Schmidt decompositions of parametric processes II: Vector four-wave mixing

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C. J. McKinstrie,[1,*] **J. R. Ott**[2] **and M. Karlsson**[3]

[1]*Bell Laboratories, Alcatel–Lucent, Holmdel, New Jersey 07733, USA*
[2]*Department of Theoretical Physics, University of Geneva, 1211 Geneva 4, Switzerland*
[3]*Department of Microtechnology and Nanoscience, Chalmers University of Technology, SE-41296 Gothenburg, Sweden*

[*]*mckinstrie@alcatel-lucent.com*

Abstract: In vector four-wave mixing, one or two strong pump waves drive two weak signal and idler waves, each of which has two polarization components. In this paper, vector four-wave mixing processes in a randomly-birefringent fiber (modulation interaction, phase conjugation and Bragg scattering) are studied in detail. For each process, the Schmidt decompositions of the coupling matrices facilitate the solution of the signal–idler equations and the Schmidt decomposition of the associated transfer matrix. The results of this paper are valid for arbitrary pump polarizations.

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1. Introduction

Parametric (wave-mixing) processes provide a variety of signal-processing functions required by classical communication systems [1, 2] and quantum information experiments [3, 4]. Such processes are governed by coupled-mode equations (CMEs) of the forms

$$d_z X_1 = iJ_1 X_1 + iKX_2^*, \ \ d_z X_2 = iJ_2 X_2 + iK^t X_1^*, \tag{1}$$

where $d_z = d/dz$ is a space derivative, $X_1 = [x_{1j}]$ and $X_2 = [x_{2j}]$ are $m \times 1$ mode-amplitude vectors, J_1, J_2 and K are $m \times m$ coefficient matrices, and the superscripts $*$ and t denote complex conjugate and transpose, respectively. The self-action (-coupling) matrices J_1 and J_2 are Hermitian, whereas the cross-coupling matrix K is arbitrary. Equations (1) can be rewritten in the compact form

$$d_z X = iLX, \tag{2}$$

where the $2m \times 1$ mode vector and $2m \times 2m$ coefficient matrix are

$$X = \begin{bmatrix} X_1 \\ X_2^* \end{bmatrix}, \ \ L = \begin{bmatrix} J_1 & K \\ -K^\dagger & -J_2^* \end{bmatrix}, \tag{3}$$

respectively. Because Eq. (2) is linear in the mode vector, its solution can be written in the input–output (IO) form

$$X(z) = T(z)X(0), \tag{4}$$

where the transfer (Green) matrix satisfies Eq. (2) and the input condition $T(0) = I$. The mathematical properties of this evolution equation and its solution were studied in detail in [5] and papers cited therein. It was shown that the transfer matrix has the Schmidt decomposition

$$T(z) = \begin{bmatrix} V_1 D_\mu U_1^\dagger & V_1 D_\nu U_2^t \\ V_2^* D_\nu U_1^\dagger & V_2^* D_\mu U_2^t \end{bmatrix}, \tag{5}$$

where U_1, U_2, V_1 and V_2 are unitary matrices, $D_\mu = \mathrm{diag}(\mu_j)$ is a positive diagonal matrix, $D_\nu = \mathrm{diag}(\nu_j)$ is a non-negative diagonal matrix and $j = 1, \ldots, m$. The columns of U_j are input

Schmidt mode-vectors, the columns of V_j are output Schmidt mode-vectors, and the entries of D_μ and D_ν are Schmidt coefficients that satisfy the auxiliary equations $\mu_j^2 - \nu_j^2 = 1$. By using the columns of U_1 and U_2^* as bases for the input vectors $X_1(0)$ and $X_2^*(0)$, respectively, and the columns of V_1 and V_2^* as bases for the output vectors $X_1(z)$ and $X_2^*(z)$, one obtains the CMEs

$$\overline{x}_{1j}(0) = \mu_j(z)\overline{x}_{1j}(0) + \nu_j(z)\overline{x}^*_{2j}(0), \quad \overline{x}^*_{2j}(0) = \nu_j(z)\overline{x}_{1j}(0) + \mu_j(z)\overline{x}^*_{2j}(0), \tag{6}$$

where the Schmidt mode-amplitudes $\overline{x}_{1j}$ and $\overline{x}^*_{2j}$ are the components of X_1 and X_2^* relative to the aforementioned bases. The physical significance of this result is that every parametric processes (no matter how complicated), can be decomposed into a collection of independent two-mode (stretching and squeezing) processes, about which much is known [6, 7].

