Institutionen för vattenbyggnad
Chalmers Tekniska Högskola

Department of Hydraulics
Chalmers University of Technology

ISSN 0348 - 1069

ON THE STABILITY OF GRADUALLY VARIED FLOW IN SEWERS

Anders Sjöberg

Contribution to
Water Resources Engineering Studies
dedicated to professor Gunnar Lindh.

Report No 3061 from
Department of Water Resources Engineering,
University of Lund, Sweden 1982

Report
Series B : 33

Göteborg 1982

Institutionen för Vattenbyggnad
Chalmers Tekniska Högskola

Department of Hydraulics
Chalmers University of Technology

ON THE STABILITY OF GRADUALLY VARIED FLOW IN SEWERS

Anders Sjöberg

Contribution to
Water Resources Engineering Studies
dedicated to professor Gunnar Lindh.

Report No 3061 from
Department of Water Resources Engineering,
University of Lund, Sweden 1982

Report
Series B:33 Göteborg 1982

Adress: Institutionen för Vattenbyggnad
Chalmers Tekniska Högskola
412 96 Göteborg

Telefon: 031/81 01 00

ON THE STABILITY OF GRADUALLY VARIED FLOW IN SEWERS

by

Anders Sjöberg, Professor
Department of Hydraulics
School of Civil Engineering
Chalmers University of Technology
S-412 96 Gothenburg, Sweden

ABSTRACT

A derivation of the well-known Vedernikov stability criteria for uniform channel flow is presented. The theoretical results are shown to be applicable also to gradually varied flow, for example, flood waves generated by storm-water runoff in sewers.

INTRODUCTION

It may be shown (see, for example, Henderson 1966) that if the channel slope I_o and the friction slope I_f are neglected, any positive wave, however gentle, must eventually form a surge with an abrupt wave front. However, if I_o and I_f are taken into account, a positive wave will attenuate as it propagates downstream if certain conditions are fulfilled. This can be demonstrated for small positive disturbances in initially uniform flow by purely algebraic arguments (Lighthill and Whitham 1955). The theory presented in this paper is based on their technique.

The theory results in stability criteria based on the so-called Vedernikov number (Ve), which is essentially a function of the Froude number. If Ve > 1, the flow is unstable, which means that growing waves will appear spontaneously. If Ve < 1, we have two possibilities. In case of an initial steepness of the disturbance greater than a certain critical value, again a growing wave is obtained. On the other hand, if the steepness is below the critical value, the disturbance will attenuate as it moves downstream.

Jolly and Yevjevich (1974) have shown that the Vedernikov stability criterion can be used also to determine whether the peak of a gradually varied single-peaked flood wave will amplify or not, in spite of the fact that the theory has been developed for small disturbances in an initially uniform flow.

The purpose of this paper is to give experimental evidence on the applicability of the criterion to a flood wave in a sewer also in case of Ve < 1. It will thus be demonstrated that if the initial steepness of the flood wave is of the same order as given by the theory, the front will steepen and a bore will be formed.

DYNAMIC WAVE VELOCITY

Gradually varied unsteady flow in an open channel may be described by the de Saint Venant equations

$$\frac{\partial Q}{\partial x} + B\frac{\partial Y}{\partial t} = 0 \tag{1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial Y}{\partial x} = gA\,(I_o - I_f) \tag{2}$$

where Q = the discharge, B = the channel width at the surface of the flow, Y = the water depth, x = the length along the channel, t = the time, g = the gravitational acceleration, A = the cross-sectional area of the flow, I_o = the bottom slope of the channel, and I_f = the friction slope. By applying the methods of characteristics to these equations, one can show that the front of a small disturbance (dynamic wave) travels downstream at the propagation rate

$$c_f^+ = V + \sqrt{gD} \tag{3}$$

where V = the mean velocity, Q/A and D = the mean depth A/B. However, the disturbance will also spread in the upstream direction at a speed

$$c_f^- = -V + \sqrt{gD} \tag{4}$$

provided that $c_f^- > 0$.

GROWTH AND DECLINE OF DISTURBANCES IN STATIONARY FLOW

Let us consider a small positive dynamic wave (for example generated by a small increase in the discharge) in a prismatic channel with an initially uniform and stationary flow with a normal depth Y_n and mean velocity V_n. The front of this wave will have a downstream speed according to Eq. (3).

