of the classes (ii) and (iii) yield similar metrics on $\mathscr{M}_3 = \mathbb{R} \times \mathbb{R}^3$ with a regular center followed by an infinite sequence of black holes.

(c) The second class of solutions could also be generalized by adding matter shells around the black holes. The mass parameters then have to be adjusted.

(d) When crossing the cosmological horizon or the event horizon of a black or white hole the Killing vector ∂_t changes from being timelike to spacelike. This means that the maximally extended spacetime contains both static and dynamic regions that are alternating. This holds for all constructed classes.

Proof. We outline now the construction of the spacetimes given in the previous theorem. For the first two classes of spacetimes we consider solutions of the Einstein–Vlasov system with a regular center. Let $(\mu_\Lambda, \lambda_\Lambda, f_\Lambda)$ be a static solution of the spherically symmetric Einstein–Vlasov system with positive cosmological constant Λ defined for $r \in [0, r_C)$ such that the support of the matter quantities is bounded by a radius $0 < R_{0\Lambda} < r_C$. The radius r_C denotes the cosmological horizon. On $[R_{0\Lambda}, r_C)$

there is vacuum and the metric is given by the Schwarzschild-deSitter metric (6.4) with mass M as the mass parameter. The mass M is then given by

$$M = 4\pi \int_0^{R_{0\Lambda}} s^2 \varrho_\Lambda(s) \mathrm{d}s. \tag{6.1}$$

If $9M^2\Lambda < 1$, the polynomial $r^3 - \frac{3}{\Lambda}r + \frac{6M}{\Lambda}$ has one negative zero and two positive ones. The largest zero of this polynomial is defined to be the cosmological horizon r_C. Moreover, r_n is the negative zero and $r_{B\Lambda}$ the smaller positive one. In terms of the mass M and the cosmological constant Λ these zeros can be calculated explicitly. Note that the Buchdahl inequality for solutions with $\Lambda \neq 0$ [6] implies $r_{B\Lambda} < R_{0\Lambda}$.

Case (i). Consider Figure 2. This spacetime can be obtained in an analogous way to the standard procedure to compactify the deSitter space as described, for example, in [16]. In the following, this procedure is carried out in detail. The metric is given as a nonvacuum solution of the Einstein–Vlasov system for $r \in [0, r_C)$, corresponding to region I in Figure 2, as discussed in Theorem 3.8. In this region we have

$$\mathrm{d}s^2 = -e^{2\mu(r)}\mathrm{d}t^2 + e^{2\lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2(\vartheta)\mathrm{d}\varphi^2. \tag{6.2}$$

In the first step we introduce coordinates U_I, V_I, which transform the region $\mathbb{R} \times [0, r_C) \times S^2$ into the left triangle (region I) in Figure 2. The coordinates usually used to compactify the vacuum deSitter metric, as for example described in [16], will suffice. They are given by

$$U_\mathrm{I} = \sqrt{\frac{r_C - r}{r_C + r}}e^{-\frac{t}{r_C}}, \quad V_\mathrm{I} = -\sqrt{\frac{r_C - r}{r_C + r}}e^{\frac{t}{r_C}} \tag{6.3}$$

and can be compactified via the transformations $p_\mathrm{I} = \arctan(U_\mathrm{I})$, $q_\mathrm{I} = \arctan(V_\mathrm{I})$. The left part of Figure 5 shows the transformed region $\mathbb{R} \times [0, r_C)$ in the p_I, q_I coordinates.

