THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Off-shell, maximal supersymmetry
&
exceptional geometry

ANNA KARLSSON

Department of Fundamental Physics
CHALMERS UNIVERSITY OF TECHNOLOGY
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ANNA KARLSSON
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Department of Fundamental Physics
Chalmers University of Technology
SE–412 96 Göteborg, Sweden
Phone: +46 (0)31–772 1000

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ANNA KARLSSON
Department of Fundamental Physics
Chalmers University of Technology

Abstract

Two lines of research are presented in this thesis, both with a focus on fundamental properties within the supersymmetric theories of high energy physics. The first features analyses based on the actions for maximal supersymmetric Yang–Mills theory and supergravity, provided by the pure spinor formalism. The second is the development of the theory of exceptional geometry.

The analyses within maximally supersymmetric Yang–Mills theory and supergravity, benefitting from the pure spinor formalism, centres around the investigations of the UV divergences of the amplitude diagrams, where the case of maximal supergravity is subject to ongoing research. There is currently an interesting development in connection to the four-dimensional theory, regarding a possible finiteness, indicated in the pure spinor formalism setting of the research articles appended to this thesis. The results are contrary to the expected divergence, currently of amplitude diagrams with more than six loops.

Exceptional geometry is an extension of supergravity that is constructed to incorporate U-duality. It is a geometric formulation with an extended space, in a way similar to in doubled geometry, where T-duality is the symmetry accommodated for. Constituting a rather recent area of research, the theory is under development with respect to the inherent symmetries, the tensor formalism, etc., regarding the properties affected by the extended space. Its recognised features, construction and concepts to be investigated are the objects of interest in this thesis.

Keywords: Maximal supersymmetry, maximal supergravity, pure spinors, loop amplitudes, exceptional geometry, U-duality, high energy particle physics

List of appended research papers

This thesis is based on the work contained in the following publications, referred to by Roman numerals in the text.

I. *Pure spinor superfields and Born–Infeld theory*
M. Cederwall and A. Karlsson
J. High Energy Phys. **11** (2011) 134
arXiv:1109.0809 [hep-th]

II. *Loop amplitudes in maximal supergravity with manifest supersymmetry*
M. Cederwall and A. Karlsson
J. High Energy Phys. **03** (2013) 114
arXiv:1212.5175 [hep-th]

III. *Exceptional geometry and tensor fields*
M. Cederwall, J. Edlund and A. Karlsson
J. High Energy Phys. **07** (2013) 028
arXiv:1302.6736 [hep-th]

IV. *Ultraviolet divergences in maximal supergravity from a pure spinor point of view*
A. Karlsson
J. High Energy Phys. **04** (2015) 165
arXiv:1412.5983 [hep-th]

V. *Pure spinor indications of ultraviolet finiteness in $D = 4$ maximal supergravity*
A. Karlsson
arXiv: 1506.07505 [hep-th]

Other publications

I have also written the following conference contribution, not included in this thesis:

- *Loop amplitude diagrams in manifest, maximal supergravity*
  A. Karlsson
  Springer Proc. Phys. **153** (2014) 95

Ongoing research projects are mentioned in chapter 1, 4 and 5.

Acknowledgements

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to my supervisor, Martin Cederwall, for being so inspiring and warm-hearted, and having such a great sense of humour,

to my mother and late grandmother, for believing in me,

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and to Johanna and Anton, for your helpful comments on the drafts of this thesis.

Anna

To my mother
& grandparents

Contents

Learn from yesterday, live for today, hope for tomorrow.
The important thing is not to stop questioning.

Albert Einstein

1 Introduction

Within mathematical high energy physics, there is an angle at the supersymmetric theories required for unifying quantum field theory and gravity: to examine their most fundamental properties in terms of cause and effect. Barring string theory, this leads to the maximally supersymmetric Yang–Mills theory (SYM) [1,2] and supergravity (SUGRA) [3–6]. Their formulations are tricky due to issues with keeping the symmetry present off-shell, but there is one way to construct the actions: in superspace [4, 5, 7–15], in the pure spinor formalism [16–30].

The pure spinor formalism provides a way to examine what maximal supersymmetry means to a theory. A similar situation is the object of interest in exceptional geometry [31–52], but for U-duality [53–56] instead of supersymmetry. There, SUGRA is altered to capture the dualities of the five string theories, a symmetry absent in string perturbation theory, but retained in the low energy limit SUGRA shows up in — both theories in perturbative limits of an elusive M-theory, desirable to capture.

Main points

A key feature in the work of the Papers is that of manifesting characteristics — in general symmetries, but truly any generating property or main feature — with the goal of a better understanding of the theories involved. Firstly, in taking advantage of the maximal SYM and SUGRA formalisms using pure spinors to, in a compact and relatively transparent way, approach the theories and the properties thereof, primarily in terms of the

— ultraviolet (UV) divergences in maximal SUGRA amplitude diagrams,

but as an aside, also in the context of Born–Infeld theory [57], describing string theory D-branes [58–68] and with maximal SYM in a perturbative limit [10–13, 15, 69].

Secondly, in developing the theory of exceptional geometry: SUGRA with manifest (inherent and explicit) U-duality, obtained through an extension of the spacetime geometry and the concepts connected to it.

1.1 Key results with backgrounds

The backbone of the major theme — off-shell, maximal supersymmetry — is the pure spinor formalism. It is not restricted to maximal SYM and SUGRA; in fact, it was first constructed in string theory, but its key feature is the introduction of a (ghost) spinor that encodes the type of structure displayed by the maximally supersymmetric theories. Regrettably, it is undervalued and deserving of more attention. Within the high energy physics community, it is regarded as complicated or difficult by many, at the same time as there is a general misconception of the realisation of off-shell maximal supersymmetry as practically impossible.

Off-shell, maximal SUGRA

The most surprising result of the Papers has to do with a

— controversial possibility of UV finiteness of $D = 4$ maximal SUGRA.

The issue of divergences in maximal SYM and SUGRA is due to the perturbative aspect with respect to energy which they are characterised by. The question of if the theories yield finite results, and under what circumstances, has been central to many an investigation [24, 28, 70–112]. While maximal SYM proved finite in $D \leq 4$ [79–81], maximal SUGRA has been a tougher nut to crack.

Investigations [70–75, 89, 95] have confirmed a SUGRA behaviour equal to that of SYM for low-loop amplitudes, but pointed towards a first divergence in $D = 4$ at 7 loops for the 4-point amplitude [97, 98, 101, 102], and with a subtlety, not for the higher-point ones. The result of Paper IV points towards a cut-off of the loop dependence, not only in a higher-than-seven loop sense, but also (and only) for the 7-loop 4-point amplitude.

The coincidence is striking, but in need of verification, most prominently in (or in relation to) a setting outside the pure spinor formalism. The situation has been further analysed in Paper V, in addition reproducing the general amplitude behaviour in the pure spinor setting, but the drawback of the too compact formulation remains: what displays the results hides the commonplace interpretation. It becomes difficult to establish key, controversial results as true.

Part of the matter is that there ought not be two inequivalent, consistent quantum theories with gravity. M-theory certainly is one, which implies $D = 4$ SUGRA to be ruled out. Perhaps a quotation from Niels Bohr is in place:

> *'How wonderful that we have met with a paradox.*
> *Now we have some hope of making progress.'*

The first step of which is either to verify or discard the results of Paper IV–V.

Exceptional geometry

The second interesting research area is

— extending concepts into exceptional geometry to get at U-duality.

In this, the subject of Paper III is just a small part of a much larger process. Exceptional geometry is SUGRA ($D < 11$) with a fibre bundle custom-made to accommodate the U-duality usually hidden in $D = 11$ SUGRA, but visible in terms of the string theory dualities. Due to the common origin of the theories, and the certain limits the theories amount to, the latter can be analysed in terms of the former. The process constitutes a way of capturing a property of M-theory, and in extension represents a step towards constructing that very theory.

In exceptional geometry, it is the extended space that is of interest, encoding the U-duality, and to exploit the construction the geometric approach must be formulated in terms of it. To begin with, the Lie derivative is generalised, which means that the precise meaning of the transformations of tensors must be reinterpreted, etc., in combination with many other features of the geometric formulation. This process is not yet altogether settled.

The theory constitutes a rather recent area of research, of increasing interest these last years. It is a more general version of the established doubled geometry [31, 113–145], concerning a subset of U-duality: T-duality [146–148]. Both are extended geometries.

Off-shell, maximal SYM

Last, and least, comes the example(s) applicable to off-shell maximal SYM. For the purposes of this thesis, its pure spinor setting is mostly used as a point of reference for the maximal SUGRA examinations, since it constitutes a simpler yet similar version. However, it can equally well be used for investigations within SYM, for example in relation to

— constructing the pure spinor action for the D-brane Born–Infeld theory,

which was initialised in Paper III and continued in [149,150]. Such constructions are, of course, made in the hopes of that the pure spinor formalism, through its virtues, will shed more light on the properties connected to the areas of research.

1.2 Thesis overview

With an attempt at some self-containment, this thesis will start out by introducing basic supersymmetry and superspace concepts, in chapter 2. Certain conventions regarding notation and the supersymmetry algebra are to be found there, as well as a slight discussion on the usage of flat (the setting of the pure spinor formalism) or curved space, especially in the construction of SUGRA.

In chapter 3, the pure spinor formalism is introduced. The maximal SYM and SUGRA component theories, not sporting off-shell supersymmetry, are presented with an explanation of how an extension to superspace lays the ground for the introduction of the pure spinor. How the correct, off-shell maximal supersymmetry properties are captured is detailed, as well as how the component theory shows up; a highly relevant check of consistency. Finally, the attributes of the formalism are discussed. Another extensive review is to be found in [151].

Following this, the investigations of UV divergences in maximal SYM and SUGRA are described. General procedures are set out in terms of SYM prior to a review of the maximal SUGRA examinations, including the explicit calculations for the 4-graviton amplitude, the counterterm approaches, and the pure spinor formalism studies. In addition, the results of maximal SYM are described prior to the analysis of the current situation in maximal SUGRA, with respect to the UV divergences, for a thorough understanding of the principles involved.

Chapter 5 deals with exceptional geometry, beginning with U-duality and how it is made manifest in SUGRA. The necessary reconsiderations of the geometric treatments are introduced in terms of doubled geometry, followed by the exceptional, slightly more involved, extension. Moreover, the so far established (extended) concepts within exceptional geometry are described, with some analysis regarding applications and further areas of interest.

The final chapter constitutes a short comment on the general ideas behind the type of research presented in the Papers, of the values of keeping properties manifest as far as possible and the advantages thereof. It also contains a mention of the Born–Infeld theory, so far as it will be presented.

Paper (contribution) overview

Concerning the Papers, the two last are wholly mine, as is most of Paper II, barring the initial idea and the analysis of the algebraic properties of the non-minimal variables, e.g. in connection to the projection operator in the generalised regularisation. In the first Paper, I contributed to the calculations and recognised the relations between the operators (essentially the Δ_a base), enabling the identification of the abelian action, whereas I in Paper III helped with the calculations in connection to the compatible (affine) connection, and took part in the general discussions.

1.3 Outlook

The two main lines of research presented in this thesis, the investigations of UV divergences in maximal SUGRA and the development of exceptional geometry, are very much subject to continued research. The former needs a consensus with regard to what happens in four dimensions. The latter is at a relatively initial stage, with every possibility of providing new information on M-theory characteristics, etc.; the investigations of different properties that currently are the objects of interest in the pure spinor formalisms remain to be addressed.

Finally, it is important not to forget the advantages of manifest symmetries, or constructions which to various degrees aim at inherent and explicit formulations, thereby facilitating research in general.

It is because simplicity and vastness are both beautiful that we seek by preference simple facts and vast facts; that we take delight, now in following the giant courses of the stars, now in scrutinising with the microscope that prodigious smallness which is also a vastness, and now in seeking in geological ages the traces of a past that attracts us because of its remoteness.

Henri Poincaré

2 Supersymmetry basics

The incompatibility of quantum field theory and gravity in combination with the desire for one fundamental theory for both, or rather the physics of the universe, calls for extra symmetries in the former. Although the basic principles of quantum field theory have been discussed in terms of consistency in relation to the firewall paradox of black holes, the theory remains our best description of the so far observed physics, excepting gravity. Despite the mathematical beauty called for in the Standard Model of particle physics, and issues with dark matter etc., the theory predicts experimental results uncannily well. Nevertheless, in the search for an extension, the most promising candidate is supersymmetry — also known from string theory, where gravity is caused by the string dynamics.

In the setting of this thesis, the original field theory properties are retained and central to the description. It is not necessary to limit the investigations to fields — other options include e.g. strings and particles — but a field theory description is both exhaustive and the natural choice of an extension.

2.1 Supersymmetry

Supersymmetry [152–154] is an additional symmetry of quantum field theory, incorporating fermionic symmetry generators transforming under the Lorentz algebra. Together with conformal[1] symmetry, it represents the only spacetime symmetry compatible with quantum field theory and the symmetries thereof:

[1]Invariant under local, angle-preserving transformations, e.g. including scale invariance.

the translations[2] (P_m) and Lorentz transformations (M_{mn}). Any other extension (excepting internal symmetries in trivial combinations with the spacetime symmetries) fails to give a result consistent with Lorentz invariance, as stated by the Coleman–Mandula theorem [155]. An attempt at extending quantum field theory therefore naturally includes the introduction of supersymmetry.

The supersymmetry operators (Q_α) open up the possibility to alter the spin and the statistics of a field. For example, an operator acting on some field ψ with spin j results in the following:

$$Q_\alpha \psi_j \rightsquigarrow \psi'_{j\pm\frac{1}{2}}. \tag{2.1}$$

The description is that of how to turn fermions and bosons into one another, relating matter particles (fermions) to force particles (bosons), in particular within certain sets of superpartners. Through the introduction of supersymmetry in quantum field theory, each particle of the original theory is assigned a superpartner, representing the other half in the pair within which the supertransformations take place.

These properties still remain to be observed in nature; the predictions of at which energy levels their presence might be detectable have yet to be verified. There are hopes for positive results in relation to the experiments at the Large Hadron Collider (LHC) at CERN, Switzerland, where the presence of supersymmetry e.g. would result in additional Brout–Englert–Higgs (BEH) bosons [156–158] to the one so far observed [159,160]. However, it is uncertain whether it is possible to find superparticles at the energy levels of the LHC, or if they would be recognised by the experiments, if indeed present. Regardless, the construction is of theoretical value, although the question has been raised of its practical use if, indeed, the energy levels at which it occurs are too high.

2.1.1 The supersymmetry algebra

The conjecture of a symmetry between fermions and bosons includes the symmetry being compatible with the symmetries of the original theory, and the operators of it obeying an algebra compatible with the same. The anticommutators of the fermionic creation (Q_α) and annihilation ($Q^\dagger_\beta$) operators, raising and lowering the spin, are required to give results in terms of the symmetry operators of the theory. This is usually expressed in terms of flat space, where the anticommutator of two supersymmetry operators must be expressible in the original Poincaré algebra. This leaves but one choice of an algebra, up to a constant, the only non-zero algebra component (among the anticommutators) of which is:

$$\{Q_\alpha, Q^\dagger_\beta\} = c\gamma^a{}_{\alpha\beta}\partial_a, \quad c \in \mathbb{C}, \tag{2.2}$$

[2]Note that in the conventions of this thesis and the Papers, bosonic indices are denoted by minuscules and fermionic by greek minuscules. $(m \ldots, \mu \ldots)$ denote curved indices while the corresponding for $(a \ldots, \alpha \ldots)$ is flat.

with ∂_a denoting the momenta, equivalently replaceable with P_a, in total describing a general coordinate transformation. The convention in this thesis is $c = 2$, although the variant of $2i$ is common outside the context of section 3.3, effectively that of Paper I,II and IV. In this setting of a real algebra, $Q^\dagger_\beta \equiv Q_\beta$.

The ensuing, total algebra goes by the name of super-Poincaré. When it holds, it includes both supersymmetry and Lorentz symmetry. Moreover, under certain conditions it can be extended to include several sets of supersymmetry operators, obeying the supersymmetry algebra within each set (Q^I_α) while opening up for non-trivial relations otherwise. The algebra is then termed to be $\mathcal{N}$-extended, with $\mathcal{N}$ denoting the number of sets of supersymmetry operators.

Supersymmetry in curved space

A general setting of a supersymmetric theory is not restricted to flat space, as the supersymmetry algebra given in the previous paragraph. Yet the discussion of this thesis will be limited to just that, when not treating the $D = 11$ maximal SUGRA component theory in a general setting; that is where the pure spinor formalisms are formulated.

For SYM, this is a common choice. The dynamics does not depend explicitly on the background, which merely enters in terms of the geometry, and so it is suitable to investigate the perturbation theory in terms of flat space. Also, as for most investigations, examinations with respect to supersymmetry are most easily performed in the absence of curvature, and a good understanding of the flat case is required for any attempt at a more extensive description. In curved space, the known formulations containing supersymmetry are limited to certain backgrounds.

For SUGRA, the choice of a certain background is limiting; the gravity fluctuations are central to the theory. However, the pure spinor formulation ought to be of value and feasible for other backgrounds. It is possible to note that the restriction, in terms of the flat background of the pure spinor formalism, really only concerns the topology — the *diffeomorphisms*[3] are present in the formulation. By their presence, the fields deform the geometry to include fluctuations, despite the appearance of a flat background. Effectively, a background invariance of the theory is implied, much like in string perturbation theory in flat space, where there exist fluctuations in the perturbative degrees of freedom. However, the diffeomorphism symmetry in the SUGRA pure spinor formalism is comparable to the maximal supersymmetry in that it is not quite manifest, compare section 3.4.4. Both exist in terms of *cohomology*[4], and while the diffeomorphisms seem to include what is necessary for background invari-

[3]Invertible function between smooth manifolds, with f and f^{-1} smooth: preserving the differentiable structure.

[4]For an explanation of this concept, turn to section 3.4.4.

ance, strictly manifest diffeomorphisms (the geometrical interpretation in superspace) have to be identified for the general statement.

Anyway, for the purposes of the Papers and this thesis, flat space is an excellent setting for investigations.

2.2 Superspace

The introduction of the supersymmetry algebra by default extends the bosonic, curved spacetime (x^m) with fermionic components (θ^μ), the total of which represents superspace, described by $z^M = (x^m, \theta^\mu)$. The spinors of this space are governed by the Clifford algebra, spanned by the irreducible representations of the relevant Lie algebra, and a convenient notation for the tensors is that of differential forms. The following section consists of a selection of superspace concepts of value to the discussion in the following chapters.

The Clifford algebra and additional properties of the spinors

In D dimensions, the fermionic space consists of $2^{\lfloor D/2 \rfloor}$ Dirac spinor components, with the exponent rounded off to the closest (lower) integer and $\lceil x \rceil$ instead referring to the *higher* equivalent. Additional constraints are present in some dimensions, sporting Majorana and/or Weyl spinors, reducing the number of components further.

The algebra of the spinors is that of the Clifford algebra, associated with the Lorentz group, where the generating elements are the Dirac matrices γ^m:

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}\mathbb{1}, \tag{2.3}$$

with two (free) spinor indices implicit. In general, the algebra is complex, but in a Majorana setting, valid in $D = 2$, 3, 4 mod 8, it can be described in terms of real components. It is then possible to set the spinors to be real, due to the equivalence of the descriptions.

Likewise, in even dimensions the Weyl representation is valid, splitting the Dirac matrices into two off-diagonal blocks described by the Weyl matrices σ^m and $\bar{\sigma}^m$. The algebra then splits into two (in some sense) equivalent sets, resulting in the presence of two chiralities of the spinors, presenting dual descriptions where one may be chosen consistently. Effectively, the degrees of freedom of the spinors are reduced by half. However, the change of matrices in the formulation, from Dirac to Weyl, will remain implicit in this thesis. Where applicable, γ denotes σ.

Moreover, the Lie algebra is described by up to $\lfloor D/2 \rfloor$ antisymmetrised (Dirac or Weyl) matrices, and only half of the $\lfloor D/2 \rfloor$-antisymmetrisation if D is even, e.g. presenting the way for how tensors are expanded in θ^μ. The antisymmetrisations of $(D-p)$ and p matrices are dual with respect to the encoded degrees of freedom, with the presence/absence of the p indices filling the same function.

2.2.1 Differential forms

A useful notation of vectors, in various mathematical contexts, is that of differential forms. They are usually defined in a purely bosonic setting, but the concepts can equally well be extended to fermions and used for objects on superspace. However, when not otherwise specified, the bosonic form is what is intended.

A differential form is (implicitly) coupled to the basis elements of the space under consideration, which encode the properties of the form. In superspace, with the local coordinates $z^M = (x^m, \theta^\mu)$, a general p-form is described by

$$\Omega_{(p)} = \frac{1}{p!} \mathrm{d}z^{M_p} \wedge \mathrm{d}z^{M_{p-1}} \wedge \ldots \wedge \mathrm{d}z^{M_1} \Omega_{M_1 \ldots M_p}, \tag{2.4}$$

where the form indices can be replaced by $[M_1 \ldots M_p)$ to explicitly denote the symmetrisation of the bosonic and fermionic indices: antisymmetrisation between all indices except pairs of fermionic ones, which are symmetrised. This follows from the properties of the wedge product:

$$\mathrm{d}z^M \wedge \mathrm{d}z^N = \begin{cases} \mathrm{d}z^N \wedge \mathrm{d}z^M , & (M,N) = (\mu, \eta) \\ -\mathrm{d}z^N \wedge \mathrm{d}z^M , & \text{otherwise} \end{cases} . \tag{2.5}$$

It also represents a choice in that it sets the convention for what is incorporated in the (anti-) symmetrisation of the tensor indices to

$$A_{[m} B_{n]} = \frac{1}{2}(A_m B_n - A_n B_m). \tag{2.6}$$

A quantity Ω_{mn} implicitly has the indices antisymmetrised unless otherwise specified, such as for the contraction of indices through δ_{ab} and η_{ab}, where the indices are symmetrised. In addition, $*\Omega_{(p)}$ denotes the dual $(D-p)$-form.

The product of two forms is

$$\Omega_{(p)} \wedge \Omega'_{(q)} = \frac{1}{(p+q)!} \mathrm{d}z^{M_p} \ldots \mathrm{d}z^{M_1} \wedge \mathrm{d}z^{N_q} \ldots \mathrm{d}z^{N_1} \binom{p+q}{p} \Omega'_{[N_1 \ldots N_q} \Omega_{M_1 \ldots M_p)}, \tag{2.7}$$

whereas contractions of indices are described by the interior product, for a vector:

$$\imath_X \Omega_{(p)} = \frac{1}{(p-1)!} \mathrm{d}z^{M_p} \wedge \ldots \wedge \mathrm{d}z^{M_2} X^{M_1} \Omega_{[M_1 \ldots M_p)}. \tag{2.8}$$

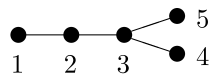
For example, the exterior derivative acts according to

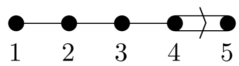
$$d\Omega_{(p)} = \Omega_{(p)} \wedge \underleftarrow{d}\,. \tag{2.9}$$

The ordering can however be simplified in a purely bosonic setting, due to the absence of fermionic quantities in the forms.

2.2.2 Irreducible representations

As to the Lie algebra, the most convenient way to denote the irreducible representations of the Lorentz group is by the highest weight Dynkin labels. The standard enumeration is described by the Dynkin diagrams, e.g. [161]

$D = 10, \quad D_5:$  (2.10a)

$D = 11, \quad B_5:$  (2.10b)

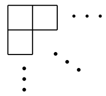
both of which have Dynkin indices presented by a vector with five entries. There, the first entry represents the number of constituent (bosonic) 1-forms (symmetrised, but not contracted) etc. up to the third and fourth entry, respectively. In $D = 10$ (00011) denotes a 4-form, because the fourth and fifth entries correspond to the two different spinor chiralities, of which there is only one in $D = 11$. In eq. (2.10a), the Weyl spinors are clearly visible. The 5-form is denoted by the value of 2 at the position of the spinor (of the chosen chirality). For higher forms, the duality of n-forms and $(D-n)$-forms needs to be taken into consideration. At most one spinor index is free; pairs are interpreted in terms of projections into n-forms. In total, the consideration of constellations of indices constitutes Lie group computations, with rearrangements of the spinor indices given by the Fierz identities. For further details on that, look to the appendices of the Papers.

Lie algebras & groups

Implicit in all this is that the symmetries are described in terms of Lie groups and Lie algebras, the properties of which are presented in e.g. [161, 162]. A detailed analysis will not be performed here; but for the central concepts, look to chapter 5. Due to the theme of that chapter — the extension of spacetime and effectively a reformulation of the ordinary Lie algebra — e.g. the Lie derivative is best presented there, in relation to the change it is subject to in extended geometry.