The conditions close to the wave front can be described in the coordinate system (ξ, u)

$$\xi = c_f^+ \cdot t - x$$
$$u = t \tag{5}$$

The position of the front is thus defined by $\xi = 0$.

If these new independent variables are substituted into the de Saint Venant equations and the dependent variable Q is replaced by the mean velocity V, these equations take the form

$$\frac{\partial Y}{\partial u} + (c_f - V)\frac{\partial Y}{\partial \xi} - D\frac{\partial V}{\partial \xi} = 0; \tag{6a}$$

$$\frac{\partial V}{\partial u} + (c_f - V)\frac{\partial V}{\partial \xi} - g\frac{\partial Y}{\partial \xi} = g\,(I_o - I_f); \tag{6b}$$

We now assume that V and Y in the vicinity of the front ($\xi = 0$) can be approximated by means of a Taylor series,

$$\begin{aligned} V &= V_n + \xi v_1(u) + \xi^2 v_2(u) + \ldots \\ Y &= Y_n + \xi y_1(u) + \xi^2 y_2(u) + \ldots \end{aligned} \tag{7}$$

where (correspondingly for v_1 and v_2).

$$y_1 = \left[\frac{\partial Y}{\partial \xi}\right]_{\xi=0^+}; \qquad y_2 = \frac{1}{2}\left[\frac{\partial^2 Y}{\partial \xi^2}\right]_{\xi=0^+}; \tag{8}$$

v and y are functions of u but independent of ξ. The first derivatives of V and Y with respect to x and u are assumed to be discontinuous for $\xi = 0$. We will consider only the conditions upstream from the front, $\xi > 0$. At the front we have

$$y_1 = \left[\frac{\partial Y}{\partial \xi}\right]_{\xi=0^+}; \qquad y_2 = \frac{1}{2}\left[\frac{\partial^2 Y}{\partial \xi^2}\right]_{\xi=0^+} \tag{9}$$

The slope of the front ($-\partial Y/\partial x$) is thus given by $y_1(u) = y_1(t)$ (Figure 1). Substitution of Eq. (7) into Eq. (6a) also gives for $\xi = 0$ and $V = V_n$.

$$v_1 = \frac{c_f - V_n}{D_n}\,y_1 \tag{10}$$

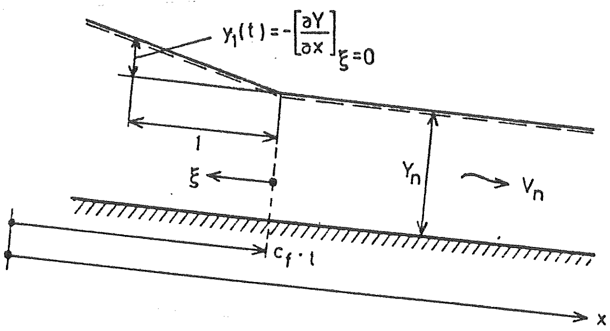

Figure 1 The wave front is described as a discontinuity in the first derivatives of V and Y with respect to x.

The slope of the wave front will increase or decrease as the wave propagates downstream in the channel. If the slope increases, the flow is unstable; otherwise, it is stable. The time history of y_1 is obtained by taking the derivative of Eqs. (6) with respect to ξ and substituting Eqs. (7) and (10). We also use the following general formulation of the friction slope,

$$I_f = I_o (V/V_n)^k \; (R_n/R)^m \tag{11}$$

where R = the hydraulic radius A/P and P = the wetted perimeter. If the Manning formula is used, k = 2 and m = 4/3. After some algebra we then have

$$\frac{dy_1}{dt} = (1 + \frac{1}{2}\frac{dD_n}{dY})\sqrt{\frac{g}{D_n}}\; y_1^2 + \frac{gI_o k}{2V_n}\,(Ve - 1)y_1; \tag{12}$$

where

$$Ve = (1 - R\frac{dP}{dA})\,\frac{m}{k}\,Fr_n \qquad \text{(Vedernikov number)}$$