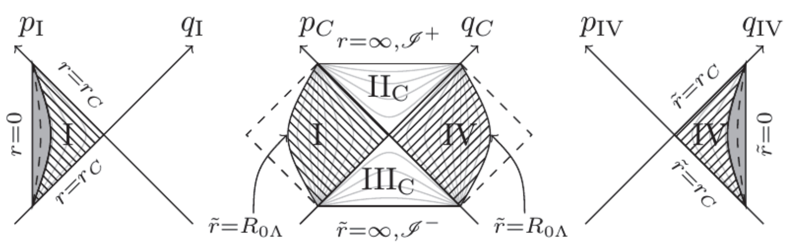

Fig. 5. *Construction of the spacetime shown in Figure* 2. *We use three coordinate charts to compactify the spacetime. Regions that are shaded in the same direction are covered by two of the coordinate charts simultaneously, thus their coordinates can be changed. The gray areas are matter regions and the dashed lines correspond to* $r = r_{B\Lambda}$. *We distinguish between* r *and* $\tilde{r}$ *to emphasize that there are different spacetime regions that cannot be covered by a single chart* $(t, r, \vartheta, \varphi)$. *All coordinates* p *and* q *take values in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The support of the matter (i.e., the matter distribution f) ends at a radius $R_{0\Lambda}$. For $r \geq R_{0\Lambda}$ the metric is merely given by the Schwarzschild–deSitter metric

$$\mathrm{d}s^2 = -\left(1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}} + r^2\mathrm{d}\Omega^2, \tag{6.4}$$

where $R_{0\Lambda} \leq r < r_C$. At $r = r_C$ there is a coordinate singularity of the metric that we want to pass. For this purpose we express the metric in other coordinates that do not have a singularity at $r = r_C$ being defined on the region where $r \in [R_{0\Lambda}, r_c)$ (region I in the middle part of Figure 5). These coordinates are given by

$$(6.5)\qquad \begin{aligned} U_C &= \sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_{B\Lambda})^{\gamma}}}\, e^{-\frac{t}{2\delta_C}} > 0, \\ V_C &= -\sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_{B\Lambda})^{\gamma}}}\, e^{\frac{t}{2\delta_C}} < 0, \end{aligned}$$

where $\delta_C = \frac{r_C}{\Lambda r_C^2 - 1} > 0$ and $\gamma = \frac{r_{B\Lambda}}{(1-\Lambda r_{B\Lambda}^2)\delta_C}$, $0 < \gamma < 1$.[2] They are used in the standard compactification procedure of the Schwarzschild–deSitter metric. For details, see [12] or [13]. In the new coordinates the line element of the Schwarzschild–deSitter metric (6.4) reads

$$(6.6)\qquad \mathrm{d}s^2 = -\frac{4\Lambda\delta_C^2}{3r}(r - r_n)^{2-\gamma}(r - r_{B\Lambda})^{1+\gamma}\mathrm{d}U_C\mathrm{d}V_C + r^2\mathrm{d}\vartheta^2 + r^2\sin^2(\vartheta)\mathrm{d}\varphi^2,$$

where $r \geq R_{0\Lambda}$. Note that here r is seen as a function of U_C and V_C. The coordinates only take values in $\{(u, v) \in \mathbb{R}^2 \mid u > 0, v < 0\}$. We extend them to $\mathbb{R}^2$. This extension goes beyond r_C. Again, the spacetime region covered by the coordinates U_C and V_C can be compactified using the transformation $p_C = \arctan(U_C)$, $q_C = \arctan(V_C)$. The middle part of Figure 5 shows the region covered by U_C and V_C, each taking values in $\mathbb{R}$, in the p_C, q_C coordinates. The line element (6.6) can be extended to the whole area covered by U_C and V_C in an analytic way. In the region where $r \in [R_{0\Lambda}, r_C)$ the coordinate charts (6.3) and (6.5) overlap and one can change coordinates (the shaded areas in the left and middle parts of Figure 5). The transformation law is given by

$$(6.7)\qquad \begin{aligned} U_C(U_{\mathrm{I}}) &= \sqrt{\frac{(r_C + r)(r - r_n)^{\gamma-1}}{(r - r_{B\Lambda})^{\gamma}}}\, e^{\frac{3-2\Lambda r_C^2}{r_C}} U_I, \\ V_C(V_{\mathrm{I}}) &= \sqrt{\frac{(r_C + r)(r - r_n)^{\gamma-1}}{(r - r_{B\Lambda})^{\gamma}}}\, e^{-\frac{3-2\Lambda r_C^2}{r_C}} V_I. \end{aligned}$$

