Supplementary Material:
Optical Flow Estimation on Sequences with Differently Exposed Frames

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1 Description of the document

This document contains supplementary material to the paper "Optical Flow Estimation on Sequences with Differently Exposed Frames". It is not fully self-contained and should therefore be read with the paper at hand. Some references are given to the original paper for further details of certain expressions and experimental setups. Many authors that publish on variational Optical Flow estimation methods provide the Euler-Lagrange (E-L) equations that correspond to their specific cost functional, one influential example being Brox et al. [1]. In this document, we derive the E-L equations with more detail than what is customary. Our specific cost functional is stated as the starting point. Then, the contributions of each of its terms to the overall E-L equations are derived. The document is concluded with details on how to iteratively solve the E-L equations, including a pseudo-algorithm and numerical aspects.

2 Cost funtional, corresponding Euler-Lagrange equations, and implementation details

In order to estimate a given flow field, the objective is to find the horizontal and vertical flow functions of $\mathbf{w}_f = (u_f, v_f)$, $\forall f \in \{1, \ldots, \mathcal{F}-1\}$, assuming without loss of generality that the first frame included is indexed $f = 1$, that minimize the cost functional

$$E(\{\mathbf{w}_f\}) = \int_\Omega F d\mathbf{x} = \int_\Omega (F_D + \alpha_S F_S + \alpha_T F_T) d\mathbf{x}. \tag{1}$$

The flow field of interest is typically $\mathbf{w}_{f_{ref}}$, which is the case throughout the paper, with $f_{ref} = 2$ and $\mathcal{F} = 4$ (to estimate the subsequent flow, take $f_{ref} = 3$ and the next set of $\mathcal{F}$ images in a sliding window approach). A necessary condition for a minimizer is that the *first variation* of E with respect to each of its arguments is equal to zero [2]. An equivalent condition is given by

$$\frac{\delta E}{\delta u_{f_0}} = 0 \Leftrightarrow \frac{\partial F}{\partial u_{f_0}} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{f_0 x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_{f_0 y}} = 0, \quad \forall f_0, \tag{2a}$$

$$\frac{\delta E}{\delta v_{f_0}} = 0 \Leftrightarrow \frac{\partial F}{\partial v_{f_0}} - \frac{\partial}{\partial x}\frac{\partial F}{\partial v_{f_0 x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial v_{f_0 y}} = 0, \quad \forall f_0, \tag{2b}$$

where the left hand sides are the first variations of the functional (1) w.r.t. u_{f_0} and v_{f_0}, $f_0 \in \{1, \ldots, \mathcal{F}-1\}$, and the right hand sides are the (strong form) Euler-Lagrange partial differential equations, for which each of the terms $u_{f_0}, u_{f_0 x}, u_{f_0 y}, v_{f_0}, v_{f_0 x}, v_{f_0 y}$ are treated as independent variables. Due to the equivalences in (2), the flow that minimizes the total cost functional is obtained by the solution to the E-L equations. With $F = F_D + \alpha_S F_S + \alpha_T F_T$, we make use of the linearity of the derivative operator and evaluate the respective terms

$$\begin{aligned} F_D &= \sum_{n=1}^{N} \theta_{p_n q_n} F_{D p_n q_n}, \\ F_{D p_n q_n} &= \Psi((I_{q_n}(\mathbf{x} + \mathbf{W}_{q_n}) - I_{p_n}(\mathbf{x} + \mathbf{W}_{p_n}))^2), \end{aligned} \tag{3a}$$

$$F_S = \Psi(\sum_{f=1}^{\mathcal{F}-1}(u_{fx}^2 + u_{fy}^2 + v_{fx}^2 + v_{fy}^2)), \tag{3b}$$

$$F_T = \sum_{f=1}^{\mathcal{F}-2} \Psi((u_{f+1} - u_f)^2 + (v_{f+1} - v_f)^2), \tag{3c}$$

part by part for F in (2) in the following sections of the document. The contributions are added together at the end. Note that F_S only depends on the derivatives of u_{f0}, v_{f0} and that F_D, F_T only depend on the constant terms. Thus, for each of F_D, F_S, F_T, some of the terms in (2) vanish directly, simplifying the respective evaluations. All the derivations are given for (2a), with respect to u_{f_0} (for a specific f_0). The derivation of the E-L equations in (2b) w.r.t. v_{f_0} is analogous. Due to the general choices allowed for p_n, q_n and f_{ref}, and furthermore that the expression is nonlinear in the unknown flow, the evaluation of the F_D part is somewhat cumbersome. The regularization terms are relatively straight-forward to treat, and are therefore given first. If the interval between the sampling instances is not 1 second, or particularly if the sampling intervals are non-uniform which is the case for the camera prototype setup in Experiment 2, a scale factor is necessary for each flow increment. While it is included in our implementation, it is left out of the derivation as it is a straight-forward extension.

