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Finite-Blocklength Channel Coding Rate Under a Long-Term Power Constraint

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Abstract—This paper investigates the maximal channel coding rate achievable at a given blocklength n and error probability ϵ, when the codewords are subject to a long-term (i.e., averaged-over-all-codeword) power constraint. The second-order term in the large-n expansion of the maximal channel coding rate is characterized both for AWGN channels and for quasi-static fading channels with perfect channel state information at the transmitter and the receiver. It is shown that in both cases the second-order term is proportional to $\sqrt{(\log n)/n}$.

I. Introduction

With *long-term* power constraint, one refers to the setup where the average power of the transmitted codewords, averaged over all possible codewords, is limited. This is in contrast to the conventional *short-term* power constraint, where the power of each transmitted codeword is limited. A long-term power constraint is useful in situations where the power limitation comes from energy efficiency considerations. For example, it captures the relatively long battery life of mobile terminals (at least compared to the duration of a codeword) in the uplink of cellular communication systems [1]. The notion of long-term power constraint is widely used in the wireless communication literature (see, e.g., [2]–[4]) as it allows for a dynamic allocation of power and rate.

In this paper, we study the maximal channel coding rate $R^*(n, \epsilon)$ achievable at a given blocklength n and average error probability ϵ, when the codewords are subject to a long-term power constraint. Two channel models are considered: the AWGN channel, and the quasi-static fading channel, i.e., a channel where the fading gain is random but remains constant during the transmission of each codeword. For the quasi-static case, we also assume that both the transmitter and the receiver have perfect channel state information (CSI). Under this assumption, a codeword is allowed to depend both on the message and on the channel realization, and—under a long-term power constraint—the average power is obtained by taking the expectation of the norm squared of the codewords with respect to both messages and channel realizations.

Previous Results: For AWGN channels subject to the short-term, i.e., per-codeword, power constraint[1] ρ, it was

shown in [5] that

$$R^*_{\text{awgn}}(n, \epsilon) = C(\rho) - \sqrt{\frac{V(\rho)}{n}} Q^{-1}(\epsilon) + \mathcal{O}\left(\frac{\log n}{n}\right). \quad (1)$$

Here, $Q^{-1}(\cdot)$ denotes the inverse of the Gaussian Q-function,

$$C(\rho) \triangleq \log(1 + \rho) \quad (2)$$

denotes the channel capacity,[2] and

$$V(\rho) \triangleq \frac{\rho(\rho + 2)}{(\rho + 1)^2} \quad (3)$$

denotes the *channel dispersion* [5, Def. 1]. The behavior of the maximal channel coding rate changes drastically if the codewords are subject to a long-term power constraint (denoted again by ρ to facilitate comparisons). Specifically, [6, Th. 77][3]

$$R^*_{\text{awgn,lt}}(n, \epsilon) = C\left(\frac{\rho}{1 - \epsilon}\right) + \mathcal{O}\left(n^{-1/3}\right). \quad (4)$$

The expansion (4) implies that the strong converse [7, p. 208] does not hold for AWGN channels subject to a long-term power constraint.

For quasi-static fading channels subject to the short-term power constraint ρ, it was shown in [8] that under mild conditions on the probability distribution of the fading gain G,

$$R^*_{\text{qs}}(n, \epsilon) = C\left(\rho F_{\text{inv}}(\epsilon)\right) + \mathcal{O}\left(\frac{\log n}{n}\right). \quad (5)$$

Here, $F_{\text{inv}} : [0, 1] \to \mathbb{R}^+$ is defined as

$$F_{\text{inv}}(t) \triangleq \sup\{g : \mathbb{P}[G < g] \le t\}. \quad (6)$$

For the case of long-term power constraint, it follows from [9, Props. 1 and 4] that[4]

$$R^*_{\text{qs,lt}}(n, \epsilon) = C\left(\rho/\bar{g}_\epsilon\right) + o(1) \quad (7)$$

where

$$\bar{g}_\epsilon \triangleq \mathbb{E}\left[\frac{1}{G} \mathbb{1}\{G > F_{\text{inv}}(\epsilon)\}\right] + \frac{\mathbb{P}[G \le F_{\text{inv}}(\epsilon)] - \epsilon}{F_{\text{inv}}(\epsilon)} \quad (8)$$

[1]Note that in [5], the short-term power constraint is referred to as *maximal power constraint* and the long-term power constraint is referred to as *average power constraint*.