Region IV in Figure 2 corresponds to a second universe that also can be equipped with Schwarzschild coordinates $(\tilde{t}, \tilde{r})$. We distinguish between r and $\tilde{r}$ to emphasize that the charts (t, r) and $(\tilde{t}, \tilde{r})$ cover different regions of the spacetime. Geometrically appear these regions appear equal. This will be different for the second class of spacetimes (6.1). In the region $\tilde{r} \in [R_{0\Lambda}, r_C)$ (region IV in the middle part of Figure 5), in terms of the $\tilde{t}$, $\tilde{r}$ coordinates U_C and V_C are given by

$$(6.8)\qquad \begin{aligned} U_C &= -\sqrt{\frac{(r_C - \tilde{r})(\tilde{r} - r_-)^{\gamma-1}}{(\tilde{r} - r_{B\Lambda})^{\gamma}}}\, e^{-\frac{\tilde{t}}{2\delta_C}} < 0, \\ V_C &= \sqrt{\frac{(r_C - \tilde{r})(\tilde{r} - r_-)^{\gamma-1}}{(\tilde{r} - r_{B\Lambda})^{\gamma}}}\, e^{\frac{\tilde{t}}{2\delta_C}} > 0. \end{aligned}$$

[2]The signs of these expressions can be checked with the equality $1 - \frac{r_C^2\Lambda}{3} - \frac{2M}{r_C} = 0$.

To get a compactification of the whole region IV, including $\tilde{r} < r_B$, we introduce coordinates similar to (6.3), namely

$$U_{\mathrm{IV}} = -\sqrt{\frac{r_C - \tilde{r}}{r_C + \tilde{r}}}e^{-\frac{\tilde{t}}{r_C}}, \quad V_{\mathrm{IV}} = \sqrt{\frac{r_C - \tilde{r}}{r_C + \tilde{r}}}e^{\frac{\tilde{t}}{r_C}} \tag{6.9}$$

covering the region characterized by $\tilde{r} \in [0, r_C)$. This region can again be compactified via $p = \arctan(U)$, $q = \arctan(V)$. This yields the right part of Figure 5. For $\tilde{r} \in [R_{0\Lambda}, r_C)$ the coordinates can be changed using a law which is analogous to (6.8). On the spacetime region represented by the middle part of Figure 5 the line element can be expressed by (6.6). Since in both regions I and IV the metric can be brought into the form (6.2) via coordinate transformations, the energy densities are also identical in these regions. This of course implies that in both regions the mass parameter is equal.

Case (ii). Now we come to the spacetimes characterized by Figure 3. For the construction of a C^2-extension of the metric (6.2) at least five coordinate charts are necessary. Figure 6 illustrates this construction.

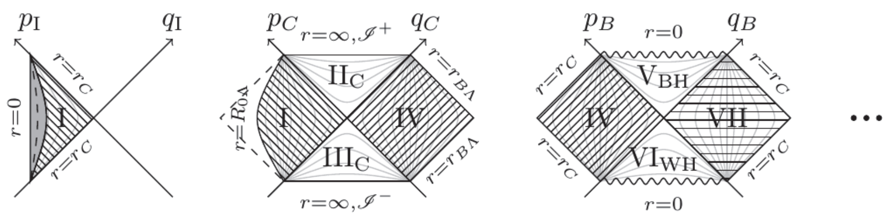

FIG. 6. *Construction of the spacetime shown in Figure* 3. *On regions that are shaded in the same direction two coordinates are defined and one can change between them. All coordinates p, q take values in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Again we begin with the region $r \in [0, r_C)$ where the metric is given by (6.2). In the same way as described above one, expresses the line element in other coordinates p_C, q_C that avoid the singularity at $r = r_C$ and cover the region $r_{0\Lambda} < r < r_C$. The line element as given by (6.6) can be analytically[3] extended onto Regions I–IV in Figure 6. From now on the procedure differs from the one above. Regions I and IV are not supposed to be geometrically identical but region IV will be a vacuum region, so the metric will be given by the Schwarzschild-deSitter solution everywhere. Certainly, the line element (6.6) of the Schwarzschild–deSitter metric being given in terms of the coordinates U_C, V_C now shows a singularity at $r = r_{B\Lambda}$.[4] This coordinate singularity can be overcome by virtue of the coordinates

