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Higher-Order Finite Element Solver for Maxwell's Equations

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Abstract

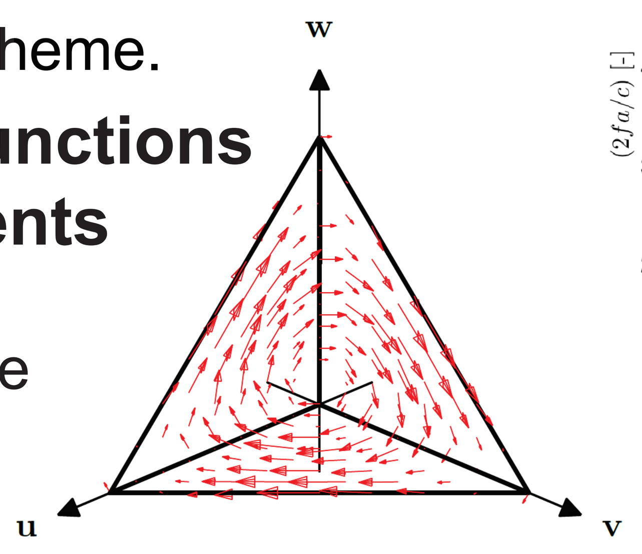
We present a finite element formulation equipped with higher-order basis functions for the electric and magnetic field, which are used together to approximate the electromagnetic field in Maxwell's equations.
The first type of basis functions are formulated on hexahedral elements, where mass lumping is feasible for the special case of brick-shaped elements. Our implementation allows for automatic generation of arbitrary order p for the field approximation, where the lowest-order approximation is the linear representation with $p = 1$.
The second type of basis functions are formulated on tetrahedral elements, which allows for meshing of arbitrary geometries. These basis functions are of hierarchical type and are implemented for orders $p = 1$ to 4 for complete order spaces as well as incomplete (gradient reduced) order spaces.

We test our basis functions on eigenvalue problems and find that the eigenvalues are

i. reproduced with the correct multiplicity
ii. converge towards the analytical result with an error that is proportional to h^{2p} where h is the element size

The solver can be adopted to a time-domain formulation and it allows for higher-order explicit time-stepping on brick-shaped elements, which is a generalization of the well-known leap-frog scheme featured in the finite difference time-domain scheme.

Hierarchical Basis Functions on Tetrahedral Elements



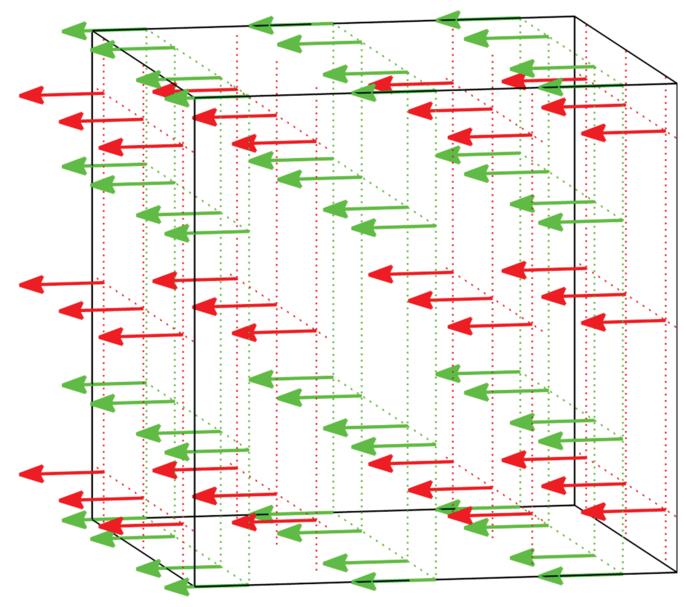
Electrial field solution for the lowest mode in the reference tetrahedron with PMC walls, single element, $p = 4$ complete

The hierarchical basis functions provide separate representation of the gradient and rotational parts of the vector fields. They allow for different orders to be used in the same mesh, which can be used for a hp-adaptive finite element scheme.
The hierarchical basis functions are constructed such that the higher-order basis functions vanish when projected onto a lower-order finite element space with the Nédélec interpolation operator. This makes the basis functions well suited for use in efficient multilevel solvers.

Mass Lumping for Brick-Shaped Elements

We construct an interpolating basis of order p for the electric and magnetic field using a product quadrature rule that consists of a Gauss-Lobatto rule with $p + 1$ points and a Gauss rule with p points for a brick-shaped element.
The specified quadrature rules and corresponding weigths yield a mass lumped scheme, which gives a diagonal mass matrix in the finite element formulation below.