[2]In this paper, $\log(\cdot)$ stands for the natural logarithm.

[3]The $\mathcal{O}\left(n^{-1/3}\right)$ term can be replaced by $\mathcal{O}\left(n^{-1/2+\delta}\right)$ for every $\delta > 0$ in the achievability proof of (4).

[4]This holds at the points where $C(\rho/\bar{g}_\epsilon)$, or equivalently, $F_{\text{inv}}(\epsilon)$, is continuous in ϵ.

with $\mathbb{1}\{\cdot\}$ standing for the indicator function. As shown in [9], (7) can be achieved by concatenating a fixed Gaussian codebook with a power controller that works as follows: it performs channel inversion when the fading gain G is above $F_{\text{inv}}(\epsilon)$; it turns off transmission when the fading gain is below $F_{\text{inv}}(\epsilon)$. This power-control scheme is sometimes referred to as *truncated channel inversion* [3] [10, Sec. 4.2.4].

Contribution: We characterize the second-order coding rate of both AWGN channels and quasi-static fading channels subject to a long-term power constraint. For AWGN channels, we show that the asymptotic expansion (4) can be refined as

$$R^*_{\text{awgn,lt}}(n, \epsilon)$$
$$= C\left(\frac{\rho}{1 - \epsilon}\right) - \sqrt{V\left(\frac{\rho}{1 - \epsilon}\right)}\sqrt{\frac{\log n}{n}} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \quad (9)$$

For quasi-static fading channels, we show that

$$R^*_{\text{qs,lt}}(n, \epsilon) = C\left(\frac{\rho}{\bar{g}_\epsilon}\right) - \sqrt{V\left(\frac{\rho}{\bar{g}_\epsilon}\right)}\sqrt{\frac{\log n}{n}} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). (10)$$

The asymptotic expansions (9) and (10) are obtained by deriving converse and achievability bounds that match up to the second-order term. Our converse bound on $R^*_{\text{awgn,lt}}(n, \epsilon)$ is based on the meta-converse theorem [5, Th. 26] with an auxiliary channel that depends on the transmitted codewords only through their power. In deriving the bound, we also exploit that the solution of the following minimization problem

$$\inf_{W \sim P_W} \mathbb{E}\left[Q\left(\sqrt{n}\frac{C(W) - \gamma}{\sqrt{V(W)}}\right)\right] \quad (11)$$

where γ is a positive number and the infimum is over all probability distributions P_W on $\mathbb{R}^+$ satisfying $\int w dP_W(w) \leq \rho$, is a two-mass-point distribution, provided that γ is chosen appropriately and n is sufficiently large. The minimization in (11) arises when optimizing the ϵ-quantile of the information density over all power allocations. Our achievability bound on $R^*_{\text{awgn,lt}}(n, \epsilon)$ is obtained by refining the proof of [6, Th. 77]. The basic idea is to add all-zero codewords to a Gaussian codebook designed for a short-term power constraint. The proof of (10) builds upon the proof of (9). In particular, the achievability part of (10) is based on the truncated channel inversion scheme (with an appropriate threshold), which transforms the quasi-static fading channel into an AWGN channel. This implies that truncated channel inversion achieves the first two terms in the large-n expansion of $R^*_{\text{qs,lt}}(n, \epsilon)$.