Young tableaux

A alternative way of displaying the form properties is through Young tableaux



where each square denotes a bosonic index, of which horizontal rows denote symmetrisation and vertical antisymmetrisation. In the absence of the dots, the Young tableau above corresponds to (11000) in terms of eq. (2.10). However, this does not capture the presence of a free spinor index.

2.2.3 Torsion & curvature

While spacetime is described by a metric, the concept fails to strictly translate into superspace. However, the superspace vielbein $E_M{}^A(z^M)$ provides mappings between different local reference frames, just as in spacetime, so that the concepts of spacetime carry over to superspace, with an extension to the superspace indices. The analogy holds, e.g. for the central concept of torsion.

Parallel transport along a curve may twist an initial reference frame. How is defined by the torsion, given by the covariant exterior derivative acting on the vielbein:

$$T^A \equiv DE^A = dE^A + E^B \wedge \Omega_B{}^A, \tag{2.11a}$$

with the 1-form spin connection Ω encoding curvature.

In flat space, this is reduced to

$$T^A = dE^A \Leftrightarrow [\partial_A, \partial_B\} = -T_{AB}{}^C \partial_C : \quad T_{ab}{}^C = 0, \tag{2.11b}$$

where the derivatives are required to be covariant. To fulfil this, the spinor derivative must be altered, a procedure which will be addressed in the next chapter. To be precise, the flat space is only required to have zero torsion in the above specified sense.

The parallel transport is equally described by the affine connection Γ, showing up in the covariant derivative (in a geometric setting) as

$$D_M \sim \partial_M + \Gamma_M, \tag{2.12}$$

with the affine connection representing a matrix $(\Gamma_M)_N{}^P$, where one of the matrix indices gets contracted with the tensor the derivative acts on. As in ordinary geometry, the curvature can be derived from this entity (Γ) as well as Ω: [163]

$$R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C \wedge \Omega_C{}^B. \tag{2.13}$$

2.3 Supersymmetric gravity & Yang–Mills theories

At the introduction of supersymmetry, and hence the presence of spinors, the supersymmetry operators play a central role. The number of them that act non-trivially on states differ between theories. For the object of interest in this thesis, maximal supersymmetry, the states are massless, obeying some Bogomol'nyi–Prasad–Sommerfield (BPS) condition, which means that the type of multiplet under consideration is short (BPS multiplets). Only half of the supersymmetry operators are relevant for the supersymmetry transformations.

In this setting, the number of non-trivial supersymmetry operators equals the degrees of freedom of the spinors (obeying the Dirac equation): half of the initial amount of spinors (n_s) set by the Clifford algebra [164]. For consistency in the formation of superpartner pairs, the degrees of freedom of the bosons furthermore must amount to the same number, a circumstance restricting the type of theory under consideration to certain dimensions. In specific, this requirement — if fulfilled *on-shell*[5] — also holds off-shell provided the supersymmetry algebra holds likewise. In Yang–Mills theory, with $D = (d + 1)$, the bosonic degrees of freedom are given by $(D - 2)$, while the corresponding count in gravity is slightly more involved.

Regarding the maximal change of spin obtainable through the application of the supersymmetry operators acting on a field or a particle, it is a quarter of the non-trivial number of supersymmetry operators ($n_s/8$), as half of that number is constituted by creation operators, and the rest by annihilation operators, each yielding a change in spin of $1/2$. Depending on the dimension and the maximal spin of the theory under consideration, a different number of sets of supersymmetry operators is required in order to fully capture the possible transformations of the supersymmetrisation of the theory.

In Yang–Mills theory, constituting a gauge theory, the maximal spin is 1, given by the presence of the photon. In the presence of gravity, however, the maximal spin is 2, due to the graviton. Hence, the former theory is spanned by 4 creation and 4 annihilation operators, whereas the equivalent for the latter is 8, as illustrated in fig. 2.1. No more supersymmetry operators can be fitted into the theories.

2.3.1 $\mathcal{N}$-extended supersymmetry theories

In many dimensions, there is room for more than one set of supersymmetry operators in the theory, in total not surmounting the upper limit on their number, set by the maximal spin. If a supersymmetry theory is extended in this way, it is termed $\mathcal{N}$-extended. For example $D = 4$ SUGRA can have up to $\mathcal{N} = 8$, with

[5] A theory is termed on-shell if the equations of motion are fulfilled, as opposed to off-shell.

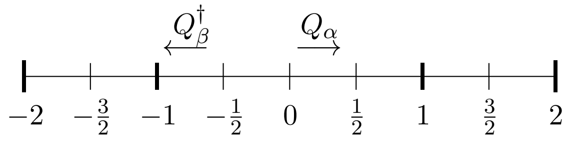


Figure 2.1: Spin step illustration. Effectively, there exists one creation and annihilation operator for each step in spin when the theory is fully supersymmetric in the sense that all possible spin states can be reached. The amount required differs between theories of different maximal spin: Yang–Mills theory has a maximal spin of 1 and requires 8 operators to capture the full transition between the spin states of -1 and 1. Gravity includes a maximal spin of 2 and a corresponding set of 16 supersymmetry operators.

the $\mathcal{N}$s denoting different theories. At the point of $\mathcal{N} = 8$, the number of supersymmetry operators equals the maximum of 16 and maximal supersymmetry is obtained. Importantly, not only does $\mathcal{N} = 8$ in the sense of presenting maximal supersymmetry constitute the maximal value of $\mathcal{N}$, but covers the full set of spin transitions in the theory.

Maximal, $\mathcal{N} = 1$ supersymmetry

Under certain conditions on the dimension of a theory, maximal supersymmetry is obtained with a single set of supersymmetry operators. These theories, with $D = 10$ in the case of Yang–Mills theory and $D = 11$ for SUGRA, are especially interesting as they 'encode' lower dimensional, $\mathcal{N}$-extended supersymmetric theories, for all $\mathcal{N}$, in that the dimensions can be compactified and, through further procedures, the ensuing extended supersymmetry broken. The (overall) properties observed in $\mathcal{N} = 1$ maximally supersymmetric theories therefore in some restricted senses (unbroken supersymmetry: in general) limit what may be valid for those supersymmetric theories, e.g. in the sense of providing perturbatively finite theories, see chapter 4. By extension, they also set the actual properties, although that requires extensive analyses. This makes them especially interesting for diverse investigations, even more than what is due to the crucial simplifications associated with maximal supersymmetry, despite the fact that only broken supersymmetry is expected to be observed in nature.

It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the ensemble and the details.

Henri Poincaré

3 Maximal supersymmetry

The formulation of an action in gravity or Yang–Mills theory, in the presence of maximal supersymmetry and with that symmetry as an inherent part of the formalism, at present requires *pure spinors*. Without the introduction of such spinors the theories formed out of the corresponding field contents are represented by (component) actions giving rise to supersymmetry variations that only close on-shell.

This situation is unique to the maximally supersymmetric theories. The following chapter describes the formulation of the actions for maximal $\mathcal{N} = 1$ SYM and SUGRA, beginning with the properties of the component actions, the desired equations of motion and how the on-shell theories are described in a superspace formalism. It continues with how the pure spinor formalism is tailor-made to fit the original, flat theories while allowing for off-shell maximal supersymmetry. Finally, the actions in the pure spinor formalisms are described with respect to their inherent parts and distinctive features, including how the characteristics correspond to those of the original theories, where applicable (on-shell).

In total, the pure spinor formalism enables a powerful formulation, encoding theories of lower dimension. In some respects, the identification of properties inherent to the theory are facilitated. However, the dense formulation, where component fields are reduced to parts of a series expansion of a single superfield, can be difficult to decipher in terms of the conventional component field theory approach. In addition, the inherent maximal supersymmetry is perhaps not to be termed manifest in SUGRA, as it is present as a part of the cohomology rather than explicitly, see section 3.4.4.

3.1 The component, maximally supersymmetric theories

The component actions of $\mathcal{N} = 1$ maximal SYM [1,2] and SUGRA [3–5] are

$$S = \frac{1}{g_{YM}^2} \int \mathrm{d}^{10}x \, \mathrm{tr}\Big(-\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} \psi \gamma^a D_a \psi \Big), \tag{3.1a}$$

$$\begin{aligned} S =& \frac{1}{2\kappa^2} \int \mathrm{d}^{11}x \sqrt{-g} \Big[R(\omega) - \frac{1}{24} H^{mnpq} H_{mnpq} - \psi_m \gamma^{mnp} D_n \Big(\frac{\omega + \hat{\omega}}{2} \Big) \psi_p - \\ & - \frac{\sqrt{2}}{96} (\psi_n \gamma^{abcdnp} \psi_p + 12 \psi^a \gamma^{bc} \psi^d)(H + \hat{H})_{abcd} \Big] - \frac{\sqrt{2}}{3\kappa^2} \int C \wedge H \wedge H. \end{aligned} \tag{3.1b}$$

The first action, describing SYM, contains terms representing the Maxwell Lagrangian and the Lagrangian for a massless Majorna–Weyl spinor, in the presence of the gauge coupling parameter[1] g_{YM} of classical Yang–Mills theory. In addition, the Lie algebra indices, with respect to which the trace is taken, are implicit.

The second action, describing SUGRA, contains the terms belonging to a graviton, a 3-form C_{mnp} and a gravitino (the first three terms), with the gravitational constant κ, followed by terms necessary for the supersymmetry to be present. Depending on the conventional constant in eq. (2.2) the end result varies slightly, e.g. compare with [3,165]. The present description follows from a constant of 2, which is used in the pure spinor formalism.

Moreover, in contrast to theories with less supersymmetry, these actions cannot be altered through an introduction of auxiliary fields G in the shape of additional terms $G^\dagger G$, which constitutes a frequent procedure to achieve off-shell supersymmetry. In maximally supersymmetric theories, nothing can represent such a field.

The field contents of the theories are

$$\begin{aligned} SYM: \;& A_a, \psi^\alpha && \text{(gauge field, spinor)}, \\ SUGRA: \;& g_{mn}, \psi_m^\alpha, C_{mnp} && \text{(graviton, gravitino, 3-form)}, \end{aligned} \tag{3.2}$$

where the SYM fields take values in the Lie algebra. Of these, g_{mn} can be represented by the vielbein $e_m{}^a$ since

$$g_{mn} = e_m{}^a e_n{}^b \eta_{ab}. \tag{3.3}$$

The remaining fields, such as the field strengths $F = dA + A \wedge A$ and $H = dC$, as well as the Ricci scalar, are constructed out of these fields. For example in SYM,

[1] The fields can also be set to incorporate g_{YM} and its equivalent in SUGRA; a reinterpretation which will not be performed here, or in relation to the pure spinor formalism.

the additional components in the action are

$$\begin{aligned} D_a &= \partial_a + A_a, \\ F^I_{ab} &= 2\partial_{[a}A^I_{b]} + ([A_a, A_b])^I = 2\partial_{[a}A^I_{b]} + f^I{}_{JK}A^J_a A^K_b, \end{aligned} \tag{3.4a}$$

where $f^I{}_{JK}$ are the structure constants of the Lie algebra[2]. In SUGRA, the covariant derivative in contrast incorporates the Lorentz connection $\omega(e)$, and the short notation used in the action is

$$\begin{aligned} \omega_m{}^{ab} &= \omega_m{}^{ab}(e) + K_m{}^{ab}, \\ K_{mab} &= -(\psi_m\gamma_b\psi_a - \psi_a\gamma_m\psi_b + \psi_b\gamma_a\psi_m) + \frac{1}{2}\psi_n\gamma_{mab}{}^{np}\psi_p, \\ \hat{\omega}_{mab} &= \omega_{mab} - \frac{1}{2}\psi_n\gamma_{mab}{}^{np}\psi_p, \\ \hat{H}_{mnpq} &= H_{mnpq} + 3\sqrt{2}\psi_{[m}\gamma_{np}\psi_{q]}. \end{aligned} \tag{3.4b}$$

The hatted quantities are supercovariant, i.e. at a local supersymmetry variation of the fields, they do not contain a derivative ∂_m acting on the supersymmetry parameter; compare to eq. (3.6b).

The equations of motion of the SYM theory is

$$\begin{aligned} D_a F^{ab} - \psi\gamma^b\psi &= 0, \\ (\not{D}\psi)_\alpha &= 0, \end{aligned} \tag{3.5a}$$

easily obtainable at a variation of the fields. The first equation corresponds to δA_a: the equation of motion for the gauge field; the latter to $\delta\psi_\alpha$. The SUGRA equivalent for the graviton, the gravitino and the 3-form (in that order) is

$$\begin{aligned} &R_{mn}(\hat{\omega}) - \frac{1}{2}g_{mn}R(\hat{\omega}) = \frac{1}{24}(4\hat{H}_{mabc}\hat{H}_n{}^{abc} - \frac{1}{2}g_{mn}\hat{H}_{abcd}\hat{H}^{abcd}), \\ &\gamma^{mnp}D_n(\hat{\omega})\psi_p - \frac{\sqrt{2}}{144}\gamma^{mnp}(\gamma_n{}^{abcd} - 8\eta^a_n\gamma^{bcd})\psi_p\hat{H}_{abcd} = 0, \\ &D_m(\hat{\omega})\hat{H}^{mnpq} - \frac{\sqrt{2}}{4!4!}\varepsilon^{npqabcdijkl}\hat{H}_{abcd}\hat{H}_{ijkl} = 0, \end{aligned} \tag{3.5b}$$

although the ones most frequently encountered are the bosonic parts:

$$\begin{aligned} &R_{mn} - \frac{1}{2}g_{mn}R = \frac{1}{24}(4H_{mabc}H_n{}^{abc} - \frac{1}{2}g_{mn}H^2), \\ &* \imath_d H - \sqrt{2}H \wedge H = 0. \end{aligned}$$

[2]The Lie algebra indices will be kept implicit from here on.

3.1.1 The supersymmetry algebra of the component theory

The supersymmetry transformations turning bosonic and fermionic fields into one another, essentially $\delta_\epsilon \sim \epsilon^\alpha Q_\alpha$, are limited to certain expressions by dimensional properties, spin and statistics. Typically, the dimension of the variable x is set to -1, so that the field strength has dimension 2 in SYM. With the same convention, the dimension of C is set to zero in SUGRA, at which point the dimensions of the other components follow from the action being dimensionless. For example, an ordinary spinor variable such as ϵ^α is fermionic and of dimension $-1/2$. Moreover, the action being invariant under the supersymmetry variations yields that they in SYM are restricted to the global transformations of

$$\begin{aligned} \delta_\epsilon A_a &= \epsilon\gamma_a\psi, \\ \delta_\epsilon \psi^\alpha &= \frac{1}{2}F_{ab}(\gamma^{ab}\epsilon)^\alpha, \end{aligned} \tag{3.6a}$$

whereas in the case of SUGRA, they are *local*: [3–5]

$$\begin{aligned} \delta_\epsilon e_m{}^a &= 2\epsilon\gamma^a\psi_m, \\ \delta_\epsilon \psi_m^\alpha &= D_m(\hat\omega)\epsilon + \frac{\sqrt{2}}{144}(\gamma_m{}^{abcd} - 8\eta_m^a\gamma^{bcd})\hat H_{abcd}\epsilon, \\ \delta_\epsilon C_{mnp} &= -\frac{3}{\sqrt{2}}\epsilon\gamma_{[mn}\psi_{p]}. \end{aligned} \tag{3.6b}$$

Here, it is possible to note that the identification $\delta_\epsilon\psi_m^\alpha = \hat D_m(\hat\omega)\psi_m^\alpha$ does not match with an identification of $(\gamma^{mnp}\hat D_n(\hat\omega)\psi_p)^\alpha = 0$ for the equation of motion for the gravitino, as is the case in [3]. For that, the coefficient in front of $\hat H$ in $\delta_\epsilon\psi_m^\alpha$ has to be purely imaginary.

The ensuing supersymmetry algebra is modulo gauge transformations, in the case of SYM the local Lorentz transformation $\Lambda \sim (\epsilon\gamma^a\epsilon')A_a$, with

$$\begin{aligned} [\delta_\epsilon, \delta_{\epsilon'}]A_a &= -2(\epsilon\gamma^b\epsilon')\partial_b A_a + D_a\Lambda, \\ [\delta_\epsilon, \delta_{\epsilon'}]\psi^\alpha &= -2(\epsilon\gamma^a\epsilon')\partial_a\psi^\alpha + [\psi^\alpha, \Lambda] + f(\epsilon,\epsilon')^{\alpha\beta}(\not{D}\psi)_\beta, \end{aligned} \tag{3.7a}$$

where f is some function and δ_ϵ has been set to $(\epsilon Q) : [\delta_\epsilon, \delta_{\epsilon'}] = -\epsilon^\alpha\epsilon'^\beta\{Q_\alpha, Q_\beta\}$. The first term in each can be reinterpreted in terms of the Lie derivative, compare to eq. (5.4). In SUGRA, $(\gamma_m)_{\alpha\beta} = e_m{}^a(\gamma_a)_{\alpha\beta}$ needs to be identified throughout the variations and the gauge transformation is Λ_{ab}. The algebra itself is slightly more complicated than in SYM, but follows the same pattern of

$$[\delta_\epsilon, \delta_{\epsilon'}] \sim \mathcal{L}_{-2(\epsilon\gamma^a\epsilon')} + \delta_\Lambda + \delta_{\text{EOM}}, \tag{3.7b}$$

where the equations of motion enter only for, and in relation to, the gravitino. Importantly, the supersymmetry algebras only close on-shell, since the form displayed in eq. (2.2) only is achieved when eq. (3.5) is fulfilled.

3.2 The superspace constructions of the on-shell theories

The failure of the component action (SYM or SUGRA) to describe off-shell, maximal supersymmetry leaves one possible way to achieve that very thing: a superspace formulation of the theory. In order to treat the fermions and bosons involved in the supersymmetry equally, a spinor variable θ^α (of dimension $-1/2$) is included alongside x, and the fields are generalised to also carry spinor indices. To capture the relevant component theory, the general fields in it (that may be present in superspace) are considered and examined. With respect to their properties, it is possible to provide a formulation describing the physics of the component formalism, as will be outlined here. The equations of motion correspond to those of eq. (3.5) with the addition of one, which when imposed brings about the rest through the Bianchi identities (BIs).

The procedure furnishes an on-shell theory equivalent to that of the (corresponding) component action, and in SYM the maximal supersymmetry is respected (present off-shell), but no action has been possible to construct in either theory. The formulation thus remains incomplete; in examinations of the maximally supersymmetric theories, not only the presence of maximal supersymmetry is crucial for a full analysis, but also the existence of an action respecting the maximal super-Poincaré symmetry. However, in contrast to the component theory, the superspace formulation can be used as a starting point for the construction of a theory defined by an action respecting maximal super-Poincaré symmetry; that of the pure spinor formalism, which is the subject of the next section.

The superspace extension

The superspace formulation considers all possible fields belonging to a theory, so that the 1-form in SYM is $A_A = (A_a, A_\alpha)$, the 2-form F_{AB}, etc. The fields are dependent on the superspace variables $z^M = (x^m, \theta^\mu)$ and the symmetry accommodated for by default is that of supersymmetry. In flat space, the form

$$Q_\alpha = \partial_\alpha + (\gamma^a\theta)_\alpha \partial_a : \quad \{Q_\alpha, Q_\beta\} = 2\gamma^a_{\alpha\beta}\partial_a \tag{3.8}$$

of the supersymmetry operator *sets* the algebra to that of eq. (2.2), in contrast to what is true in the component theory, where the result of Q_α acting on a field is set by what can be expressed in terms of the fields of the theory. In superspace, the maximal supersymmetry algebra always holds.

The superfield formulation has a few fundamental issues to begin with:

1. The spinor derivative does not map superfields into dittos: $\{Q_\alpha, \partial_\alpha\} \neq 0$.
2. A redundancy of field components. For example, there are two different 1-forms in SYM. The situation in SUGRA is similar, with multiple fields

filling the same function. The extra fields need to be removed for the theory not to be degenerate, i.e. constraints need to be introduced.

3. The BIs must be respected by the changes brought about by (2).

4. The equations of motion must match the component theory. In this, an alternative object of interest is the abelian[3] theory. Although incomplete, it represents the starting point for the pure spinor formalism.

The first issue is remedied by replacing the spinor derivative ∂_α (except for in Q_α) with a covariant spinor derivative, D_α:

$$D_\alpha = \partial_\alpha - (\gamma^a\theta)_\alpha\partial_a : \quad \{Q_\alpha, D_\beta\} = 0, \quad \{D_\alpha, D_\beta\} = -2(\gamma^a)_{\alpha\beta}\partial_a, \tag{3.9}$$

where the last entity in flat space sets the only non-zero torsion as specified by eq. (2.11b).

The second one leads to a search for constraints on the fields compatible with the third point. In particular, one such constraint (in SUGRA, a set of constraints) shows up which in the absence of curvature sets the BIs to be equivalent to the equations of motion of the flat component theory of SYM and linearised SUGRA. In addition, a second, separate set of constraints in SUGRA likewise gives the full component theory in curved space. Each condition can be identified to compose of two parts; one necessary for the BIs to hold, the other interpretable as an equation of motion for one of the superfields unique to superspace. It represents the only equation of motion necessary to consider, as the BIs in its presence give rise to the rest.

3.2.1 The SYM superspace

In SYM, maximal supersymmetry in the $\mathcal{N} = 1$ setting takes place in a $D = 10$ theory with Majorana-Weyl spinors. Consequently, the spinor space is split into two dual chiralities with 16 degrees of freedom each, with Dirac matrices of up to five antisymmetrised indices (and only half of the 5-form projection) as a valid basis of the Lie algebra. The superspace is described by $z^M = (x^m, \theta^\mu)$, and the subsequent theory was recognised in [7,8,12].

The gauge theory has the gauge field $A = E^A A_A$ ($E^A = \mathrm{d}z^M E_M{}^A$) and its 2-form field strength F, with the BI $DF = 0$, and is typically examined in flat space, compare to section 2.1.1. Moreover, the description is degenerate as it contains two 1-forms, A_a and the A'_a in the series expansion of A_α:

$$A_\alpha = c_\alpha + (\theta\gamma^a)_\alpha A'_a + \dots, \tag{3.10}$$

one of which needs to be eliminated (expressed in terms of the other). The best way to attempt this while respecting the BIs is through a condition on F, the part

[3]The free theory, also called linearised: no interactions present.

which ought to be unphysical in the sense of the component theory aimed at: that of dimension 1. Except for A_a, there are no physical fields of that dimension, and A_a does not count as it does not transform homogeneously under Lorentz transformations. So $F_{\alpha\beta}$ better be zero.

As it turns out,

$$F_{\alpha\beta} = 0 \tag{3.11a}$$

in flat space not only respects the BIs, but ensures that they make up the equations of motion. That of dimension $3/2$ is trivial, 2 describes the existence of a spinor field, $5/2$ the equation (3.5a) of motion of that field and 3 the BI of the 2-dimensional field strength. Another covariant (spinor) derivative acting on the spinor equation of motion gives the field equation for the vector field, a procedure which when repeated yields a pattern: the result proportional to the equations of motion, alternating, and any other information trivial. In this way, the condition of eq. (3.11a) in flat superspace exactly corresponds to formulating the on-shell component theory.

However, a softer condition, the *conventional constraint* [12, 13]

$$(\gamma^a)^{\alpha\beta} F_{\alpha\beta} = 0 \tag{3.11b}$$

is consistent with the BIs. It sets A_a to be a function of A_α. Moreover, it defines the full, flat superspace theory, with interactions, to consist of A_α and z^A, with the equation of motion and subsequent gauge invariance:

$$(\gamma^5)^{\alpha\beta} D_{(\alpha} A_{\beta)} = 0, \quad \delta_\Lambda A_\alpha \overset{(3.9)}{=} D_\alpha \Lambda. \tag{3.12}$$

The flat superspace formulation thus (through the BIs) coincides with the component theory when the equation of motion is fulfilled, corresponding to the part of the theory desirable to describe that is known to be correct. Furthermore, global maximal supersymmetry is present as superfields transform into superfields; it is given by Killing vectors on the background superspace. The problem is that it is unclear how to formulate an action encoding the structure without extending the superspace with the pure spinor.

3.2.2 The SUGRA superspace

In SUGRA, maximal supersymmetry in the $\mathcal{N} = 1$ setting occurs in $D = 11$. The theory has symplectic spinors, making it convenient to (in flat space) consider all spinor indices to be subscripts, and contracted through the antisymmetric $\varepsilon^{\alpha\beta}$. Order is of vital importance[4]. There are 32 fermionic degrees of freedom

[4] In this setting, a raised index (without further notation) strictly does not make any sense. The graviton ψ_m^α is best interpreted as $\psi_{m,\alpha}$, with $\psi_m{}^\beta$ (connecting to the right) equal to $\psi_{m,\alpha}\varepsilon^{\alpha\beta}$. However, in a discussion on both SYM and SUGRA, this difference will be implicit.

with Dirac matrices of up to five antisymmetrised indices as a valid basis of the Lie algebra, and the superspace is described by $z^M = (x^m, \theta^\mu)$.