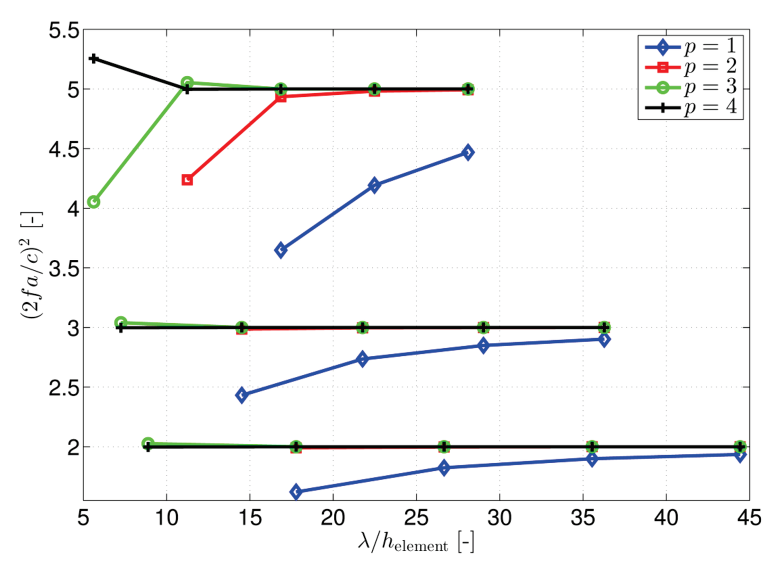
$$\begin{bmatrix} 0 & \mathrm{S} \\ -\mathrm{S}^T & 0 \end{bmatrix} \begin{bmatrix} E \\ c_0 B \end{bmatrix} = \frac{j\omega}{c_0} \begin{bmatrix} \mathrm{M}_E & 0 \\ 0 & \mathrm{M}_B \end{bmatrix} \begin{bmatrix} E \\ c_0 B \end{bmatrix}$$


Degrees of freedom for E_x in green and H_x in red for $p = 3$ for the reference hexahedral element

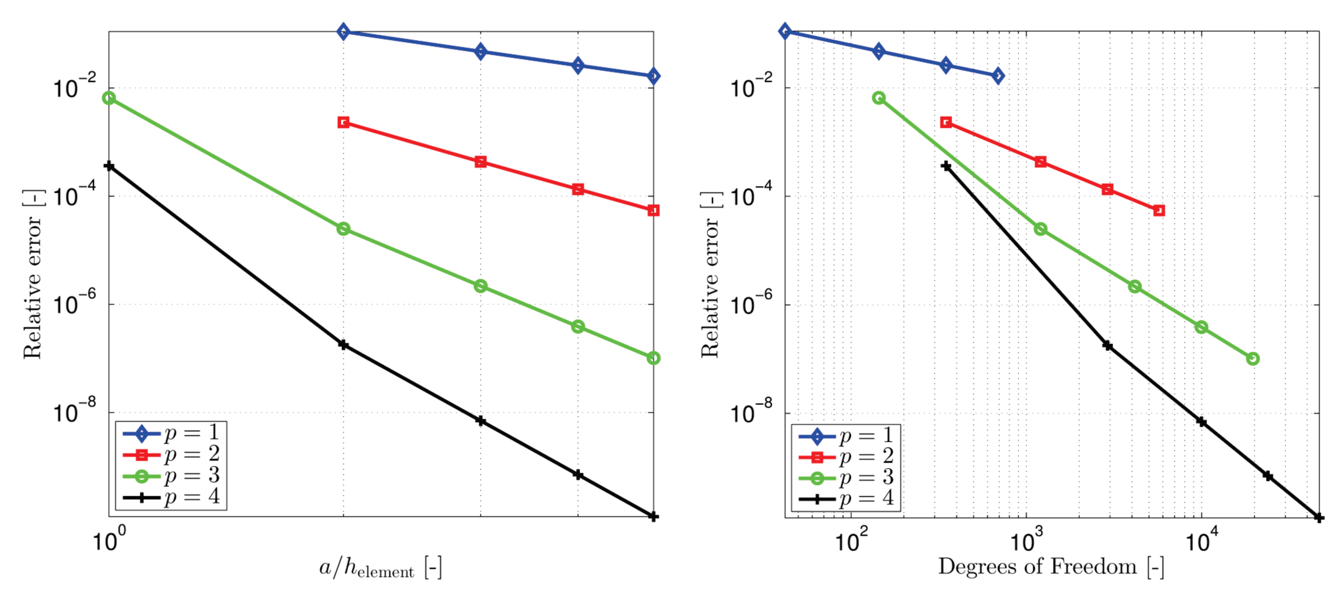
where S is the finite element representation of the curl and M_E and M_B are the diagonal mass matrices for the electric and magnetic field.

Results for Eigenvalue Problem

We test our implementation of the hexahedral based basis functions for a cubic cavity with side length a and PEC walls.