In a previous paper [5], two specific examples were discussed: Scalar (inverse) modulation interaction (MI) and phase conjugation (PC). Although these examples were sufficient to illustrate the general results, they involved only one or two complex modes: For such processes, the Schmidt decomposition is an elegant, but unnecessary, tool. This paper is the first in a sequence of papers on four-mode parametric processes. Such processes are more complicated than their one- and two-mode counterparts, and their analyses showcase the benefits of Schmidt decompositions. In this paper, vector four-wave mixing (FWM) in a randomly-birefringent fiber is considered [8–10].

2. Modulation interaction

Light-wave propagation in a randomly-birefringent fiber is governed by the vector nonlinear Schrödinger equation (NSE)

$$\partial_z A = i\beta(i\partial_t)A + i\gamma(A^\dagger A)A, \tag{7}$$

where ∂_z and ∂_t are space and time derivatives, respectively, $A = [x,y]^t$ is the two-component amplitude vector, $\gamma = 8\gamma_K/9$ is proportional to the Kerr nonlinearity coefficient γ_K and the superscript † denotes Hermitian conjugate. In the frequency domain, the dispersion function $\beta(\omega) = \sum_{n=1}^{\infty} k_n \omega^n/n!$, where the k_n are dispersion coefficients evaluated at some reference (carrier) frequency, and ω is the difference between the actual frequency and this carrier frequency. One converts from the frequency domain to the time domain by replacing ω with $i\partial_t$. Equation (7) is the simplest equation that models the effects of convection, dispersion, nonlinearity and polarization, and is sometimes called the Manakov equation [11–17]. It is written in a frame that rotates with the birefringence axes of the fiber, and is based on the assumption that the FWM length is much longer than the length over which the birefringence strength and axes change due to random fiber nonuniformities (1–100 m). Although this condition is barely satisfied for fibers shorter than 1 Km, the predictions of the Manakov equation agree with the results of many recent FWM experiments. The Manakov equation does not account for polarization-mode dispersion [18], which can reduce the FWM efficiency [19, 20].

In the degenerate FWM process called modulation interaction (MI), one strong pump wave (p) drives weak signal (s) and idler (r) waves (sidebands), subject to the frequency-matching condition $2\omega_p = \omega_r + \omega_s$, which is illustrated in Fig. 1(a). By substituting the three-frequency ansatz

$$A(z,t) = A_p(z)\exp(-i\omega_p t) + A_r(z)\exp(-i\omega_r t) + A_s(z)\exp(-i\omega_s t) \tag{8}$$

in Eq. (7) and collecting terms of like frequency, one obtains the MI equations

$$d_z A_p = i\beta_p A_p + i\gamma(A_p^\dagger A_p)A_p, \tag{9}$$

$$d_z A_r = i\beta_r A_r + i\gamma(A_p^\dagger A_p + A_p A_p^\dagger)A_r + i\gamma(A_p A_p^t)A_s^*, \tag{10}$$

$$d_z A_s = i\beta_s A_s + i\gamma(A_p^\dagger A_p + A_p A_p^\dagger)A_s + i\gamma(A_p A_p^t)A_r^*, \tag{11}$$

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where the wavenumbers $\beta_j = \beta(\omega_j)$ and $j = p$, r or s. For reference, this procedure is described in [9, 10]. Notice that the weak sidebands do not affect the strong pump, which is undepleted. The right sides of Eqs. (9)–(11) contain the scalar operator $A_p^\dagger A_p = |A_p|^2 I$, which produces self-phase modulation (PM) and cross-PM, and the tensor operator $A_p A_p^\dagger$, which produces cross-polarization rotation (PR). Notice that $(A_p^\dagger A_p)A_p = (A_p A_p^\dagger)A_p$, so one can write the operator in Eq. (9) as a PM or a PR operator, whichever is more convenient. Notice also that in Eqs. (10) and (11) the self-coupling operators (matrices) are Hermitian, and the cross-coupling operators (matrices) satisfy the equation $A_p A_p^t = (A_p A_p^t)^t$, as required by Eqs. (1). Because the pump vector A_p depends on z, so also do the coupling matrices.

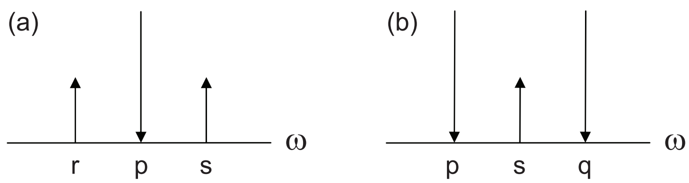

Fig. 1. Frequency diagrams for (*a*) modulation interaction and (*b*) inverse modulation interaction. Long arrows denote pumps (*p* and *q*), whereas short arrows denote sidebands (*r* and *s*). Downward arrows denote modes that lose photons, whereas upward arrows denote modes that gain photons.

