On the observation with polarised light of a granular medium under stress

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Abstract. Whereas a two dimensional assembly of photoelastic discs shows clear coloured fringes when viewed with polarised light, a three dimensional granular medium composed of glass beads or grains shows structured patterns in the form of stripes of light of varying intensity which appear to have orientation related to the overall direction of major principal stress. Any light ray passes through many individual grains, each of which contributes to the retardation of the polarised light which emerges from the sample. The retardation within a grain is the result of nonuniform stress states within that grain which themselves result from the distribution of the contact forces between neighbouring grains in the assembly. In this paper we combine the framework of Jones calculus and the stress optic law in order to calculate and predict the effect that the randomly distributed contact forces on randomly distributed grains should have on the intensity of the light emerging from such a complex three dimensional assembly.

Stress; Granular medium; Photoelasticity; Jones Calculus

INTRODUCTION

Full field measurement techniques have become popular for probing the internal fabric changes in a granular medium. Particularly popular are two dimensional (2D) plane observations through a transparent window which capture the evolution of particle movements with digital photography (e.g. White & Bolton 2004, Hall et al 2010). X-ray tomography techniques have been employed to deduce three dimensional internal particle movements (e.g. Desrues et al. 1996, Lenoir et al. 2007). In both cases the image data are processed to extract kinematic evidence using 2D or 3D Digital Image Correlation (or Particle Image Velocimetry), particle tracking, stereophotogrammetry or image subtraction techniques (Sutton et al. 1983, Hall et al 2010, Desrues and Viggiani 2004, Ochiai et al. 2006, Rosenbrand & Dijkstra 2012). Although these methods offer new insights in the kinematics of a granular medium, the (local) stress information inside the sample remains unknown.

In an attempt to discover some of this missing information researchers have used various implementations of the photoelastic measurement technique. The granular medium is replaced by discs, beads, or grains of a birefringent material and the incident light is (circularly) polarised to enable qualitative and quantitative observations on the stress in the sample (Dantu 1957, Wakabayashi 1957, Drescher and De Josselin de Jong 1972, Drescher 1976, Rossmanith and Shukla 1982, Allersma 1982, Howell et al. 1999a, 1999b, Lesniewska and Sklodowski 2005, Dijkstra and Broere 2010, Lesniewska and Muir Wood 2011, Muir Wood and Lesniewska 2011).

For 2D plane stress samples, composed of circular discs, the stress information can be unambigously retrieved. Upon loading the discs of a birefringent continuum material classical fringes are observed in polarised light. The emergent light intensity resulting from these fringes can be linked to the fundamental properties of the polarised light tensor which in turn is linked to the stress in the disc by the stress optical law (Neumann 1841). However, in 3D assemblies of photoelastic particles structured patterns of stripes of light of varying intensity are observed. The latter cannot be rigorously linked to quantitative observations of stress in the assembly. In this paper the effect of multiple layers of birefringent discs along the light ray and its effect on the emergent light intensity will be studied using the concept of (extended) Jones Calculus for polarised light in order to gain insight into the origin of these light stripes.

MODEL FOR OPTICAL COMPONENTS

Optical system for idealised optical medium

Fig. 1 shows a cross section along the light ray for an idealised granular medium composed of multiple particle layers within a polariscope. The light passes the linear polariser and quarter wave plate of the polariscope before entering the medium, which is modelled as a stack of retarders with arbitrary properties. Thus the angle of retardation and orientation of principal axis of each layer are chosen randomly and there is no correlation between successive layers.

The Jones Calculus approach (e.g. Theocaris and Gdouto 1979) is used to calculate the emerging light intensity from the system of optical components. The effect of each optical element, i.e. linear polariser, quarter wave plate and arbitrary retarder, on the emergent light intensity is represented by a corresponding Jones matrix and the polarisation state of the light by a Jones vector. The emerging light intensity is subsequently calculated from the Jones vector for the emerging light $\boldsymbol{a_e}$:

$$I = \overline{\mathbf{a}_e}\mathbf{a}_e \tag{1}$$

The emerging light intensity I is a scalar and is generally $I \leq 1$. The Jones vector for the emerging light $\mathbf{a_e}$. can be obtained by multiplying the Jones matrices of each optical element in the chain with the Jones vectorof the unpolarised incident light $\mathbf{a_i}$. Here we will consider a dark light circular polariscope with two linear polarisers with their fast axis respectively at $\phi = \pi/2$ and $\phi = 0$ and two quarter wave plates with fast axis at $\phi = -\pi/4$ and $\phi = \pi/4$ and a retardation angle $\delta = \pi/2$.:

$$\mathbf{a}_e = \mathbf{P}_0\mathbf{R}_{\pi/4}(\pi/2)\mathbf{R}_{\phi}(\delta)\mathbf{R}_{-\pi/4}(\pi/2)\mathbf{P}_{\pi/2}\mathbf{a}_i \qquad (2)$$

The Jones matrix for the middle Retarder with its fast axis at φ and its retardation angle of δ is the object of this study. This matrix is either a single Jones matrix for one optical component or a product of multiple (arbitrary) retarders, corresponding to the grains in a granular medium:

$$\mathbf{R} = \mathbf{R}^n\mathbf{R}^{n+1}...\mathbf{R}^{n+100} \qquad (3)$$

The optical properties within the grain, the isoclinic angle ϕ and retardation angle δ are directly linked to the stress state within the grain by the stress optical law:

$$\delta = C_o(\sigma_1 - \sigma_2) \qquad (4)$$

where C_o is a stress-optical material constant. The isoclinic angle ϕ is the angle of rotation of the fast axis of a retarder in an arbitrary frame of reference and in the theory coincides with the principal stress direction. As a result the shear stress is linked to both the isoclinic angle ϕ and the retardation angle δ:

$$\sigma_{xy} = \frac{\delta\sin(2\phi)}{2C_0} \qquad (5)$$

Jones matrices for optical components

The normalised effect of each individual optical component on the light polarisation state is established in Jones matrix notation. The propagation matrix for an idealised polariser yields:

$$\mathbf{P}_{\mathbf{p}} = \begin{pmatrix} p_x & 0 \\ 0 & p_y \end{pmatrix} \qquad (6)$$

where p_x and p_y are respectively the normalised horizontal and vertical polarisation amplitude (0...1). Similarly, the propagation matrix for an idealised retarder follows from:

$$\mathbf{P}_{\mathbf{R}}(\delta) = \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix} \qquad (7)$$

where δ represents the retardation phase angle between the fast and slow polarisation wave. In order to use these propagation matrices for the polariser and retarder in an arbitrary rotated frame of reference a transformation matrix is used:

$$\mathbf{K}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \qquad (8)$$

where ϕ represents the in plane orientation of the fast axis of the polariser or the retarder. Multiplication of the transformation matrix and the propagation matrix yields the Jones matrix for an arbitrary polariser and retarder:

$$\mathbf{P}(\phi) = \mathbf{K}(-\phi)\mathbf{P_P}\mathbf{K}(\phi) \text{ and } \mathbf{R}(\phi,\delta) = \mathbf{K}(-\phi)\mathbf{P_R}\mathbf{K}(\phi) \tag{9}$$

In the current calculations an idealised linear polariser with $p_x = 1$, $p_y = 0$ has been used for the propagation matrix.

Internal reflections

The previous transformation matrices are a special case where light is orthogonally incident on the retarder. A more complete description should also account for the leakage of light through imperfect refraction at grain boundaries. Such effects can be included in an extended Jones calculus (Yeh 1982) by including an additional transmission matrix. The effects are especially pronounced for a stack of retarders with less than ideal contact: this is always the case in granular media where precise matching of the refractive index of the pore liquid and the grains is hard to accomplish. The transmission matrices for the incident and emergent light are:

$$\mathbf{T}_i(\theta) = \begin{pmatrix} t_s & 0 \\ 0 & t_p \end{pmatrix} \tag{10}$$

where

$$t_s = \frac{2n\cos\theta}{n\cos\theta + n_o\cos\theta_o} \text{ and } t_p = \frac{2n\cos\theta}{n\cos\theta_o + n_o\cos\theta} \tag{11a \& 11b}$$

and

$$\mathbf{T}_e(\theta) = \begin{pmatrix} t_s' & 0 \\ 0 & t_p' \end{pmatrix} \tag{12}$$

where

$$t_s' = \frac{2n_o\cos\theta_o}{n_o\cos\theta + n\cos\theta_o} \text{ and } t_p' = \frac{2n_o\cos\theta_o}{n_o\cos\theta + n\cos\theta_o} \tag{13a \& 13b}$$