$$Fr_n = V_n/\sqrt{gD_n} \qquad \text{(Froude number)}$$

If the quantities

$$b = gI_o k(1-Ve)/(2V_n) \tag{13}$$

$$G = b\sqrt{D_n/g}/(1 + \frac{1}{2}\frac{dD_n}{dY}) \tag{14}$$

are introduced, Eq. (11) is simplified to

$$\frac{dy_1}{dt} = \frac{b}{G}\, y_1 (y_1 - G) \tag{15}$$

Eq. (15) describes the growth of the steepness of the wave front. If the Vedernikov number Ve > 1 (e.g. b<0, G<0), it thus follows that the slope of any positive wave (y_1>0), no matter how small, will increase without limits and after some distance the wave breaks and becomes a bore. As the flow is always subject to small disturbances, growing waves (so-called roll waves) will always appear if Ve > 1. The flow is then said to be unstable.

Roll waves are frequently observed in storm-water runoff on steep paved surfaces. They consist of a series of surges at regular intervals, separated by regions in which the depth reaches a minimum just downstream of the surge front. However, the waves must travel a certain distance until they are readily recognized. A measure of this distance is the distance, x_{br} to the point where the waves break, and it may be estimated by integrating Eq. (15),

$$y_1(t) = \frac{y_o}{\frac{y_o}{G} + \left(1-\frac{y_o}{G}\right) e^{bt}} \tag{16}$$

where y_o is the initial steepness of the disturbance, $y_o = y_1(0)$. Putting the denominator in Eq. (16) equal to zero, we obtain

$$x_{br} = t_{br} \cdot \left(V_n + \sqrt{gD_n}\right) =$$

$$= \frac{\left(1+\frac{2}{3Ve}\right)}{(1-Ve)} \cdot \frac{V_n^2}{gI_o} \ln \frac{y_o}{y_o - G} \tag{17}$$

which has also been derived by Montuori (1963) (see Figure 2).