2.1 Spatial regularization term

The contribution from the spatial term to the E-L equations of (2a) is given by

$$
\begin{aligned}
-\frac{\partial}{\partial x}\frac{\partial F_S}{\partial u_{f_0 x}} - \frac{\partial}{\partial y}\frac{\partial F_S}{\partial u_{f_0 y}} &= -2\mathrm{div}(\Psi'_S \nabla u_{f_0}), \\
\Psi'_S \triangleq \Psi'(\sum_{f=1}^{\mathcal{F}-1}(u_{fx}^2 + u_{fy}^2 + v_{fx}^2 + v_{fy}^2)), &
\end{aligned} \tag{4}
$$

where $\Psi'(z^2) = (1/2)(z^2+\epsilon^2)^{-1/2}$, due to

$$
\begin{aligned}
\frac{\partial F_S}{\partial u_{f_0}} = 0, \qquad \frac{\partial}{\partial x}\frac{\partial F_S}{\partial u_{f_0 x}} &= \frac{\partial}{\partial x}(\Psi'_S 2 u_{f_0 x}), \\
\frac{\partial}{\partial y}\frac{\partial F_S}{\partial u_{f_0 y}} &= \frac{\partial}{\partial y}(\Psi'_S 2 u_{f_0 x}).
\end{aligned} \tag{5}
$$

Although short notation is used, the reader is reminded that $u_{f_0 x}$ and $u_{f_0 y}$ are functions of (x, y). An interesting observation is that the contribution (4) of the spatial term to the E-L equations has the form of a nonlinear (due to the dependence of Ψ'_S on the unknown parameters) diffusion, commonly used for edge-preserving image denoising [3, 4]. Here, however, this term is balanced against the other included terms.

2.2 Temporal regularization term

Because the temporal regularization term (3c) does not contain any partial derivatives of the flow functions in its expression, the contribution to (2a) is given directly as

$$
\begin{aligned}
&\frac{\partial F_T}{\partial u_{f_0}} = \\
&= \begin{cases}
\Psi'_{TI}\cdot 2(u_2 - u_1)\cdot(-1), & f_0 = 1, \\
\Psi'_{TII}\cdot 2(u_{f_0} - u_{f_0-1})\cdot(+1)+ & \\
+\Psi'_{TI}\cdot 2(u_{f_0+1} - u_{f_0})\cdot(-1), & 1 < f_0 < \mathcal{F}-1, \\
\Psi'_{TII}\cdot 2(u_{\mathcal{F}-1} - u_{\mathcal{F}-2})\cdot(+1), & f_0 = \mathcal{F}-1,
\end{cases} \\
&\Psi'_{TI} \triangleq \Psi'((u_{f_0+1} - u_{f_0})^2 + (v_{f_0+1} - v_{f_0})^2), \\
&\Psi'_{TII} \triangleq \Psi'((u_{f_0} - u_{f_0-1})^2 + (v_{f_0} - v_{f_0-1})^2).
\end{aligned} \tag{6}
$$

2.3 Data term

Similarly to the temporal regularization term, the data term does not contain any partial derivatives of the flow in its expression, thus its contribution to (2a) is

$$\frac{\partial F_D}{\partial u_{f_0}} = \sum_{n=1}^{N} \theta_{p_n q_n} \frac{\partial F_{D p_n q_n}}{\partial u_{f_0}}, \tag{7}$$

where for each specific n,

$$\frac{\partial F_{D p_n q_n}}{\partial u_{f_0}} = \Psi'((I_{p_n q_n})^2) \cdot 2 I_{p_n q_n} \cdot \frac{\partial I_{p_n q_n}}{\partial u_{f_0}}, \tag{8}$$

and where $I_{p_n q_n} \triangleq I_{q_n}(\mathbf{x} + \mathbf{W}_{q_n}) - I_{p_n}(\mathbf{x} + \mathbf{W}_{p_n})$. To further evaluate (8), which is nonlinear in the flow functions u_{f_0} and v_{f_0} contained in $\mathbf{W}_{q_n}$ and $\mathbf{W}_{p_n}$, successive linearizations about the current flow estimates are employed in an iterative scheme. This is what is referred to as a warping scheme in the papers cited in the introduction of the original paper, including citation [1]. The warping scheme typically relies on a coarse-to-fine multiresolution strategy to avoid local minima, with implications discussed in the introduction. The flow functions are separated into the current estimate at iteration (k) and a flow update term, according to