SIMULATION RESULTS

Simulation conditions

These equations have been implemented in a MATLAB script to investigate the effect of the number of retarders, (the thickness of the sample), and their properties, (the stress state in the granular medium), on the emerging light intensity. The large 3D stress fluctuations in each layer of discs are modelled by randomly varying the retardation and isoclinic angle for each layer of grains which are directly linked to stress state by the stress optic law. The thickness of the sample along the light ray is increased by progressively adding further retarders with random properties).

The remaining research question is: do the light stripes observed in an experiment on a 3D assembly of photoelastic particles actually represent some quantifiable photoelastic response of the sample? The current working hypothesis is that with increasing sample thickness such quantitative stress information must be lost if only because of the random distribution of contact forces and the inevitable slight mismatch in refractive index between the grains and the pore fluid, resulting in even the slightest internal reflections.

Four cases are investigated: (1) constant stress in all grains, represented by 100 similar retarders $\varphi = \pi/2$ and $\delta = \pi/2$; (2) pure random fluctuating stress, represented by 100 retarders with randomly sampled values $-\pi/2 < \varphi < \pi/2$ and $-2\pi < \delta < 2\pi$; (3) constant base stress with a fluctuating amplitude, represented by 100 retarders with $\varphi = \pi/4 \pm \text{rand()}*\pi/4$, $\delta = \pi/2 \pm \text{rand()}*\pi/2$, where rand() is the pseudorandom generator embedded in MATLAB. (4) pure random fluctuating stress, but now with internal reflections from a refractive index mismatch of ($n_0 - n = 0.2$) represented by 100 retarders with randomly sampled values $-\pi/2 < \varphi < \pi/2$ and $-2\pi < \delta < 2\pi$.

Results

The results of cases 1-4 are shown in Figs. 2 – 5. Only case 4 shows a clear decay in light intensity as a function of the number of retarders (thickness of the sample). The fluctuating nature of the emergent light intensity in the other cases as a function of the number of retarders is surprising at first. However the system alternates around the value of one ideal retarder ($I = 0.5$) between near extinction, (the emergent polarisation is nearly linearly vertically polarized) or full output, (near linearly horizontally polarized light vector). The use of pure random retarders (case 2) in principal shows the same trend, only the extremes are more erratic. When a base level stress is added, representing some correlation between the stress state between the previous and next disc, a similar trend is still seen. However, the spatial fluctuation with depth, (the number of retarders), seems to be larger when compared to the other two cases. As in these situations no energy is lost by light leakage, the emergent light intensity will never decay and remain to fluctuate around 0.5.

When internal reflections are taken into account, however, it is not surprising that there is a significant decay in the light intensity with the rate of decay being dependent on the number of retarders (Fig 6).

With the chosen leakage rate dictated by the difference in refractive index some 250 layers of grains are required for full extinction.

DISCUSSION & CONCLUSIONS

When stacking (arbitrary) retarders a modulation with clear periodicity is seen when non arbitrary retarders are used. This periodic behaviour results because no energy is lost in the system (no internal reflections) and still polarisation is maintained (idealised description of elements). Adding a random aberration or complete randomness of the retarder maintains a fluctuation of the signal emerging from the stack of retarders but makes it more erratic. In order to get a decay in light intensity for random retarder matrices the theory needs to be extended to incorporate internal refraction of the light.

If the theory for ideal contact is hypothesised for fluctuation within the grain, and the extended theory for the stack of such grains, some limitations of plane strain analysis of a 3D stack of particles arise from grain scale effects and (changing) structural properties of the assembly. At present the variations in light intensity cannot be uniquely linked to change of retardation, and hence change of stress state, in the grain material. It is shown that within samples with an adequate number of grains within the thickness of the sample for an averaged plane strain stress interpretation a mismatch in internal reflections will significantly attenuate the signal emerging from the sample, thus obscuring any information concerning the stress state in the sample. The present analysis does not throw much light on the source of light stripes regularly reported in observation of stressed granular assemblies viewed with polarised light.