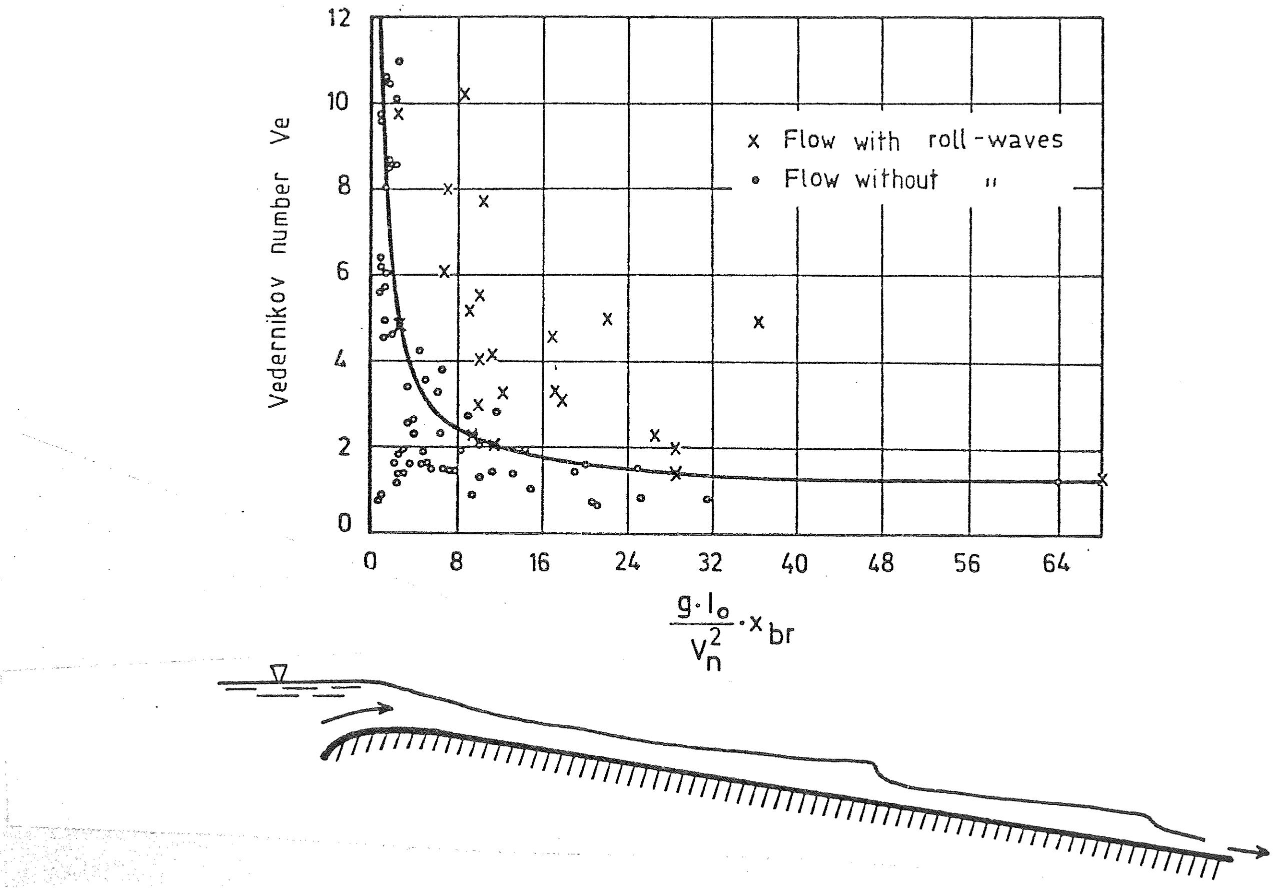

Figure 2 Development of roll waves in stationary steep channel flow. Required distance x_{br} to form breaking roll-waves. The line is Eq. (17) with $y_o/(y_o-G)=10^{-4}$. After Mountuori (1963).

If Ve < 1, we have two possibilities. In case of an initial steepness y_o > G, again a steepening wave front is obtained. On the other hand, if y_o < G, the distrubance will attenuate as it moves downstream.

The Vedernikov stability criterion Ve < 1 is thus a necessary but insufficient condition for stable stationary flow. If a sufficiently steep disturbance is created, a steepening wave front will develop, irrespective of the actual value of Ve.

STABILITY CRITERIA FOR PIPE FLOW

As the Vedernikov number is a function of the Froude number, Fr, the curve corresponding to neutral stability, Ve = 1, may also be expressed as a function of Fr. In figure 3 such a curve is shown for a circular pipe.

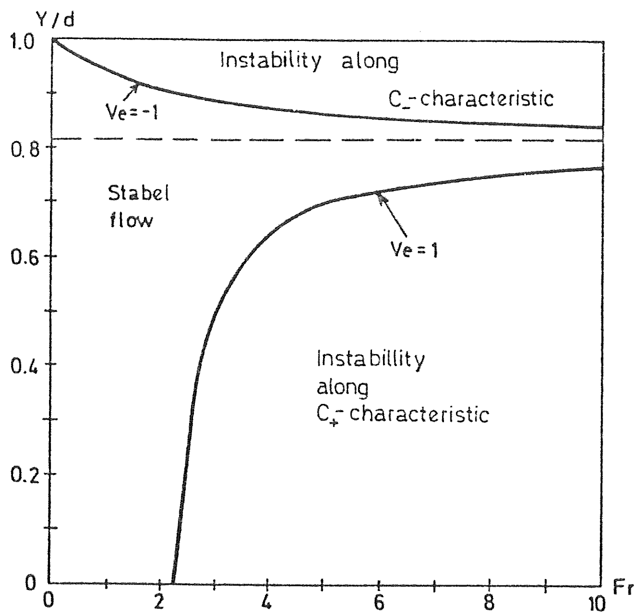

Figure 3 Stability conditions for stationary flow in a circular pipe

For Y/d > 0.82, the stability analysis has to be performed on a dynamic wave moving in the upstream direction at a speed corresponding to Eq. (4), which leads to the stability criteria Ve <-1.

A comparison of figures 3 and 4 shows that the flow is always unstable for relative depths Y/d close to 1.0. This instability is frequently observed in practice.