$$U_B = \sqrt{\frac{(r - r_{B\Lambda})(r - r_n)^{\beta-1}}{(r_C - r)^{\beta}}}e^{\frac{t}{2\delta_B}}, V_B = -\sqrt{\frac{(r - r_{B\Lambda})(r - r_n)^{\beta-1}}{(r_C - r)^{\beta}}}e^{-\frac{t}{2\delta_B}}, \tag{6.10}$$

where $\delta_B = \frac{r_{B\Lambda}}{1-\Lambda r_{B\Lambda}^2} > 0$ and $\beta = \frac{r_C}{(\Lambda r_C^2 - 1)\delta_B} > 1$. The coordinates are defined on the middle part of Figure 6. This is part of the standard compactification procedure of the Schwarzschild–deSitter metric; see [12] or [13]. By alternating the coordinate

[3]In matter regions the regularity of the metric is C^2 as provided by Theorem 3.8; in vacuum regions the metric is analytic.

[4]By abuse of notation we use r for the radius coordinate in every region of the spacetime $\mathscr{M}_1$.

charts (U_C, V_C) and (U_B, V_B) this procedure can be continued an arbitrary amount of times, extending the spacetime to additional black hole and cosmological regions. This periodic extension stops if for $r < r_C$ the metric is not given by a vacuum solution of the Einstein equations but again by the solution (6.2) of the Einstein–Vlasov system. There is no coordinate singularity at $r = r_{B\Lambda}$ and there is a regular center at $r = 0$. So a regular expression of the line element by the coordinates (6.3) is again possible, leading to region X in Figure 3. This region is now geometrically identical to region I in Figure 3 (and also in Figure 6). In the extension procedure described above the expressions for the coordinates (6.8) and (6.10) used to pass the coordinate singularities at $r = r_{B\Lambda}$ and $r = r_C$ in the vacuum regions of the spacetime $\mathscr{M}_1$ depend on Λ and M. So the identification of corresponding regions in the different coordinate charts, e.g., I or IV in Figure 6, is only possible if the parameters Λ and M are equal in all regions of $\mathscr{M}_1$. In terms of the notation of Figure 3 this implies $M_1 = M_2$.

Case (iii). A maximal extension of a solution to the Einstein–Vlasov system on the manifold $\mathscr{M}_2$ as characterized by Figure 4, i.e., spacetimes in class (6.1), can be obtained in a similar way. The starting point is the region $r_{B\Lambda} < r < r_C$. On this interval the existence of a unique solution to a given ansatz for f is established by Theorem 5.5. The solution on hand can be understood as a Schwarzschild–deSitter spacetime with an immersed shell of Vlasov matter supported on an interval $(r_{+\Lambda}, R_{0\Lambda})$. Two mass quantities are important. On the one hand one has the mass parameter M_0 of the black hole at the center; on the other hand there is M that is defined to be

$$M = M_0 + M_\varrho, \quad M_\varrho = 4\pi \int_{r_{+\Lambda}}^{R_{0\Lambda}} s^2 \varrho_\Lambda(s) \mathrm{d}s. \tag{6.11}$$

This quantity represents the sum of the mass of the black hole and the shell of Vlasov matter. As constructed in Theorem 5.5, for $r_{B\Lambda} < r \leq r_{+\Lambda}$ the metric is given by a shifted Schwarschild–deSitter metric

$$\mathrm{d}s^2 = -C\left(1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{C\left(1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}\right)} + r^2\mathrm{d}\Omega^2, \tag{6.12}$$

where $r_{B\Lambda} < r \leq r_{+\Lambda}$ with the mass M_0 of the black hole as the mass parameter and the shift $C > 0$. For $R_{0\Lambda} \leq r < r_C$ the metric is given by the Schwarzschild–deSitter metric (6.4) with mass parameter M.