$$\mathbf{w}_f \rightarrow \mathbf{w}_f^{(k+1)} = \mathbf{w}_f^{(k)} + \mathbf{dw}_f^{(k)}, \tag{9}$$

such that

$$\begin{aligned} & I_{q_n}(\mathbf{x} + \mathbf{W}_{q_n}) \rightarrow I_{q_n}(\mathbf{x} + \mathbf{W}_{q_n}^{(k+1)}) \approx \\ & \approx I_{q_n}(\mathbf{x} + \mathbf{W}_{q_n}^{(k)}) + I_{q_n x}^{(k)} dU_{q_n}^{(k)} + I_{q_n y}^{(k)} dV_{q_n}^{(k)}, \\ & I_{p_n}(\mathbf{x} + \mathbf{W}_{p_n}) \rightarrow I_{p_n}(\mathbf{x} + \mathbf{W}_{p_n}^{(k+1)}) \approx \\ & \approx I_{p_n}(\mathbf{x} + \mathbf{W}_{p_n}^{(k)}) + I_{p_n x}^{(k)} dU_{p_n}^{(k)} + I_{p_n y}^{(k)} dV_{p_n}^{(k)}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} I_{q_n}^{(k)} = I_{q_n}(\mathbf{x} + \mathbf{W}_{q_n}^{(k)}), \quad & I_{q_n x}^{(k)} = \frac{\partial}{\partial x}\Big(I_{q_n}(\mathbf{x} + \mathbf{W}_{q_n}^{(k)})\Big), \\ & I_{q_n y}^{(k)} = \frac{\partial}{\partial y}\Big(I_{q_n}(\mathbf{x} + \mathbf{W}_{q_n}^{(k)})\Big), \end{aligned} \tag{11}$$

and similarly for $I_{p_n}^{(k)}, I_{p_n x}^{(k)}, I_{p_n y}^{(k)}$. Expressions of the type $I_{q_n}(\mathbf{x} + \mathbf{W}_{q_n}^{(k)})$ are computed by means of bicubic interpolation. Using the substitution in (10), we get for (8) that

$$\frac{\partial I_{p_n q_n}^{(k)}}{\partial u_{f_0}} = \begin{cases} I_{q_n x}^{(k)} - I_{p_n x}^{(k)}, & f_0 \geq r, \quad q_n > f_0, \quad p_n > f_0, \\ I_{q_n x}^{(k)}, & f_0 \geq r, \quad q_n > f_0, \quad p_n \leq f_0, \\ I_{p_n x}^{(k)} - I_{q_n x}^{(k)}, & f_0 < r, \quad p_n \leq f_0, \quad q_n \leq f_0, \\ I_{p_n x}^{(k)}, & f_0 < r, \quad p_n \leq f_0, \quad q_n > f_0, \\ 0, & else, \end{cases} \tag{12}$$

where $q_n > p_n, \forall n$, by construction. For $\partial I_{p_n q_n}^{(k)} / \partial v_{f_0}$ in the E-L equations (2b) w.r.t. v_{f_0}, all partial derivatives of (12) are w.r.t. y instead of x.

2.4 Pseudo-algorithm for the minimization procedure

The full Euler-Lagrange equations stated in (2) are given by summing the contributions (4), (6) and (8) together (using the weights α_S, α_T). The flow terms in all expressions are replaced, for the sake of the iterative solution scheme, by a current estimate and an update term according to (9), as showed for the data term in the previous section. However, there still remains a non-linearity in the E-L equations, due to the expression of Ψ', that should be dealt with. In order to obtain a linear expression of the E-L equations, an inner iteration loop over iteration index l is added in which the Ψ'-terms in (4), (6) and (8) lag behind the flow updates $\mathbf{dw}_f^{(k,l+1)} = (du_f^{(k,l+1)}, dv_f^{(k,l+1)})$. The notation $\Psi'\{\cdot\}^{(k,l)}$ is thus introduced to refer to any of the Ψ'-terms with the flow update terms in its argument taken from the previous iteration (l), i.e. $(du_f^{(k,l)}, dv_f^{(k,l)})$. Several papers take a similar iterative approach, with an outer and an inner loop to successively linearize the problem. We refrain from typing out the full linearized E-L equations, and suggest a study of other publications where the final expressions are not as involved, e.g. [1], in order to ease into the methodology. At a given iteration, (k, l), a set of linear equations that interconnects all E-L equations is formed and solved numerically on the pixel grid of the reference image to yield discrete approximations of $(du_f^{(k,l+1)}, dv_f^{(k,l+1)}), \forall f$. The pseudo-algorithm for the full iterative scheme is given in Table 1.