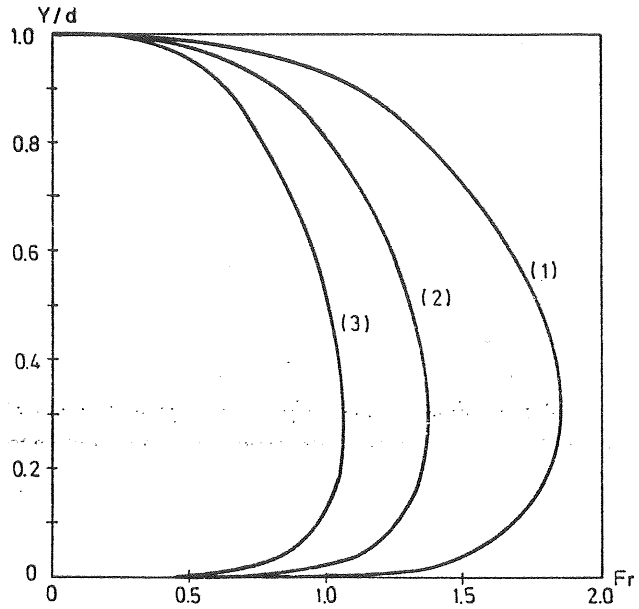

Figure 4 *Froude number as a function of relative depth Y/d for three different pipes*

(1): $d = 225$ mm $I_0 = 10‰$ $Q_{full} = 65$ l/s

(2): $d = 500$ mm $I_0 = 8‰$ $Q_{full} = 360$ l/s

(3): $d = 1000$ mm $I_0 = 4‰$ $Q_{full} = 1560$ l/s

STABILITY OF FLOOD WAVES IN SEWERS

A flood wave created by, for example, storm-water runoff does not fulfill the requirement of stationary flow, nor that of small disturbances. In spite of this the Vedernikov stability criteria seem to be applicable also to such waves.

By means of numerical experiments (computer calculations based on the method of characteristics) Jolly and Yevjevich (1974) have demonstrated how the Vedernikov number can be used to determine whether the peak of a gradually varied single-peaked flood wave will amplify or not. They conclude:

- when Ve < 1 over the wave profile, the wave attenuates
- when Ve < 1 on the top portion of the wave profile and greater than one on the bottom portion, the wave attenuates
- when Ve > 1 over the entire wave profile, the wave amplifies, and if the channel is long enough, the front of the wave will steepen and eventually become a bore.

They did not study the possibility of obtaining a steepening wave front in case of an initial steepness $y_o > G$ for Ve < 1. However, as will be demonstrated below, also this result of the stability analysis seems to be applicable to gradually varied flood waves.

In order to verify numerical solutions of de Saint-Venant equations for partly filled storm sewers, a limited number of experiments were performed with an 85 m long pipe with a diameter of 0.105 m and a slope of 5%. The full capacity of the pipe was 6.3 l/s.

In one of the experiments the input hydrograph to the pipe accidently had a rather steep front, and a bore was formed about 70 m from the pipe inlet (Figure 5).