II. THE AWGN CHANNEL

In this section, we consider the AWGN channel

$$\boldsymbol{Y} = \boldsymbol{x} + \boldsymbol{Z}. \quad (12)$$

Here, $\boldsymbol{x} \in \mathbb{C}^n$ is the transmitted codeword and $\boldsymbol{Z}$ denotes the additive noise vector, which has independent and identically distributed (i.i.d.) circularly symmetric complex Gaussian entries $\mathcal{CN}(0, 1)$. An $(n, M, \epsilon)_{\text{lt}}$ code for the AWGN channel (12) consists of:

1) an encoder $f: \{1, \ldots, M\} \to \mathbb{C}^n$ that maps the message $J \in \{1, \ldots, M\}$ to a codeword $\boldsymbol{x} \in \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_M\}$ satisfying the power constraint

$$\frac{1}{M}\sum_{j=1}^{M}\|\boldsymbol{c}_j\|^2 \leq n\rho. \quad (13)$$

2) A decoder $g: \mathbb{C}^n \to \{1, \ldots, M\}$ satisfying the average error probability constraint

$$\mathbb{P}[g(\boldsymbol{Y}) \neq J] \leq \epsilon. \quad (14)$$

Here, J is assumed to be equiprobable on $\{1, \ldots, M\}$, and $\boldsymbol{Y}$ denotes the channel output induced by the transmitted codeword according to (12).

We shall refer to the constraint (13) on the codewords as long-term power constraint [9], as opposed to the more stringent short-term power constraint

$$\|\boldsymbol{c}_j\|^2 \leq n\rho, \quad j = 1, \ldots, M. \quad (15)$$

The maximal channel coding rate is defined as

$$R^*_{\text{awgn,lt}}(n, \epsilon) \triangleq \sup\left\{\frac{\log M}{n} : \exists (n, M, \epsilon)_{\text{lt}} \text{ code}\right\}. \quad (16)$$

As reviewed in Section I, the ϵ-capacity of the AWGN channel is [6, Th. 77]

$$\lim_{n \to \infty} R^*_{\text{awgn,lt}}(n, \epsilon) = C\left(\frac{\rho}{1 - \epsilon}\right), \quad 0 < \epsilon < 1. \quad (17)$$

Note that if we replace (13) with (15) or the average error probability constraint (14) with the maximal error probability constraint

$$\max_{1 \leq j \leq M} \mathbb{P}[g(\boldsymbol{Y}) \neq J \,|\, J = j] \leq \epsilon \quad (18)$$

the strong converse applies and (17) ceases to be valid.

Theorem 1 below characterizes the second-order term in the asymptotic expansion of $R^*_{\text{awgn,lt}}(n, \epsilon)$ for fixed $0 < \epsilon < 1$, and large n.

Theorem 1: For the AWGN channel (12) subject to the long-term power constraint ρ, the maximal channel coding rate $R^*_{\text{awgn,lt}}(n, \epsilon)$ is

$$R^*_{\text{awgn,lt}}(n, \epsilon) = C\left(\frac{\rho}{1 - \epsilon}\right) - \sqrt{V\left(\frac{\rho}{1 - \epsilon}\right)}\sqrt{\frac{\log n}{n}}$$
$$+ \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), \quad \forall 0 < \epsilon < 1 \quad (19)$$

where $C(\cdot)$ and $V(\cdot)$ are defined in (2) and (3), respectively.

Proof: Due to space limitations, we only provide a sketch of the proof. For full details, see [11].

Converse: Consider an arbitrary $(n, M, \epsilon)_{\text{lt}}$ code. Let $P_{\boldsymbol{X}}$ denote the probability distribution on the channel input $\boldsymbol{X}$ induced by the code. To upper-bound $R^*_{\text{awgn,lt}}(n, \epsilon)$, we use the meta-converse theorem [5, Th. 26] with the following *auxiliary* channel $Q_{\boldsymbol{Y} \,|\, \boldsymbol{X}}$:

$$Q_{\boldsymbol{Y} \,|\, \boldsymbol{X} = \boldsymbol{x}} = \mathcal{CN}\left(\boldsymbol{0}, \left(1 + \|\boldsymbol{x}\|^2/n\right)\mathsf{I}_n\right). \quad (20)$$

