Thesis for the Degree of Licentiate of Philosophy

# Three applications of graph-theoretic methods

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# ABSTRACT

This thesis is based on three papers in different areas of Discrete Mathematics, namely additive combinatorics, percolation and combinatorial group theory. Although in general these fields are rather distant from each other, graph-theoretic considerations play a major role in each paper as the name of the thesis suggests.

Paper I studies arithmetic progressions of maximal length in product sets. It is proved that for a set of complex numbers $B$ with $|B| = n$ the length of an arithmetic progression in the product set $B.B = \{bb'|b, b' \in B\}$ cannot exceed $O(\frac{n \log^2 n}{\log \log n})$. The main argument of the paper is based on an inequality which gives a bound on the sizes of elements of the arithmetic progression, provided a certain incidence graph contains a cycle of even length. The actual result of the paper then follows from bounds of extremal graph theory and optimization of parameters.

In Paper II a variation of the Grimmett random-orientation percolation model on the square lattice proposed by Hegarty is considered. The crucial difference between such random-orientation models and classical bond percolation is the absence of Harris-type correlation inequalities and sharp threshold bounds. It is shown that both models are to some extent equivalent to the random-orientation model confined to a quadrant where a phase transition is expected to occur at exactly one point $p_c = \frac{1}{2}$, though it is not if even known if the critical point is unique. As a corollary, a non-trivial lower bound for $p_c$ is obtained, assuming it exists. The proof is based on a purely topological argument utilising planarity of the square lattice and rotational symmetry of the model.

In Paper III a family of random finite groups is constructed which we believe provides a counterexample to a conjecture of Iranmanesh and Jafarzadeh saying that, if connected, the commuting graph of arbitrary finite group is of bounded diameter. The construction is based on a uniformly chosen random bilinear map $\phi : V \times V \to H$ where $V$ and $H$ are linear spaces over $\mathbb{F}_2$. With appropriately chosen dimensions of $V$ and $H$ the commuting graph of the corresponding group is similar to the Erdős–Rényi graph $G_{n,p}$ with $p = n^{-1+\epsilon}$ and $\epsilon > 0$ small. It is known that in this regime the diameter of the Erdős–Rényi

graph is concentrated on the single value $\lceil \frac{1}{\epsilon} \rceil$ so presumably the same phenomenon occurs for the random commuting graph in question. However, we only prove that the diameter of the commuting graph grows unboundedly with high probability, though it is still not known if the graph stays connected. Thus, the original conjecture of Iranmanesh-Jafarzadeh is not completely disproved this way. However, it was disproved by Guidici and Parker by a deterministic example based on the construction presented in the paper.

**Keywords:** Product set, arithmetic progression, random orientations, percolation, square lattice, random group, commuting graph, diameter

# Preface

We present three papers:

I. **D. Zhelezov**, Product sets cannot contain long arithmetic progressions, (accepted by *Acta Arith.*).

II. **D. Zhelezov**, On a Property of Random-Oriented Percolation in a Quadrant (to appear in *J. Stat. Phys.*, available online at DOI:10.1007/s10955-013-0856-z).

III. P. Hegarty and **D. Zhelezov**, On the diameters of commuting graphs arising from random skew-symmetric matrices (submitted to *Combin. Probab. Comput.*).

# Acknowledgments

First of all, I would like to thank my advisor Peter Hegarty for being what is called a "defensive pessimist" when it was needed – in particular, when I showed up with yet another vague and messy proof which usually turned out to be simply wrong after all. On the other hand, he was encouragingly optimistic when a problem seemed completely hopeless. He always supports even my most desperate ideas.

I thank Olle Häggström for the excellent course on percolation and Jeff Steif for scrupulous proofreading of an early version of Paper II. Without these people (and Peter of course, who proposed the question) this paper would never have been published.

I thank all my friends, both those whom I met in Göteborg and those who now live in St. Petersburg and other cities[1] for reminding me from time to time that life is not limited to mathematics.

Last, but not least, I am grateful to my parents for nurturing my interest in mathematics since childhood.