The theory of gravity consists of the superfields $E_M{}^A$ (vielbein incorporating $e_m{}^a$ and ψ_m^α) and C_{MNP} (3-form). Its degeneracy can be treated in two different ways:

1. Through a supergeometric approach [4, 5, 9]. For two reasons, this construction centres around the torsion $T = DE^A$ and its BI [163]

$$DT^A = E^B \wedge R_B{}^A, \tag{3.13}$$

rather than the other fields initially present: R and C. Firstly, the BI for the curvature R is encoded by the torsion equivalent. Secondly, it is unnecessary to consider C in order to capture the equations of motion through the BIs. The field strength H shows up anyway, in the torsion and the curvature.

To constrain the theory, the torsion therefore is the object of interest. The condition which makes the BIs equivalent to the equations of motion of the component theory is

$$T_{\alpha\beta}{}^c = 2(\gamma^c)_{\alpha\beta}, \tag{3.14a}$$

as the only non-vanishing torsion component. A softer condition in relation to this also allows for the modules [10, 11, 15]

$$T_{\alpha\beta}{}^c \in \ydiagram{2,1} \oplus \ydiagram{2,1,1,1,1}\,, \tag{3.14b}$$

which brings about the elimination of all fields unique to the superspace formulation except for the γ-traceless part of $E_\mu{}^a$. It represents the only field of interest, in the sense of A_α in SYM, but in the presence of curvature.

Nevertheless, an action cannot be formulated from this. For that, the presence of C is required, in contrast to what is true for the equations of motion.

2. Through an approach based on C [4, 5], where the linearised theory is obtained. While the superspace geometry usually enters the BIs of the 3-form in the shape of torsion, complicating the third point of section 3.2, this is not the case in flat space, in the absence of interactions, where

$$D_{(\alpha}C_{\beta\gamma\delta)} = 0 \tag{3.15}$$

gives the on-shell, linearised component theory. Moreover, the constraint can be softened into a conventional constraint, with the theory described by z^A and $C_{\alpha\beta\gamma}$, the field equation and gauge of which is [15]

$$\begin{aligned} D_{(\alpha}C_{\beta\gamma\delta)}\Big|_{\ydiagram{2,2}\oplus\ydiagram{2,2,1,1,1}\oplus\ydiagram{2,2,2,2,2}} &= 0, \\ \delta_\Lambda C_{\alpha\beta\gamma} = D_{(\alpha}\Lambda_{\beta\gamma)} : \quad (\gamma^1)^{\alpha\beta}\Lambda_{\alpha\beta} &= 0. \end{aligned} \tag{3.16}$$

At this point, however, the formulation of the corresponding action is just as problematic as in SYM.

In these constructions, both the geometric and 3-form superspace formulations fail to provide maximal supersymmetry; that local structure is absent, compare to section 3.4.4, which is what is required, in contrast to what is true in SYM. The main advantage of the reformulations is the compactness of the superspace theories, each contained in a single superspace field through the conventional constraints and the BIs, and the possible extension to the pure spinor formalism.

3.3 The pure spinor formalism construction

A closer look at the equations of motion and the gauge variations of the SYM and SUGRA superspace formulations show a peculiar pattern. It is most easily recognised for abelian SYM, where the condition on $D_{(\alpha}A_{\beta)}$ is that the 5-form vanish on-shell. Off-shell, a 5-form current $J_{\alpha\beta}$ is allowed. In combination with the gauge, this indicates a series of modules connected by a projection operator:

$$\Lambda \overset{PD_\alpha}{\to} A_\alpha \overset{PD_\beta}{\to} J_{\alpha\beta} \to \dots, \tag{3.17}$$

with the projection symmetrising the indices into 5-form irreducible representations, while respecting the gauge invariance (and supersymmetry). Equivalently (and more generally), the process can be regarded as symmetrising the indices while removing the 1-form, effectively forcing $D_\alpha D_\beta$ to zero (generating the gauge) and removing torsion components. Compared to the equation of motion and gauge invariance, the fields in each module are physical only if they are unconnected to the other modules. [12]

This pattern is visible in SUGRA as well. The linearised theory sports an equation of motion where the 1-form projections of the indices are absent, i.e. already set to zero by the conventional constraint, and similar restrictions apply to the gauge. Meanwhile, the non-abelian theory contains a 1-form torsion on-shell while (in addition) 2- and 5-form modules are present off-shell. As such, the off-shell physics desirable to capture is centred around the symmetric modules of the irreducible representations that do not contain a 1-form.

The wish for a reformulation of the superspace theory then naturally centres around the projection operator encoding the physics, and it is the recognition of it and the ensuing theory that gives rise to the pure spinor formalism. A formalism characterised by a focus on the end properties being correct rather than the procedure being conventional.

That said, the pure spinor was initially introduced as a means to provide a super-Poincaré covariant quantisation of the ten-dimensional superstring in [16]. The quantisation method was then applied to the scalar superparticle in [20] to provide a simpler case for studies of the method, to better understand it. It is this latter case which naturally represents the $D = 10$ maximal SYM theory. However, for clarity in the context of maximal SYM and SUGRA, the primary focus of this chapter, the pure spinor formalism will be presented as it figures there, rather than as it once was identified. The superstring formulation will merely be touched upon in the next section, to illustrate how the principles were recognised and to give a better insight into the different pure spinor approaches of the next chapter.

The formulation of the pure spinor formalism starts out with the recognition of that the addition of a pure spinor to the superspace makes it possible to capture the properties of the projection operator through a Becchi–Rouet–Stora–Tyutin (BRST) operator, with the theory (SYM or linearised SUGRA) possible to be reinterpreted as a BRST formalism. The key property of this formulation is the presence of an abelian action respecting maximal supersymmetry, see section 3.4.4. The second most important feature is that it represents a formulation with a classically recognised extension to a theory of interactions in the Batalin–Vilkovisky (BV) formalism [166, 167], respecting the same symmetries. Moreover, through incorporating the properties of the interactions (observed in each superspace theory) into the BV formalism, a specific part of the on-shell formulation corresponds to that of the non-abelian component theory. In this way, the introduction of the pure spinor furnishes a way to formulate a theory of interactions, in terms of an action, respecting the super-Poincaré symmetry, for both maximal SYM and SUGRA.

3.3.1 The covariant quantisation of the superstring

The Green–Schwarz action [168], invariant under super-Poincaré symmetry, can be quantised in a number of different ways, for example in the light-cone gauge. With vanishing Ramond–Ramond (RR) field strengths, the Ramond–Neveu–Schwarz (RNS) formalism [169, 170] is another option, to which the Ramond states can be added afterwards. Either way, it is impossible — at least so far as is known — to keep the relevant (maximal) symmetries inherent to the description (with subsequent simplifications) without proceeding according to the pure spinor formalism of [16]. The previously described situation in SYM and

SUGRA, representing the low energy limits of open and closed superstrings, naturally mirrors this setting.

The construction originates in an analysis of the situation in conformal gauge investigated by Siegel [171] in comparison with other formulations, in an attempt to keep the covariance of the first while obtaining a working formalism. In specific, it is possible to observe a demand on the Lorentz current of the vertex operator[5] for the massless open superstring conjectured by Siegel in order for it to correspond to that of the gluon vertex operator of the RNS formalism. An additional term N^{mn} is required, providing a formulation encompassing the two different quantisations. The properties of this term can be constructed out of a pure spinor λ^α, as it will be described in the next section (N^{mn} is composed of λ^α and its derivative). In the presence of this additional, ghost variable, a formalism with maximal supersymmetry is available. [16]

The recognition of the pure spinor makes it possible to define a new formalism with vertex operators constrained to belong to the cohomology (of the same ghost number as λ) of a nilpotent operator [16]

$$Q = \oint [\mathrm{d}z] \lambda^\alpha(z) d_\alpha(z). \tag{3.18}$$

with d_α as the covariant string spinor derivative. The subsequent formalism, checked for consistency in [17–19, 21, 22], contains (in an extended version) the integrated vertex conjectured by Siegel and provides a way to compute scattering amplitudes. Meanwhile, all the desired symmetries remain inherent to the formalism, regardless of the background, including curved ones with RR flux. The trade-off for this efficient formulation is the visibility of what goes on in the calculations, which is why the case of the scalar superparticle soon followed as an illustrative example.

3.3.2 The BRST formalism — mimicking the free theory

The properties of the physical fields in abelian SYM (A_α) and SUGRA ($C_{\alpha\beta\gamma}$) can be captured through the introduction of a bosonic spinor (of dimension $-1/2$) obeying the condition [16, 20]

$$\lambda^\alpha : \quad \lambda\gamma_a\lambda = 0, \tag{3.19}$$

and thus henceforth being referred to as *pure*. The key point is that λ^2 removes the 1-form projection of the spinor indices it is attached to, in line with the ob-

[5] A vertex operator represents an emission of states from the world surface of a string. Effectively, it may be interpreted as splitting one state into several others. For open strings, the only one present is the 3-point vertex, by default termed unintegrated (simply the vertex). An integrated variant in addition describes the propagation of one of the associated states.

servations for the superspace theory. With this spinor, an operator

$$Q = \lambda^\alpha D_\alpha \quad Q^2 \overset{(3.9)}{=} 0 \tag{3.20}$$

can be formed and used to reformulate the theory in flat space in combination with an alteration of the fields under consideration to

$$SYM: \quad \psi = \lambda^\alpha A_\alpha, \tag{3.21a}$$

$$SUGRA: \quad \psi = \lambda^\alpha \lambda^\beta \lambda^\gamma C_{\alpha\beta\gamma}. \tag{3.21b}$$

The theories are then equally described, instead of by eq. (3.12) and (3.16), by

$$Q\psi = 0, \quad \delta_\Lambda \psi = Q\Lambda. \tag{3.22}$$

In this, the conventional constraints are fulfilled by default in the sense of the entities falling out of the theory, compare to eq. (3.11) and (3.15). Torsion similarly drops out, with exactly the right components retained for interactions in SUGRA, although that is a comment running ahead of the discussion. In this setting, the only field under consideration is ψ, with its BI captured by the nilpotency of Q and an encoding of the component theory fields, see section 3.4.4, in an abelian sense due to Q constituting a linear operator.

By endowing the pure spinor with a ghost number (1) the nilpotent operator Q can be interpreted as a BRST operator and the theory as that of a BRST formulation. Consequently, the physical fields matching the component theory have ghost number zero and the field ψ ghost number (1,3). Its dimension is $(0, -3)$. A general ψ is furthermore, in the BRST reinterpretation of the theory, considered to contain a general set of fields (of any ghost number different from zero) apart from the contents of eq. (3.21). It then represents the pure spinor superfield, containing all possible ghosts and antifields while encoding the abelian SYM and SUGRA theories at ghost number zero. [16, 20]

The reinterpretation of the theory as described by a BRST formalism (naturally abelian) gives the abelian pure spinor formalism, containing $z^A_\lambda = (x^a, \theta^\alpha, \lambda^\alpha)$ and the superfield ψ, with the field equation of eq. (3.22). It is desirable as

1. The BRST operator Q respects the supersymmetry present through

$$\{Q, Q_\alpha\} = 0, \tag{3.23}$$

in the same way as D_α in the superspace formalism. It also encodes it (locally) as gauge symmetry, see section 3.4.4. As a result, maximal supersymmetry is present in the BRST formalism and the constructions based on it, as long as gauge invariance is respected.

2. The BRST formalism provides an action for the theory:

$$S \sim \int \psi Q \psi [\mathrm{d}Z], \tag{3.24}$$

granted the existence of a working integral measure $[\mathrm{d}Z]$. It stems from the requirement of a variation $\delta\psi$ in the action to give a vanishing expression on-shell, through the field equation displayed in eq. (3.22).

In combination, these two points yield the key feature of the pure spinor formalism: the action describes a theory (either SYM or SUGRA) where off-shell, maximal supersymmetry has been realised. The problem of the BRST formalism only describing the abelian theories is furthermore easily overcome due to the existence of a natural extension to a theory of interactions: the BV formalism.

3.3.3 The BV formalism — extending to a theory of interactions

The extension of an abelian theory to a non-abelian setting, in consistency with the gauge invariance, is typically not a trivial procedure. In classical field theory it is dealt with through the BV (or antifield) formalism, where the BRST symmetry is central [166, 167]. Essentially, the BRST structure can consistently be generalised to the BV one.

Both formalisms are characterised by a replacement of the symmetries of the theory under consideration with a rigid symmetry. In the BRST formalism, the BRST charge (Q) is constructed to be nilpotent and respecting the symmetry variations, and ghosts and antifields are introduced to that end as well as for a full action (invariant under the BRST symmetry). In the BV formalism, describing interactions, the symmetry does not act linearly on the fields, but is described by the generalised action, acting on fields through an antibracket:

$$(F, G) = \frac{\delta^R F}{\delta \Phi^A} \frac{\delta^L G}{\delta \Phi^*_A} - \frac{\delta^R F}{\delta \Phi^*_A} \frac{\delta^L G}{\delta \Phi^A}, \tag{3.25a}$$

with R and L referring to the standard right and left derivatives. This illustrates the classical case, where ghosts and antifields need to be introduced at the BV reformulation: F and G are functionals of the fields and antifields (Φ^A, Φ^*_A), with a hidden dependence on x in the indices so that integration is included. However, in the pure spinor formalism those fields are already present in ψ, constituting its own antifield — no more fields can be introduced. The generalisation from the BRST formalism to the BV one then is restricted to a replacement of the BRST charge (Q) with the BV charge (S), acting through the only antibracket possible to formulate (in the presence of one, single field) [29]:

$$(A, B) \sim \int \frac{\delta A}{\delta \psi} \frac{\delta B}{\delta \psi} [\mathrm{d}Z]. \tag{3.25b}$$

In addition, a check on the component fields in the cohomology verifies this construction as the correct one, compare to section 3.4.4.

The BV formalism describes the theory through

$$(S, \psi) = 0, \quad \delta_\Lambda \psi = (S, \Lambda), \tag{3.26}$$

the first equation constituting the equation of motion and the second the gauge, with a variation of the field: $\delta\psi = (S, \psi)$. The identical terms are typical for BV formalisms. Meanwhile, the nilpotency of the BRST symmetry translates into

$$(S, S) = 0, \tag{3.27}$$

which represents the master equation for the action, the requirement for the theory to capture the interaction counterpart to the initial BRST formulation. It sets the form of the action, in combination with the shape of the interactions given by the superspace formulation.

3.3.4 The integral measure & the non-minimal superspace

For the BRST and BV formalisms to be valid, a working integral measure must be present. It turns out to be difficult to construct such a non-degenerate entity from the superspace variables $(x^a, \theta^\alpha, \lambda^\alpha)$ alone. To begin with, the $[\mathrm{d}Z]$ must have ghost number $(-3, -7)$ and dimension $(-6, -3)$ due to the shape of the abelian action and the coupling constants of the component theories, which are of dimension $(-3, -9/2)$. The action itself is of course required to be dimensionless and of ghost number zero. These criteria are not met by

$$[\mathrm{d}z_\lambda] = \mathrm{d}^D x \, \mathrm{d}^n\theta \, [\mathrm{d}\lambda], \quad n = (16, 32), \tag{3.28}$$

since an examination of the effects of the pure spinor constraint on the volume form of the pure spinor gives [23,25,27,29]

$$[\mathrm{d}\lambda]\lambda^{\alpha_1} \ldots \lambda^{\alpha_i} = \star\bar{T}^{\alpha_1 \ldots \alpha_i}{}_{\beta_1 \ldots \beta_j} \mathrm{d}\lambda^{\beta_1} \wedge \ldots \wedge \mathrm{d}\lambda^{\beta_j}, \; (i, j) : \begin{cases} (3, 11) & SYM \\ (7, 23) & SUGRA \end{cases} \tag{3.29}$$

where T projects into the irreducible representations of (00003) for SYM and (02003) for SUGRA, and j reflects the degrees of freedom of the pure spinor. The integration over λ absorbs λ^i from the integrand, giving $[\mathrm{d}z_\lambda]$ the correct ghost number, but it does not have the required dimensional properties. Nor does an addition of a function of the variables help in this; the resulting measure would be degenerate, excluding most of ψ by effectively chopping it off at some point in the series expansion in θ. Although there is a formulation with so-called picture-changing operators which allows for the construction of a correct integral measure [24], the most versatile option is to introduce new variables in the spinor space.

The *non-minimal* superspace contains $Z^A = (x^a, \theta^\alpha, \lambda^\alpha, r_\alpha, \bar{\lambda}_\alpha)$. In addition to the pure spinor and the superspace variables, there are two counterparts to (θ, λ): $(r, \bar{\lambda})$. The former is a fermionic spinor of ghost number 0, the latter a bosonic spinor of ghost number -1. Both have dimension $1/2$, and obey

$$(\bar{\lambda}\gamma^a\bar{\lambda}) = (\bar{\lambda}\gamma^a r) = 0, \tag{3.30}$$

which results in that their (respective) degrees of freedom equal that of the pure spinor. The advantage of the new set of variables is that

$$[\mathrm{d}\bar{\lambda}]\bar{\lambda}_{\alpha_1}\dots\bar{\lambda}_{\alpha_i} = \star T_{\alpha_1\dots\alpha_i}{}^{\beta_1\dots\beta_j}\,\mathrm{d}\bar{\lambda}_{\beta_1}\wedge\dots\wedge\mathrm{d}\bar{\lambda}_{\beta_j}, \tag{3.31a}$$

$$[\mathrm{d}r] = \bar{\lambda}_{\alpha_1}\dots\bar{\lambda}_{\alpha_i}\star\bar{T}^{\alpha_1\dots\alpha_i}{}_{\beta_1\dots\beta_j}\frac{\partial}{\partial r_{\beta_1}}\dots\frac{\partial}{\partial r_{\beta_j}}, \tag{3.31b}$$

with $[\mathrm{d}Z]$ presenting the right properties for the integral measure. [27]

However, the introduction of the non-minimal formalism has consequences beyond the existence of a well-defined integral measure. Effectively, all fields become dependent on the non-minimal variables, so to retain the physics, the previous concept of cohomology must be interpreted as a subclass of the extended theory: [27]

$$Q = (\lambda D) + (r\bar{\omega}), \quad \bar{\omega}^\alpha = \frac{\partial}{\partial\bar{\lambda}_\alpha}. \tag{3.32}$$

While incorporating the extended theory to its fullest, this allows for an extraction of the original theory, present in each cohomology class as independent of the non-minimal variables. The desired properties remain, the theory is enlarged with what is necessary, and finally it is reinterpreted from the subsequent point of view, in line with what previously has been described for the pure spinor, the BRST and the BV formalisms.

3.3.5 The actions

The first term in the BV action is the abelian action in eq. (3.24). It represents a natural starting point, the abelian theory, to which the correct interactions must be added in compliance with the master equation. The remaining question is how to add the interactions, expected to show as 3-point couplings ($\sim \psi^3$) and possibly higher terms. In part, what goes into the action is limited by what is possible to formulate, with the correct ghost and dimensional properties, in the theory. However, it also has to capture the non-abelian physics disregarded at the linearisation in section 3.2.2 and 3.3.2.

The gauge theory

In SYM, to include interactions merely consists of the recognition of the theory as similar to Chern–Simons theory. The fields are all part of the superfield ψ, and the only possible additional term in the action is proportional to ψ^3. The subsequent BV action [16, 20]

$$S_{SYM} = \frac{1}{g_{YM}^2} \int [\mathrm{d}Z] \mathrm{tr} \left(\frac{1}{2} \psi Q \psi + \frac{1}{3} \psi^3 \right), \tag{3.33}$$

with implicit Lie algebra indices on the superfield, obeys the master equation. In that way the only BV extension of the abelian theory, consistent with the non-abelian theory, has been identified. Since the BV extension classically yields the correct interactions and symmetry invariance, this does as well.

Supergravity

The case of SUGRA is slightly more involved than that of SYM. While the ψ built from the 3-form C contains the linearised theory, the full superspace theory is only obtainable from $E_\mu{}^a$, as outlined in section 3.2.2. Therefore, with the action to be formulated in flat space, there is an additional presence of $E_\alpha{}^a$ in the full theory, and one more field than ψ needs to be considered as part of the pure spinor BV formulation: the field Φ^a containing $\lambda^\alpha E_\alpha{}^a$ as one of its components (the only minimal, of set ghost number). Naturally, Φ^a is extended to contain ghosts and antifields in the same manner as ψ, but the property of $E_\alpha{}^a$ being γ-traceless translates into the equivalence

$$\Phi^a \approx \Phi^a + (\lambda \gamma^a \rho), \quad \rho : \text{ any spinor} \tag{3.34}$$

for the field. Typically, this type of relation is referred to as a 'shift symmetry' [29, 30, 172, 173], which constitutes a key concept in Paper I.

The presence of Φ^a complicates matters, apart from presenting new opportunities at forming interaction terms in the BV action. There would seem to be two pure spinor fields present in the theory. However, to a certain degree the physics overlaps: the cohomology of ψ and Φ^a are partially identical. Moreover, the pure spinor field ψ is more fundamental (as indicated for the linearised theory) in that Φ^a is expressible as [29]

$$\Phi^a = R^a \psi, \tag{3.35}$$

with the recognition of a new operator in the theory:

$$\begin{aligned} R^a = {} & \eta^{-1} (\bar\lambda \gamma^{ab} \bar\lambda) \partial_b - \eta^{-2} (\bar\lambda \gamma^{ab} \bar\lambda)(\bar\lambda \gamma^{cd} r)(\lambda \gamma_{bcd} D) \\ & - 16 \eta^{-3} (\bar\lambda \gamma^{a[b} \bar\lambda)(\bar\lambda \gamma^{cd} r)(\bar\lambda \gamma^{e]f} r)(\lambda \gamma_{fb} \lambda)(\lambda \gamma_{cde} \omega), \end{aligned} \tag{3.36a}$$

where $\eta = (\lambda \gamma^{ab} \lambda)(\bar\lambda \gamma_{ab} \bar\lambda)$. This operator represents the necessary addition to the linearised theory in order to capture the interactions through a BV formulation.

The BV 3-point coupling possible to form in the presence of R^a is limited by gauge invariance[6], ghost and dimensional properties to [29,30]

$$(\lambda\gamma_{ab}\lambda)\psi R^a\psi R^b\psi,$$

a term which at its introduction in the action by the master equation implies the additional presence of a 4-point coupling. The unexpected property of this is that with the introduction of a second operator [30]

$$T = 8\eta^{-3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}r)(rr)(\lambda\gamma_{ab}\omega), \tag{3.36b}$$

the 4-point coupling can be formulated as the final part of the action: the master equation is fulfilled. With no other possible terms in the action, it turns out to be [29,30]:

$$S_{SUGRA} = \frac{1}{\kappa^2}\int [\mathrm{d}Z]\Big(\frac{1}{2}\psi Q\psi + \frac{1}{6}(\lambda\gamma_{ab}\lambda)\Big(1 - \frac{3}{2}T\psi\Big)\psi R^a\psi R^b\psi\Big), \tag{3.37}$$

where it is possible to verify the correct encoding of the component theory in terms of section 3.4.4.

In total, eq. (3.33) and (3.37) describe the pure spinor formulation of maximal SYM and SUGRA. The actions provide theories with inherent, maximal supersymmetry (present both on- and off-shell) and retainable interpretations in terms of the component theories; the so desired reformulations of eq. (3.1).

3.4 Features of the pure spinor formalism

The pure spinor formalism calls for a reinterpretation of many concepts as well as a recognition of the new features displayed, e.g. concerning BRST equivalence, the procedure for gauge fixing and how to regularise divergent integrands. Another integral feature is how the correct and desirable physics of the maximal SYM and SUGRA theories shows up in the construction —- as cohomology in the case of the component fields, and as gauge in the case of the supersymmetry variations and the diffeomorphisms. Although powerful, the pure spinor formulation complicates matters in that it shows neither supersymmetry nor component fields explicitly.