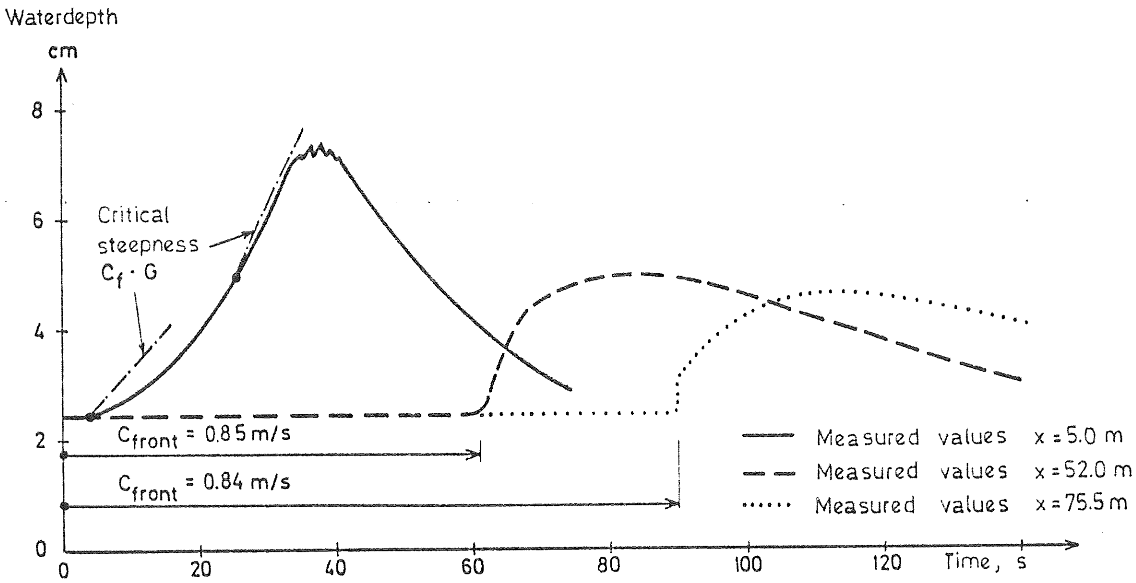

Figure 5 Water depths measured at different distances x from the inlet point. The inflow hydrograph had a triangular form with a base flow of 0.7 l/s and peak flow of 6.1 l/s. The duration was 70 s. From Sjöberg (1976). (The critical slope in the time domain is given by $c_f^{\pm} \cdot G$).

Unfortunately, a detailed analysis of this experiment was not undertaken until the test series was completed. Thus, the significance of the result was realized too late, and therefore no further tests with similar input hydrographs are available.

As shown in figure 5, the measured steepness of the flood wave at 5 m from the inlet is close to the critical value. The G- and $c_f^{\pm}$-values have been calculated for depths of 2.3 and 5.0 cm (see Table 1).

Table 1: Summary of characteristic values calculated for the input hydrograph according to figure 5

y_n (m)	G	c_f^+ (m/s)	$c_f^+ \cdot G$ (m/s)	Ve	V_n (m/s)	x_{br} 1)	x_{br} 2) (m)	x_{br} (3) (m)
2.3	0.0014	0.81	0.0011	0.49	0.48	38	52	100
5.0	0.0024	1.32	0.0032	0.40	0.71	82	109	210

1) with $y_o/G = 1.20$
2) with $y_o/G = 1.10$
3) with $y_o/G = 1.01$

According to Eq. (17), the distance required for a bore to be formed is a function of the initial slope y_o of the front. If we put $y_o = G$, we get $x_{br} = \infty$. Thus, we have to use a value of y_o greater than G to get a limited value of x_{br}. As shown in Table 1, we arrive at reasonable estimates of x_{br} compared to the measured value $\approx$ 70 m for $1.01 < y_o/G < 1.20$. The experimental result thus agrees well with the theory.

The development of the steep front has also been verified by numerical calculations based on an implicit solution of the full de Saint Venant equations (Figure 6).

CONCLUSIONS

The Vedernikov stability criterion, which has been developed for small disturbances in initially uniform and stable flow, has been proved to be valid for gradually varied flow. This includes also the development of a steepening front of a flood wave in case of Ve < 1 and an initial steepness greater than a certain critical value given by the theory.