Dmitrii Zhelezov
Göteborg, November 2013

[1] Amsterdam, Berlin, Leipzig, Moscow, Trondheim, to name a few

*To Anya*

# Contents

# Part I

# Extended Summary

# 1
# Introduction

The present thesis consists of three different notes in Discrete Mathematics, loosely connected by the methods used. The topics covered by the papers (in the order they appear in the thesis) are additive combinatorics, percolation and group theory. In what follows we provide a brief introduction to the aforementioned topics, narrowed of course to subjects most relevant to the questions under consideration.

## 1.1 Sets with additive and multiplicative structures

### 1.1.1 The sum-product phenomenon

One of the central topics of additive combinatorics is the so-called sum-product phenomenon which arises in many different settings with applications to cryptography, analytic number theory, functional analysis and other seemingly distant branches of mathematics. The main objects in question are the product set $B.B = \{bb'|b, b' \in B\}$ and the sumset $B + B = \{b + b'|b, b' \in B\}$ of a set of real numbers $B$. A priori, either $|B + B|$ or $|B.B|$ can be as small as $O(|B|)$ which can be achieved by an arithmetic or geometric progression respectively. However, by looking at this example it seems intuitively plausible that for both $B + B$ and $B.B$ to be small, $B$ should be both additively and multiplicatively structured in some sense. The sum-product phenomenon asserts that the latter is impossible.

The first quantitative result on the sum-product phenomenon is due to Erdős and Szemerédi [ES83] who proved that

$$\max(|B.B|, |B + B|) \gg |B|^{1+\delta},$$

for some $\delta > 0$. Henceforth we will use the Vinogradov notation ("$\gg$", resp. "$\ll$") for big-$\Omega$ (resp. big-$O$), with optional parameters in the subscript indicating which ones the constant may depend on. In the same paper they posed what is now called the Erdős–Szemerédi conjecture that $\delta$ can be taken arbitrarily close to one. Since then this problem has received significant attention but remains wide open. We mention here some important contributions, but this list is by no means complete.

1. Solymosi [Sol09] proved that $\delta$ can be taken arbitrary close to $\frac{1}{3}$ for sets of real numbers. This is the best exponent up to date.

2. Konyagin and Rudnev [KR13] extended Solymosi's bound $\delta \geq \frac{1}{3} - o(1)$ to sets of complex numbers

3. Chang [Ch03] and Elekes, Ruzsa [ElRu03] ruled out the cases when either $|B.B|$ or $|B+B|$ is comparable to $|B|$, that is, if $B$ is multiplicatively or additively small then the Erdős–Szemerédi conjecture holds.

4. Studies of sum-product estimates in finite fields were pioneered by Bourgain, Katz, Tao [BKT04] and later their result was improved by Bourgain, Glibichuk, Konyagin [BGK06]. It was proven that for a given $\delta' > 0$ and a prime $p$, if $B \subset \mathbb{F}_p$ with $|B| < p^{1-\delta'}$, then

$$\max(|B.B|, |B+B|) \gg |B|^{1+\epsilon},$$

for some $\epsilon(\delta') > 0$.

Additional information on applications of sum-product estimates can also be found in the survey post of Tao [T07].

### 1.1.2 Arithmetic progressions in product sets

In Paper I we investigate a different sort of relationship between the additive and multiplicative properties of a set. In contrast to the sum-product problem we are concerned with additive structure in an arbitrary product set rather than with the tradeoff between the sizes of the sum and product sets. Namely, we want to bound the maximal length of an arithmetic progression in a product set $A.A$ in terms of the size of the original set $A$. The result is as follows.

**Theorem 1.1.1.** Suppose that $B$ is a set of $n$ complex numbers. Then the longest arithmetic progression in $B.B$ has length $O(\frac{n \log^2 n}{\log \log n})$.

A lower bound is provided by the following

**Theorem 1.1.2.** Given a integer $n > 0$ there is a set $B$ of $n$ natural numbers such that $B.B$ contains an arithmetic progression of length $\Omega(n \log n)$.

The proof of Theorem 1.1.1 is divided into two parts, bootstrapping with the case when $B$ is a subset of the positive integers and then extending it to the case of complex numbers. However, the methods we use in the integer and non-integer settings are completely different from each other – the former case is treated by means of a combination of graph-theoretical and arithmetic arguments, while the latter argument is more algebraic. We now consider them in more detail.