The choice of letting the auxiliary channel in (20) depend on the transmit codeword through its power, is inspired by a similar approach used in [6, Sec. 4.5] to characterize $R^*(n,\epsilon)$ for the case of parallel AWGN channels, and in [8] for the case of quasi-static MIMO fading channels under a short-term power constraint. With this choice, we have [5, Th. 26]

$$\beta_{1-\epsilon}(P_{\boldsymbol{XY}}, P_{\boldsymbol{X}}Q_{\boldsymbol{Y}|\boldsymbol{X}}) \leq 1 - \epsilon' \qquad (21)$$

where $\beta_{(\cdot)}(\cdot,\cdot)$ is defined in [5, Eq. (100)] and ϵ' is the average probability of error incurred by using the selected $(n,M,\epsilon)_{\text{lt}}$ code over the auxiliary channel $Q_{\boldsymbol{Y}|\boldsymbol{X}}$.

Next, we lower-bound the left-hand side (LHS) of (21). Under $P_{\boldsymbol{XY}}$, the random variable $\log \frac{dP_{\boldsymbol{XY}}}{d(P_{\boldsymbol{X}}Q_{\boldsymbol{Y}|\boldsymbol{X}})}$ has the same distribution as

$$S_n(W) \triangleq nC(W) + \sum_{i=1}^n \left(1 - \frac{\left|\sqrt{W}Z_i - 1\right|^2}{1+W}\right) \qquad (22)$$

where $W \triangleq \|\boldsymbol{X}\|^2/n$ and $Z_1, ..., Z_n$ are i.i.d. $\mathcal{CN}(0,1)$ distributed and independent of W. Using [5, Eq. (102)] and (22), we obtain the desired lower bound

$$\beta_{1-\epsilon}(P_{\boldsymbol{XY}}, P_{\boldsymbol{X}}Q_{\boldsymbol{Y}|\boldsymbol{X}}) \geq e^{-n\gamma}\left(\mathbb{P}[S_n(W) \leq n\gamma] - \epsilon\right) \quad (23)$$

which holds for every $\gamma > 0$. The right-hand side (RHS) of (21) can be upper-bounded as follows [11]

$$\log(1 - \epsilon') \leq -\log M + \mathcal{O}(\log n). \qquad (24)$$

Substituting (23) and (24) into (21) we obtain

$$\log M \leq n\gamma - \log\left(\mathbb{P}[S_n(W) \leq n\gamma] - \epsilon\right) + \mathcal{O}(\log n). \quad (25)$$

Note that the RHS of (25) depends on the code only through the probability distribution that the code induces on $W = \|\boldsymbol{X}\|^2/n$. We denote this probability distribution by P_W.

Let Ω be the set of probability distributions P_W on $\mathbb{R}^+$ that satisfy

$$\int w\, dP_W(w) \leq \rho. \qquad (26)$$

Maximizing the RHS of (25) over all $P_W \in \Omega$ and then dividing both terms by n, we conclude that for every $\gamma > 0$

$$R^*_{\text{awgn,lt}}(n,\epsilon)$$
$$\leq \gamma - \frac{1}{n}\log\left(\inf_{P_W \in \Omega} \mathbb{P}[S_n(W) \leq n\gamma] - \epsilon\right) + \mathcal{O}\left(\frac{\log n}{n}\right). (27)$$

To evaluate $\mathbb{P}[S_n(W) \leq n\gamma]$, we note that—given W—the random variable $S_n(W)$ is the sum of n i.i.d. random variables with mean $C(W)$ and variance $V(W)$. An application of the Berry-Esseen theorem [5, Th. 44] yields [11]

$$\mathbb{P}[S_n(W) \leq n\gamma] \geq \mathbb{E}[q_{n,\gamma}(W)] - \frac{6 \cdot 9^{3/4}}{\sqrt{n}} \qquad (28)$$

where

$$q_{n,\gamma}(w) \triangleq Q\left(\sqrt{n}\frac{C(w) - \gamma}{\sqrt{V(w)}}\right). \qquad (29)$$

To eliminate the dependency of the RHS of (28) on P_W, we next minimize the first term on the RHS of (28) over all P_W

in Ω, i.e., we solve the optimization problem given in (11). The following lemma, proven in [11], gives the solution of (11).