3.4.1 BRST equivalence

The pure spinor formalism holds one key, novel property. The fact that any external state is on-shell and freely propagating (the only physical kind) brings

[6] A term $\sim \lambda^2\Phi^5$ is ruled out.

with it that a calculation concerning them — effectively any calculation — only is defined up to BRST equivalent terms. Considering how the solution to eq. (3.22) remains unchanged under a transformation

$$1 \leftrightarrow 1 + \{Q, \chi\}, \quad \chi : \begin{array}{l} \text{fermionic scalar} \\ \text{dimension } 0 \\ \text{ghost number } -1 \end{array}, \tag{3.38a}$$

provides the reason why: the extra terms do not affect the end result. Because of this, alterations as in eq. (3.38a) can be performed in any expression, at any time. This freedom of the theory constitutes a property known as BRST or Q-equivalence, which is surprisingly useful. It is most known in the shape of the special case of *regulators*

$$e^{\{Q,\chi\}}, \tag{3.38b}$$

which are introduced frequently. The expression is especially versatile as it captures the entire series expansion of the exponential function. In addition, the BRST equivalence constitutes a reminder of the (most likely) many equivalences present in the pure spinor formalism, yet to be pinned down.

3.4.2 Gauge fixing

The standard way for gauge fixing in the BV formalism is that of introducing a gauge fixing fermion χ to eliminate the antifields. Through

$$\Phi^*_A = \frac{\delta\chi}{\delta\Phi^A}, \tag{3.39}$$

the physical quantities are set to be independent of the choice of gauge. However, this is not an option in the pure spinor formalism. The single field present, the pure spinor superfield, contains both the fields and the antifields and ought not be split, as previously mentioned at the extension of the BRST formalism to the BV one. Instead, the process of gauge fixing is performed in a manner borrowed from string theory.

In the pure spinor formalism, the Siegel gauge [174] for a scalar particle is imitated through the introduction of a b-ghost figuring in the free propagator

$$\frac{b}{p^2} \tag{3.40a}$$

and required to obey a condition

$$\{Q, b\} = \partial^2 \quad \Rightarrow \quad \{b, b\} = 0, \tag{3.40b}$$

where the implied property should hold by default, at least in a Q-equivalent sense. In this setting, the choice of gauge is

$$b\psi_{\text{on-shell}} = 0. \tag{3.40c}$$

3.4.3 Singularities & general regularisations

Integration is only well-defined in the absence of divergencies. These ought not arise at all, in a well-defined theory, but especially not from the presence of introduced variables, such as $(\lambda, \bar{\lambda}, r)$. With respect to this, the bosonic spinors are troublesome, in the non-minimal formalism. Their presence in the integrands threaten to cause divergences for both large and 'small' $(\lambda, \bar{\lambda})$, though only superficially so, as regulators can be applied in the manner of the general regularisations to provide convergence.

The limit of infinity

The first type of divergence is the one most easily remedied. An introduction of a regulator [27]

$$e^{\{Q,\chi\}} = \left[\chi = -\bar{\lambda}\theta\right] = e^{-\lambda\bar{\lambda}-r\theta} \tag{3.41}$$

ensures a good convergence in the limit of infinity for $(\lambda, \bar{\lambda})$: the divergence of any polynomial in either is suppressed. At the same time, the regulator provides a way to saturate the fermionic integrals $[\mathrm{d}\theta]$ and the $[\mathrm{d}r]$ in eq. (3.31). This is important for integrands containing fewer of the fermionic spinors than their degrees of freedom, which are an integral part of the theory. An insensitivity to those parts of an integrand on behalf of the integration would make the theory void, which is perhaps most notable in the case of r, on which the minimal part of the theory does not depend. However, it is equally important with respect to θ. Without the option to saturate the fermionic integrals for the minimal integrands, key information would be disregarded and the theory would fail to capture the physics of maximal SYM and SUGRA.

Singular subspaces

The second type of divergence is connected to singularities with respect to $(\lambda, \bar{\lambda})$, originating in that scalars of $(\lambda, \bar{\lambda})$ may appear in the denominator of an integrand. There are two kinds of such scalars: [27,29]

$$\xi = (\lambda\bar{\lambda}), \quad \eta = (\lambda\gamma^{ab}\lambda)(\bar{\lambda}\gamma_{ab}\bar{\lambda}) \sim \xi^2\sigma^2, \tag{3.42}$$

of which the latter only exists in SUGRA. Clearly, it carries a possible singularity not only for small $(\lambda, \bar{\lambda})$, but also for a subspace where either $(\lambda\gamma^{ab}\lambda)$ or its counterpart (both $\propto \sigma$) is zero separately from ξ. Despite this, it is convenient to refer to the type of divergence in a manner relatable to SYM, which is why the additional subspace is implicitly incorporated in the 'limit of small $(\lambda, \bar{\lambda})$' of Paper IV.

Meanwhile, the integral measure contains

$$\mathrm{d}^i\lambda\,\mathrm{d}^i\bar{\lambda} \overset{SUGRA}{\propto} \mathrm{d}^{14}\sigma, \quad i = (11, 23), \tag{3.43a}$$

since the submanifold has real codimension 14 in SUGRA. Moreover, as $[\mathrm{d}Z]$ extracts $\lambda^7 \sim \sigma^2$ in SUGRA, compare to eq. (3.29), it behaves like:

$$[\mathrm{d}Z] \sim \xi^k \mathrm{d}\xi \, \sigma^l \mathrm{d}\sigma, \quad (k,l): \begin{cases} (10,0) & SYM \\ (22,11) & SUGRA \end{cases}. \tag{3.43b}$$

An integrand is convergent (and an expression well-behaved) with respect to the singularities if it contains less than $(11, 23)$ inverse ξ and 12 inverse σ. In addition, any divergent expression composed of well-behaved operators is BRST equivalent to a convergent expression, since each part can be regularised through the introduction of a new set of variables with a matching regulator and integral measure [28], a procedure generalised to SUGRA in Paper II. As each operator in the theories is well-behaved with respect to the singularities, this means that any integrand can be regularised into convergence.

The regularisation removing the singularities has the key feature of introducing a new set (or any number of sets) of variables $z'^A = (f^\alpha, \bar{f}_\alpha, g^\alpha, \bar{g}_\alpha)$ which make up counterparts to $(\lambda^\alpha, \bar{\lambda}_\alpha, \theta^\alpha, r_\alpha)$. Naturally, each set comes with its own integral measure, and so removes a singularity corresponding to that of eq. (3.43b) while a regulator converts variables of the old set into the new, and a second regulator removes divergences with respect to large $(f, \bar{f})$. The principle follows

$$O_{\text{reg}}(\lambda, \bar{\lambda}) = \int [\mathrm{d}z'] e^{-\{Q, \bar{f}g\}} e^{i\varepsilon\{Q, gW + \bar{f}V\}} O(\lambda, \bar{\lambda}), \tag{3.44a}$$

with (W, V) representing gauge invariant operators acting on (λ, r), and ε a constant. Implicitly, the regulators act on the operator and Q has been extended by the addition of z', much in the same way as when the non-minimal variables were introduced, so that the first exponential represents a regulator with a function corresponding to that of eq. (3.41). In this way, what effectively takes place is a change

$$\begin{aligned} O(\lambda, \bar{\lambda}) \rightsquigarrow & \int [\mathrm{d}f][\mathrm{d}\bar{f}] e^{-f\bar{f}} e^{i\varepsilon(fW + \bar{f}\bar{W})} O(\lambda, \bar{\lambda}) = \\ & = \Big[\lambda' = e^{i\varepsilon f W} \lambda \, , \ \bar{\lambda}' = e^{i\varepsilon \bar{f}\bar{W}} \bar{\lambda} \Big] = \\ & = \int [\mathrm{d}f][\mathrm{d}\bar{f}] e^{-f\bar{f}} O(\lambda', \bar{\lambda}'), \end{aligned} \tag{3.44b}$$

with $(W, \bar{W})$ as gauge invariant versions of $(\omega, \bar{\omega})$. Effectively, part of the singularities are 'smeared out' in the manner of a heat kernel regularisation in quantum mechanics. The procedure can be performed any number of times, each increasing the tolerance by what was initially allowed for. [28]

Regularising expressions

The general regularisations have one final thing in common: their existence is crucial, but they are not performed in practice. Although the theory would be divergent in their absence, once they are applied the analysis of an integrand is severely complicated. Especially the regularisation of the singularities is opaque, because once applied, the different sets of variables are related in ways not explicitly captured by the formulation. The description as such becomes unnecessarily rigid. Due to these circumstances, regularisation is typically left off to the very last, merely figuring as an existing concept connected to integration, which in practice is not performed. They take care of what needs to be taken care of, end of story.

Important to note in this is that if an integrand under scrutiny contains parts, in total divergent with respect to the singularities, which seem to combine into zero by the properties of the variables subject to the regularisation $(\lambda, \bar{\lambda}, \theta, r)$, that zero is superficial. In order to actually examine such a part, enough of the general regularisation for the expression to represent a convergent entity[7] must be considered. This is for example the situation in the examinations of the UV divergences in the theories. There however, the complications brought about by the general regularisation are utilised/dodged (as far as possible) by a reinterpretation of the new set of variables as loop variables.

A key feature is, however, that convergent parts of an integrand retain their individual pre-regularisation properties during the procedure of regularisation. All alterations are Q-exact, i.e. of the type in eq. (3.38a).

3.4.4 The relevant physics captured

The main feature of the pure spinor formalism is that the associated actions respect maximal supersymmetry. This is achieved by construction: first through requiring that the operators (anti)commute with the supersymmetry operator, then by putting supersymmetry on an equal footing with the gauge symmetry (pure spinor construction), and finally carried through by a restriction to gauge invariant components in the theory.

Worthwhile to note in this is that gauge invariance with respect to the pure spinor stipulate a dependence on ω modulo $(\lambda\gamma^i)A_i$: A any 1-form. That is why all pure spinor derivatives must show only in the combinations of

$$(\lambda\omega) \quad (\lambda\gamma^{ab}\omega), \tag{3.45}$$

usually denoted by N and N^{ab}. Expressions of other combinations may be gauge invariant also, but are then possible to reformulate in terms of eq. (3.45), as shown for R^a in Paper IV.

[7]Note that the entity under discussion might be a very small fraction of the final integrand. The point made is that any convergent part can be examined on its own, without loss of information.

However, the first crucial verification in the pure spinor formalism is that the relevant component theory is encoded. Provided that the field contents are correct, the relevant equations of motions can be verified in relation to eq. (3.22), with a starting point in the field contents: the cohomology of ψ, also accommodating the supersymmetry variations and the diffeomorphisms.

Cohomology & partition functions

In the BRST formalism, the physical fields show up as cohomology of ψ. The concept arises from the presence of a nilpotent operator (Q), appears as [175]

$$H^p = \text{Ker}(Q_p)/\text{Im}(Q_{p-1}), \tag{3.46}$$

and concerns different modules V_p, where the cohomology H^p is the kernel of Q in V_p modulo the images from V_{p-1}. Essentially, the solutions to eq. (3.22) in the presence of a field of general ghost number.

In the pure spinor formalism, the module of interest naturally is the one of ψ. Here, it is possible to check that the theory truly describes maximal SYM or SUGRA — the component fields must reside in the cohomology. This is indeed the case, as illustrated in table 3.1, where the (minimal) cohomology of ψ has been solved for in the absence of a dependence on x. The equivalent of Φ^a is also displayed. As a physical field, it obeys the same equation of motion as ψ: $Q\Phi^a = 0$ or $(S, \Phi^a) = 0$ modulo gauge, and despite its carrying no original information ($\Phi^a = R^a\psi$), the cohomology is of interest e.g. at an analysis of the non-abelian equations of motion.

A second way of checking the cohomology is through an examination of why it is non-trivial, which in the case of $Q = \lambda D$ amounts to an examination of the pure spinor, in specific the effects of the constraint, without which the cohomology would have stopped at a scalar. To this purpose, the object of interest is the pure spinor partition function.

In their simplest shape, partition functions encode the number of states of a quantum number (most commonly energy levels). For spinor components, this looks like:

$$P(t) = 1 + \sum_{i=1}^{\infty} c_i t^i, \tag{3.47}$$

with $|c_i| = \text{dim}(R_n)$ denoting the number of states for i spinors, deductible by the irreducible representations R_n they show up in. If the overall sign of the term is negative, the complex is fermionic, and the initial constant represents the bosonic state in the absence of any spinor whatsoever.

While a fermionic spinor component is described by $(1 - t)$ and a bosonic by

dim \ gh #	1	0	-1	-2
0	(00000)			
$\frac{1}{2}$	•	•		
1	•	(10000)	•	
$\frac{3}{2}$	•	(00001)	•	•
2	•	•	•	•
$\frac{5}{2}$	•	•	(00010)	•
3	•	•	(10000)	•
$\frac{7}{2}$	•	•	•	•
4	•	•	•	(00000)

Table 3.1a: The zero-mode, minimal cohomology in ψ for maximal SYM [20]. It represents the solutions to $Q\psi = 0$ (abelian field equation) modulo gauge, compare to eq. (3.22), when the dependence on $(x, \bar{\lambda}, r)$ of the field is disregarded, which is preferable for a qualitative assessment of the cohomology. The superfield $\psi(Z)$ contains fields beyond the superfield A_α (of ghost number 0), fields of nonzero ghost number. The contents can be displayed through a series expansion in $(\theta^\alpha, \lambda^\alpha)$:

$$\psi_{\text{on-shell}}(\theta, \lambda) = c + (\lambda\gamma^a\theta)A_a + (\lambda\gamma^a\theta)(\theta\gamma_a\chi) + \ldots.$$

The horizontal direction represents the expansion of the superfield in terms of λ, that of the vertical is θ and λ. The irreducible representations of the component fields are listed with their Dynkin indices (compare to section 2.2.2) at positions describing their ghost numbers and dimensions. For example, the spinor χ^α of ghost number 0 and dimension $3/2$ in the series expansion above shows as (00001). This corresponds to a convention for the chiralities with a lowered spinor index in (00010). Note that the spinor and the gauge field of the component theory (ghost number 0) are present ($\chi^\alpha \sim \psi^\alpha$), with the addition of the scalar c and the antifields. The latter mirror the properties of the fields with respect to a slanting line (ne-sw) through the point $(0.5, 2)$ in the diagram. Compare with the situation in SUGRA, table 3.1b, where ghost fields other than the trivial c show up.

dim \ gh #	3	2	1	0	-1	-2	-3	-4
-3	(00000)							
$-\frac{5}{2}$	•	•						
-2	•	(10000)	•					
$-\frac{3}{2}$	•	•	•	•				
-1	•	•	(01000) (10000)	•	•			
$-\frac{1}{2}$	•	•	(00001)	•	•	•		
0	•	•	•	(00000) (00100) (20000)	•	•	•	
$\frac{1}{2}$	•	•	•	(00001) (10001)	•	•	•	•
1	•	•	•	•	•	•	•	•
$\frac{3}{2}$	•	•	•	•	(00001) (10001)	•	•	•
2	•	•	•	•	(00000) (00100) (20000)	•	•	•
$\frac{5}{2}$	•	•	•	•	•	(00001)	•	•
3	•	•	•	•	•	(01000) (10000)	•	•
$\frac{7}{2}$	•	•	•	•	•	•	•	•
4	•	•	•	•	•	•	(10000)	•
$\frac{9}{2}$	•	•	•	•	•	•	•	•
5	•	•	•	•	•	•	•	(00000)

Table 3.1b: The zero-mode, minimal cohomology in ψ for maximal SUGRA, following the principles outlined in table 3.1a [14]. Note the parts of zero ghost number: the 3-form C, the (graviton) $g_{(ab)} = \tilde{g}_{(ab)} + c_1\eta_{ab}\tilde{g}$ (both of dimension 0) and the (gravitino) $\chi^a_\alpha = \tilde{\chi}^a_\alpha + c_2(\gamma^a\tilde{\chi})_\alpha$ (of dimension $1/2$), with c_i as some constants. Compare to eq. (3.2).

dim \ gh #	1	0	−1	−2	−3	−4
-1	(10000)					
$-\frac{1}{2}$	(00001)	•				
0	•	(20000)	•			
$\frac{1}{2}$	•	(00001) (10001)	•	•		
1	•	(00010) (10000)	•	•	•	
$\frac{3}{2}$	•	•	(00001) (10001)	•	•	•
2	•	•	(00000)(00002) (00100)(01000) (10000)(20000)	•	•	•
$\frac{5}{2}$	•	•	•	•	•	•
3	•	•	•	(00000)(00002) (00100)(01000) (10000)(20000)	•	•
$\frac{7}{2}$	•	•	•	(00001) (10001)	•	•
4	•	•	•	•	(00010) (10000)	•
$\frac{9}{2}$	•	•	•	•	(00001) (10001)	•
5	•	•	•	•	(20000)	•
$\frac{11}{2}$	•	•	•	•	•	(00001)
6	•	•	•	•	•	(10000)

Table 3.1c: The zero-mode, minimal cohomology in $\Phi^a = R^a\psi$ in maximal SUGRA, following the principles outlined in table 3.1a [14]. As the expansion there concerns $\lambda^\alpha A_\alpha$ for the components of zero ghost number, here it concerns $\lambda^\alpha E_\alpha{}^a$. In a comparison with table 3.1b, note the presence of $\tilde{g}_{(ab)}$ and χ_a^α, with the addition of the 4-form H of dimension 1. As Φ^a is not the fundamental field, some extra, 'irrelevant' components are present, like the Weyl scalar, compare to [9].

$1/(1-t)$, the pure spinor in SYM has [26]

$$P(t) = \frac{1-10t^2+16t^3-16t^5+10t^6-t^8}{(1-t)^{16}} = \frac{(1+t)(1+4t+t^2)}{(1-t)^{11}}, \tag{3.48}$$

where the first expression describes the constraints on a bosonic spinor of 16 components, while its most simple form shows the presence of 11 bosonic degrees of freedom. The constraints represent a scalar (1), a vector (10) and a spinor (16) with characters opposite to the ones displayed due to the fermionic character of ψ.

In total, the partition function provides information in unison with table 3.1 down to the accompanying number of θ, if not λ itself. This is also true in SUGRA, although that is a more complicated case. Additional information, compared with $P(t)$, can be provided through the formal partition function [26, 176–178]

$$\mathscr{P}(t) = \bigoplus_{n=0}^{\infty} R_n t^n. \tag{3.49}$$

Equations of motion

In table 3.1 the minimal field contents at ghost number zero of SYM and SUGRA were identified to correspond to the component theories. In the zero-mode abelian theory this corresponds to an identification of what is allowed in a series expansion in (λ, θ), while yielding a vanishing result when acted on by $\lambda^\alpha \partial_\alpha$, yet not constituting a product of that operator on another entity. The equations of motion of the abelian theory are reconstructible, in a similar way, with an extension to a dependence on x of the zero-mode components.

Take for example the $A_a(x)$ in maximal SYM. The equation of motion for that part of the field gives a non-zero entity $(\lambda\gamma^b\theta)(\lambda\gamma^a\theta)\partial_{[a}A_{b]}$ possible to match with $\lambda^\alpha\partial_\alpha$ acting on $(\theta\gamma^{abc}\theta)(\lambda\gamma_a\theta)F_{bc}$ (not a gauge). The field equation then states the existence of $F_{ab} \sim \partial_{[a}A_{b]}$ in the abelian theory:

$$\begin{array}{lccccccc} \psi: & & \lambda^1\theta^1 A & & \lambda^1\theta^3 F & & \lambda^1\theta^5 T & \\ & & \swarrow \quad \searrow & & \swarrow \quad \searrow & & \swarrow \quad \searrow & , \\ Q\psi: & 0 & & \lambda^2\theta^2(F-\partial A) & & \lambda^2\theta^4(\partial F - T) & & 0 \end{array} \tag{3.50}$$

modulo constants (rescalings are possible). The entity $\lambda^2\theta^4\partial F$ can be matched with a tensor T, but only in (11000) as the $\lambda^1\theta^5$ does not support the other '3-index' configurations: (00100) and (10000). Consequently, $Q\psi = 0$ at $\lambda^2\theta^4$ represents the BI $\partial_{[a}F_{bc]} = 0$ and the abelian equation of motion $\partial^a F_{ab} = 0$. It also sets T to the remaining ∂F, representing cohomology as the term at $\lambda^1\theta^5$ in ψ is not expressible as gauge. Finally, the chain stops as $(\lambda\gamma^a\theta)(\lambda\gamma^b\theta)\partial_{[a}\partial_{b]} = 0$.

This illustrates the general mechanisms for how the BIs and the equations of motion at ghost number zero show up. In the presence of interactions, the non-linear action of the BV charge must be considered in terms of (S, ψ), rendering the procedure more delicate in terms of gauge, etc. In SYM, this translates to

$$Q\psi + \psi^2 = 0, \tag{3.51a}$$

with e.g. $dA + A \wedge A$ to be matched by F. Considering the hidden Lie indices in the equation of motion, the zero-mode cohomology and the abelian situation described above, this is quite apparently fulfilled. The ∂A in $Q\psi$ shows up at $\lambda^2\theta^2$ and the part at the same expansion in ψ^2 is A^2. At $\lambda^2\theta^4$, where the BI and the equation of motion of F were recognised, additional terms show up from the $\lambda^1\theta^2$ in ψ (with a total of χ^2) and the combination of the terms at $\lambda^1\theta^1$ and $\lambda^1\theta^3$ (AF), in unison with the equation of motion for the gauge field in eq. (3.5a), concerning $\partial F + AF - \chi^2$ in (10000).

A similar check can be done for the spinor, both of which can be extended to a thorough examination. Naturally, the cohomology changes not only with the introduction of a dependence on x, but also with the generalisation to interactions. At a qualitative assessment, however, it is possible to check what the non-abelian cohomology[8] accommodates for in terms of the (zero-mode) abelian situation, as the latter is contained in the former and further corrections show up at a higher order in terms of the number of fields incorporated.

In SUGRA, in a comparison with eq. (3.5b), it is simplest to look to the gravitino and the field strength. With χ at $\lambda^3\theta^4$ in ψ, $\partial\chi$ shows up at $\lambda^4\theta^5$ in $Q\psi$. The BV equation of motion in a minimal setting,

$$Q\psi + (\lambda\gamma_{ab}\lambda)\Phi^a\Phi^b = 0, \tag{3.51b}$$

then shows the presence of additional terms at $\lambda^4\theta^5$ through the zero-mode cohomology of Φ. The contributions from the ghost number zero components reside at $\lambda^1\theta^3$ and $\lambda^1\theta^2$, yielding χH and accommodating for $\partial\chi + \chi H = 0$. In terms of the field strength, $\partial H \sim \partial^2 C$ sits at $\lambda^4\theta^6$ in $Q\psi$ with interaction additions in terms of zero-mode cohomology only from $\lambda^1\theta^3$ in Φ: H^2, consistent with $\partial H + H^2 = 0$. The further components in terms of eq. (3.4b), e.g. a generalisation to $\hat{H} \sim H + \chi^2$, ought to show up in a further analysis of the non-abelian cohomology in Φ, e.g. of what sits at $\lambda^1\theta^3$.

Consequently, the zero-mode cohomology in table 3.1 is in accordance with equations of motion matching eq. (3.5). It provides the possibility to confirm the correctness of the embedding of the component theories at ghost number zero in the minimal setting; the feature contained by construction. Strictly speaking,

[8]Cohomology is generally used in terms of a nilpotent operator acting linearly on fields. However, the BV charge is a concept extended from this, acting consistently in a nilpotent way albeit non-linearly, presenting a cohomology in a non-abelian sense. This is what is referred to here.

though, what is contained in A_a etc. of the pure spinor formalism is not quite restricted to the component theory entities. Much like the operators of Paper I, they fill the correct niche but might be just a bit more general than that.

Supersymmetry and diffeomorphisms in the pure spinor formalism

The maximal supersymmetry figures in two different ways in the pure spinor formalism, depending on the theory. In SYM, the fact that the covariant spinor derivative commutes with the supersymmetry operator makes the entire construction invariant under the supersymmetry; Killing vectors on the background give the global maximal supersymmetry.

Anyhow, it is possible to see the equivalence of a field and its variation in the cohomology. A field *redefinition*

$$A_a \approx A_a + (\epsilon\gamma_a\chi), \tag{3.52}$$

is allowed, with any coefficient and spinor χ, as long as the correct dimension, spin and statistics are demonstrated by the extra term and no further dependence on (λ, θ) exists. Under these conditions, the term exists as cohomology and is equivalent to A_a, representing a gauge, though not in terms of $Q\Lambda$. It is an extra symmetry accommodated for. Note that the supersymmetry variations are equally observable through $\delta_\epsilon\psi$, which is no surprise, as $\psi' = \psi + \delta_\epsilon\psi$ is a superfield, just as ψ. With $\delta_\epsilon = \epsilon^\alpha Q_\alpha$ this exactly reproduces the structure in eq. (3.6a): δ_ϵ encode dimension etc. by (λ, θ), same as Q; except for the pure and the infinitesimal spinors, the difference is the sign in front of the ∂. For example in the zero-mode cohomology, χ at $\lambda^1\theta^2$ is brought down to $\lambda^1\theta^1$, where A_a is, by $\epsilon^\alpha\partial_\alpha$, with the element at $\lambda^1\theta^1$ in $\psi + \delta_\epsilon\psi$ as the right hand side of eq. (3.52). The option of a general, real coefficient corresponds to the cohomology accommodating any real supersymmetry algebra. At a rescaling of Q_α, the minimal Q undergoes the same redefinition through eq. (3.9), but remains nilpotent. Consequently, any real, maximal supersymmetry algebra is respected in the formalism.