Let $B$ be a set of natural numbers of size $n$ such that $B.B$ contains an arithmetic progression $A = \{r + id\}, i = 0, \cdots, N$ of length much larger than $n = |B|$. Then for every prime $p$ less than $N$ and coprime with $d$ there exists an element of $A$ divisible by $p$. On the other hand, if we can prove that all such prime divisors are evenly spread across $A$ and do not concentrate on a few elements, then they must come from different elements of $B$. Then we can argue that $|B|$ is of order $N/\log N$ by the Prime Number Theorem. Such a "non-concentration" result is therefore the heart of the proof and this is where graph-theoretic arguments come into play.

Let $G$ be the incidence graph of $A$, meaning that its vertex set is identified with $B$ and for each $a \in A$ there is a unique edge $(b_i, b_j)$ such that $b_i b_j = a$. One may choose the smallest pair in the lexicographical order if the are many such representations. It turns out that by analyzing the structure of $G$, one can obtain constraints on $A$ merely by the fact that $A$ is an arithmetic progression and that every edge is the product of its vertices. In particular, if $G$ contains an even cycle of length $2k$, then every element of the arithmetic progression is bounded by $N^{k+1}$ where $N$ is the length of $A$. On the other hand, such a bound guarantees that each element of $A$ can have at most $k + 1$ prime divisors of order $N$, which indeed implies that prime divisors do not concentrate provided that $N/\log N \gg kn$. It then remains to optimize $k$ using known bounds from extremal graph theory on the number of edges in $C_{2k}$-free graphs.

It is worth mentioning that Erdős and Pomerance [EP80] asked back in 1980 if it is true that, for a large enough $c$, every interval of length $cn$ contains a number divisible by precisely one prime in $(n/2, n]$? While the question seems to be still wide open, a positive answer would give an essentially sharp upper bound $O(n \log n)$ for Theorem 1.1.1.

To extend Theorem 1.1.1 to the case of complex numbers an algebraic approach is used. It turns out that if the incidence graph $G$ contains a cycle, then algebraic relations between vertices of the cycle force elements of $A$ to be rational. One can then prove that all elements of $B$ can be made rational, while preserving the length of the longest arithmetic progression in the product set.

This argument is in some sense in tune with the paper of Vu, Wood and Wood [VWW11] which says that a set $S$ in a characteristic zero integral domain can be mapped to $\mathbb{Z}/p\mathbb{Z}$, while preserving all algebraic incidences in $S$, for an infinite set of primes $p$. In the opposite direction, Grosu [G13] proved that every sufficiently small subset of $\mathbb{F}_p$ can be mapped to $\mathbb{C}$ while preserving polynomial relations of bounded degree and with bounded coefficients. The fact that the product set contains an arithmetic progression of a certain length can obviously be encoded as a set of polynomial relations of such type, namely that consecutive differences of products are equal. We then prove that it is possible to map such specific relations from $\mathbb{C}$ to $\mathbb{Z}$, reducing the problem to the case of integer numbers[1]. However, in contrast to the results of Vu et al. and Grosu we do not pose any size restrictions.

[1] In our proof we assume that $G$ contains a cycle, but the claim is trivial if $G$ is a forest

## 1.2 Non-monotonic models in percolation

### 1.2.1 The classical model

The first mathematically rigorous model of percolation was given by Broadbent and Hammersley in 1957 [BH57] to describe the process of soaking of a porous material, e.g. a stone. Their model can be described as a random process on the integer lattice $\mathbb{Z}^2$ as follows. Fix a parameter $0 \le p \le 1$ and for each edge $e$ of the lattice (which is usually called a *bond*) run an independent Bernoulli trial $X_e$ with success probability $p$. If $X_e = 1$ the edge is declared open and closed otherwise. The state of the edge is understood as whether or not the liquid can pass through the corresponding part of the stone, which is identified with a large subgraph of the lattice. The main question of percolation theory is whether the center of the stone is wet if it is fully immersed in the liquid, or in mathematical terms, if there is a path from a boundary vertex to the center of the subgraph which consists solely of open edges (it is called an *open path*). Of course, for a finite subgraph the probability that such a path exists is always positive, so to make the problem meaningful one considers the infinite lattice $\mathbb{L} = \mathbb{Z}^2$ with random i.i.d. Bernoulli variables $X_e$ representing the states of the edges as described above. The central question is then for which $p$ the probability $\theta(p)$ that there exists an infinite open path from $(0, 0)$ to infinity is positive.