Lemma 2: Let $\gamma > 0$ and assume that $n \geq 2\pi(e^{2\gamma}-1)\gamma^{-2}$. Let $w_0 \in [e^\gamma - 1, \infty)$ be the unique number that satisfies

$$\frac{q_{n,\gamma}(w_0) - 1}{w_0} = q'_{n,\gamma}(w_0) \qquad (30)$$

where $q'_{n,\gamma}(\cdot)$ stands for the first derivative of $q_{n,\gamma}(\cdot)$. Then, the infimum in (11) is a minimum and the probability distribution P^*_W that minimizes $\mathbb{E}_{P_W}[q_{n,\gamma}(W)]$ has the following structure:
1) if $w_0 \geq \rho$, then P^*_W has two mass points, one located at 0 and the other located at w_0. Furthermore, $P^*_W(0) = 1 - \rho/w_0$ and $P^*_W(w_0) = \rho/w_0$.
2) If $\rho > w_0$, then P^*_W has only one mass point located at ρ.

We now use Lemma 2 to further lower-bound the RHS of (28), and, hence, further upper-bound the RHS of (27). Let w_0 be as in Lemma 2. Assume that γ in (23) is chosen from the interval $(C(\rho/(1-\epsilon))-\delta, C(\rho/(1-\epsilon))+\delta)$ for some $0 < \delta < C(\rho/(1-\epsilon))$. For such a γ, we have $w_0 \geq e^\gamma - 1 > \rho$ provided that δ is chosen sufficiently small. Using Lemma 2, we conclude that for sufficiently large n

$$\inf_{P_W \in \Omega} \mathbb{E}[q_{n,\gamma}(W)] = 1 - \frac{\rho}{w_0} + \frac{\rho}{w_0}q_{n,\gamma}(w_0). \qquad (31)$$

Substituting (31) into (28) and then (28) into (27), we obtain

$$R^*_{\text{awgn,lt}}(n,\epsilon) \leq \gamma - \frac{1}{n}\log\left(1 - \frac{\rho}{w_0} + \frac{\rho}{w_0}q_{n,\gamma}(w_0)\right.$$
$$\left. - \frac{6 \cdot 9^{3/4}}{\sqrt{n}} - \epsilon\right) + \mathcal{O}\left(\frac{\log n}{n}\right). (32)$$

We choose now γ as the solution of

$$1 - \frac{\rho}{w_0} + \frac{\rho}{w_0}q_{n,\gamma}(w_0) - \frac{6 \cdot 9^{3/4}}{\sqrt{n}} - \epsilon = \frac{1}{\sqrt{n}}. \qquad (33)$$

Solving (30) and (33) for w_0 and γ we obtain [11]

$$\gamma = C\left(\frac{\rho}{1-\epsilon}\right) - \sqrt{V\left(\frac{\rho}{1-\epsilon}\right)}\sqrt{\frac{\log n}{n}} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \quad (34)$$

Observe now that γ belongs indeed to the interval $(C(\rho/(1-\epsilon)) - \delta, C(\rho/(1-\epsilon)) + \delta)$ for sufficiently large n. The proof of the converse part of Theorem 1 is concluded by substituting (33) and (34) into (32).

Achievability: The proof is a refinement of the method used to establish [6, Th. 77]. Let $(n,M_n,\epsilon_n)_{\text{st}}$, where $\epsilon_n = 2/\sqrt{n\log n}$, be a code for the AWGN channel (12), whose codewords $\{\boldsymbol{c}_1, ..., \boldsymbol{c}_{M_n}\}$ satisfy the short-term power constraint

$$\frac{1}{n}\|\boldsymbol{c}_l\|^2 \leq \rho_n \triangleq \rho\frac{1-\epsilon_n}{1-\epsilon}, \quad l = 1, ..., M_n. \qquad (35)$$