It is convenient to define the operator O_p, which satisfies the evolution equation

$$d_z O_p = i(\beta_p + \gamma A_p A_p^\dagger) O_p \tag{12}$$

and the input condition $O_p(0) = I$. Because the pump equation conserves the products $|A_p|^2$ and $A_p A_p^\dagger$, the operator on the right side of Eq. (12) is constant. It is also Hermitian. Hence, the operator

$$O_p(z) = \exp[i(\beta_p + \gamma A_p A_p^\dagger)z], \tag{13}$$

which is unitary. O_p describes linear PM and nonlinear PR, which in Stokes space [18] is a rotation about the Stokes vector of the pump by the angle $2\gamma|A_p|^2 z$ [9, 21].

It is also convenient to define the transformed amplitude vectors

$$A_j(z) = O_p(z) B_j(z). \tag{14}$$

By substituting the first of these definitions in Eq. (9) and using Eq. (13), one finds that $d_z B_p = 0$: The transformed pump vector is constant. By substituting the other definitions in Eqs. (10) and (11), one obtains the transformed MI equations

$$d_z B_r = i(\beta_r - \beta_p + \gamma|B_p|^2)B_r + i\gamma(B_p B_p^t)B_s^*, \tag{15}$$

$$d_z B_s = i(\beta_s - \beta_p + \gamma|B_p|^2)B_s + i\gamma(B_p B_p^t)B_r^*. \tag{16}$$

Notice that the self-coupling matrices are still Hermitian and the (common) cross-coupling matrix is still symmetric, but all three matrices are now constant. By measuring the phases of B_p, B_r and B_s relative to a common reference phase (which could be the input phase of one of the components of B_p), one can remove common phase factors from Eqs. (15) and (16).

Every complex matrix M has the Schmidt decomposition $M = VDU^\dagger$, where U and V are unitary matrices and D is a non-negative diagonal matrix. The columns of U (input Schmidt vectors) are the eigenvectors of $M^\dagger M$, the columns of V (output Schmidt vectors) are the eigenvectors of $MM^\dagger$, and the entries of D (Schmidt coefficients) are the square roots of the (common) eigenvalues of $M^\dagger M$ and $MM^\dagger$. Because the cross-coupling matrix $K = \gamma B_p B_p^t$ is symmetric, it has the simpler Schmidt decomposition $K = VD_\gamma V^t$. Let $E_\parallel$ and $E_\perp$ denote unit vectors that are parallel and perpendicular (orthogonal) to the pump vector B_p. Then, in the context of MI, the columns of V are $E_\parallel$ and $E_\perp$, and the diagonal entries of D_γ are $\gamma|B_p|^2$ and 0 (parallel sidebands couple to the pump, whereas perpendicular sidebands do not couple). The self-coupling matrices are proportional to the identity matrix, which has the unitary decomposition $I = VV^\dagger$. Notice that the polarization properties of MI are determined completely by the Schmidt vectors of the cross-coupling matrix.

By substituting the decompositions

$$B_r = \textstyle\sum_j b_{rj} V_j, \quad B_s = \sum_j b_{sj} V_j, \tag{17}$$

which are based on the same Schmidt vectors, in Eqs. (15) and (16), one obtains the scalar equations

$$d_z b_{rj} = i\delta_r b_{rj} + i\gamma_j b_{sj}^*, \quad d_z b_{sj} = i\delta_s b_{sj} + i\gamma_j b_{rj}^*, \tag{18}$$

where $j = \parallel$ or $\perp$. The wavenumber mismatches $\delta_r = \beta_r - \beta_p + \gamma|B_p|^2$ and $\delta_s = \beta_s - \beta_p + \gamma|B_p|^2$, and the coupling coefficients $\gamma_\parallel = \gamma|B_p|^2$ and $\gamma_\perp = 0$. Equations (18) describe two-mode stretching and squeezing. Their solutions, which are well known, can be written in the IO forms

$$\begin{aligned} b_{rj}(z) &= e(z)\mu_j(z) b_{rj}(0) + e(z)\nu_j(z) b_{sj}^*(0), &(19)\\ b_{sj}(z) &= e^*(z)\mu_j(z) b_{sj}(0) + e^*(z)\nu_j(z) b_{rj}^*(0), &(20)\end{aligned}$$

where the transfer functions and phase factor are

$$\begin{aligned} \mu_j(z) &= \cos(k_j z) + i\delta_a \sin(k_j z)/k_j, &(21)\\ \nu_j(z) &= i\gamma_j \sin(k_j z)/k_j, &(22)\\ e(z) &= \exp(i\delta_d z), &(23)\end{aligned}$$

respectively. In these formulas, the mismatch average $\delta_a = (\delta_r + \delta_s)/2$, the mismatch difference $\delta_d = (\delta_r - \delta_s)/2$ and the MI wavenumbers $k_j = (\delta_a^2 - \gamma_j^2)^{1/2}$. Notice that $k_\parallel$ can be imaginary, so the parallel process is conditionally unstable (as required for amplification). For the perpendicular process $\gamma_\perp = 0$, so $k_\perp = \delta_a$, $\nu_\perp(z) = 0$, $e(z)\mu_\perp(z) = \exp(i\delta_r z)$ and $e^*(z)\mu_\perp(z) = \exp(i\delta_s z)$.