Of course, there is a lot of room for generalizing this model – one can consider different lattices, arbitrary graphs or even random plane tilings (*Voronoi percolation*, see [BR06]), higher dimensions, models where vertices rather than edges are declared open or closed (*site percolation*) and so on. Many important results rest on two crucial ideas which we now consider in the classical setting (and historically they were introduced that way), though they appear to be general enough to be applicable to most of the models we just mentioned.

The first important idea is to consider the dual graph of the lattice. Let $G$ be a planar graph. The *planar dual* graph $G^d$ is constructed as follows. The vertices are the faces of $G$ and two vertices of $G^d$ are adjacent if and only if the corresponding faces of $G$ share an edge. It is easy to see that the dual of $\mathbb{L}$ is simply $\mathbb{L}^d = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$, so it isomorphic to a translation of its dual. There is a one-to-one correspondence between the edges of $\mathbb{L}$ and $\mathbb{L}^d$ provided by the fact that for each edge $e$ of $\mathbb{L}$ there is exactly one edge $e^d$ of $\mathbb{L}^d$ which crosses it and vice versa. Further, for each percolation realization on $\mathbb{L}$ one canonically obtains a realization on $\mathbb{L}^d$ by declaring $e^d$ open (resp. closed) if $e$ is closed (resp. open).

This construction already provides a way to prove that $\mathbb{P}_c(\mathbb{L}) < 1$ using a so-called Pierels argument, which amounts to the observation that, if there is no open path from the origin to infinity, then there must exist an open circuit in the dual lattice surrounding the origin. Then, by an annuli-counting argument it can be shown that, for $p$ sufficiently close to 1, the probability that there exists an open dual annulus is small.

The second, perhaps more important, property is monotonicity, which we define in the following sense. Let $A$ be an event which is determined by a finite number of i.i.d. Bernoulli variables $X_i$, so one can write

$$I_A = f(X_1, ..., X_n),$$

where $I_A$ is the indicator function of $A$ and $f$ is a Boolean function defined on $\{0,1\}^n$. Then $A$ (as well as the function $f$) is called *increasing* if

$$f(X_1, ..., X_n) \leq f(X'_1, ..., X'_n)$$

whenever $X_i \leq X'_i$ for all $i = 1, \cdots, n$. A typical example of an increasing event is the existence of an open path inside a finite region $R$, say from the origin to the boundary of the box $B_n$ with corners at $(-n,-n)$, $(-n,n)$, $(n,n)$ and $(n,-n)$. Indeed, this event is completely determined by the states of the edges inside $B_n$ and by opening additional edges inside $B_n$ an open path from the origin to the boundary cannot cease to exist. With this example in mind one can prove that the percolation probability $\theta(p)$ is monotone and thus there is a critical value $p_c$, i.e. $\theta(p) = 0$ for $p < p_c$ and $\theta(p) > 0$ for $p > p_c$.

However, perhaps even stronger consequences follow from the correlation inequality for increasing events due to Harris.

**Theorem 1.2.1.** (The Harris Inequality) Let $A$ and $B$ be increasing events. Then

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B).$$

The power of this inequality becomes apparent when $A$ and $B$ are events that certain paths are open. Then the Harris inequality allows one to obtain an effective bound that *both* paths are present simultaneously. Using merely this inequality and multiple ingenious symmetry arguments applied to $\mathbb{L}$ and $\mathbb{L}^d$, Harris [Ha60] was able to prove that $p_c(\mathbb{L}) \geq \frac{1}{2}$ which was one of the most important conjectures in the field at that time.

It required twenty years before Kesten [Ke80] managed to settle the original conjecture in full and prove that $p_c(\mathbb{L}) = \frac{1}{2}$. It is remarkable that the crucial step of Kesten's proof hinges upon another deep property of increasing Boolean functions. Roughly speaking, he showed that the probability that there is an open path from the left to the right side of the box $B_n$, which lies entirely within, viewed as a function of $p$, must exhibit a sudden jump around $p = \frac{1}{2}$ when $n$ is large.