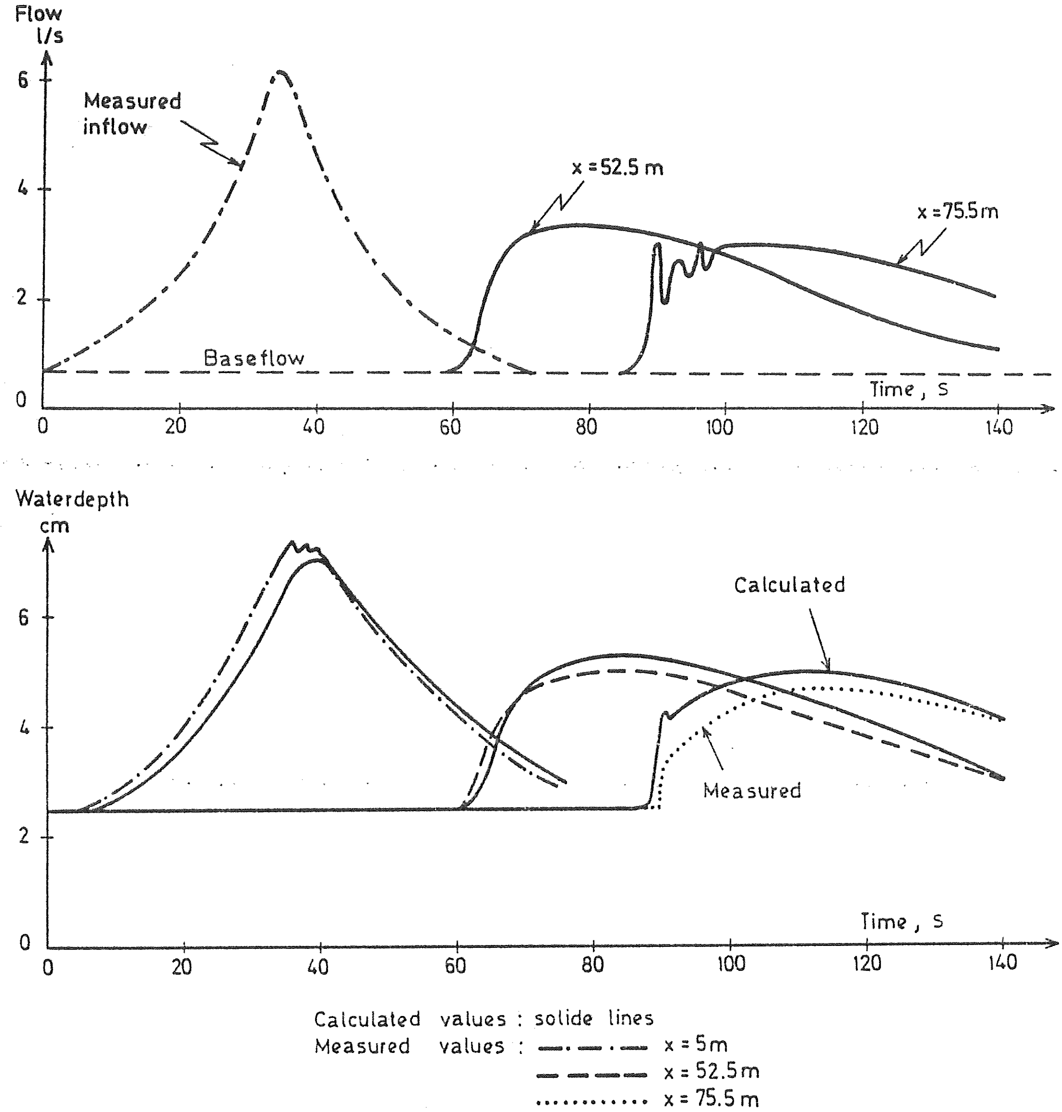

Figure 6 *Comparison between measured and calculated water depths for a straight pipe (d=0.105 m; I_o= 5‰ ; Q_{full}= 6.3 ℓ/s; Δt=1.2 s; Δx=1 m). From Sjöberg (1981).*

REFERENCES

Henderson, F.M., "Open Channel Flow". The MacMillan Company, New York, 1966.

Lighthill, M.I. and Whitham, G.B.' "On Kinematic Waves: I - Flood Movement in Long Rivers". Proc. Roy. Soc. (London) vol. 229, no. 1178. May 10, 1955.

Sjöberg, A., "Calculation of Unsteady Flows in Regulated Rivers and Storm Sewer Systems". Report Series A:6. Dept. of Hydraulics, Chalmers University of Technology, Gothenburg, Sweden, 1976 (in Swedish).

Sjöberg, A., "The Sewer Network Models DAGVL-A and DAGVL-DIFF". (Contribution to the Second International Conference on Urban Storm Drainage, Univ. of Illinois, Illinois, USA, June 14-19, 1981). Report Series B:28. Dept. of Hydraulics, Chalmers University of Technology, Gothenburg, Sweden, 1981.

Moutuori,C., Discussion to "Stability aspects of flow in open channels" by Escoffier, F.F. and Boyd, M.B. Journal of the Hydraulics Division, ASCR, Vol. 89, No. HY4, July 1963.

Department of Hydraulics
Chalmers University of Technology

Report Series A

A:1 Bergdahl, L.: Physics of ice and snow as affects thermal pressure. 1977.

A:2 Bergdahl, L.: Thermal ice pressure in lake ice covers. 1978.

A:3 Häggström, S.: Surface Discharge of Cooling Water. Effects of Distortion in Model Investigations. 1978.

A:4 Sellgren, A.: Slurry Transportation of Ores and Industrial Minerals in a Vertical Pipe by Centrifugal Pumps. 1978.

A:5 Arnell, V.: Description and Validation of the CTH-Urban Runoff Model. 1980.