In SUGRA, the supersymmetry is local, and the variations only present in terms of the field redefinitions, e.g. for the 3-form as

$$C_{abc} \approx C_{abc} + (\epsilon\gamma_{[ab}\chi_{c]}), \tag{3.53}$$

under the conditions already mentioned in connection to SYM. Just as there, they correspond to eq. (3.6b) and can be analysed in terms of $\delta_\epsilon\psi$, etc. In addition, the same type of equivalence allows for

$$(\lambda^3\theta^3)^{abc}C_{abc}(x^a) \approx (\lambda^3\theta^3)^{mnp}C_{mnp}(x^m), \tag{3.54}$$

i.e. a change of the supervielbein from one smooth manifold to another, describing diffeomorphisms. As all indices are contracted, the pure spinor formulation is insensitive to the alteration. Contrary to what is true in SYM, the diffeomorphisms change the fields, which are background dependent, and in this way background invariance is implied in SUGRA, despite an initial formulation on a flat background. As noted in section 3.3.2, a presence of the soft condition torsion modules in eq. (3.14b) is captured.

The important point is that the maximal supersymmetry of SUGRA is present in the cohomology: respected by the formulation, which also goes for the diffeomorphism symmetry. They are inherent in the formalism, but not explicitly so — the equivalence classes are a rather circumvent feature, bundled up with gauge symmetries etc., so that the symmetries are just short of manifest, a concept used a bit too liberally in Paper II and IV. In fact, even the matter of manifest supersymmetry in SYM can be questioned in terms of the general regularisations and the subsequent dependence on θ (in an equivalence class). That action does, however, present manifest supersymmetry. Compare to the discussion in chapter 6; any advantage is best taken care of. At present, the importance of the pure spinor formalism lies in it constituting the only known way of describing an action respecting maximal supersymmetry. How to achieve a formulation with it present *explicitly* is another thing entirely: completely unknown, as well as the possible benefits thereof. Be it SYM or SUGRA, in terms of explicitness, the Q-equivalence falls short of what might be wished for.

Interestingly, the structure in SYM and linearised SUGRA is present already in the superspace formulation. The pure spinors merely encode the symmetry properties.

Discerning properties in terms of the component theory

The opaqueness of the pure spinor formalism with respect to the symmetries has an equivalent concerning the fields of the theory. They show up in a series expansion of the pure spinor superfield, and are in many ways best left alone there. Any attempt at tracking a part of the field ψ, or extracting it from calculations, will prove troublesome; a feature especially regrettable when it comes to comparisons between the pure spinor formalism and other approaches, as will be illustrated in the next chapter.

However, when left alone, the sheer compactness of the theory is what makes it so attractive. In many ways it simplifies the observation of properties, which is a key feature in Paper I, II and IV. Compared, for example, to the examinations of the UV divergences in SYM, the corresponding procedure in the pure spinor formalism is remarkably straightforward.

The known is finite, the unknown infinite; intellectually we stand on an islet in the midst of an illimitable ocean of inexplicability. Our business in every generation is to reclaim a little more land, to add something to the extent and the solidity of our possessions.

Thomas Huxley

4 Ultraviolet divergences

The theories of SYM and SUGRA describe interactions, and it is natural to wonder at what those look like. Key features can be illustrated and analysed in terms of 'super-Feynman rules' and amplitude diagrams; a process possible to fashion in a number of ways, e.g. through the use of the pure spinor actions in maximal SYM and SUGRA. Importantly, the search connected to the supersymmetric theories does not end at the actions, for what they contain is the true object of interest.

Integral to any examination of these theories, representing low energy limits, is whether or not the results are perturbatively finite. Ideally, perturbative finiteness is provided by default in a formulation. In its absence, it is either necessary to consider non-perturbative treatments, or to restrict the examinations to areas where perturbative finiteness is ensured, for accurate results. In this, the dimension of four is of special relevance, as it represents the type of theory directly related to our experienced reality; what is desirable to describe.

Regarding the maximally supersymmetric theories, perturbative finiteness has been a subject of interest for quite some time. The issue can be divided into two categories: the ultraviolet (UV) and the infrared (IR) regimes, where the theories may threaten to diverge. In specific, it concerns the integrations over free momenta (corresponding to x) present in loop amplitudes as loop momenta, and the ensuing dependence on the momentum cut-off. However, the IR divergences are comparatively well-known [179–193] and possible to regularise through restoring the momentum dependence of the amplitude diagrams [91, 194], as only diagrams independent of the momenta display IR divergences. Therefore, the subject of this chapter is the UV divergences.

The UV divergences in maximal SYM have long been well-known [79–81]. The theory is perturbatively finite in up to four dimensions, and as such constitutes a very well-behaved theory, for the purposes of investigations related to the physical world. The situation in maximal SUGRA is more complex, quite unsurprisingly considering the fundamental differences between the two theories. In general, SYM at best is a guideline for what happens in SUGRA, though usually a highly important one. The UV divergences in maximal SYM provide a general understanding for what to expect in maximal SUGRA — a dependence on the number of loops present, and the dimension of the theory under consideration. The higher either of those is, the more divergent the corresponding amplitude diagram. The SYM investigations also provide the groundwork for some of the examinations of the SUGRA equivalent.

The UV divergences in maximal SUGRA still remain to be determined in full. The final result may range anywhere between that of SYM and a theory convergent in up to two dimensions; the first limit arises naturally as SYM is part of SUGRA, and the second, since the theory of gravity in two dimensions is trivial. The key question is what happens in four dimensions.

In fact, maximal SUGRA is not expected to prove finite in $D = 4$. That would correspond to the theory constituting a well-defined quantum theory, on its own. The same is expected for M-theory, and the presence of two consistent, inequivalent field theories would present a strange situation. It is currently supposed that maximal SUGRA needs to be altered in some way, through taking a non-perturbative treatment of the underlying (M-)theory into account, for it to make sense. The ongoing investigations of $\mathcal{N} = 8$, $D = 4$ SUGRA points towards a first possible divergence at 7 loops, but the results of Paper IV and V in addition show a removal of the divergent terms at and above 7 loops. Due to this, there is a possibility of perturbative finiteness in $D = 4$, which warrants further investigations. The situation is highly interesting.

This chapter starts out from the UV divergences in maximal SYM; the associated general methods and features are described in order to facilitate the outline of the situation in maximal SUGRA. The subsequent presentation of the investigations in maximal SUGRA, past and present, includes an outline of the general investigations using counterterms, the explicit calculations which are performed for the four-graviton amplitude, and the investigations based on the pure spinor formalism. Finally, the current results are summarised.

4.1 Maximally supersymmetric Yang–Mills theory

Maximal SYM is perturbatively finite in $D \leq 4$, as first proven in [79–81], and the means to get to the proofs adequately illustrate the key properties of the UV divergences in the maximally supersymmetric theories; the effects of a restriction on a finite, non-perturbative theory. They are visible as a dependence of the amplitude diagrams on the momentum cut-off, and are at times superficial in that they cancel out due to symmetry properties. Some cases are perturbatively finite, others not. In determining which are which, the starting point is in the generalisation of what takes place in ordinary space; the situation of quantum field theory naturally provides a basis for the workings of the superfield theories. There, the divergences are described in terms of counterterms.

Counterterms

So-called counterterms occur in quantum field theory as local, physical operators that are products of the procedure of renormalisation — one of the ways to remove the UV divergences of a theory. They constitute the extra terms in the renormalised action (compared to the initial one) that appear as the convergent and divergent terms are separated. In this, the counterterms represent the divergent parts of the theory: one type of divergence for each counterterm. The renormalised perturbation theory includes Feynman rules with counterterms, but the divergences can be suppressed through an alteration of the coefficients in the counterterms. The procedure is especially suitable for analysing and removing the superficial UV divergences of multi-loop diagrams. [164]

In the supersymmetric theories, the concept of counterterms is simply extended to the supersymmetric setting, by default connected to the UV divergences in the loop amplitudes in the same way as is the case in ordinary quantum field theory. However, a theory cannot be altered into finiteness. Instead, the relevance of the counterterms lies in the identification of the perturbative behaviour of a theory. The object of interest in the analysis of whether a theory is perturbatively finite or not is the set of counterterms that may be formulated non-trivially in it, as that equals an identification of the unavoidable UV divergences. If no such counterterms can be formulated, the theory is perturbatively finite. In a step-by-step approach, each type of diagram is proven perturbatively finite through a proof of the absence of counterterms.

Miraculous cancellations

A typical trait of the supersymmetric theories is unexpected cancellations among the UV divergences. Some UV divergences that initially might be considered unavoidable because of a presence of befitting counterterms in a context analogous to quantum field theory, simply vanish in the superfield perturbation theory. This is e.g. the case in the Wess–Zumino model [195–197], which contains no mass or interaction counterterms. The simplifications are caused by the supersymmetry present in the theories, making it integral to keep the supersymmetry as a natural part of the formulation during investigations. Without it, a loss of predictability occurs, or at least, predictions are rendered more difficult by far. This is doubly true for maximally supersymmetric theories, constrained by 'extra' symmetries as far as possible.

Utilising the symmetries

Since it is of interest to keep as many as possible of the symmetries of a theory manifest during investigations, for maximal simplicity, in (flat) maximal SYM the super-Poincaré symmetry preferably should be an inherent part of the formulation. Moreover, for an overall proof, an action is required. In its presence, e.g. the equivalent of Feynman rules can be identifies and utilised, to all order of perturbation theory. In the absence of an action, the divergences need to be addressed order by order.

However, there is no action for any maximally supersymmetric theory with strictly manifest maximal supersymmetry. More importantly, at the time of the maximal SYM proof of perturbative finiteness in $D \leq 4$, even an action with nearly manifest maximal supersymmetry (as provided by the pure spinor formalism) remained to be formulated. Consequently, the only way to get to an overall proof was to keep as much of the symmetry as possible.

In [79–81], for the proof of perturbative finiteness in $D = 4$ maximal SYM, manifest Lorentz symmetry was abandoned for the presence of maximal supersymmetry, in line with or perhaps as additional proof of the statement in the section on miraculous cancellations: that of the supersymmetry to be the central piece in the puzzle. The proofs were performed in two different ways:

- In [79, 80], the light-cone frame was used for the formulation of an action, with the subsequent amplitudes given by supergraph Feynman rules in a certain gauge, where the amplitudes could be proven finite (to any order of perturbation theory) through a power counting of the constituent momenta. The gauge invariance of the UV divergences (by them being a product of on-shell external states) then set the theory to be finite.
- In [81], the $\mathcal{N} = 4$, $D = 4$ theory was captured through an $\mathcal{N} = 2$ SYM theory coupled to an $\mathcal{N} = 2$ matter theory in the adjoint representation. In that setting, the non-renormalisation theorem (limiting how renormalisation can change a theory) set the UV divergences originating in two loops or more to correspond to gauge invariant functionals of the background fields, which by symmetry properties vanish. In addition, convergence was shown for the 1-loop structures. Consequently, no non-trivial counterterms existed, yielding a finite theory.

From this, finiteness in the dimensions lower than four is given by default. However, $D = 4$ also represents the upper limit on the perturbative finiteness of maximal SYM. In $D = 5$, the theory is UV divergent, as conjectured in [85] and shown in [86], both approaches which are quite different from the ones just presented. The latter uses harmonic superspace [198] whereas the former is a unitarity method developed for the examination of the maximal $D = 4$ SUGRA theory, though based on SYM investigations [84, 85, 100, 106, 107].

The methods of investigation relevant for the UV divergences in maximal SYM that have been developed since the $D = 4$ proof will be described in more detail in the following section on maximal SUGRA, apart from the one in [86]. This includes, besides the unitarity method, approaches based on the pure spinor formalism, which are equally applicable to the maximal SYM and SUGRA theories. In the detailed situation in maximal SYM now to be addressed, the inclusion of the pure spinor results e.g. is relevant in a comparison with SUGRA. Partly, it also helps in establishing the relevance of the results (as they are in accordance with the results produced by other means) and partly in that the pure spinor results in some ways are more specific than that.

4.1.1 Factors influencing the UV behaviour of the amplitudes

The limit on maximal SYM for perturbative finiteness, [79–81]

$$D \leq 4, \tag{4.1a}$$

can be specified in more detail with respect to different amplitude diagrams, or parts thereof. Naturally, a general statement concerns all possible amplitude diagrams involved, but in understanding the nature of the UV divergences, it is equally important to note how they arise. This is doubly true in a comparison with SUGRA; in order to specify the overall UV behaviour, it is necessary to identify the factors that determine it, and how they do so.

The loop dependence

The most obvious factor contributing to the UV divergences of an amplitude diagram is the number of loops present in it. An amplitude with no loops is convergent in any dimension, while perturbatively finite in the presence of L loops if [78, 85, 86]

$$\begin{aligned} &D < 8 && L = 1, \\ &D < 4 + \frac{6}{L} && L > 1. \end{aligned} \tag{4.1b}$$

The overall constraint on the maximal SYM theory in eq. (4.1a) is here due to the open limit of $L \to \infty$. For separate amplitudes, the higher the number of loops present, the more severe the constraint on the dimension. The first divergence

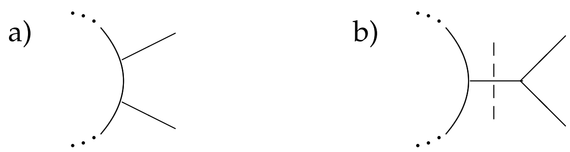


Figure 4.1: A one-particle irreducible 2-point connection to a loop structure (a) and the by default (if allowed by loop integration) accompanying one-particle reducible 2-point connection (b). The cut indicated divides the diagram in (b) into two separate one-particle irreducible parts, impossible to divide further (non-trivially) by a single cut. The subsequent tree diagram part in (b) could equally be replaced with a second loop structure mirroring the first, provided no more connections exist between the two.

in five dimensions occurs for the 6-loop diagram, as stated in [86], and explicitly calculated in [107].

The limits originate in the power of the momentum cut-off being required to be negative:

$$\begin{aligned} \Lambda^{DL-8L} &\ll 1 \qquad L = 1, \\ \Lambda^{DL-4L-6} &\ll 1 \qquad L > 1, \end{aligned} \tag{4.1c}$$

where DL shows up by default due to the integration over D loop momenta in each loop. Effectively, this means that the higher the dimension, the more divergent the theory (or less convergent, when the requirements are fulfilled). The same also goes for an increase in the number of loops under consideration.

The relevant parts of an amplitude diagram

Most investigations are performed for the 4-point amplitude: a completely general set of amplitude diagrams, with four external particles. This represents the lowest number of external particles that may be present, since tree diagrams naturally have this lower limit[1] and loop diagrams are required to be characterised similarly, by the non-renormalisation theorem, for finiteness. Their vanishing is typically confirmed separately in each setting for the investigations of the amplitudes, for example as in [24, 104]. Moreover, the general n-point amplitudes are not expected to behave any worse than the 4-point versions. As for the absence of the lowest n-point diagrams, the arguments for this take different shapes in different settings.

A general amplitude diagram can be divided into one-particle irreducible parts, as illustrated in fig. 4.1. In this setting, the requirement of four external particles is not to be confused with the required connections to a one-particle irreducible subset of the amplitude diagram. Naturally, any overall number of

[1]Simply a 3-point vertex between external particles (on-shell) is trivial.

legs free to connect to a diagram may do so in any configuration, e.g. as illustrated in fig. 4.1, but the different parts thereof are separate. In the one-particle irreducible setting, the absolute lower bound for an n-point (strictly speaking: part of a) diagram is 2 instead of 4. The former allows for the latter[2] through vertices external to the one-particle irreducible loop structure, while avoiding the total derivative a single propagator would constitute on the expression, thereby representing a valid, non-trivial entity.

A somewhat subtle point, and the reason for considering one-particle irreducible loop structures, is that the UV divergences of those parts of an amplitude diagram are independent of one another. This is observed in general for maximal SUGRA, but is perhaps most obvious in a pure spinor approach. The loop momenta shared between different loops is what sets the UV divergences, and loop momenta are not shared between one-particle irreducible subsets of a diagram. As such, amplitude diagrams may be split into their one-particle irreducible parts before an analysis of the subsequent UV divergences, and a diagram does not diverge any more than what is set by the worst behaved one-particle irreducible part of it.

The UV behaviour of the one-particle irreducible parts of an amplitude diagram

For a one-particle irreducible loop structure, the number of loops is not the only factor that sets the UV divergence. When the number of legs (j) connected to it is considered, it is possible to observe that the UV behaviour changes to the better for higher j. It is most commonly noted e.g. for the 4-point 3- and 4-loop amplitude diagrams of one single one-particle irreducible loop structure that, depending on the configuration of the loops, demand different minimum js. The result is that the UV divergences differ between the different configurations. In Paper IV and V, this was mentioned and the conditions for finiteness *for such a part* implied to alter from eq. (4.1b) into

$$\begin{array}{lll} D < 2j & L = 1 & 4 \leq j \leq 10, \\ D < 8 + 2\left\lceil \dfrac{j-1}{2} \right\rceil & L = 1 & j > 10, \\ D < 6 + \dfrac{2j-6}{L} & 1 < L \leq 3 & L + j \leq 7, \\ D < 4 + \dfrac{4 + 2\lceil j/2 \rceil}{L} & L \geq 5 & 2L + j > 12, \\ D < 4 + \dfrac{2\lceil (L+j)/2 \rceil}{L} & \text{otherwise,} & \end{array} \tag{4.1d}$$

[2] And the latter is required through varying properties, e.g. in the pure spinor formalism for the integration over the fermionic variables to be non-zero (additional terms required), as noted in [104].

with additional conditions: $j \geq 4$ for $L = 2$, $j \geq 3$ for $L = 3$ and $j \geq 2$ for $L \geq 4$. That is, in the 3-loop 4-point amplitude diagram, there are two types of configurations with 3-loop one-particle irreducible structures, different with respect to their UV divergences: the 4-point and 3-point diagram parts. The first is perturbatively finite in $D < 20/3$, the second in $D < 6$, as observed in e.g. [103, 104]. It is also possible to note that the leading divergence at $L = 4$ ($D < 11/2$) is caused by $j = 2$, just as observed in [100].

Summary

What is of interest for the examinations of the UV divergences is the behaviour of the L-loop one-particle irreducible parts that go into the amplitude diagrams. In maximal SYM, the power of the momentum cut-off increases with the dimension and the number of loops, but decreases with a higher number of legs attached. Therefore, the lowest number of j allowed for sets the overall behaviour of the L-loop, n-point one-particle irreducible diagram. The overall UV divergence of an amplitude is effectively that of the most divergent one-particle irreducible constituent part.

4.2 Methods of investigation in maximal supergravity

There exists quite a number of approaches to deducing the UV divergences of maximal SUGRA. Most investigations (but not those using pure spinors) take place in $\mathcal{N} = 8$, $D = 4$ SUGRA [6]. The most common object of investigation is furthermore the existence or absence of counterterms: that of what limits their presence. However, regardless of the method, the results can always be interpreted in terms of counterterms.

In this, some approaches are set apart from the others in terms of their unique methods or the importance of their results, such as the explicit calculations for the four-graviton amplitude, the U-duality and counterterm arguments for the UV behaviour up to seven loops, and the examinations using pure spinors. In particular, it may be noted that the pure spinor field theory approach, that of Paper II, IV and V, does not build on a deduction of counterterms, although the result may be interpretable in terms of them, in a way similar to the situation in [103, 104].

Possible counterterms

The behaviour of a theory (i.e. of its amplitudes) is a product of on-shell external states. As such, the UV behaviour ought to be gauge invariant and independent of the diffeomorphisms of the theory in question. For maximal SUGRA, with graviton external states, this leaves a possible dependence of the counterterms on the Riemann tensor and the Ricci tensor and scalar, transforming homogenously under coordinate transformations [88]. In addition, further restrictions occur due to dimensional properties. The counterterms are usually denoted by covariant derivatives acting on *Riemann tensors*, schematically:

$$\mathcal{D}^{2k} R^{L+1-k}, \tag{4.2a}$$

where a loop dependence on the interpretation of the counterterm is visible, occurring due to dimensional properties. The first possible divergence for each loop configuration, the logarithmic divergence, is for the 4-point diagrams described by

$$\mathcal{D}^{2(L-3)} R^4. \tag{4.2b}$$

With an increase in k compared to this, the counterterms describe increasingly worse UV divergences. In total, the behaviour of a diagram is partly set by the tendencies towards divergence of each loop in it, and partly by the overall loop configuration. [88]

4.2.1 General summary of the investigation methods

There has been a number of different approaches to restricting the counterterms further, apart from what is set by gauge invariance and dimensional properties. Light-cone, non-Lorentz covariant, harmonic and conventional superspace approaches, as for maximal SYM, have traded one type of manifest symmetry for another in different configurations, for an accurate description of the interactions through Feynman rules etc. Naturally, the removed property is utilised to reduce the derived counterterms (of the restricted theory) in retrospect or separately, in want of a complete formalism more like the kind provided by the pure spinor. Furthermore, the string theory dualities have been applied to different degrees, in order to catch the properties of the SUGRA theory that are hidden in the low energy limit representation, and otherwise lost in approaches in $D < 11$.

The finiteness up to three loops [70–75, 89] can e.g. be explained in terms of the amplitudes being colourless[3] and exhibiting crossing symmetry [90]:

- At the investigation of an n-point amplitude, it is necessary to consider a summation over all possible orderings of the external legs, as well as over all possible diagram contributions. Hence, additional cancellations of the divergences, not exhibited by the solitary diagrams and external leg configurations, may occur.

or investigated in terms of helicity amplitudes [92]. At that point, it also becomes apparent that U-duality[4] is present as a relevant mechanism in the amplitude calculations. A review of the situation for up to three loops can be found

[3] In gravity, there is no concept of colour, in contrast to what is true in quantum chromodynamics (QCD).

[4] Chapter 4 provides an introduction to this. In four dimensions, the relevant symmetry is $E_{7(7)}$.

in [93], although the at the time conjectured limits for UV finiteness at higher loop order differ from the current situation, described in section 4.3. In addition, [96] tried to determine whether or not the 3-loop finiteness could be completely explained by $E_{7(7)}$ invariance, without a conclusive result.

The proof of perturbative finiteness for 4-loop amplitudes [95], through explicit calculations for the four-graviton amplitude, caused renewed interest in the perturbative behaviour of maximal SUGRA. Supersymmetry and U-duality arguments were combined to further narrow down the possible counterterms. [99] found the non-linearised theory to be non-anomalous with respect to continuous U-duality, to all orders in perturbation theory:

- Counterterms must respect the $E_{7(7)}$ symmetry.

This requirement was used in [97, 101, 102] to exclude counterterms in the 6-point $D = 4$, $\mathcal{N} = 8$ SUGRA amplitudes up to 6 loops, in accordance with the 4-point results of [98], obtained through arguments for that the UV behaviour from 4 loops and upwards should be determined by the factorisation of the $\mathcal{D}^8 R^4$ operator. At 7 loops, the only valid counterterm was recognised to be precisely $\mathcal{D}^8 R^4$, in summary a

- $D = 4$ first possible counterterm at 7 loops (4-point), encoding logarithmic divergence.

Since a counterterm invariant under both supersymmetry and U-duality had been recognised to exist at eight loops early on [76, 77], [98, 102] effectively implied the question of UV divergence in four dimension to be between 7 and 8 loops.

Meanwhile, the pure spinor approaches merely (or at best, depending on the amount of properties considered) confirmed the results of the analyses performed through other means, up until the results of Paper IV, where an interesting cut-off at seven loops was recognised. This, as well as the aforementioned results, will be further discussed in the following section on the current state of affairs for the UV divergence in maximal SUGRA.

4.2.2 Explicit calculations for the four-graviton amplitude

A systematic search for the UV divergences of maximal SUGRA is to be found in the explicit calculations performed for the four-graviton amplitude in [89, 91, 95, 105], where [89, 95] provided the proofs of 3- and 4-loop finiteness. The procedure is meticulous in its execution, working its way up from the amplitude diagrams with the lowest number of loops, and at each new step drawing upon the previous results. It is also highly time-consuming and dependent on the available computer power, and increasingly so with an increasing number of loops in the amplitude diagrams, as the number of different diagrams possible to construct increase drastically with L. The latest computations could not have been performed a few decades ago, nor is it very reasonable to believe them to be carried on very far — the related goal is to determine the deviations from the maximal SYM case and when the maximal SUGRA diagrams get UV divergent, as well as what causes those additional divergences. It is, on the whole, a thoroughly impressive procedure.