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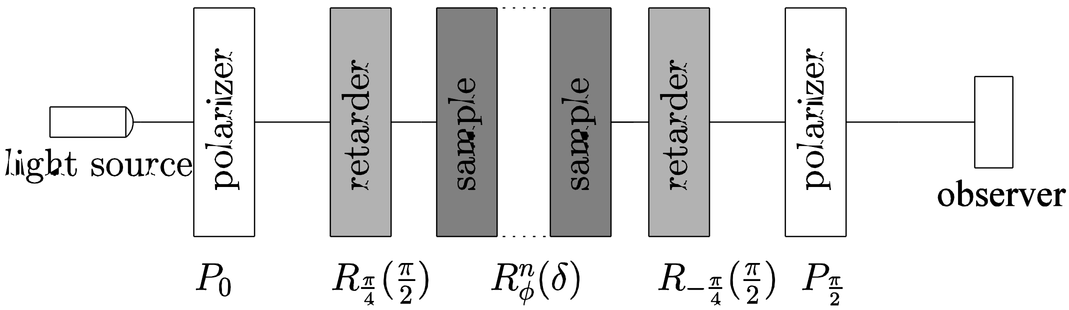

Fig. 1.
Optical system for a medium of N stacked arbitrary retarders in a dark light polariscope.

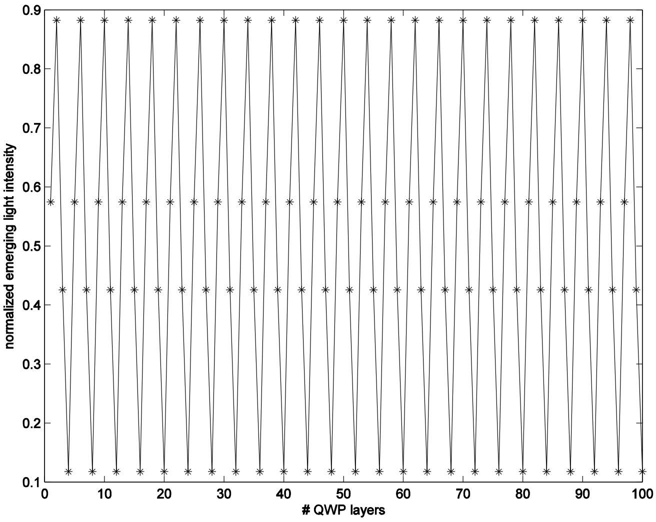

Fig. 2.
Emerging light intensity from case 1: 100 similar stacked retarders φ = π/2 and δ = π/2.

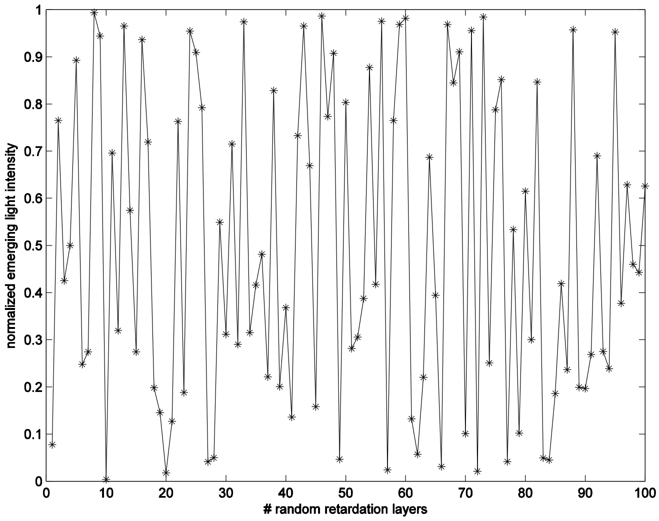

Fig. 3.
Emerging light intensity from case 2: 100 retarders with randomly sampled values between $-\pi/2 < \varphi < \pi/2$ and $-2\pi < \delta < 2\pi$.