A:6 Sjöberg, A.: Calculation of Unsteady Flows in Regulated Rivers and Storm Sewer Systems. (in Swedish). 1976.

A:7 Svensson, T.: Water Exchange and Mixing in Fjords. Mathematical Models and Field Studies in the Byfjord. 1980.

A:8 Arnell, V.: Rainfall Data for the Design of Sewer Pipe Systems. 1982.

Report Series B

B:1 Bergdahl, L.: Beräkning av vågkrafter. 1977. Ersatts med 1979:07

B:2 Arnell, V.: Studier av amerikansk dagvattenteknik. 1977.

B:3 Sellgren, A.: Hydraulic Hoisting of Crushed Ores. A feasibility study and pilot-plant investigation on coarse iron ore transportation by centrifugal pumps. 1977.

B:4 Ringesten, B.: Energi ur havsströmmar. 1977.

B:5 Sjöberg, A. and Asp, T.: Brukar-anvisning för ROUTE-S. En matematisk modell för beräkning av icke-stationära flöden i floder och kanaler vid strömmande tillstånd. 1977

B:6 Annual Report 76/77.

B:7 Bergdahl, L. and Wernersson, L.: Calculated and expected Thermal Ice Pressures in Five Swedish Lakes. 1977.

B:8 Göransson, C-G. and Svensson, T.: Drogue Tracking - Measuring Principles and Data Handling.

B:9 Göransson, C-G.: Mathematical Model of Sewage Discharge into confined, stratified Basins - Especially Fjords.

B:10 Arnell, V. and Lyngfelt, S.: Beräkning av dagvattenavrinning från urbana områden. 1978.

B:11 Arnell, V.: Analysis of Rainfall Data for Use in Design of Storm Sewer Systems. 1978.

B:12 Sjöberg, A.: On Models to be used in Sweden for Detailed Design and Analysis of Storm Drainage Systems. 1978.

B:13 Lyngfelt, S.: An Analysis of Parameters in a Kinematic Wave Model of Overland Flow in Urban Areas. 1978.

B:14 Sjöberg, A. and Lundgren, J.: Manual for ILLUDAS (Version S2). Ett datorprogram för dimensionering och analys av dagvattensystem.

B:15 Annual Report 78/79.

B:16 Nilsdal, J-A. and Sjöberg, A.: Dimensionerande regn vid höga vattenstånd i Göta Älv.

B:17 Stöllman, L-E.: Närkes Svartå. Hydrologisk inventering. 1979.

B:18 Svensson, T.: Tracer Measurements of Mixing in the Deep Water of a Small, Stratified Sill Fjord.

B:19 Svensson, T., Degerman, E., Jansson,B. and Westerlund, S.: Energiutvinning ur sjö- och havssediment. En förstudie.

Report Series B

B:20 Annual Report 1979

B:21 Stöllman, L-E.: Närkes Svartå. Inventering av vattentillgång och vattenanvändning. 1980.

B:22 Häggström, S. och Sjöberg, A.: Effects of Distortion in Physical Models of Cooling Water Discharge. 1979.

B:23 Sellgren, A.: A Model for Calculating the Pumping Cost of Industrial Slurries. 1981.

B:24 Lindahl, J.: Rörelseekvationen för en kabel. 1981

B:25 Bergdahl, L. och Olsson, G.: Konstruktioner i havet. Vågkrafter, rörelser. En inventering av datorprogram.

B:26 Annual Report 1980.

B:27 Nilsdal, J-A.: Teknisk-ekonomisk dimensionering av avloppsledningar. En litteraturstudie om datormodeller. 1981.

B:28 Sjöberg, A.: The Sewer Network Models DAGVL-A and DAGVL-DIFF. 1981.

B:29 Moberg, G.: Anläggningar för oljeutvinning till havs. Konstruktionstyper, dimensioneringskriterier och positioneringssystem. 1981

B:30 Sjöberg, A. och Bergdahl, L.: Förankringar och förankringskrafter. 1981

B:31 Häggström,S. och Melin,H.: Användning av simuleringsmodellen MITSIM vid vattenresursplanering för Svartån

B:32 Bydén, S. och Nielsen, B.: NÄRKES SVARTÅ. Vattenöversikt för Laxå kommun. 1982.

B:33 Sjöberg, A.: On the stability of gradually varied flow in sewers. 1982