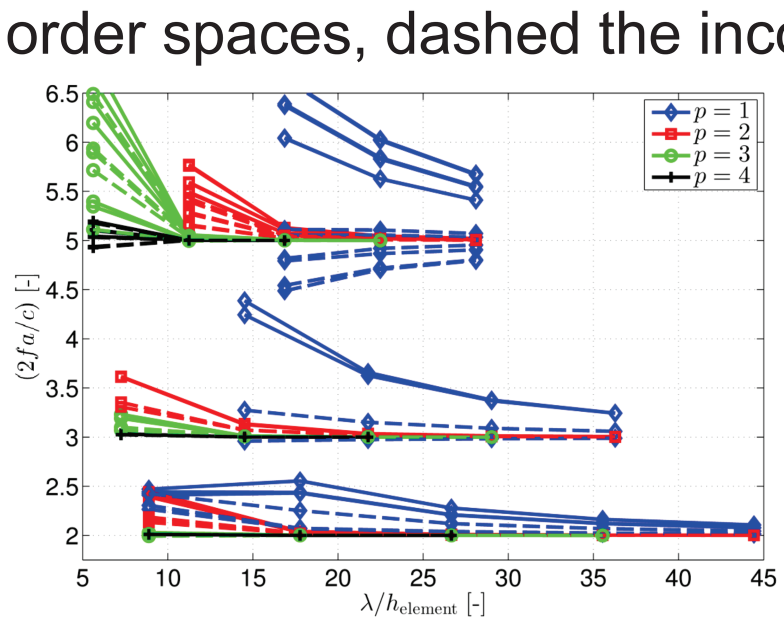


Computed scaled resonant frequency versus elements per wavelength, convergence to analytical values and correct multiplicity is achieved

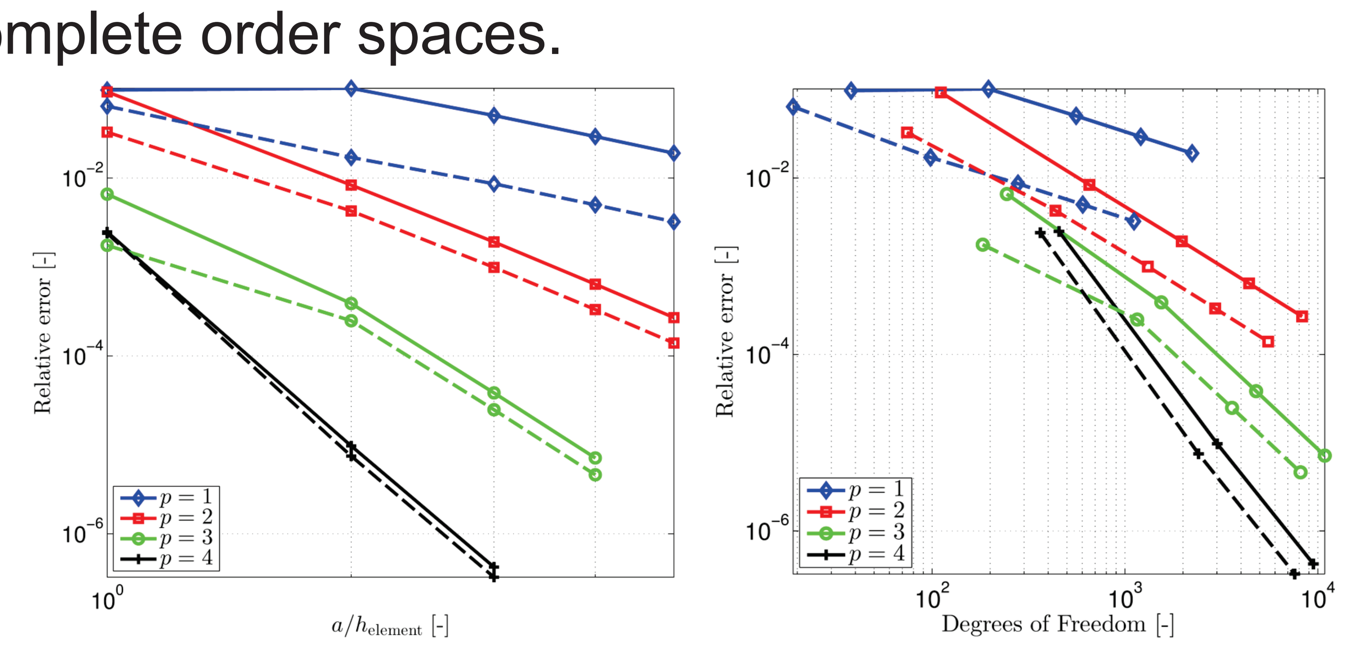


Relative error for the lowest eigenfrequency verus element size in logscale to the left and total number of degrees of freedom to the right, we find that the error is proportional to h^{2p}

We test our implementation of the tetrahedron based basis functions for a cubic cavity with side length a and PMC walls. Solid line represents the complete order spaces, dashed the incomplete order spaces.



Computed scaled resonant frequency versus elements per wavelength, convergence to analytical values and correct multiplicity is achieved



Relative error for the lowest eigenfrequency verus element size in logscale to the left and total number of degrees of freedom to the right, we find that the error is approximately proportional to h^{2p}

Conclusions

- Higher-order basis functions for brick-shaped and tetrahedral elements
- Efficient and accurate mass lumped scheme for brick-shaped elements
- Hierarchical basis on tetrahedrons can be used for a hp-adaptive scheme and meshing of arbitrary geometries
- Correct multiplicity is achieved with spurious free spectrum for eigenvalue problems, and optimal order of convergence for cases without sharp corners or edges

Vetenskapsrådet

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