Set

$$M = M_n\frac{1-\epsilon_n}{1-\epsilon} \qquad (36)$$

and assume that n is large enough so that $M > M_n$. We construct a code with M codewords for the case of long-term power constraint by adding $(M - M_n)$ all-zero codewords to

the codewords of the $(n, M_n, \epsilon_n)_{\mathrm{st}}$ code. The resulting code satisfies the long-term power constraint. Indeed,

$$0 \cdot \frac{M - M_n}{M} + \rho_n \cdot \frac{M_n}{M} = \rho. \qquad (37)$$

At the same time, the average probability of error is upper-bounded by

$$1 \cdot \frac{M - M_n}{M} + \epsilon_n \cdot \frac{M_n}{M} = \epsilon. \qquad (38)$$

Therefore, by definition,

$$R^*_{\mathrm{awgn,lt}}(n, \epsilon) \geq \frac{\log M}{n} = \frac{\log M_n}{n} + \mathcal{O}\left(\frac{1}{n}\right) \qquad (39)$$

where the last step follows from (36).

To conclude the proof, it suffices to show that

$$\frac{\log M_n}{n} \geq C\left(\frac{\rho}{1 - \epsilon}\right) - \sqrt{V\left(\frac{\rho}{1 - \epsilon}\right)} \sqrt{\frac{\log n}{n}} + o\left(\frac{1}{\sqrt{n}}\right). \qquad (40)$$

The proof of (40), which is omitted for space limitations and can be found in [11], uses the $\kappa\beta$ bound [5, Th. 25] and a Cramer-Esseen-type central limit theorem [12, Th. VI.1]. Note that a weaker version of (40), with $o(1/\sqrt{n})$ replaced by $o(\sqrt{\log(n)/n})$, follows directly from [6, Th. 96]. ∎

III. THE QUASI-STATIC FADING CHANNEL

We move now to the quasi-static fading case. The channel input-output relation is given by

$$\boldsymbol{Y} = H\boldsymbol{x} + \boldsymbol{Z} \qquad (41)$$

where H denotes the complex fading coefficient, which is random but remains constant for all n channel uses. We assume that the realizations of H are known to both the transmitter and the receiver. We denote the channel gain by $G = |H|^2$. An $(n, M, \epsilon)_{\mathrm{lt}}$ code for the quasi-static fading channnel (41) consists of:

1) an encoder $f\colon \{1, \ldots, M\} \times \mathbb{C} \to \mathbb{C}^n$ that maps the message $J \in \{1, \ldots, M\}$ and the channel coefficient H to a codeword $\boldsymbol{x} = f(J, H)$ that satisfies the long-term power constraint

$$\mathbb{E}\left[\|f(J, H)\|^2\right] \leq n\rho. \qquad (42)$$

Here, J is equiprobable on $\{1, \ldots, M\}$, and the expectation in (42) is with respect to the joint probability distribution of J and H.

2) A decoder $g\colon \mathbb{C}^n \times \mathbb{C} \to \{1, \ldots, M\}$ satisfying the average error probability constraint

$$\mathbb{P}[g(\boldsymbol{Y}, H) \neq J] \leq \epsilon \qquad (43)$$

where $\boldsymbol{Y}$ is the channel output induced by the transmitted codeword $\boldsymbol{x} = f(J, H)$ according to (41).

The maximal channel coding rate is defined as

$$R^*_{\mathrm{qs,lt}}(n, \epsilon) \triangleq \sup\left\{\frac{\log M}{n} \colon \exists (n, M, \epsilon)_{\mathrm{lt}} \text{ code}\right\}. \qquad (44)$$

As discussed in Section I, the ϵ-capacity of the quasi-static fading channel (41) is [9]

$$\lim_{n \to \infty} R^*_{\mathrm{qs,lt}}(n, \epsilon) = C\left(\rho/\bar{g}_\epsilon\right). \qquad (45)$$

If we replace (42) with the short-term power constraint

$$\|f(j, h)\|^2 \leq n\rho, \quad \forall j = 1, \ldots, M, \ \forall h \in \mathbb{C} \qquad (46)$$

then CSI at the transmitter is ineffectual and (45) ceases to be valid [13]. Differently from the AWGN case, (45) holds also if (43) is replaced by the maximal error probability constraint

$$\max_{1 \leq j \leq M} \mathbb{P}[g(\boldsymbol{Y}, H) \neq J \mid J = j] \leq \epsilon. \qquad (47)$$

Theorem 3 below characterizes the second-order term in the asymptotic expansion of $R^*_{\mathrm{qs,lt}}(n, \epsilon)$ for fixed $0 < \epsilon < 1$ and large n.