The two critical horizons, $r_{B\Lambda}$ and r_C, can be given explicitly as zeros of the expression $1 - \frac{r^2\Lambda}{3} - \frac{2m(r)}{r}$. But it is important to note that the mass parameter $m(r)$ does not stay constant throughout the whole interval $(r_{B\Lambda}, R_{0\Lambda})$. The black hole horizon $r_{B\Lambda}$ is characterized by M_0 and the cosmological horizon r_C by M. This has to be kept in mind when choosing coordinates to construct an extension of the metric on $\mathscr{M}_2$, as illustrated in Figure 7.

We distinguish between the zeros of $1 - \frac{r^2\Lambda}{3} - \frac{2m(r)}{r}$ when $m(r) \equiv M_0$ and $m(r) \equiv M$ and call them r_{B0}, r_{C0} or r_B, r_C, respectively. Note that $r_{B0} = r_{B\Lambda}$. Consider the metric on the region $r_{B0} < r < r_C$, being part of region VII in Figure 4 or the middle part of Figure 7. The metric shall be extended to the left (regions IV, $\mathrm{V_B}$, $\mathrm{VI_W}$) and to the right (regions $\mathrm{VIII_C}$, $\mathrm{IX_C}$, X) as a vacuum solution until the next matter shell appears. So the coordinate transformations have to be chosen with respect to the radii r_B and r_C belonging to the current mass parameter in the respective spacetime region. Three coordinate charts are needed to extend the metric beyond the black hole

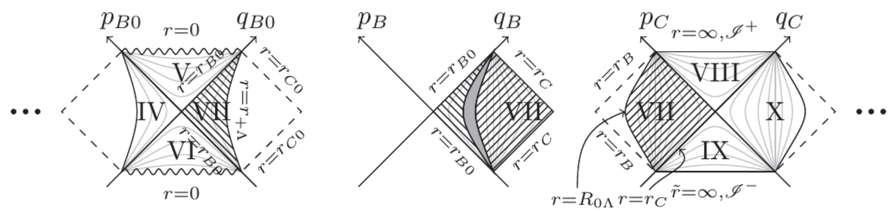

FIG. 7. *Construction of the spacetime shown in Figure 4. The middle part shows a Schwarzschild–deSitter spacetime with an immersed matter shell for $r_{B\Lambda} = r_{B0} < r < r_C$. The left and right parts show the adjacent vacuum region containing several coordinate singularities. On regions that are shaded in the same direction two coordinates are defined and one can change between them. All coordinates p, q take values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.*

and the cosmological horizon. First we compactify the region $r_{B\Lambda} = r_{B0} < r < r_C$ using the coordinates

$$
\begin{aligned}
U_B &= \sqrt{\frac{(r - r_{B0})(r - r_n)^{\beta-1}}{(r_C - r)^{\beta}}}\, e^{\frac{t}{2\delta_{B0}}}, \\
V_B &= -\sqrt{\frac{(r - r_{B0})(r - r_n)^{\beta-1}}{(r_C - r)^{\beta}}}\, e^{-\frac{t}{2\delta_{B0}}},
\end{aligned}
\tag{6.13}
$$

where $\delta_{B0} = \frac{r_{B0}}{1-\Lambda r_{B0}^2} > 0$ and $\beta = \frac{r_C}{(\Lambda r_C^2 - 1)\delta_{B0}} > 1$. These coordinates give rise to $p_B = \arctan(U_B)$ and $q_B = \arctan(V_B)$. This region is depicted in the middle part of Figure 7. The spacetimes characterized by Figure 4 show two types of connected vacuum regions. The first type is characterized by $r \le r_{+\Lambda}$ (inside the matter shell) and the second by $r \ge R_{0\Lambda}$ (beyond the matter shell). To extend the metric to the region inside the matter shell (and the black hole) one uses the coordinates