The investigations concern the 4-point amplitude and take place directly in the field theory, with two key concepts to make the whole feasible: the unitarity method and the Kawai–Lewellen–Tye (KLT) relations. These are applied to simplify the field theory description of the interaction amplitudes, so that the complicated loop structures (causing the divergences) can be termed in tree diagram parts in maximal SYM. Still with an encoding of the UV divergences, of course. Because of this, the (computer) calculations include the investigations of the maximal SYM case, previous to the SUGRA investigation, which is why results from this approach were brought up in the section on maximal SYM. The simplifications are vital for the feasibility of the investigations, the first to get rid of the complicating loop structures with subsequent loop momenta, etc., the second to end up performing the calculations in a well-known setting, easier to deal with: maximal SYM. A detailed introduction can e.g. be found in [88].

The unitarity method

Since the scattering matrix is unitary, unitary relations can be applied to the description of the amplitude diagrams with the end result unaltered. In specific, this is relevant due to the existence of unitary relations between structures of a certain genus — number of loops — and those with lower genus, effectively all the way down to tree level. Provided an amplitude fulfils certain conditions, such as is the case in the $D = 4$ maximally supersymmetric theories, it is so-called cut-constructible. That is, the amplitude is constructible from information on its cuts and the intermittent tree diagram parts. The computation of the interaction amplitudes, or that of possible counterterms present, in that way simplifies to that of tree computations, which is of key value in the investigations of e.g. the UV divergences in maximal SUGRA. However, it is not to to be confused with the property of perturbative finiteness of the tree diagrams, which does not hold for the loop diagrams. [82, 83]

The Kawai–Lewellen–Tye relations

The KLT relations [199] concern closed and open string tree amplitudes: the former can always be expressed as a sum of the latter. Since the maximal SYM and SUGRA theories represent the low energy limits of these superstring theories [78], SYM from the type I open superstring and SUGRA from the type II closed superstring, the KLT relations infer relations between maximal SUGRA and SYM. With them, it is possible to interpret supergravity tree amplitudes in terms of SYM tree amplitudes.

Progress

The explicit calculations for the four-graviton amplitude have so far reached four loops, up to which point there is no deviation in the UV divergences of maximal SUGRA from the maximal SYM results. The latter have been performed for up to five loops, also including non-planar loops: loops not possible to represent by diagrams in a plane, although one investigation also has addressed the maximal SYM $D = 5, L = 6$ situation. The 5-loop calculation in maximal SUGRA is currently underway, but perhaps to some degree at a standstill. Not only is the analysis time-consuming, but there has also been an issue with the solution ansatz not being general enough, while seemingly containing what ought to be enough for the description. It is a complicated issue and not the only project of the associated researchers. Nevertheless, the results are of true relevance and eagerly awaited.

4.2.3 U-duality & $\mathcal{D}^8 R^4$ arguments

The presently conjectured behaviour of loop diagrams with $5 \leq L \leq 7$ in maximal SUGRA is the result of [98,102], representing two different approaches. The first is based on what type of counterterm, representing the worst possible UV divergence, might be allowed in the amplitudes based on the situation at four loops. The second is due to an analysis of what possible counterterms might be invariant under the required $E_{7(7)}$ symmetry.

$\mathcal{D}^8 R^4$ arguments

In [98], the analysis is based on the counterterm responsible for the worst UV divergence for the 4-loop amplitude in $D = 4$ maximal SUGRA: the $\mathcal{D}^8 R^4$. This is the counterterm responsible for the worst divergence (all dimensions) that may exist above 3 loops, and it is not loop dependent above $L = 4$. Depending on L, the divergence it causes sets different limits on the dimension, for perturbative finiteness of the theory, through a connection to the momentum cut-off according to

$$\Lambda^{(D-2)L-6-2\beta_L} \mathcal{D}^{2\beta_L} R^4, \tag{4.3}$$

with β_L restricted [95] to be at least 4 for $L \geq 4$. A similar analysis of the dimensional restrictions can be made for $L \leq 4$ in accordance with the results obtained regarding the counterterms, which restrict β_L to be at least L, compare to eq. (4.4).

U-duality restrictions on counterterms

The results of [102] concern the possible counterterms for 5- to 9-loop diagrams with up to 6 external states. The key test is what counterterms are invariant under $E_{7(7)}$, and to obtain the candidate counterterms that may exist in maximal SUGRA, closed string amplitudes obtained from the open string amplitudes of [200–202] through the KLT procedure are analysed. The matrix elements are acquired from the α'-expansion of the closed string tree amplitude rather than e.g. through Feynman rules, which would contain too may complications. Furthermore, averaging etc. over the external states is performed before a final analysis of the U-duality invariance of the counterterm candidate, which is analysed in the soft scalar limit of the obtained next-to-maximally-helicity-violating (NMVH) amplitudes.

4.2.4 Pure spinor approaches

Despite the ingenuity of analyses unconnected to actions with maximal supersymmetry, the importance of methods based on such actions is not to be neglected. As pointed out in the section on maximal SYM, inherent symmetries are expected to simplify the analysis to the outmost, which due to the complicated setting becomes all the more relevant in maximal SUGRA. It is not easy to determine the UV divergences without a full description.

The pure spinor investigations were initiated in [24,28,94]. As already mentioned, the pure spinor itself was initially introduced to provide a covariant description of the superstring and the superstring amplitudes. The tree amplitude results of [16], found to give at hand perturbatively finite 1-loop diagrams in [87], was extended to loop amplitudes in [24], reformulated in the presence of the non-minimal variables in [28], and further developed with respect to integrated vertex operators etc. in [94].

These investigations, continued and to some degree improved in [103, 104], are characterised by their string theory approach, in contrast to what is true for the field theory approach of Paper II, IV and V. The fundamental difference lies in the construction of the amplitude diagrams for the scalar particle. In the string theory approach, the amplitude diagrams are determined by arguments from string theory, in a way similar to how operators and characteristics of the pure spinor formalism were identified. Typically, the inherent maximal supersymmetry is lost, and it is necessary to check BRST invariance of the results to verify their consistency.

In the field theory approach, however, the rules for the amplitude diagrams are deduced from the action in question, SYM or SUGRA, in eq. (3.33) and (3.37). In this way, the symmetry properties are kept inherent in the formulation throughout the investigations. Arguably, this invokes the symmetry properties of the theory to a maximal extent, at least in the absence of an action with (truly) manifest supersymmetry.

The string theory approach (1st quantised version)

The string theory approach of Berkovits centres around the vertex operators and the pure spinor properties of the open superstring description along with the propagator obtained from a gauge fixing for the scalar particle. The amplitudes (for the scalar particle) are built from from these components. For example, a propagating field can be split into two states by a vertex operator acting on it. The ensuing states can be described to propagate further by adding propagators acting on them, etc. In this, there is a choice of expressing the process in unintegrated or integrated vertex operators, the latter of which encodes the propagation of one of the two states in the just mentioned example, and is possible to describe in terms of an unintegrated vertex, a propagator and an integral. The description of any amplitude is in this way built piece by piece.

The resulting amplitudes initially figure in a kind of maximal SYM setting, as the chosen propagator for the open superstring is the same as for maximal SYM. However, the inherent BRST equivalence of the formalism is by no means guaranteed in the ad hoc (though by no means strange in a pure spinor context) recognition of the amplitude construction. Therefore, the results need to be checked for BRST invariance, which typically is present. The encodement as such seems satisfactory and complete.

For the SUGRA results, the SYM procedure is generalised to describe maximal SUGRA in $D = 10$ through an imitation of the transition from the open string to the closed string, a process which includes doubling all fields except for x^a and its momenta. This alters the vertex operators, the degrees of freedom of the variables, etc., and in the end provides results for maximal SUGRA in $D \leq 10$.

The extensions of [103, 104] in this setting mainly is a matter of how to deal with the problems with loops that occur in the pure spinor formalism. The propagator is too local (proportional to a delta function) with respect to the pure spinors to work in a context with loop integrations, and needs to be altered in a BRST equivalent way for the formation of loops. This is feasible in an ingenious combination of general regularisations and loop properties: the loop momenta are introduced as one set of the general regularisation variables, introduced for the loop in question to render the propagator non-local enough [28]. This, done for each loop in a diagram, takes care of the loop integration in a way consistent with the pure spinor formulation.

However, there is a slight problem with relying on the results of [103, 104] in the light of Paper IV, due to the absence of certain non-minimal variable constellations which was not previously recognised. The difference for the SYM results can be expected to be less to (most probably) none due to the small change introduced as well as the nature of it, though the results reasonably cannot change, either, since they correspond to the results for maximal SYM established through other methods than the pure spinor approach. It is, however, difficult to conclude the implications for the SUGRA results, despite their reproduction of the expected results from the explicit 4-graviton calculations and [98, 102].

The field theory approach (2nd quantised version)

This pure spinor approach was developed in Paper II and represents a complete field theory approach originating in the pure spinor action for SUGRA of eq. (3.37). What corresponds to the Feynman rules in quantum field theory has been derived from the action and the propagator, by consistency demands such as that the overall ghost number of an amplitude diagram ought to be zero for it to be physical, etc. The loop integration is that of [103, 104], representing a BRST equivalent alteration of the theory. This process was originally performed for maximal SYM, which was not addressed in Paper II since the results were the same as those in [103, 104]. In Paper IV, on the other hand, some new insights regarding the non-minimal variables were presented, and the SYM case displayed both as an illustrative example and for a consistency check of the SUGRA results.

The marked similarities between the observed workings of the string and field theory approaches in SYM at an analysis of the UV divergences is unsurprising: the building blocks are effectively very much the same. However, in SUGRA marked differences show in the shape of the propagator (the field theory version is nothing like the SYM ditto) and the vertices. Most likely, these differences make it easier to capture the maximal SUGRA properties, observed in Paper IV and V in the ($D = 11$) field theory setting.

In contrast, the processes of the two different field theories (SYM and SUGRA) in principle differs but little from each other (due to dimensional properties and different degrees of freedom) but for the existence of operators in the vertices in SUGRA. This, clearly visible already in the difference between the SYM and SUGRA 3-point and 4-point interactions in eq. (3.33) and (3.37), changes the properties of the theory, which is not surprising — theories of gravity are fundamentally different from gauge theories.

Still, SUGRA in many ways follows suit after SYM, disregarding a more allowing situation for divergent terms resulting in more severe UV divergences, except for a circumstance of the restrictions brought on by the non-minimal variables, as observed in Paper IV (which furthermore made the conjectured UV divergences of Paper II void for $L > 1$). This is discussed in more detail in the next section.

It is difficult to tell in what way the formulation would benefit from manifest supersymmetry. Due to the construction, the maximal supersymmetry is an inherent part of the description. Nor does SYM require a formulation more to the point than provided by the pure spinor formalism. In addition, the further properties, unique to SUGRA, that remained to be identified after Paper IV in order to give rise to the (known) overall UV behaviour, was identified in Paper V. Perhaps further simplifications would occur in a manifest setting, but it is likely that any result is equally observable in the pure spinor formalism.

4.3 The state of affairs in maximal supergravity

The analysis of the UV divergences in maximal SUGRA has a long history, and so have the numerous results. The ones here presented are deemed to be the most integral to the current limits on the perturbative finiteness of the maximal SUGRA theory. For a more extensive coverage of the subject, the reader e.g. is referred to the proceedings[5] of the workshop on 'Breaking of Supersymmetry and Ultraviolet Divergences in Extended Supergravity' that was held in 2013, the volume of the 'other publication' in the list of appended research papers of this thesis.

4.3.1 Summary of the results concerning the UV divergences

The results can be divided into two categories:

1. The counterterms explicitly determined by 4-graviton calculations (though not initially for $L = 1, 2$). These are set for $L \leq 4$.
2. The counterterms constrained by the symmetries of maximal SUGRA, setting a limit on the worst possible UV divergences.

Implicit in this is as a rule of thumb that the investigations concern 4-point amplitudes. Not much effort has been spent on analysing n-point diagrams with $n > 4$, although the exception to the rule [102] will prove most intriguing, and an increased interest in 5-point amplitudes has been visible since Paper IV. In specific, the second point above can be divided into three areas of interest:

- $L < 7$ where divergences cannot occur in $D = 4$.
- $L = 7$ where supersymmetry and U-duality properties allow for a counterterm in $D = 4$ for the 4-point amplitude [98, 102] *but not* for the n-point one-particle irreducible amplitude with $5 \leq n \leq 6$ [102]. Meanwhile, the result of Paper IV states that the 4-point diagram vanish, and Paper V indicates those of $n > 4$ to be convergent in $D = 4$.

[5] Presentations given at the workshop can also be found at *http://www.lnf.infn.it/theory/buds*.

- $L > 7$ where supersymmetry and U-duality properties allow for counterterms for all n-point diagrams ($n \geq 4$) [76,77,102], while the results of Paper IV (and V) state that all such one-particle irreducible diagrams vanish, limiting the divergences to those of $L \leq 7$.

Up until recently, the $L = 7$ amplitude was regarded as the amplitude to set the limit of the perturbative finiteness of $D = 4$ maximal SUGRA: either the perturbative theory was to diverge at $L = 7$, or at $L = 8$. However, in combination with the results of Paper IV, the results of [102] (reproduced in Paper V for all n), specific with respect to the n-point dependence of the possible counterterms, open up for the scenario of $D = 4$, $\mathcal{N} = 8$ SUGRA as perturbatively finite. Even when disregarding the result of Paper V, it is unlikely that counterterms show up for 7-point diagrams, or higher, when absent for the lower ones. This is extremely interesting, calling for further investigations in the different settings, due to the opaque workings of the pure spinor formalism.

An overview of the explicit results is illustrated in fig. 4.2, the properties of which will now be discussed in detail.

At $L \leq 4$

The counterterm results of the $L \leq 4$ amplitude diagrams set the limit on the UV perturbative finiteness to [70–75,89,95]

$$\begin{aligned} &D < 8 && L = 1 \\ &D < 4 + \frac{6}{L} && 2 \leq L \leq 4 \end{aligned} \tag{4.4}$$

for the 4-point amplitude: exactly the same (overall) behaviour as in maximal SYM. In fact, the coincidence with SYM carries surprisingly far with respect to the number of loops. A deviation from the SYM case was expected earlier, and has now been postponed to the $L = 5$ amplitude, where the actual case remains to be explicitly addressed.

At $5 \leq L \leq 6$

The key point of interest in this regime is when maximal SUGRA deviates from the SYM case. The loops above $L = 4$ and below $L = 7$ are characterised by a UV behaviour somewhere in-between the limits of [98,102]

$$2 + \frac{14}{L} \leq D_c \leq 4 + \frac{6}{L} \qquad 4 \leq L \leq 6, \tag{4.5}$$

where D_c is the critical, lowest dimension where UV divergences may occur. The situation in SYM provides the best case scenario, while the worst possible

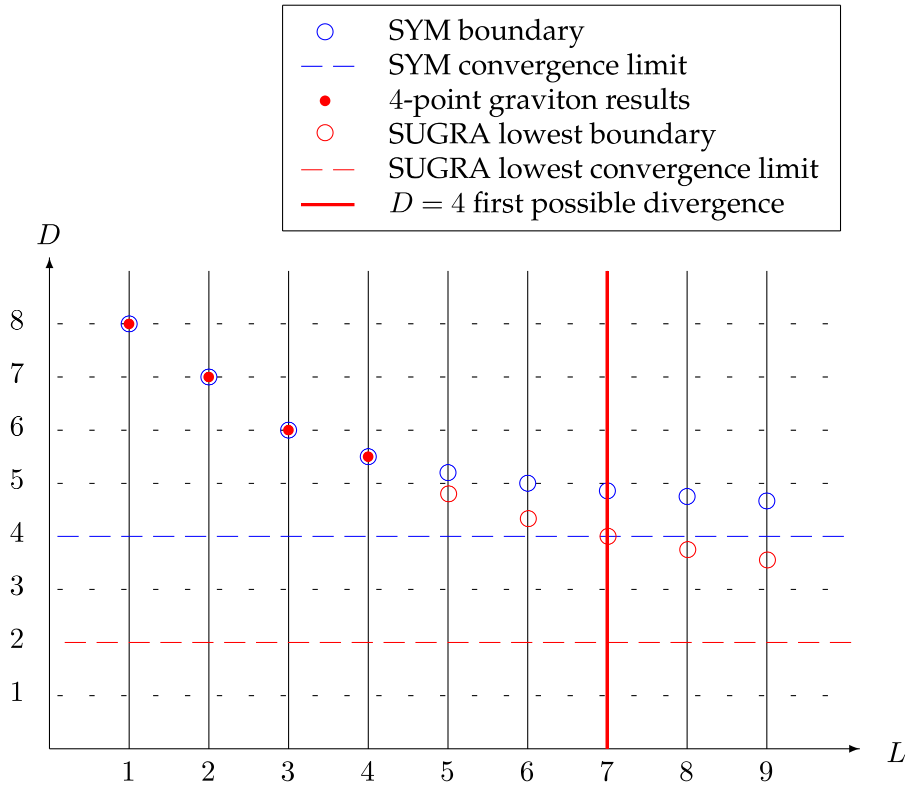


Figure 4.2: The limit on the dimension (D) of a maximal SUGRA theory for it to be perturbatively finite, as deduced through different studies. It is presented for the 4-point amplitude, as a function of the number of loops (L) present. The blue colour refers to the situation in maximal SYM (compare to section 4.1.1) where the requirement is $D < 4 + 6/L$ for $L > 1$. This SYM boundary (the lowest dimension for divergence) represents the upper bound for what may be the case in SUGRA. The dashed line represents the highest dimension in which maximal SYM is perturbatively finite (all loop configurations considered). The red colour denotes various SUGRA investigations, in analogy with the SYM markings. The results from the four-graviton calculations are in unison with the results by the counterterm approaches and Paper V, otherwise displayed as red circles. There are two highly interesting points here, the situations for $L = 5$ and $L = 7$. The first is where SUGRA currently is expected to deviate from the SYM case, a limit that previously has been lower. When a deviation occurs, and how it appears, will provide an indication for what to expect at higher L, in specific when the $D = 4$ theory turns divergent. At present, any deviation is expected to result in a curve below the SYM boundary, possibly tangent to the SUGRA lower boundary, forcing the overall limit below four dimensions. The first possible divergence then occurs at 7 loops. However, the pure spinor results of Paper IV state that the loop dependence ceases after 7 loops, making the $L = 7$ situation integral. Since the same results state that the 7-loop configurations strictly are not allowed in a 4-point diagram, the relevant examination is that of the 5-point amplitude, with [102] and Paper V implying convergence. This opens up for a situation where maximal SUGRA is perturbatively finite in $D = 4$.

behaviour is bounded from below only by $D \leq 2$ in an extended argument, for the overall number of loops, as also was the result of [103,104]. The corresponding result of Paper II was, however, rendered void by Paper IV, in the absence of constellations of $(\bar{\lambda}, r)$ with $r^x : x > 15$, but successfully analysed in Paper V to give both the results of eq. (4.4) and (4.5).

At $L = 7$

The amplitude with 7 loops presents the crucial point for the present analysis. It is where the first UV divergences in $D = 4$ may occur, with a limit on the dimension of the theory according to [98, 102] and Paper V:

$$D < 2 + \frac{14 + 2s}{L} \quad s : \begin{cases} 0 & \text{if} \quad j = 4 \\ 1 & \text{if} \quad 4 < j \leq 6 \end{cases} \qquad L = 7, \tag{4.6}$$

with the s as an extension[6] of the argument in [98] in the presence of the [102] results, which specify results for n-point amplitudes, (presumably) directly connected to the number of outer legs of one-particle irreducible loop diagrams.

Usually, this relation is shown for the 4-point amplitude, which was the subject of investigation in [98]. There, a logarithmic divergence in $D = 4$ was shown to be possible. [102] instead specifies the divergences with respect to up to 6-point amplitudes, with the 4-point amplitude connected to a logarithmic divergence in $D = 4$, while the others are found to be perturbatively finite. This corresponds to the result of Paper V, not limited to $n \leq 6$.

On the other hand, according to the results of Paper IV and V, 4-point diagrams with 7 loops vanish. For the presence of one-particle irreducible structures (each 4-point by requirement) with 7 loops, 5 outer legs are required. This implies a restriction to diagrams that by [102] and Paper V are not as divergent as the 4-point version. The general n-point diagrams have of course not been investigated outside the pure spinor formalism, but even so, higher point diagrams ought not present worse divergences than the lower point ones. Consequently:

- The logarithmic divergence seems to be completely avoidable at 7 loops[7].

Above $L = 7$

Traditionally, UV perturbative finiteness in $D = 4$ has been regarded as highly unlikely (or impossible) above 7 loops. Counterterms valid in the presence of

[6]Regardless of the accuracy, the important point is the addition of some positive number on the right hand side of the relation, for $4 < j < 7$.

[7]Note that this conclusion was not drawn in Paper IV, due to a too heavy focus on the 4-point amplitude results which overlooked the 5-point result of [102].

maximal supersymmetry and U-duality at 8 loops were recognised early on [76, 77] and the behaviour effectively conjectured to be that of eq. (4.5) extended to a higher number of loops. There, the overall lowest limit of perturbative finiteness in $D = 2$ is unsurprising, obtainable from a general power counting of the loop momenta.

The results of Paper IV and V, on the other hand, state a cut-off of the loop dependence at 7 loops. That is, one-particle irreducible diagrams may not have more than 7 loops, setting the overall behaviour of the higher loop diagrams (made out of several one-particle irreducible diagrams) to be that of the 7-loop diagrams. In that way, the UV divergences are, in combination with [98, 102], identified as entirely absent in $D = 4$.

4.3.2 The cancellations of Paper IV and V

There are two parts to the results of Paper IV and V; one controversial, the other one not. The first is the cut-off of the loop dependence, the second the limit on the possible counterterms already observed (effectively) in [98, 102]. A first, logarithmic divergence at 7 loops, as discussed in e.g. [109], is a well-established scenario. Even a postponement to 8 loops would not be especially unthinkable, but there has been nothing identified outside the setting of Paper IV and V to indicate a cut-off of the loop dependence, especially not exactly at the crucial point of divergence in $D = 4$. The closest would be arguments for finiteness in $D = 4$ like [108], by no means definitely established.

In Paper IV and V, the cut-off of the loop dependence is due to

— a restriction on the effectively present parts of R in the vertices, so as not to constitute a total derivative.

— an inherent restriction on the non-minimal spinor variable r, which only shows up in certain constellations, in combination with a duality between r and $\lambda\bar{\lambda}D$, not fully captured by the integrations over the loop momenta.

Importantly, the investigations concern parts convergent with respect to the singularities of $(\lambda, \bar{\lambda})$, allowing for BRST equivalent analyses of the properties without having to take the general regularisations into account[8]. The first restriction is integral as it defines $R \propto r$, effectively, in loops, so that there are always two r on each loop-specific propagator, in contrast to what is true in e.g. SYM. The second restriction is a consequence of this: there are rs on the loop-specific propagators, which if too many in number have to figure in their dual entity $\lambda\bar{\lambda}D$. However, the integrations over the loop momenta, equivalently figuring on the loop-specific propagators, render expressions with these additional momenta to zero.

[8]Except for with respect to the general UV limits, equally analysed through a power-counting of D prior to a general regularisation anyway, compare to Paper V.

Perhaps the final step has to do with U-duality (momenta/brane charge duality). The loop equivalence between r and covariant spinor derivatives (originating in the loop regularisation) in Paper V provided the limit on the dimension to those presented (for $L > 1$), partly 'normally' deduced through U-duality arguments. Effectively, the vanishing of amplitude diagrams in Paper IV and V may be interpreted as being due to an overall limit on the encoded momenta/coordinate duality in the loops, in terms of what can be accepted by the integrations over the loop momenta. What that would correspond to in terms of U-duality is more difficult to divine, but there would seem to be a contradiction between required dualities and accepted entities.

The most crucial point is, of course, if the analysis truly holds in a BRST equivalent setting, but the extended analysis of Paper V certainly seems to verify that.

4.3.3 Outlook

The combined results of [98, 102], Paper IV and V imply UV perturbative finiteness in $D = 4$ maximal SUGRA, which is a highly interesting and controversial result. Further investigations are of interest:

- An extended analysis in settings outside the pure spinor formalism, for a confirmation of the results and a better understanding of what causes the counterterm cancellation. In [102] the $E_{7(7)}$ symmetry is found to be increasingly restrictive for higher loop and point diagrams. The possibility of the $\mathcal{D}^8 R^4$ counterterm for the 7-loop 4-point amplitude in their analysis is the product of a remarkable cancellation. Perhaps it is possible to extend the analysis and obtain results in line with Paper IV.
- It might be desirable to see an explicit proof of the 7-loop 4- and 5-point amplitudes in the pure spinor formalism setting of Paper IV and V. However, the overall limit of Paper V in practice already has established the worst UV behaviour of the one-particle irreducible loop structures, and the workings of the 'counterterm cancellations' have been identified.