Theorem 3: Assume that the input of the quasi-static fading channel (41) is subject to a long-term power constraint ρ. Assume that CSI is available at both the transmitter and the receiver. Let $0 < \epsilon < 1$ be the average probability of error and assume that

1) $F_{\mathrm{inv}}(\cdot)$ defined in (6) is continuous at ϵ;
2) $\mathbb{E}[G] < \infty$, where $G = |H|^2$ is the channel gain.

Then

$$R^*_{\mathrm{qs,lt}}(n, \epsilon) = C\left(\frac{\rho}{\bar{g}_\epsilon}\right) - \sqrt{V\left(\frac{\rho}{\bar{g}_\epsilon}\right)} \sqrt{\frac{\log n}{n}} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \qquad (48)$$

where $C(\cdot)$ and $V(\cdot)$ are defined in (2) and (3), respectively, and $\bar{g}_\epsilon$ is given in (8).

Remark 1: The asymptotic expansion (48) holds also if the average error probability constraint (43) is replaced by the maximal error probability constraint (47).

Proof: For simplicity of presentation, we assume that G is a continuous random variable, i.e., the probability density function of G exists. Under this assumption, $\bar{g}_\epsilon$ takes the following simpler form

$$\bar{g}_\epsilon = \mathbb{E}\left[\frac{1}{G}\mathbb{1}\{G > F_{\mathrm{inv}}(\epsilon)\}\right]. \qquad (49)$$

The proof can be easily extended to the case where G is not continuous by proceeding as in the proof of [9, Prop. 4].

Due to space limitations, we only provide the proof of the achievability part of Theorem 3. The proof of the converse part follows closely that of the converse part of Theorem 1 and can be found in [11].

Let

$$\epsilon_n \triangleq \frac{2}{\sqrt{n \log n}} \quad \text{and} \quad \epsilon'_n \triangleq \frac{\epsilon - \epsilon_n}{1 - \epsilon_n}. \qquad (50)$$

For sufficiently large n, we have $\epsilon'_n > 0$. Let

$$\bar{g}_n = \mathbb{E}\left[\frac{1}{G}\mathbb{1}\{G > F_{\mathrm{inv}}(\epsilon'_n)\}\right]. \qquad (51)$$

Furthermore, let $\rho_n \triangleq \rho/\bar{g}_n$. We define the following power-allocation function for each $g \in \mathbb{R}^+$:

$$w(g) \triangleq \frac{1}{g}\mathbb{1}\{g > F_{\mathrm{inv}}(\epsilon'_n)\}. \qquad (52)$$

As we shall see, this power allocation function corresponds to truncated channel inversion: the fading channel is inverted if the gain is above $F_{\mathrm{inv}}(\epsilon'_n)$. Otherwise, transmission is silenced. Let M_n denote the maximal number of length-n codewords that can be decoded with maximal probability of error not exceeding ϵ_n over an AWGN channel subject to the short-term power constraint ρ_n. Let the corresponding code be $(n, M_n, \epsilon_n)_{\mathrm{st}}$ and its codewords be $\{c_1, \ldots, c_{M_n}\}$.