The exact formulation of this property of an increasing Boolean function (usually called a *sharp threshold phenomenon*) is a bit technical, so we refer the interested reader to an excellent exposition in [BR06] where a modern proof of the Kesten theorem is presented. We just mention here that, years later, it was understood (by Kahn, Kalai, Linial, (1988); Bourgain, Katznelson (1992); Talagrand (1994); Friedgut, Kalai (1996) and others) that sharp threshold phenomena can be observed under much milder conditions than were initially applied by Kesten.

### 1.2.2 The Grimmett conjecture and the $H$-model

Now we turn to the model that was studied in Paper II which is somewhat different from the classical setting. In the first edition of his book on percolation [Gr89] Grimmett formulated the following problem which remains open even today. Consider the square lattice $\mathbb{L}$ and let each vertical edge be directed upwards with probability $p \in [0, 1]$ and downwards otherwise. Analogously, each horizontal edge is directed rightwards with probability $p$ and leftwards otherwise. The percolation probability $\theta_G(p)$ is defined as the probability that there is an infinite oriented path from the origin. Grimmett conjectured that $\theta(p) > 0$ whenever $p \neq \frac{1}{2}$. Of course, by an obvious symmetry we have $\theta_G(p) = \theta_G(1 - p)$ so it is sufficient to consider only $p > \frac{1}{2}$.

Although the Grimmett model has translational symmetry, which, for example, gives the uniqueness of the infinite oriented cluster (see [Xia1] for details), it seems natural to reformulate the problem such that for small $p$ one the model a.s. does not percolate while for $p$ close enough to one it is supercritical. Hegarty proposed a model where $p$ controls the orientation bias away from the origin, such that for small $p$ most of the edges are likely to be oriented towards the origin so the model is subcritical. The model is defined as follows. For each edge $e$ of the lattice $\mathbb{L}$ assign a direction away from the origin with probability $p$, and towards the origin otherwise. We say that a directed edge from $x$ to $y$ is oriented *inwards* if $\|x\| > \|y\|$ (which occurs with probability $1 - p$) and *outwards* otherwise (which occurs with probability $p$), with the usual Euclidean norm. We denote by $\theta_H(p)$ the corresponding probability that there exists an infinite directed path from the origin and will call this model the $H$-model.

The Grimmett and Hegarty models are identical when confined to the North-East quadrant. We will call this specialisation the $NE$-model and write $\theta_{NE}(p)$ for the corresponding percolation probability. By deleting all the edges oriented towards the origin, one gets a realization of *directed* or *oriented* percolation with parameter $p$. In this model each edge of the quadrant is oriented away from the origin at the beginning, and then each edge is retained with probability $p$, otherwise it is erased. As before, the model percolates if there is an infinite oriented path emanating from the origin. This model is well understood, it is monotone so it has a single critical point $\vec{p}_c$. It is proved that $\vec{p}_c < 0.6735$, [BBS94], and believed that $\vec{p}_c \approx 0.6447$. Thus, both $\theta_H(p)$ and $\theta_G(p)$ are positive for $p > \vec{p}_c$. Since it is not known that $\theta_H(p)$ is non-decreasing, let us denote by $p^H_{c-} = \inf\{\theta_H(p) > 0\}$ and $p^H_{c+} = \sup\{\theta_H(p) = 0\}$[2] the "lower" and "upper" critical points. We then have the bound $p^H_{c+} \leq \vec{p}_c < 1$.

On the other hand, if $p$ is sufficiently close to zero it is not hard to show that $\theta_H(p) = 0$ by a path counting argument, therefore $p^H_{c+} > 0$ so a phase transition must occur on the interval $[p^H_{c-}, p^H_{c+}]$, perhaps an infinite number of times. It is not the only complication caused by the absence of monotonicity. In fact, all textbook proofs of the Kesten theorem (and numerous adaptations to different models) rely both on the Harris inequality and sharp threshold re-

[2]We will use similar notation for the $NE$-model, namely $p^{NE}_{c-}$ and $p^{NE}_{c+}$

sults which hold only for increasing events. The only tools that seem to fit to the random-orientation model as is are self-duality and various symmetry arguments. In Paper II such purely topological considerations give the following result (Theorem 1).

**Theorem 1.2.1.** If $p$ is such that $\theta_{NE}(1-p) > 0$ then $\theta_H(p) = 0$.

Although Theorem 1.2.1 is a conditional result, it already gives a non-trivial lower bound for $\theta_H(p)$.