Provided the results of Paper IV and V hold, the development of the perturbative finiteness in maximal SUGRA has taken an unexpected turn. Interestingly, due to the simplifications of the analysis provided by the inherent symmetries of the field theory pure spinor formalism. Perhaps the workings of the process would be more transparent when formulated in a setting with manifest U-duality, e.g. in $D = 4$ with explicit parallels to the other investigations. Such a pure spinor formalism yet remains to be constructed. However, the $D = 11$ theory already encodes this, effectively making the observations in Paper V possible, in part representing the powerfulness of the formulation.

We have to remember that what we observe is not nature herself,
but nature exposed to our method of questioning.

Werner Heisenberg

5 Exceptional geometry

A non-perturbative formulation of string theory has long been an object of interest in theoretical high energy physics. The superstring theories[1], originating in different quantisations of the closed string, are known to be related by dualities [53, 203], effectively representing different perturbative versions of one single theory [203–205]: the so-called M-theory. This theory e.g. arises in the strong coupling limit of type IIA string theory, with the other string theories in different limits of low coupling. Importantly, its low energy limit is $D = 11$ SUGRA.

In understanding the properties of the elusive M-theory, examinations of $D = 11$ SUGRA represent key investigations. Through a better understanding of the low energy properties, an increased knowledge of what ought to go into the description of M-theory is obtained. Integral to the subject of this chapter is the U-duality [53–56] properties displayed by $D = 11$ SUGRA compactified on tori. This duality encompasses both T- and S-duality and so relates all of the string theories. The process represents one example of how to, in the low energy limit of M-theory, identify a property important in its other perturbative expansions (the string theories).

Essentially, U-duality represents a quantum symmetry of M-theory, and an attempt can be made at making it manifest in the low energy limit formulation. The subsequent theory is termed extended geometry and exists in two versions: doubled and exceptional geometry, with manifest T- (double) and U-duality (exceptional), although the former represents a special, far more investigated, case of the latter. The point of these formulations is partly to better understand

[1]The type I, IIA and IIB theories, and the heterotic $SO(32)$ and $E_8 \times E_8$ string theories.

the effects of U-duality in general, but can also be regarded as a step towards M-theory. A consistent description at low energy, enlarged in lower dimensions ($D < 11$) to encompass the U-duality usually broken there, might be possible to extend to higher energy levels or give a clue to how that is to be done. Such attempts have been made in other contexts before, especially for T-duality [146–148, 206, 207], but also concerning theories describing M-branes [208–211]. For example, a current and important venue of research concerns the M5 theory (in $D = 6$) which relates the concepts of $D = 4$ gauge theory and $D = 2$ conformal field theory (CFT) [212–214].

This chapter will start out with a description of how U-duality shows up in $D = 11$ SUGRA and what it is characterised by, mainly following [56]. Subsequently, the geometric construction for including manifest U-duality in SUGRA will be described (initially in terms of T-duality), followed by the current situation for exceptional geometry and the investigations therein, partly related to Paper III. For a full, detailed and rather recent review of the field, [215] is recommended.

5.1 U-duality

The dualities relating the superstring theories partly figure in a context where brane charges and momenta are treated on an equal basis. To (by construction) get to this property, it is necessary to consider a spacetime with directions conjugate to the brane charges. At the same time, the symmetries of the SUGRA theory, a version of eq. (3.1b), must be preserved. Both criteria are fulfilled by a compactification of the $D = 11$ SUGRA theory on a torus of dimension d, where discrete momenta and brane windings on the torus form the multiplets of the U-duality groups [55], thereby capturing properties hidden in the $D = 11$ theory; the possible remnants from M-theory.

The SUGRA theory compactified in this way displays a superalgebra decomposed into the maximal superalgebra of the ensuing $D = 11 - d$ theory. The $\mathcal{N}$-extended algebra transforms under

$$SO(1, 10 - d) \times SO(d), \tag{5.1}$$

where the latter describes the symmetry relations between the different sets of supercharges, and thus represents a part of the (local) R-symmetry. For an $\mathcal{N} = 1$ theory, this symmetry is simply described by $U(1)$, but it varies with d to extend beyond $SO(d)$. The former part naturally denotes the Lorentz group in the uncompactified dimensions, so that the correct properties of the theory (in total) are preserved, in line with a valid construction as outlined above.

The interesting result of the compactification is the properties of the subsequent theory. It turns out to be invariant under a global symmetry G_d containing

D	d	$G_d = E_{d(d)}$	H_d
10	1	$\mathbb{R}^+$	1
9	2	$Sl(2,\mathbb{R}) \times \mathbb{R}^+$	$U(1)$
8	3	$Sl(3,\mathbb{R}) \times Sl(2,\mathbb{R})$	$SO(3) \times U(1)$
7	4	$Sl(5,\mathbb{R})$	$SO(5)$
6	5	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$
5	6	$E_{6(6)}$	$USp(8)$
4	7	$E_{7(7)}$	$SU(8)$
3	8	$E_{8(8)}$	$SO(16)$

Table 5.1: The symmetry groups of U-duality displayed in D dimensions after a compactification of $D = 11$ SUGRA on tori of dimension d (T^d), representing the internal symmetries of the compactified directions. These are the Cremmer–Julia symmetry groups, or equivalently: the exceptional symmetry groups $E_{d(d)}$. Their maximal compact subgroups H_d correspond to the R-symmetry of the superalgebra. [216, 217]

$$SO(d-1, d-1, \mathbb{R}) \bowtie Sl(d, \mathbb{R}), \tag{5.2a}$$

which denotes a continuous symmetry group generated by two non-commuting subgroups. The former of the two groups incorporates T-duality, whereas the corresponding for the latter is S-duality, so that the described global symmetry incorporates the dualities between the string theories. In addition, the latter group represents a subgroup of the diffeomorphism symmetry.

In total, the global symmetry G_d is that of $E_{d(d)}(\mathbb{R})$ [216, 217], as listed in table 5.1, i.e. the normal real form of the exceptional group E_d. However, in the quantum theory the gauge potentials transform non-trivially under E_d, so that the continuous symmetry cannot be valid. Effectively, the states charged under the gauge potentials are required to have quantised charges, reducing the continuous symmetry described above to the part of it that remains unbroken in the quantum theory: the discrete symmetry

$$E_{d(d)}(\mathbb{Z}). \tag{5.2b}$$

This is the symmetry group of the U-duality.

Technically, a few more proofs (of the absence of further restrictions) are required before the last statement can be made, in general. For example, the diffeomorphism symmetry in $D = 11$ SUGRA can be shown to give $Sl(d, \mathbb{Z})$ as an exact symmetry [53, 218] of the quantised theory. Another, further requirement is the fact that T-duality, described by $SO(d-1, d-1, \mathbb{Z})$, holds to all orders [219] in type IIA string perturbation theory. With $d \leq 7$, this is sufficient to prove the exact quantum symmetry to be $E_{d(d)}(\mathbb{Z})$ [53,218]. For the general statement, verifications extended all the way to $d = 11$ [220–222] are necessary. However, the interpretations, and relations, of the special cases of $d \geq 10$ are slightly different from the ones previously mentioned.

In summary, the symmetry group in eq. (5.2b) describes how U-duality relates the different superstring theories (effectively through S- and T-duality) as equivalent perturbative expansions of one and the same theory. The process of compacitification of $D = 11$ SUGRA on tori therefore correctly captures not only the equivalent treatment of momenta and brane charges first identified among the string theories, but the entire quantum symmetry connected to that equivalence. A recognition made possible by the complementary situation of the string and SUGRA theories, perturbative with respect to M-theory in different senses: the former in a weak coupling limit, the latter in a low energy limit, but neither in both senses. Most importantly, one property of M-theory, U-duality, has been uncovered.

5.2 The geometric construction

With one property of M-theory identified, it is possible to extend the low energy limit of it. That is, the maximal SUGRA theory might be altered (in $D < 11$) to comprise the $E_{d(d)}$ symmetry. The reason for this procedure, as previously mentioned, mostly is the opportunity it offers at examining what U-duality brings about in a theory. Along with the reflections in the previous chapters of this thesis, symmetries made manifest simplify investigations of different kinds in a theory. Secondly, of course, it can also be termed an intermediate step towards M-theory, which certainly must display the $E_{d(d)}$ symmetry as well as, in a low energy limit: maximal SUGRA. The resulting formulation though, is neither that of M-theory, nor in proper terms its low energy limit.

So, how to incorporate the $E_{d(d)}$ symmetry? That question was first addressed in terms of T-duality, the part of U-duality that shows up in the weak coupling limit of perturbation theory. There, the symmetry group of relevance is

$$O(d, d, \mathbb{Z}). \tag{5.3}$$

and the ensuing formulation, first examined in a purely mathematical context [113, 116, 117], is termed doubled geometry [31]. It represents a special case, more easily handled than the full U-duality, thoroughly examined [31, 113–145] and providing the guidelines for the exceptional setting.

Integral to the construction, regardless of the symmetry under consideration, is that the compactified theory (as described in the previous section) naturally separates into the compactified (internal) and uncompactified directions. The latter represents the original theory in the dimensions remaining after the T^d compactification, and is well-known. The former, on the other hand, represents an additional feature, not quite fit to capture the desired symmetry. It is what is altered in order to take care of the duality symmetry (manifesting it in the description) and, consequently, the only part of the description of special significance. Regardless of the setting, the compactified directions represent the first object of investigation.

5.2.1 T-duality construction

The concept of a geometric description of T-duality is central to doubled geometry, giving rise to the term 'generalised geometry'. It is through an alteration of the geometry that the desired symmetry invariance is captured.

In specific, the space of the (string) theory, compactified on a torus T^d, is a fibre bundle, which locally looks like the (direct) product of a base space and a fibre; at each point of the spacetime (not subject to compactification) a copy of the T^d is attached. In the geometric formulation, the space is extended with respect to the torus fibration, which introduces the concept of a generalised tangent space. A cotangent space is added to the original tangent space, so that the fibration is described not by one, but two tori, dual to each other. This shape of the generalised tangent space (doubled coordinates with a duality constraint) causes the generalised metric — i.e. the overall metric with respect to the compactified directions — to split into an $O(d, d)$ invariant metric. In this way, the added geometric (by compactification) and non-geometric (by construction) backgrounds make room for the desired symmetry in the theory.

With a metric invariant under the desired symmetry, it possible to accommodate the rest of the theory to it. The present T-duality naturally affects the gauge transformations, which means that the original infinitesimal transformations of tensors, the Lie derivative in the direction of a diffeomorphism parameter, falls short of capturing the $O(d, d)$ symmetry. For a correct geometric encoding of the diffeomorphisms (of the altered theory), the Lie bracket must be generalised appropriately, which is achieved through replacing the ordinary Lie bracket with the Courant bracket.

The change from the original Lie algebra, which for a theory of gravity is $\mathfrak{gl}(n)$, is such that, with a diffeomorphism parameter u^n and a vector v^n, an infinitesimal transformation is altered from[2]

$$\delta_u v^m = \mathcal{L}_u v^m = [u, v]^m = u^n \partial_n v^m - (\partial_n u^m) v^n \tag{5.4}$$

to that of a generalised Lie algebra with a generalised Lie derivative (a Dorfman

[2]Capital letters here refer to inner coordinates (coordinates on the generalised tangent space) in terms of the forms of the uncompactified theory, and minuscule letters to the original spacetime of the extended theory (in exceptional geometry: $(11 - d)$-dimensional). For example, the Lie derivative naturally is only altered with respect to the inner directions. Note that all forms are bosonic, despite the overlap with the superspace notation.

derivative): [42, 114, 115]

$$\begin{aligned} \mathscr{L}_U V^M &= \mathcal{L}_U V^M + Y^{MN}{}_{PQ}(\partial_N U^P) V^Q, \\ Y^{MN}{}_{PQ} &= \eta^{MN}\eta_{PQ}. \end{aligned} \tag{5.5}$$

Effectively, the original transportation term ($u^n\partial_n v^m$) is kept, while the matrix of infinitesimal transformations, through the projection $Y^{MN}{}_{PQ}$, is extended beyond $\mathfrak{gl}(n)$ to include the $O(d,d)$ transformations. For example, $\mathscr{L}_U$ preserves the $O(d,d)$ structure through

$$\mathscr{L}_U \eta_{MN} = 0. \tag{5.6}$$

However, the algebra generated by the generalised Lie derivative only closes provided a restriction on the theory. The corresponding, conventional constraint in doubled geometry, the so-called *section condition* is [114]

$$\eta^{MN}\partial_M \otimes \partial_N = 0. \tag{5.7a}$$

This version of the constraint, where the derivatives may act on arbitrary, separate objects, is termed *strong*. It is necessary for the closure of the algebra in doubled geometry, although the *weak* version

$$\eta^{MN}\partial_M\partial_N = 0 \tag{5.7b}$$

is enough to capture the level matching constraint in closed string theory, and therefore sometimes is used for preliminary investigations. Effectively, the restriction of the section condition (when solved for) puts the theory on a subspace equalling the normal space of the theory; different solutions of it represent dual theories, like the different string theories. This e.g. makes the section condition, and how it arises, interesting with respect to one of the venues of research mentioned in section 5.3.4.

In the presence of the strong section condition, the Lie algebra is valid and transforms among itself according to

$$[\mathscr{L}_U, \mathscr{L}_V] = \mathscr{L}_{[\![U,V]\!]}, \tag{5.8}$$

with the Courant bracket [115]

$$[\![U,V]\!]^M = \frac{1}{2}\big(\mathscr{L}_U V^M - \mathscr{L}_V U^M\big) = \tag{5.9a}$$

$$= U^N\partial_N V^M - (\partial_N U^M)V^N + \frac{1}{2}Y^{MN}{}_{PQ}\Big[(\partial_N U^P)V^Q - U^Q\partial_N V^P\Big]. \tag{5.9b}$$

This presents the natural way of generalising the Lie algebra, when the Lie derivative is generalised according to eq. (5.5). In total, the altered Lie algebra ensures that the theory locally displays the $O(d,d)$ symmetry.

Conceptually, this provides a geometric interpretation of T-duality, although further properties must be considered for a general reformulation of the theory, both in terms of the construction and its implications for the rest of the theory. For example, the generalised metric contains both the string theory metric and the B-field, the latter of which may have non-trivial flux, in which case a suitable alteration of the construction is required for the formulation to fully work [32]. That process of extending the theory will not be touched upon here.

In summary, the doubled geometry unifies the bosonic degrees of freedom (the fields of the theory) in the generalised metric G. It contains the string theory metric and the B-field. Meanwhile, the gauge symmetries and diffeomorphisms of the theory are encoded by the generalised Lie derivative, resulting e.g. in that an infinitesimal bosonic symmetry transformation is given by

$$\delta_U G = \mathscr{L}_U G. \tag{5.10}$$

In addition, the R-symmetry of the superalgebra translates into the maximal compact subgroup of the symmetry, rendering the actual symmetry to G_d/H_d in analogy with the situation in exceptional geometry. The modulo H describes the equivalence between the different sets of supercharges.

5.2.2 U-duality construction

Generalised geometry in the sense of capturing the U-duality properties, so-called *exceptional geometry*, was first recognised by [31,32] and represents a rather recent object of research. Although increasingly popular, its properties [31–52] are nowhere near as well understood as those of doubled geometry.

The geometric construction in the U-duality setting very much resembles the doubled geometry, with an extension of the original theory through a generalised tangent space, a generalised metric and a generalised Lie algebra. Only, the additional symmetry accommodated for is exceptional instead of double, and the theory concerned is that of supergravity. It represents a generalisation of the doubled geometry, in that it is non-perturbative in the coupling constant.

Exceptional geometry fundamentally differs from doubled geometry in that:

- The generalised tangent space (of d dimensions) added to the $D = 11 - d$ maximal SUGRA theory displays $E_{d(d)}$ invariance.
- The diffeomorphisms are invariant under $\mathfrak{e}_{d(d)}$ and a real scaling instead of $\mathfrak{gl}(n)$: the symmetry group is $E_{d(d)} \times \mathbb{R}^+$.
- The unification of the fields into the exceptional generalised metric makes manifest the equal treatment of momenta and brane charges.

The first statement effectively means that the compactified directions (the extended space) in exceptional geometry are described by something a bit more exotic than a fibration of a pair of dual tori. The generalised tangent space introduced in the geometric description of the $D < 11$ maximal SUGRA theory is constructed to accommodate the exceptional symmetry rather than the $O(d,d)$ symmetry, and varies with d in unison with the specific symmetries shown in table 5.1.

The second point brings about a different generalisation of the Lie derivative compared with the case in doubled geometry. The projection $Y^{MN}{}_{PQ}$ needs to remove the $Gl(n)$ invariance and substitute it with the new symmetry properties, a feature achieved by [34, 35, 37–40, 42]

$$\begin{aligned} \mathscr{L}_U V^M &= \mathcal{L}_U V^M + Y^{MN}{}_{PQ}\partial_N U^P V^Q, \\ Y^{MN}{}_{PQ} &= \delta^M_P \delta^N_Q - \alpha P_{(adj)}{}^M{}_Q,{}^N{}_P + \beta \delta^M_Q \delta^N_P. \end{aligned} \tag{5.11a}$$

Here, the first term in $Y^{MN}{}_{PQ}$ removes the undesired infinitesimal transformations. The second term adds the ones corresponding to the $E_{d(d)}$ through a projection onto the adjoint representation of $E_{d(d)}$, while the third is responsible for the real scaling. The (α, β) represent real scalars set by the particular theory (value of d etc.) under consideration.

Furthermore, as in doubled geometry, the generalised Lie algebra does not close on its own. The necessary and conventional (strong) section condition is

$$Y^{MN}{}_{PQ}\partial_M \otimes \partial_N = 0, \tag{5.11b}$$

a condition with a weak counterpart, just as in doubled geometry. The strong section condition needs to hold for the metric to be invariant, with some additional constraints, which restricts $Y^{MN}{}_{PQ}$ to the R_2 representation (2-forms) in $E_{d(d)}$ for $d \leq 6$ [42], compare to section 5.3.1. The Lie bracket is also altered in order to accommodate for the closure of the algebra, compare to eq. (5.8), into an exceptional Courant bracket [32] corresponding to that of eq. (5.9a).

The third statement refers to the treatment of the fields in the theory. While the symmetries (of U-duality) are taken care of by the added space and altered Lie derivative (including both S- and T-duality properties) the fields are incorporated into the extended metric, as in doubled geometry. Infinitesimal symmetry transformations are equally given by eq. (5.10). The difference is that the full dual situation is contained in the formulation: U-duality is present.

In total, there are naturally more differences between the doubled and exceptional geometries than what is listed above. In general, the properties of the latter are both generalised with respect to the former, and more complicated. There is also a definite difference in interpretation of the higher d-dimensional formulations.

5.2.3 Directions in relation to extended geometry

Once the geometric formulation of the U- or T-duality is set, the theory splits into two parts: the inner and the outer directions, where the former refers to the compactified part of the theory and, in extension, to the new physics. At this point, it is optional to extend the spacetime to the same symmetry, obtaining more than SUGRA with U-duality, by considering *extended* geometry instead of *generalised*, a distinction increasingly made these last years and not necessarily rigorous in e.g. Paper III. The investigations of the tangent space coincide. Furthermore, it is possible to investigate the consequences of the geometry, or (in extended geometry) to proceed to what a field theory would correspond to. The former is the venue of research of Paper III and consequently the one addressed (primarily) in this thesis. However, there is also a lot of research done on the field theories [48–50,114,115,120,121,137,138,141,142,144,145,223–225]. Either way, a complicating factor in exceptional geometry is the difficulties of analysing concepts with respect to a general d, due to the properties of the $E_{d(d)}$ representations. Most often, an analysis has to be performed for each value of d.

The doubled & exceptional field theories

An extension of the geometrical approach is to be found in doubled (DFT) and exceptional (EFT) field theory, where the prime interest is the fields of the theory instead of the geometry. The EFT structure is similar to that of $(11 - d)$ gauged maximal SUGRA [226, 227], except in that the fields depend on the extended space. Matters of interest are, for example, the gauge structure and the connections, the supersymmetry algebra and the form of the action. In part, the results from the field theory approaches and the investigations from a geometric point of view concern the same issues, and can then be translated from one formulation to the other. For example, the difference between investigations regarding the gauge structure and the connections on the generalised tangent space of a certain theory is mostly a matter of notation.

Comment on the generality of the construction

Some years ago, the question was raised of whether or not the global structure of exceptional geometry was that of a manifold, and tensors too constrained (e.g. to flat space) to form a consistent theory. Since then, however, progress on the intrinsic structure of the theory has been made. This includes, for example, additional knowledge of the finite transformations, soon to be described in section 5.3.1. The accompanying gerbe structure is relevant for coordinate patches through overlaps, and thereby for the associated manifold. On the whole, the theory of exceptional geometry represents a promising candidate with respect to general applications, such as the possibility to get at gauged SUGRA.

5.3 Exceptional geometry

In exploring the theory of (extended) exceptional geometry, there are a few fundamental questions concerning the theory:

1. What symmetries does it show?
2. What tensor formalism and curvature is it characterised by?
3. What is the associated dynamics?

The first question refers to both local and global symmetries, where the latter so far has proven difficult to describe in a general way. For an illustration of the concepts, the case of doubled geometry will have to suffice.

The second question has been addressed in a general way for $d \leq 7$ (the cases most easily investigated), which was the subject of Paper III, but remains to be further investigated for $d > 8$. It is a matter complicated by the presence of extra parts in the affine connection of the covariant derivative, which cannot be allowed to be part of the effective theory.

Thirdly, it is desirable to have a full classification of the dynamics and the gauge structures of the fields present in the theory. This matter is possible to approach, in a geometric setting, both from the observed structure of the tensor formalism and through an extension of ordinary gravity. Primarily, it is the bosonic sector that is subject to an analysis, as the extension to fermions is well-known.

These issues will now be described in more detail, prior to an outlook with respect to the different venues of research within exceptional geometry. Since the formulation is rather recent, there exist both many diverse investigations and concepts of interest yet to be approached.

5.3.1 Transformations within $E_{d(d)}(\mathbb{R})$

In any theory, it is desirable to have a complete classification of the constituent, local symmetries. This is of course also true in exceptional geometry. With a division of these transfromations into those finite and infinitesimally small, the latter is represented by the Lie algebra and well-known. The finite transformations, however, are far from completely classified.

Diffeomorphisms

Given the Lie derivative of eq. (5.11) the classification of the infinitesimal transformations in exceptional geometry equals the identification of the projection $Y^{MN}{}_{PQ}$ in the different d-dimensional compactifications. This tensor is completely determined by the requirement that the transformations generated by the Lie derivative represent a Lie algebra, which — in addition to the section condition — is fulfilled iff [42]

$$\Big(Y^{MN}{}_{TQ}Y^{TP}{}_{RS} - Y^{MN}{}_{RS}\delta^P_Q\Big)\partial_{(N} \otimes \partial_{P)} \tag{5.12}$$
$$\Big(Y^{MN}{}_{TQ}Y^{TP}{}_{[SR)} + 2Y^{MN}{}_{[R|T|}Y^{TP}{}_{S)Q} - Y^{MN}{}_{[RS)}\delta^P_Q - 2Y^{MN}{}_{[S|Q|}\delta^P_{R)}\Big)\partial_{[N} \otimes \partial_{P)^*}$$

all vanish. $[SR)$ denotes either the anti- or symmetrisation of the S and R indices, with any $|X|$ remaining put, while the $[NP)^*$ is opposite to the first bracket, so that the second expression really represents two different objects.

The section condition, as mentioned previously, restricts the projection tensor to a projection into 2-forms for $d \leq 6$, while it for higher d also allows for an additional term. The above displayed terms then set further restrictions, e.g. resulting in a projection for $d = 3$:

$$Y^{i\alpha,j\beta}{}_{k\gamma,l\delta} = 4\delta^{ij}_{kl}\delta^{\alpha\beta}_{\gamma\delta}\,, \tag{5.13}$$

where each inner index M for $d = 3$ is represented by a vector and a spinor index: $i\alpha$, as a result of the restriction of the 1-form of the $D = 11$ theory to the subgroup symmetry present after compactification. [42]

In this way, the infinitesimal transformations have been deduced for $d \leq 7$ [42]. In general, the $d > 7$ theories represent special cases, and this is no exception. The Lie derivative for $d = 8$ cannot be restricted into an ordinary Lie algebra [40,42], but requires further identifications [50,51] for the correct geometry [52] to be captured. The formulations with $d > 8$ remain to be addressed for a full identification of the infinitesimal symmetries.