Consider now a code for the quasi-static fading channel (41) whose encoder f has the following structure:

$$f(j, h) = \sqrt{w(|h|^2)}\, c_j, \quad j \in \{1, \ldots, M_n\}, \ h \in \mathbb{C}. \quad (53)$$

Such a code satisfies the long-term power constraint:

$$\frac{1}{M_n}\mathbb{E}\left[\sum_{j=1}^{M_n} \|f(j, H)\|^2\right] = \frac{1}{M_n}\mathbb{E}\left[w(|H|^2)\right]\sum_{j=1}^{M_n} \|c_j\|^2 \quad (54)$$

$$\leq \bar{g}_n \rho_n \quad (55)$$

$$= \rho. \quad (56)$$

Furthermore, the average error probability of the code is upper-bounded by

$$1 \cdot \epsilon'_n + \epsilon_n(1 - \epsilon'_n) = \epsilon. \quad (57)$$

Indeed, channel inversion according to (52) is performed whenever the fading gain G is larger than $F_{\mathrm{inv}}(\epsilon'_n)$, which occurs with probability $1 - \epsilon'_n$. Channel inversion transforms the quasi-static fading channel into an AWGN channel. Hence, the conditional error probability given that channel inversion is performed is upper-bounded by ϵ_n. When channel inversion is not performed, an error occurs with probability 1. This shows that the code is an $(n, M_n, \epsilon)_{\mathrm{lt}}$ code. Hence,

$$R^*_{\mathrm{qs,lt}}(n, \epsilon) \geq \frac{\log M_n(n, \epsilon_n)}{n}. \quad (58)$$

From (40) in Section II, we know that

$$\frac{\log M_n}{n} \geq C(\rho_n) - \sqrt{\frac{\log n}{n}}\sqrt{V(\rho_n)} + o\left(\frac{1}{\sqrt{n}}\right). \quad (59)$$

Furthermore, it follows from (50) after algebraic manipulations that [11]

$$\bar{g}_n = \bar{g}_\epsilon + \mathcal{O}(1/\sqrt{n \log n}). \quad (60)$$

Since $\rho_n = \rho/\bar{g}_n$, (60) implies that

$$\rho_n = \rho/\bar{g}_\epsilon + \mathcal{O}\left(1/\sqrt{n \log n}\right). \quad (61)$$

The achievability part of (48) follows by substituting (61) into (59) and by a Taylor series expansion of $C(\cdot)$ and $V(\cdot)$ around $\rho/\bar{g}_\epsilon$. ∎

IV. REMARKS

Convergence to capacity: For AWGN channels subject to a short-term power constraint, it follows from (1) that the finite-blocklength rate penalty compared to channel capacity is approximately proportional to $1/\sqrt{n}$. By contrast, Theorem 1 shows that for AWGN channels subject to a long-term power

constraint, this rate penalty is approximately proportional to $\sqrt{\log(n)/n}$. However, this does not necessarily mean that $R^*_{\mathrm{awgn,lt}}(n, \epsilon)$ converges to capacity slower in the case of long-term power constraint than in the case of short power constraint. In fact, the term $Q^{-1}(\epsilon)$ multiplying $1/\sqrt{n}$ in (1) is comparable to $\sqrt{\log n}$ for practical values of ϵ and n. For example, for $\epsilon = 10^{-3}$, $n = 1000$, and $\rho = 0$ dB, we have

$$\sqrt{\frac{V(\rho)}{n}}Q^{-1}(\epsilon) = 0.085 > \sqrt{V\left(\frac{\rho}{1-\epsilon}\right)}\sqrt{\frac{\log n}{n}} = 0.072. \quad (62)$$

To better characterize the speed of convergence to capacity, our result needs to be complemented with nonasymptotic bounds on $R^*_{\mathrm{awgn,lt}}(n, \epsilon)$ similar to the ones reported in [5, Sec. III-J].

Fading vs. noise: For quasi-static fading channels, it is well known that the typical error event in the large blocklength regime is that the channel is in outage (deep fade event). In the finite blocklength regime, however, errors may occur not only if the channel is in outage, but also if the code is not able to average out the effect of the additive noise. How do these two phenomena contribute to the error probability, as a function of the blocklength n? For the code used in Section III, the contribution to the error probability due to the additive noise goes to zero as $n \to \infty$ no slower than $1/\sqrt{n \log n}$; the remaining contribution is due to channel outage.

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