**Corollary 1.2.1.** $\theta_H(p) = 0$ for $0 < p < 1 - \vec{p}_c$.

Inserting the upper bound for oriented percolation, we get that $\theta_H(p) = 0$ for $p < 0.3265$, which is better than $\frac{1}{\mu^2} \approx 0.15$ given by the path-counting estimate. It is worth noting that the crucial property of the $H$-model is its 90-degree rotational symmetry which is absent in the Grimmett model.

Perhaps Theorem 1.2.1 is more meaningful when one conjectures that $p_{c-}^{NE} = p_{c+}^{NE} = \frac{1}{2}$ because then it follows that $p_{c-}^{H} = p_{c+}^{H} = \frac{1}{2}$ together with the Grimmett conjecture which follows trivially. This explains why the $NE$-model seems to be more general.

### 1.2.3 Monotone reformulation and higher dimensions

Since major techniques developed for percolation require monotonicity and increasing events, it is thus desirable to reformulate the original model with random orientations in a monotone setting. Grimmett proposed an extension where, instead of assigning an orientation for each edge of $\mathbb{L}$, oriented *arcs* are placed independently by random processes with parameters $\rho$, $\lambda$ and $\beta$ with $\rho + \beta + \lambda < 1$ in the following way. Let $e$ be a horizontal lattice edge for concreteness. Then

1. $e$ is oriented only rightwards with probability $\rho$.
2. $e$ is oriented only leftwards with probability $\beta$.
3. $e$ is oriented leftwards and rightwards with probability $\lambda$.
4. $e$ is absent with probability $1 - (\rho + \beta + \lambda)$.

The same rule applies for vertical edges. The key observation is that the percolation probability depends only on the *marginal* probabilities $\rho + \lambda$ and $\beta + \lambda$ that an edge has a certain orientation. This follows from the cluster-growing argument sketched next. Assign an arbitrary order to the edgeset $E$ of the lattice $\mathbb{L}$. Let $A = \{(0,0)\}$ be the cluster we are going to grow (viewed as a subgraph of $\mathbb{L}$). At step $n$ examine the least unexamined edge $e$ between $A$ and $\mathbb{L} \setminus A$ in the predetermined order, and if there is an arc on $e$ with the proper orientation, add its endpoint to $A$. Since in this exploration algorithm the state of the edge currently under examination is always independent of the previously seen edges, it is only marginal probabilities which determine if the algorithm stops or not. Thus two processes with the same marginal probabilities have identical cluster distributions. For a completely rigorous coupling argument we invite the reader to check the details in [Gr99], pp. 28, 211.

Of course, the same arguments with obvious modifications work for the $H$- and $NE$-models. In particular, by taking $\rho = \beta = 0$ and $\lambda = \frac{1}{2}$ we conclude that all the models are equivalent to the classical bond percolation with $p = \frac{1}{2}$ and thus don't percolate at this point. Write $a = \rho + \lambda$ and $b = \rho + \lambda$. Grimmett noticed that the process with parameters $(a, b)$ is dual to the one with parameters $(1-a, 1-b)$ and thus, if $a+b = 1$, the probability that there is an oriented path between two opposite sides of an arbitrarily large diamond (a square box rotated by 45 degrees) is bounded away from zero, so the model is not subcritical.

**Theorem 1.2.2.** (Grimmett, [Gr00]) Consider the following independent process on $\mathbb{L}$ with parameters $a$ and $b$: rightward and leftward (respectively, upward and downward) arcs are placed independently between each pair of horizontal (respectively, vertical) neighbors. The probability of each upward or rightward arc being placed is $a$ and the probability of each downward or leftward arc being placed is $b$.

If $a + b > 1$ then the independent process with parameters $a, b$ contains an infinite oriented self-avoiding path from 0 with strictly positive probability.

The proof relies on the exponential decay of the cluster size in the subcritical regime, which can be proven for a general class of translation-invariant monotone models, so the proof of Theorem 1.2.2 does not hold for the $H$-model or $NE$-model. This difference becomes more apparent in higher dimensions:

for $d \geq 3$ while $p_{c-}^H$, $p_{c+}^H$ (resp. $p_{c-}^{NE}$, $p_{c+}^{NE}$) remain strictly between zero and one, it is shown in the second part of Paper II that the Grimmett model already percolates in any 3-dimensional slab of height at least three.