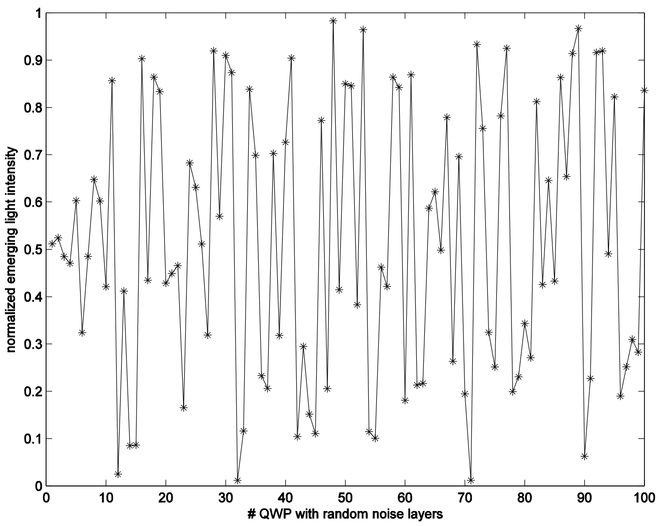

Fig. 4.
Emerging light intensity from case 3: constant base stress with a fluctuating amplitude, represented by 100 retarders with $\varphi = \pi/4 \pm rand() * \pi/4$, $\delta = \pi/2 \pm rand() * \pi/2$.

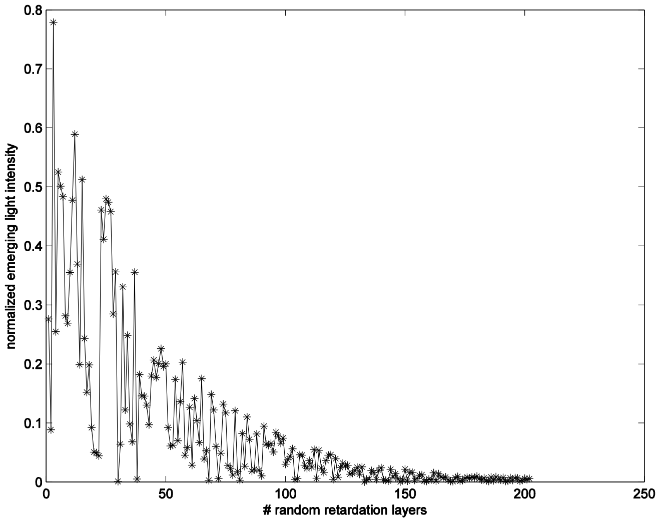

Fig. 5.
Emerging light intensity from case 4: 250 retarders with randomly sampled values between $-\pi/2 < \varphi < \pi/2$ and $-2\pi < \delta < 2\pi$; internal reflections incorporated in analysis.

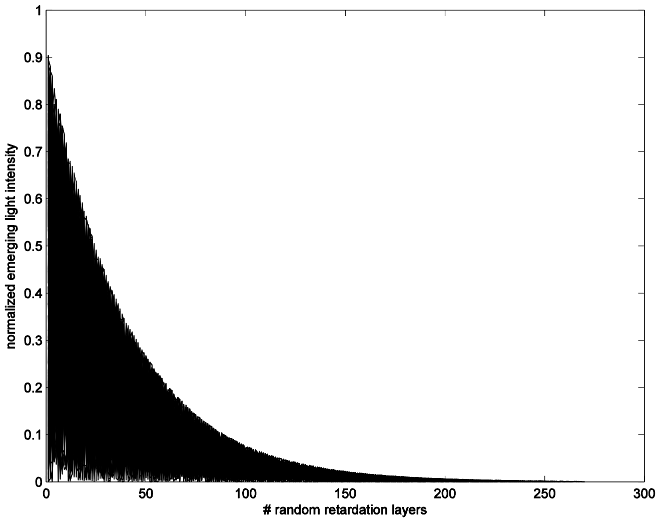

Fig. 6.
100 realisations of case 4: 250 retarders with randomly sampled values between $-\pi/2 < \varphi < \pi/2$ and $-2\pi < \delta < 2\pi$; internal reflections incorporated in analysis.