Pseudo-algorithm
initialization: $\mathbf{u}^{(0)} = 0, \mathbf{v}^{(0)} = 0$
for $k = 0, \ldots, K-1$
compute $I_{q_n}^{(k)}, I_{q_n x}^{(k)}, I_{q_n y}^{(k)}, I_{p_n}^{(k)}, I_{p_n x}^{(k)}, I_{p_n y}^{(k)}, \quad \forall n$
$\mathbf{du}^{(k,0)} = 0, \mathbf{dv}^{(k,0)} = 0$
for $l = 0, \ldots, L-1$
compute the $\Psi'\{\cdot\}^{(k,l)}$-terms in (4), (6), (8), $\forall n, f_0$
construct $\mathbf{A}^{(k,l)}, \mathbf{b}^{(k,l)}$
solve $\mathbf{A}^{(k,l)}[(\mathbf{du}^{(k,l+1)})^T, (\mathbf{dv}^{(k,l+1)})^T]^T = \mathbf{b}^{(k,l)}$
end
$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \mathbf{du}^{(k,L)}, \mathbf{v}^{(k+1)} = \mathbf{v}^{(k)} + \mathbf{dv}^{(k,L)}$
end

Table 1: Optical flow estimation by finding the (discretized) minimizer of a variational cost functional as the solution to the corresponding Euler-Lagrange equations.

A coarse-to-fine estimation strategy is used in the experiments, with $S = 10$ scale levels and a re-sampling factor of 0.85. The first scale $s = 1$ is re-sampled by a factor $0.85^{(10-1)} \approx 0.23$, which represents the coarsest resolution. For each scale, the algorithm in Table 1 is run, although for $s > 1$, $\mathbf{u}^{(0)}, \mathbf{v}^{(0)}$ are assigned values corresponding to the re-scaled solution from the previous scale.

The number of outer and inner iterations used are $K = 5$ and $L = 5$ respectively, verified to be sufficient for convergence. The flow updates at each step (k, l) is computed in closed-form in the experiments, but would also demand an iterative approach (e.g. Gauss-Seidel type methods are used by other authors) for larger image resolutions. The boldface vectors $\mathbf{u}^{(k)} = [(\mathbf{u}_1^{(k)})^T, \ldots, (\mathbf{u}_{\mathcal{F}-1}^{(k)})]^T$ and $\mathbf{v}^{(k)} = [(\mathbf{v}_1^{(k)})^T, \ldots, (\mathbf{v}_{\mathcal{F}-1}^{(k)})]^T$ in Table 1 contain all flow increments $\mathbf{u}_f$ and $\mathbf{v}_f$, each of size $M \times 1$, that in turn contain flow data from the discretized image domain in vectorized form. Thus, the dimension of the equation system that results from numerically implementing the E-L equations on the pixel grid of the reference frame is $2(\mathcal{F} - 1) \cdot M$, where M is the number of pixels per frame. The update terms $\mathbf{du}^{(k,l+1)}, \mathbf{dv}^{(k,l+1)}$ are formed similarly. All the first order derivatives are implemented with the discrete convolution kernel $[0.5, 0, -0.5]$ and $[0.5, 0, -0.5]^T$ for the horizontal and vertical case respectively. No prior low-pass filtering of the image at the given resolution scale is performed for the discrete derivative approximations. A reservation is made for the implementation of the divergence of the scaled gradient in (4), which has the form

$$\mathrm{div}(\Psi'_S \nabla u_{f_0}) = \frac{\partial}{\partial x}(\Psi'_S \frac{\partial u_{f_0}}{\partial x}) + \frac{\partial}{\partial y}(\Psi'_S \frac{\partial u_{f_0}}{\partial y}) \tag{13}$$

and is implemented with convolution kernels $[1, -1]$ and $[1, -1]^T$ for the respective partial derivatives. This choice coincides with a previously proposed implementation from a similar OF method with a non-variational formulation, where the term corresponding to the expression in (13) is referred to as a generalized Laplacian [5]. The discretization leads to the following approximation

$$\frac{\partial}{\partial x}(g \frac{\partial u}{\partial x})\Big|_{i,j} \approx g_{i,j}(u_{i,j+1} - u_{i,j}) - g_{i,j-1}(u_{i,j} - u_{i,j-1}), \tag{14}$$

where g is seen to be evaluated asymmetrically, yet gives convincing results in empirical tests against other discrete operators for the given experiments. The formulation in (14) allows for comparison with the rich research results on numerics of nonlinear diffusion, to which the interested reader is referred for a more thorough study [6].

References

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