$$
\begin{aligned}
U_{B0} &= \sqrt{\frac{(r - r_{B0})(r - r_n)^{\beta_0-1}}{(r_{C0} - r)_0^{\beta}}}\, e^{\frac{t}{2\delta_{B0}}}, \\
V_{B0} &= -\sqrt{\frac{(r - r_{B0})(r - r_n)^{\beta_0-1}}{(r_{C0} - r)_0^{\beta}}}\, e^{-\frac{t}{2\delta_{B0}}},
\end{aligned}
\tag{6.14}
$$

where $\delta_{B0} = \frac{r_{B0}}{1-\Lambda r_{B0}^2} > 0$ and $\beta_0 = \frac{r_{C0}}{(\Lambda r_{C0}^2 - 1)\delta_{B0}} > 1$, and the corresponding compactification $p_{B0} = \arctan(U_{B0})$, $q_{B0} = \arctan(V_{B0})$. These coordinates are valid for $0 < r < r_{+\Lambda}$. The black hole horizon can be crossed using the usual arguments of the extension of the Schwarzschild–deSitter metric, as done for example in [16, 12, 13]. This is illustrated in the left part of Figure 7. The region beyond the matter shell (and the cosmological horizon) can be reached via the coordinates

$$
\begin{aligned}
U_C &= -\sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_B)^{\gamma}}}\, e^{-\frac{t}{2\delta_C}}, \\
V_C &= \sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_B)^{\gamma}}}\, e^{-\frac{t}{2\delta_C}},
\end{aligned}
\tag{6.15}
$$

where $\delta_C = \frac{r_C}{\Lambda r_C^2 - 1} > 0$ and $\gamma = \frac{r_B}{(1-\Lambda r_B^2)\delta_C}$, $0 < \gamma < 1$. These coordinates extend the metric to the area $R_{0\Lambda} < r < \infty$, shown in the right part of Figure 7.

On the connected vacuum regions the metric is given by only one expression, even though the vacuum extends to several regions of $\mathscr{M}_2$, e.g., Regions VII, $\mathrm{VIII_C}$, $\mathrm{IX_C}$ and X. This implies that the coordinates U_{B0}, V_{B0} or U_C, V_C have to be given by the same expressions (6.14) or (6.15), respectively, (modulo sign; see [16, 12, 13]), which in turn implies that the mass parameter has to stay the same on these connected vacuum regions. For the vacuum region with $r \geq R_{0\Lambda}$ this implies $M_0^A + M_\varrho^{A_2} = M_0^B + M_\varrho^{B_1}$ (notation of Figure 4). On the region characterized by $r \leq r_{+\Lambda}$ this is always granted because the mass is entirely given by the black hole mass M_0. Finally the shift constants $C > 0$ of the vacuum metric have to coincide in this region (IV and VII in Figure 4). They are determined by the matter shells surrounding the black hole and are equal if these shells have the same shape which implies $M_\varrho^{A_1} = M_\varrho^{A_2}$. ∎