**Theorem 1.2.3.** The 3-dimensional Grimmett model confined to the slab $\mathbb{Z}^2 \times \{-1, 0, 1\}$ percolates for any $p \in [0, 1]$.

In contrast to this the question of monotonicity of $\theta_H(p)$ and $\theta_{NE}(p)$ in $d \geq 2$ remains open so it would be interesting to see if the methods of percolation in high dimensions are applicable to these models.

## 1.3 Commuting graphs of random finite groups

### 1.3.1 The conjecture of Iranmanesh and Jafarzadeh

In the third paper we present a family of random groups related to the conjecture of Iranmanesh and Jafarzadeh about commuting graphs of finite groups. Let $G$ be a non-abelian group. We define the *commuting graph* of $G$, denoted by $\Gamma(G)$, as the graph whose vertices are the non-central elements of $G$, and such that $\{x, y\}$ is an edge if and only if $xy = yx$. One can just as well define the graph to have as its vertices the non-identity cosets of $Z(G)$, with $\{Zx, Zy\}$ adjacent if and only if $xy = yx$ and we stick to this definition henceforth. So the conjecture of Iranmanesh and Jafarzadeh is as follows.

**Conjecture 1.3.1. (Iranmanesh and Jafarzadeh, [IJ08])** There is a natural number $b$ such that if $G$ is a finite, non-abelian group with $\Gamma(G)$ connected, then $\mathrm{diam}(\Gamma(G)) \leq b$.

The initial motivation of Paper III was to show that Conjecture 1.3.1 is false by providing a counterexample using probabilistic methods. Some partial results in favor of Conjecture 1.3.1 (see details in Paper III) were already known at the moment the work on this paper was initiated. It might seem natural to guess that for the commuting graph to be of large diameter, the group itself should be far from being abelian. However, it turns out in many cases the opposite holds and the commuting graph is connected and is of small diameter. It is thus reasonable to look at "more abelian" groups. Guidici and Pope [GPo13] were first to consider the case of $p$-groups and provided a few notable results in support of Conjecture 1.3.1.

Let us recall some basic definitions first. If $x, y$ are two elements of a group $G$, then their *commutator* $[x, y]$ is defined to be the group element $x^{-1}y^{-1}xy$. The commutator subgroup of $G$ is the subgroup generated by all the commutators and is denoted $G'$. If $G' \subseteq Z(G)$ one says that $G$ is of *nilpotence class* 2. Quite surprisingly, one of the results of Guidici and Pope was that in this case the center of the group should be of considerable size, otherwise the conjecture holds.

**Theorem 1.3.1.** If $G$ is of nilpotence class $2$ and $|Z(G)|^3 < |G|$, then $\operatorname{diam}(\Gamma(G)) = 2$.

### 1.3.2 Commuting graphs of random elementary abelian $2$-groups

Peter's insightful idea, which he came up with independently of [GPo13], was that if Conjecture 1.3.1 is false, then it should already fail among groups of nilpotence class two. Even more, one can take $G$ such that both $Z(G)$ and $G/Z(G)$ are both elementary abelian 2-groups, that is, additive groups of some vector spaces over $\mathbb{F}_2$. However, instead of constructing an explicit counterexample we are going to introduce randomness in defining commutator relations in order to study how the commuting graph of a typical group of that kind looks like. As illustrated by many applications of the probabilistic method pioneered by Erdős (see [AS] for the full treatment), the behaviour of a random object is often easer to analyze, so by adjusting parameters it is sometimes possible to provide an example with desired properties. Unfortunately, we were unable to disprove the conjecture in full in this way, but were able to produce a group whose commuting graph is of diameter $10$, which became the largest value achieved by that time.

Before we proceed with the model of random groups, let us describe the significant success which took place after the paper was submitted for publication. In [GPa13], Giudici and Parker provide explicit examples of connected commuting graphs of unbounded diameter, thus disproving Conjecture 1.3.1. Their construction is based on and inspired by the random groups presented here, though they were able to devise an explicit construction. They have checked by computer that their model produces examples of commuting graphs of every diameter between $3$ and $15$, though it appears to remain open whether every positive integer diameter is achievable. As a remarkable counterpoint to their result, Morgan and Parker [MP13] have proven that if $G$ has trivial centre then every connected component of $\Gamma(G)$ has diameter at most $10$. Note that this condition specifically excludes nilpotent groups. In contrast to these purely group-theoretical advances, we are not aware of any further progress having been made on the analysis of the random groups described below.