Finite transformations

Despite the fact that the compactification of $D = 11$ SUGRA on tori is a thoroughly understood and entirely feasible procedure, the finite transformations of the ensuing theory remain to be pinned down in their entirety. In this respect, the effects of the compactification need to be further recognised.

The situation in doubled geometry, simpler yet relevant, has been investigated in [134,139,141,142], with parallels to the exceptional case. The situation is as follows:

- In doubled geometry, any global continuous symmetry ought to also be a local symmetry, due to the string theory background, where all global symmetries are gauged [228]. As such, all the continuous symmetries should be possible to express in terms of the Lie derivative, and the other transformations correspond to discrete, finite (large) transformations. By extension, this should hold in exceptional geometry too.

- The finite transformations of tensors in doubled geometry, conjectured in [134], correspond to an exponentiation of the Lie derivative:

$$e^{\mathscr{L}_\xi} = e^{\mathcal{L}_\xi - a^t} = e^{\xi + a - a^t}. \tag{5.14}$$

  Any such exponentiation is equivalent to the finite transformations, allowing for non-translating transformations. However, the situation is not easily generalised to exceptional geometry, as that exponentiated Lie derivative contains a term different from a^t, with properties complicating the formulation of the finite transformations on a form other than that of an infinite series, contrary to what is true in doubled geometry. [139]

- The continuous, finite transformations display a so-called *gerbe* structure; two finite transformations describe a third up to non-translating transformations

$$e^{\Delta}, \qquad \Delta: \quad [\mathscr{L}_U, \mathscr{L}_V] = \mathscr{L}_{[U,V]} + \Delta_{U,V}. \tag{5.15}$$

  The deviation is possible to describe, in doubled geometry, in terms of the symmetry algebra $O(d,d)$, and is trivial for three consecutive finite transformations. The characteristics are equivalent to those displayed by a fibre bundle, which effectively is the topology of the theory once the section condition is solved for. In exceptional geometry, on the other hand, a higher gerbe structure is expected — perhaps infinite — e.g. due to the presence of higher n-form potentials in some of the compactifications. [139]

Although this constitutes a promising conceptual knowledge of the situation in exceptional geometry, a better insight into the finite transformations is required for a thorough understanding of the global 'isometries'[3].

5.3.2 Torsion, curvature & the tensor formalism

With the symmetries of the theory (partially) recognised, the next concern is for the tensor formalism. It needs to be well-defined and, preferably, easy to deal with — a feature complicated by the already mentioned difficulties in making statements for general d. The former is in turn connected to the torsion of the theory, or rather the part of it which is not allowed, and leads on to a recognition of the curvature of the theory, in total presenting a full description of the exceptional space.

The affine connection

The covariant derivative in exceptional geometry is such that

$$D_M V^N = \partial_M V^N + \Gamma_{MN}{}^P V_P, \tag{5.16}$$

[3]Non-distorting mappings.

with $\Gamma_{MN}{}^P$ representing the affine connection in the Lie algebra ($\mathfrak{e}_{d(d)} \oplus \mathbb{R}$). For consistency, the covariant derivative of a tensor must also constitute a tensor, which it does not automatically do. In ordinary differential geometry, this is solved for in terms of the compatibility condition, i.e. the requirement of the vielbein to be covariantly constant:

$$D_M E_N{}^A = \partial_M E_N{}^A + \Gamma_{MN}{}^P E_P{}^A - E_N{}^B \Omega_{MB}{}^A = 0, \tag{5.17}$$

with $\Omega_{MB}{}^A$ as the spin connection[4]. With this relation, the spin connection usually is expressed in terms of the vielbein and derivatives thereof through a restriction to zero torsion; enough of an additional constraint for the curvature to be solved for. However, in exceptional geometry additional components show up in the affine connection, complicating the procedure.

Torsion and non-torsion

In ordinary differential geometry, the torsion tensor is proportional to the antisymmetrised affine connection

$$T_{MNP} \propto \Gamma_{[MN]P}, \tag{5.18}$$

which means that the two first indices in Γ are symmetrised when the torsion is set to zero. This is a consistent constraint that fails in exceptional geometry as the affine connection contains unconventional parts, not transforming homogeneously. Instead, the relevant components need to be divided into two categories: *torsion* and *non-torsion*. The former transforms with the Lie derivative as ordinary torsion does and can be put to zero, and the latter represents the parts subject to further examinations and constraints.

In this way, a feature of exceptional geometry is that the torsion is not completely determined, with respect to the non-torsion part. Non-torsion components in Γ that are left undetermined by eq. (5.17) cannot be part of a well-defined covariant derivative, and need to drop out of the theory (action, supersymmetry variations, etc.) for consistency. Either a constraint on the affine connection needs to be imposed, or the unwanted irreducible representations thereof need to fall out of the relevant description. In specific, the Ricci tensor and scalar must be possible to construct from the affine connection in an unambiguous way, compare to eq. (5.20).

Consistent tensor formalisms

There exist different approaches to a formulation excluding the undetermined non-torsion from the theory, equivalent in their effects but slightly different in their consequences for the interpretation of the theory.

[4] In a comparison with section 2.2.3, note that the conventional notation in extended geometry here displayed neither refers to superspace components, nor implicitly antisymmetrised indices.

The most obvious approach, pursued in [40, 44], is that it is possible to deduce the relevant conditions on the affine connection for it not to contain undetermined non-torsion, and apply constraints on it in order to ensure that it does not. For $d < 8$, with a torsion set to zero, this translates into the vanishing of the affine connection in the covariant derivative, fulfilled if

$$Y_{MN}{}^{QR}\Gamma_{QR}{}^{P} + Y_{MN}{}^{PQ}\Gamma_{RQ}{}^{R} = 0, \tag{5.19}$$

as noted in Paper III.

On the other hand, it is possible to observe that the affine connection contains no undetermined non-torsion if the partial derivatives ∂_M in the Lie derivative are replaced with covariant ones. This leads to the recognition of that there is a covariant mapping of the covariant derivative between certain modules, and to the approach of Paper III ($d \leq 7$) and [45] ($d = 7$). The latter was extended with respect to d in [47], but for $d \geq 8$, the discussion in [51] gives the so far best suggestions on how to extend the Lie derivative in a desirable way, specialised further for $d = 8$ in [52].

The procedure consists of a classification of between which (pairs of) modules the covariant derivative is allowed to act; i.e. how a tensor might be altered by it in a consistent way, without the introduction of undetermined non-torsion components. The relevant restriction on the theory then becomes a restriction on the modules involved, rather than constraints on the affine connection. It presents a geometric solution to the problem in question — how to obtain a well-defined covariant derivative — and the properties of the affine connection remain untouched.

Conceptually, the first approach provides torsion-free projections of the affine connections (one for each d), unique and possible to use in a derivation of the curvature of the theory. However, the initial connections giving rise to the projections remain failing with respect to uniqueness, which ought not to be. Only in $d = 4$ has a remedy been identified [44], providing a manifestly covariant tensor formalism. The second approach, on the other hand, removes that issue. Consequently, while the effects of the two approaches (the results obtainable in the theory) are equivalent, there is a subtle difference between them, concerning the treatment of the symmetry of the theory: manifest or not, reflecting upon the fundamental understanding of what governs the theory.

A useful illustration can be provided by an analogy to Maxwell's theory, where the tensor formalism may be described either in terms of the electric and magnetic fields ($\mathbb{E}, \mathbb{B}$), or in terms of 4-tensors. Lorentz covariance is present in both settings, but while the existence of a manifestly Lorentz covariant formulation is apparent in the first approach, it is not realised in anything but the second one. In a manifest situation, all components represent tensors under the specified symmetry transformations, thus constituting a complete tensor formalism, with no further improvements to be wished for.

In exceptional geometry, the relevant symmetry is represented either by the global $E_{d(d)}(\mathbb{Z})$ transformations or (somewhat stricter) the extended diffeomorphisms, but the principle remains the same. The geometry of the theory is constructed to encode U-duality as a manifest symmetry. An approach where the theory at a tensorial level is 'reduced' to a matching between which intermediate results, obtained from the different forms present in the theory, provide the correct end result, fails to retain the manifest formulation. Such is the removal of parts of the affine connection, in contrast to a recognition of where the theory is consistent.

The value of the approaches in Paper III and [45, 47, 52] thus lies in the end result being given in terms of a tensor formalism with manifest U-duality. It is also nice to have a slightly more general formulation, valid beyond a specific d. However, most importantly, a valid tensor formalism exists, despite the non-torsion problematics, and further properties of the exceptional geometry may be examined consistently.

Curvature

The presence of a well-defined tensor formalism makes it possible to proceed with the examination of the curvature in exceptional geometry. The generalisation from ordinary geometry[5] to the exceptional extension is fairly straightforward. However, a Riemann tensor cannot be constructed, because the extended version does not represent an object transforming as a tensor. The Ricci tensor, on the other hand, is perfectly well-defined in $\mathfrak{g} \ominus \mathfrak{h}$:

$$\begin{aligned} R_{MN} = {} & \partial_{(M}\Gamma_{|P|N)}{}^P - \partial_P\Gamma_{(MN)}{}^P + \\ & + \Gamma_{(MN)}{}^Q\Gamma_{PQ}{}^P - \frac{1}{2}\Gamma_{PM}{}^Q\Gamma_{QN}{}^P - \frac{1}{2}\Gamma_{P(M}{}^Q\Gamma_{N)Q}{}^P, \end{aligned} \tag{5.20}$$

with a restriction to vanishing torsion, as pointed out in Paper III ($d \leq 7$). It is the Ricci tensor and scalar that describe the curvature in exceptional geometry.

5.3.3 Dynamics from the tensor formalism

With a consistent geometric formulation and a valid tensor formalism, as at least mostly provided for $d \leq 8$, the next relevant venue of research concerns the dynamics and the gauge structure of the existing fields. These are in need of a full classification.

The (extended) dynamics take place in G/H, with G and H representing the symmetry group and its maximal compact subgroup in the d-compactification,

[5] Any construction in exceptional geometry ought to fall back on the case of ordinary geometry under the condition $Y_{MN}{}^{PQ} = 0$. Likewise, the extended geometry represents a generalisation with respect to a non-zero projection tensor and the subsequent extended Lie derivative.

listed in table 5.1. With an appropriate tensor formalism, local H_d covariance is manifest and represents gauge. However, it is possible to proceed in slightly different ways to obtain the dynamics and the gauge structure of the theory. The actual situation is in need of further investigation for a full recognition of the inherent properties.

The starting point, in a geometric setting, can e.g. either be from the tensor structures observed in relation to the requirements concerning the affine connection, or from the identified curvature and how it ought to appear in an action to fit with the unextended theory. The former primarily focuses on the gauge fields of the compactified theory, while the latter represents an initial attempt at capturing the gravitational dynamics.

Implications from the tensor structure

The identification of between which modules the covariant, extended Lie derivative is allowed to act consistently, described in the previous section, by default brings about a recognition of the gauge structure and dynamics of the gauge fields in the compactified theory. These, as in doubled geometry [123, 126, 133], show up in the modules arising from the compactification of the theory, and the dynamics in the inner directions is naturally limited by the consistent action of the Lie derivative. From this, both gauge symmetries and field equations can be deduced.

As such, the observed tensor structure in Paper III and [45, 47, 52] provides a first recognition of the gauge structure and dynamics. The modules are R_n (n-forms in $D = 11$) with a structure given by

$$\begin{array}{c} R_{n-1} \\ \text{Field strength} \end{array} \longleftarrow \begin{array}{c} R_n \\ \text{Field} \end{array} \longleftarrow \begin{array}{c} R_{n+1} \\ \text{Gauge parameter} \end{array} . \tag{5.21}$$

As in [46], the investigation can be extended to the supermultiplets associated with these (known) fields, in a setting with manifest U-duality. The degrees of freedom in the modules between which transformations are allowed then set the bosonic matter fields, to which spinor and auxiliary fields are added in a way suitable to represent the desired supermultiplets. However, this type of procedure merely deals with the gauge fields known from compactification, and says nothing about e.g. the properties associated with gravity.

The bosonic action

In order to capture the gravitational dynamics of exceptional geometry, it is possible to look to the unextended theory for a representation of the interactions, which with an accurate extension ought to provide the correct physics. In this manner a candidate for the bosonic action is obtainable through a generalisation of gravity to $E_{d(d)} \times \mathbb{R}^+$, with the bosonic properties effectively described by the action

$$S_B \sim \int R, \quad R_{MN} = 0, \tag{5.22}$$

in line with the conjecture presented in [40]. Its precise interpretation is complicated by the difficulties in introducing an integration measure (with respect to the extended metric) without breaking the covariance of the exceptional geometry theory, with respect to the inner directions (the others are considered a separate case). For example, a covariant formulation cannot rely on a solution of the section condition, as the covariance is broken once the section condition is solved for. Consequently, the treatment of the real, physical space (in general) is complicated, as is the issue of how to perform partial integration, partly discussed in Paper III. The precise appearance of the action remains to be fully investigated.

However, with the bosonic properties determined, the extension to the fermionic ones is set, and can be determined, by the extended connection, which provides the description of the supersymmetry algebra. This was, for example, shown in the setting of projected affine connections in [43], and the argument also holds for the manifest interpretation. Because of this, the fermionic sector is often termed to follow from the bosonic one, with S_B constituting the remaining issue.

5.3.4 Matters of interest

There are many ways of approaching the (relatively) new field of exceptional geometry, and different opinions of what is important to investigate in it. Some frequently encountered settings, apart from the geometric approach here presented, are field theory descriptions concerning strings, the scalar particle, etc. Regardless of the strategy, the development is promising.

Examples of features remaining to be examined

Effectively, quite a number of interesting venues of research within exceptional geometry (primarily from a geometric point of view) have already been discussed in the previous sections of this chapter. The finite transformations, $d > 7$ examinations of concepts in general and the affine connection ($d > 8$) in particular, the action, etc., remain merely partially investigated. However, apart from those main issues, there exist interesting points to be made concerning features of the theory which have not been much investigated yet, the ranking of which of course is individual. Among these are the three examples of:

— The possibility of an underlying geometric structure, giving rise to the section condition in a way similar to an equation of motion.

As previously mentioned, the section condition is necessary for the closure of the exceptional Lie algebra, and different solutions of it provide dual theories. This raises the question of if it is a feature, not simply to be identified, but originating in a more general interpretation of the geometry. Clearly, it is a condition for the theory to be consistent, and so it may arise naturally from the formulation. The advantage of such a formulation would primarily be the associated manifest properties, but the concept is intriguing and might give a better understanding of the observed dualities.

— The description of branes in exceptional geometry and what that may say of M-branes.

At the beginning of this chapter, one of the most promising lines of investigation leading up to M-theory (apart from exceptional geometry) was described to concern the M-branes, with the M5-branes of special interest. The importance of the M-branes in string and M-theory suggests that a better understanding of them would provide essential information on the non-perturbative physics of M-theory. As such, investigations in the setting of the non-perturbative, low energy limit provided by exceptional geometry ought to be highly relevant for a better understanding of M-theory, investigations that have been initiated in terms of brane solutions in [229–231].

— The formulation of an exceptional supergeometry, geometrically extending superspace and including both U-duality and supersymmetry.

It would be interesting to see what a theory, extended like exceptional geometry but based on superspace instead of spacetime, would look like. A theory with both U-duality and supersymmetry manifest[6] would be especially interesting with respect to maximal supersymmetry. As already mentioned, this type of formulation might facilitate the identification of characteristics in SUGRA, although the $D = 11$ theory would not be subject to any change. At least, it might help in the interpretation of e.g. the UV cancellations. However, the concept and its applications are much more general. Note that it differs from the superalgebras existing in EFT much as the pure spinor formalism differs from the component theory.

At present, it is unclear how to introduce maximal supersymmetry in exceptional geometry. The corresponding pure spinor would be infinitely reducible, and attempts at introducing the concept have so far proven futile. It seems the geometric setting (of superspace) must be analysed further, perhaps initially in doubled geometry, as initiated in [232, 233].

[6] Or nearly so, in the sense of a pure spinor formalism.

General outlook

In total, exceptional geometry constitutes a promising theory growing in interest. It is still at an initial stage, with basic features under consideration and rapid development. Whereas the geometric groundwork for encoding manifest U-duality for the most part is laid, further work on higher level components is necessary. This includes the interesting results connected to tensors, field dynamics and branes, many of which have the potential to yield more information on M-theory. Importantly, M-theory characteristics or the effects thereof still remain to be identified. So far, the picture unfolding is in accordance with the construction, but reasonably new effects ought to show up as well. The theory, overall, represents a world of possibilities for the deduction of non-perturbative features of M-theory.

Science is built up of facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house.

Henri Poincaré

6 Manifesting characteristics

There is a common thread to much of the discussion in the previous chapters, discernible to the attentive reader: the theme of manifest constructions with respect to the essentials of the theories, yielding formalisms advantageous for the investigations of diverse properties. The process of identifying the underlying structure of a theory is not an easy task, but upon the recognition of the key features, it is often rewarding to introduce a formulation with the key features explicitly expressed or, failing that, present in a way where the structures can be drawn upon to access the true features of the theory. It is an issue of clarity. As cumbersome as the wrong choice of a description can be, with (insurmountable) difficulties in proving relations etc., as facilitating the right formulation can be.

There are three steps in this kind of process:

1. Identifying a key property, such as a symmetry.

2. Making that symmetry present in the descriptions in a way as explicit as possible, preferably manifest (inherent and explicit).

3. Investigating the theory, in a manner compatible with the previous point.

Of these steps, the discussion in this thesis does not particularly address the first point, except in the sense where it equals the third. At the investigation of any theory, it is important to keep an open mind to what causes the effects, as that might give clues to further key properties, or at least to possible benefits from other treatments.

Manifesting properties

The situation in exceptional geometry is a prime example of the second point above. The formulation of the theory itself is currently under development; although the geometric formulation including the U-duality has been identified, the finer points on how the theory works with respect to the alterations are in various degrees of progress towards identification. To fully access the benefits of the formulation, it is important to keep the symmetry manifest and not to disregard pointers towards possible reinterpretations and overall constructions. True, properties may be well caught by a description lacking in this sense, but that cannot be termed to represent a full investigation. As such, the geometric formulation is at the centre of chapter 5 rather than the associated field theories; not as a statement of importance, but in line with the conceptual approach of the work presented in the Papers, influenced largely by my supervisor Martin Cederwall. Recall the discussion on the tensor formalism of Paper III in section 5.3.2.

With respect to this argumentation, the pure spinor formalism falls somewhat short, as discussed in section 3.4.4. The inherent supersymmetry is not explicitly present. In addition, the construction may seem somewhat contrived. However, it does capture the maximal supersymmetry in a manifest way, barring explicitness of the actual workings of the present supersymmetry. The former is a requirement, the latter a circumstance yet to be circumvented. It is a fact that the formalism holds a lot of inexplicit symmetry, e.g. through Q-equivalence — it is difficult to spot equivalent descriptions, and to recognise them as such. For the moment, however, it represents the best formulation available, which is apparent in comparison with other approaches, treating maximal supersymmetry, and their results with respects to issues where the properties of the supersymmetry play a central role.

Investigating properties: Born–Infeld theory

With respect to the pure spinor Papers (I, II, IV and V) there are two contexts to the investigations, with new approaches enabled by the pure spinor formalism. The first has hardly been mentioned up until now, as it represents a sidetrack in comparison to the rest. Despite this, it fits completely well with the overall theme of the Papers.

Paper I concerns the extension of maximal SYM to Born–Infeld theory [57], which shows up in the low energy limit of string theory as part of the bosonic half of the theory describing D-branes [58–68]. This (full) theory is interesting with respect to higher energy physics and M-theory, which indirectly (in a perturbative setting) might be analysed in terms of the D-brane properties, a process yielding clues to what components and dualities the M-theory ought to sport. In specific, Born–Infeld theory shows up with respect to the D-branes in terms of the Dirac–Born–Infeld action [60]. By the Born–Infeld theory extension however, we here denote the full bosonic part of the action for the D-branes, which can be expressed in terms of a series expansion in the slope parameter (α') [10–13, 15, 69], with maximal SYM at the lowest order of the relevant perturbation theory.

The extension of the SYM theory to a series expansion in α' was initiated in [13] through an identification of the second order (α'^2) correction to the SYM action, representing the first non-zero correction, followed not by the third order [234, 235], but a fourth [236]. In Paper I, the result at α'^2 was reformulated in terms of operators acting on the superfield ψ, making it possible to perform a full analysis of what, by consistency of the master equation, would represent the abelian and non-abelian Born–Infeld actions in the pure spinor formalism. The procedure constitutes a prime example of how to get to the properties of one theory by extending another (paralleling the situation in exceptional geometry), in order to expand the knowledge of the first theory, a process vital for the formulation of the target theory, be it Born–Infeld or M-theory.

The Born–Infeld theory itself is only well-known in a component formalism sense with respect to the abelian theory, where the action is described by an infinite polynomial. The purpose of Paper I was to investigate the possible simplifications brought about by the pure spinor formalism, with the ultimate goal of a formulation of the non-abelian action. Although this goal was not reached, the simplifications were indeed impressive, with an abelian action of only two terms. Also worth mentioning, is that the actions presented in Paper I are not strictly proven to directly correspond to the actual Born–Infeld theories. However, they coincide up to the fourth order in α', which is a strong indication of that they represent the relevant theories.

Happily, the investigations of the non-abelian action abandoned in Paper I were continued in [149, 150]. There, the correct term $S_3 \sim \psi^3$ in the action was used to devise an iterative procedure for obtaining the infinite polynomial, total action; a very satisfying result. Note however, that quite a number of the calculations (preliminary to the S_3 discussion) presented there were given, or implicit by presentation of results, in Paper I.

In total, this nicely illustrates the simplifications brought about by the inherent supersymmetry (if not manifest).

Investigating properties: amplitude calculations

The main topic of the pure spinor investigations in the Papers is the description of amplitude diagrams in the pure spinor setting. As discussed in chapter 4, this is far from the only approach to the issue, yet promising with respect to results on the perturbative behaviour in the UV regime.

The approach is relevant simply for encoding maximal supersymmetry in a field theory setting, but the difference between Paper II and IV is also a good example of the importance of drawing upon the symmetry properties of the theory as much as possible, throughout the investigations. The key development in Paper IV was the reformulation of the propagator, in specific the expression for the b-ghost. In the two articles, these entities are Q-equivalent, yet the results easily identified in Paper IV are difficult to even discern in Paper II — they are present, but elusive. In Paper IV, the mere choice of pairing derivatives (often acting on each other with significant implications for the end result) with the pure spinor, like

$$(\lambda\gamma^{(n)}D), \tag{6.1}$$

makes the existence and vanishing of different combinations (more) explicit, partly by

— symmetries between the (covariant) spinor derivatives.

— two spinor derivatives acting on each other explicitly resulting in the combination of two pure spinors.

To spot the same characteristics by Fierz identities is a messy business, and it is significant to note that the supersymmetry constraints on the formalism are tightly connected to the pure spinor, consequently to be exploited to the outmost.

This (further) illustrates the importance of keeping the key properties of a theory manifest. Visibility is a central concept. Part of the problematics arising in the pure spinor formalism ought of course to be the result of a symmetry not truly manifest in the description. However, this only makes it all the more important to keep the symmetries (to the degree present) as discernible as possible.

Summary & outlook

In the investigation of a theory or a property thereof, it is key to catch the essence of what causes the behaviour:

- By this, it is easier to spot and identify (new) properties.
- Also, this shows an understanding of the theory, truly required for a full recognition of what it constitutes.

Although, naturally, the first requirement is to recognise cause and effect, through the intermediate, highly relevant investigations outside the manifest approach. In total, it represents how science, as we know it, is expanded.

A second relevant discussion on visibility with respect to the pure spinor formalism was brought up at the end of chapter 3: the opaqueness of what is going on with respect to the component algebra, and the subsequent difficulties in interpreting the results in terms of known concepts for comparisons etc. This is the major reason for why the results of Paper IV and V remain to be confirmed, apart from the adequateness of a second opinion. The pure spinor formalism and the results provided by it in many ways are regarded with slight scepticism in the general community of high energy physics. In that sense, a breakthrough is yet to come, though highly anticipated by e.g. Green, who expresses his hopes for an increased interest each year at the Strings conference. It remains yet to be seen when (if) he is proven right.

To a lesser degree, the establishment of an approach as a well-known method of investigation also is relevant with respect to extended geometry. It is a process equally important to the one of capturing the manifest characteristics in the first place.

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