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REFERENCES

[1] L. Andersson, R. Beig, and B. Schmidt, *Static self-gravitating elastic bodies in Einstein gravity*, Commun. Pure Appl. Math., 61 (2008), pp. 988–1023.
[2] H. Andréasson, *The Einstein-Vlasov System/Kinetic Theory*, Living Rev. Relativity, 24, 2011.
[3] H. Andréasson, *On static shells and the Buchdahl inequality for the spherically symmetric Einstein-Vlasov system*, Comm. Math. Phys., 274 (2007), pp. 409–425.
[4] H. Andréasson, *Sharp bounds on $2M/R$ of general spherically symmetric static objects*, J. Differential Equations, 245 (2008), pp. 2243–2266.
[5] H. Andréasson, *Sharp bounds in the critical stability radius for relativistic charged spheres*, Commun. Math. Phys., 288 (2009), pp. 715–730.
[6] H. Andréasson and C.G. Böhmer, *Bounds on M/R for static objects with a positive cosmological constant*, Class. Quantum Grav., 26 (2009), 195007.
[7] H. Andréasson, C.G. Böhmer, and A. Mussa, *Bounds on M/R for charged objects with positive cosmological constant*, Class. Quantum Grav., 29 (2012), 095012.
[8] H. Andréasson and G. Rein, *On the steady states of the spherically symmetric Einstein-Vlasov system*, Class. Quantum Grav., 24 (2007), pp. 1809–1832.
[9] H. Andréasson, M. Kunze, and G. Rein, *Existence of axially symmetric static solutions of the Einstein-Vlasov system*, Commun. Math. Phys., 308 (2011), pp. 23–47.
[10] H. Andréasson, M. Kunze, and G. Rein, *Rotating, stationary, axially symmetric spacetimes with collisionless matter*, Commun. Math. Phys., 329 (2014), pp. 787–808.
[11] J. Batt, W. Faltenbacher, and E. Horst, *Stationary spherically symmetric models in stellar dynamics*, Arch. Rational Mech. Anal., 93 (1986), pp. 159–183.
[12] S. L. Bażański and V. Ferrari, *Analytic Extension of the Schwarzschild-de Sitter Metric*, Il Nuovo Cimento B, 91 (1986), pp. 126–142.
[13] C. Stanciulescu, *Spherically Symmetric Solutions of the Vacuum Einstein Field Equations with Positive Cosmological Constant*, Diploma thesis, University of Vienna, Austria, 1998.
[14] M. Dafermos and A. Rendall, *An extension principle for the Einstein-Vlasov system in spherical symmetry*, Ann. Henri Poincaré, 6 (2005), pp. 1137–1155.
[15] G. F. R. Ellis, S. T. C. Siklos, and J. Wainwright, *Dynamical Systems in Cosmology*, Cambridge University Press, Cambridge, UK, 1997.
[16] G. W. Gibbons and S. W. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, Phys. Rev. D, 15 (1997), pp. 2738–2751.
[17] S. W. Hawking and H. S. Reall, *Charged and rotating AdS black holes and their CFT duals*, Phys. Rev. D, 61 (2000), 024014.
[18] P. Karageorgis and J. Stalker, *Sharp bounds on $2M/R$ for static spherical objects*, Class. Quantum Grav., 25 (2008), 195021.

[19] T. Ramming and G. Rein, *Spherically symmetric equilibria for self-gravitating kinetic or fluid models in the nonrelativistic and relativiztie case—A simple proof for finite extension*, SIAM J. Math. Anal., 45 (2013), pp. 900–914.

[20] G. Rein and A. D. Rendall, *Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics*, Math. Proc. Cambridge Philos. Soc., 128 (2000), pp. 363–380.

[21] G. Rein, *Static shells for the Vlasov-Poisson and Vlasov-Einstein system*, Indiana University Mathematics Journal, 48 (1999), pp. 335–346.

[22] G. Rein, *Static solutions of the spherically symmetric Vlasov-Einstein system*, Math. Proc. Cambridge Philos. Soc., 115 (1994), pp. 559–570.

[23] G. Rein and A. D. Rendall, *Smooth static solutions of the spherically symmetric Vlasov-Einstein system*, Ann. Henri Poincaré, 59 (1993), pp. 383–397.

[24] G. Rein and A. D. Rendall, *Global existence of solutions of the spherically symmetric Vlasov–Einstein system with small initial data*, Comm. Math. Phys., 150 (1992), pp. 561–583.

[25] H. Ringström, *On the Topology and Future Stability of the Universe*, Oxford Mathematical Monographs, Oxford University Press, Oxford, UK, 2013.

[26] M. Thaller, *Spherically Symmetric, Static Solutions of the Einstein-Vlasov System with Cosmological Constant*, Master thesis, University of Vienna, Austria, 2014.

[27] G. Wolansky, *Static solutions of the Vlasov-Einstein System*, Arch. Rational Mech. Anal., 156 (2001), pp. 205–230.