Returning to our random construction, the group is defined as follows. Let $m$, $r$ be positive integers and $V = V_m$ and $H = H_r$ be vector spaces over $\mathbb{F}_2$ of dimensions $m$ and $r$ respectively. Let $\phi : V \times V \to H$ be a bilinear map. Set $G := V \times H$ and define a multiplication on $G$ by

$$(v_1, h_1) \cdot (v_2, h_2) := (v_1 + v_2, h_1 + h_2 + \phi(v_1, v_2)). \tag{1.1}$$

Then it is easy to check that

(i) $(G, \cdot)$ is a group of order $2^{m+r}$, with identity element $(0, 0)$.

(ii) Let $\mathcal{H} := \{(0, h) : h \in H\}$. Then $\mathcal{H}$ is a subgroup of $G$ and $G/\mathcal{H} \cong V$, as an abelian group.

(iii) $G' \subseteq \mathcal{H} \subseteq Z(G)$.

(iv) $G$ is abelian if and only if $\phi$ is symmetric.

(v) The commutator of two elements is given by

$$[(v_1, h_1), (v_2, h_2)] = (0, \phi(v_1, v_2) - \phi(v_2, v_1)) \tag{1.2}$$

The map $\phi(\cdot, \cdot)$ is taken uniformly at random among all possible bilinear maps. It is then clear, due to (1.2), that, for two fixed distinct elements of $G$, their commutator becomes uniformly distributed on $\mathcal{H}$. Moreover, if we fix a basis $(v_1, ..., v_m)$ of $V$ then all the commutator relations are determined by the skew-symmetric matrix $A$ with $A_{i,j} = \phi(v_i, v_j) - \phi(v_j, v_i)$. Now we are going to define the parameters $m$ and $r$ such that the commuting graph $\Gamma(G)$ is similar to the Erdős–Rényi graph $G_{n,p}$ with $p = n^{-1+\epsilon}$, which is known to have diameter concentrated at $\lceil 1/\epsilon \rceil$ with high probability for small $\epsilon > 0$.

Let $k \geq 2$ be an integer, and $\delta \in \left(0, \frac{1}{2k(k-1)}\right)$ a real number. There is a choice of real number $\delta_1 > 0$ such that the following holds: for each positive integer $m$, if we set

$$r := \lfloor (1 - \delta_1)m \rfloor, \quad p := 2^{-r}, \quad n := 2^m - 1, \tag{1.3}$$

then, for all $m$ sufficiently large,

$$1 + \log_n p \in \left(\frac{1}{k} + \delta, \frac{1}{k-1} - \delta\right). \tag{1.4}$$

The probability that an edge of $\Gamma(G)$ is present is then $p$, as this is the probability that a uniformly chosen random element of $\mathcal{H}$ is zero. Thus one can hope that its diameter is concentrated around $k$, as it would be if the states of all edges were independent as in $G_{n,p}$.

Unfortunately, it becomes difficult to translate the known methods of $G_{n,p}$ to our setting due to large amount of dependence between edges, so we were unable to prove this correspondence in full. However, some convincing structural results appear to be amenable to the second moment method.

**Proposition 1.3.1.** Let $G_{m,k}$ be the group defined above with corresponding parameters $m, r$ and $k$. Then

(i) As $m \to \infty$, $\mathbb{P}(G' = Z(G) = \mathcal{H}) \to 1$.

(ii) There is some $\delta_3 > 0$, depending on the choices of $\delta$ and $\delta_1$, such that, as $m \to \infty$, $\Gamma(G_{m,k})$ almost surely has a connected component of size at least $n - n^{1-\delta_3}$. The diameter of $\Gamma(G_{m,k})$ is at least $k$ w.h.p., but might be infinite if it is not connected.

So in fact to provide a counterexample to Conjecture 1.3.1 it is sufficient to prove that $\Gamma(G_{m,k})$ remains connected for large $m$ and fixed $k$. We conjecture that even a more precise statement holds.

**Conjecture 1.3.2.** As $m \to \infty$, $\Gamma(G_{m,k})$ is almost surely connected and of diameter $k$.

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