By combining Eqs. (19) and (20) with Eqs. (17) and their inverses

$$b_{rj} = V_j^\dagger B_r, \quad b_{sj} = V_j^\dagger B_s, \tag{24}$$

one can write the solutions of Eqs. (15) and (16) in the vector IO forms

$$\begin{aligned} B_r(z) &= \textstyle\sum_j V_j e(z)\mu_j(z) V_j^\dagger B_r(0) + \sum_j V_j e(z)\nu_j(z) V_j^t B_s^*(0), &(25)\\ B_s^*(z) &= \textstyle\sum_j V_j^* e(z)\nu_j^*(z) V_j^\dagger B_r(0) + \sum_j V_j^* e(z)\mu_j^*(z) V_j^t B_s^*(0). &(26)\end{aligned}$$

Equations (25) and (26) can be rewritten in the compact form

$$\begin{bmatrix} B_r(z) \\ B_s^*(z) \end{bmatrix} = \begin{bmatrix} VeD_\mu V^\dagger & VeD_\nu V^t \\ V^* eD_\nu^* V^\dagger & V^* eD_\mu^* V^t \end{bmatrix} \begin{bmatrix} B_r(0) \\ B_s^*(0) \end{bmatrix}. \tag{27}$$

The transfer matrix in Eq. (27) is similar to the matrix in Eq. (5). It is in Schmidt-like form, rather than Schmidt form, because the diagonal matrices eD_μ, eD_μ^*, eD_ν and eD_ν^* are complex, rather than non-negative. Nonetheless, Eq. (27) is useful: It shows that the polarization properties of MI are determined by the single unitary matrix V, rather than the four matrices allowed by the general theory of parametric processes. Let $\phi_e = \arg(e)$, $\phi_\mu = \arg(\mu)$ and $\phi_\nu = \arg(\nu)$, and define the phase average $\phi_a = (\phi_\mu + \phi_\nu)/2$ and phase difference $\phi_d = (\phi_\nu - \phi_\mu)/2$, which depend implicitly on j. Furthermore, define the column vectors $U_j = V_j \exp(i\phi_d)$, $V_{rj} = V_j \exp[i(\phi_a + \phi_e)]$ and $V_{sj} = V_j \exp[i(\phi_a - \phi_e)]$. Then, by using this notation, one can rewrite Eq. (27) in the (canonical) Schmidt form

$$\begin{bmatrix} B_r(z) \\ B_s^*(z) \end{bmatrix} = \begin{bmatrix} V_r|D_\mu|U^\dagger & V_r|D_\nu|U^t \\ V_s^*|D_\nu|U^\dagger & V_s^*|D_\mu|U^t \end{bmatrix} \begin{bmatrix} B_r(0) \\ B_s^*(0) \end{bmatrix}, \tag{28}$$

in which the diagonal matrices $|D_\mu|$ and $|D_\nu|$ are non-negative. Notice that in Eq. (28) the output Schmidt vectors of the signal and idler are different. However, if one were to measure the output signal and idler phases relative to ϕ_e and $-\phi_e$, respectively, this difference would disappear and decomposition (28) would involve only two unitary matrices (U and V).

3. Phase conjugation

In the nondegenerate FWM process called phase conjugation (PC), two strong pumps (p and q) drive weak sidebands (r and s), subject to the frequency-matching condition $\omega_p + \omega_q = \omega_r + \omega_s$, which is illustrated in Fig. 2. By substituting the four-frequency ansatz

$$\begin{aligned} A(z,t) &= A_p(z)\exp(-i\omega_p t) + A_q(z)\exp(-i\omega_q t) \\ &\quad + A_r(z)\exp(-i\omega_r t) + A_s(z)\exp(-i\omega_s t) \end{aligned} \tag{29}$$

in Eq. (7) and collecting terms of like frequency, one obtains the PC equations

$$d_z A_p = i\beta_p A_p + i\gamma(A_p^\dagger A_p + A_q^\dagger A_q + A_q A_q^\dagger)A_p, \tag{30}$$

$$d_z A_q = i\beta_q A_q + i\gamma(A_q^\dagger A_q + A_p^\dagger A_p + A_p A_p^\dagger)A_q, \tag{31}$$

$$\begin{aligned} d_z A_r &= i\beta_r A_r + i\gamma(A_p^\dagger A_p + A_p A_p^\dagger + A_q^\dagger A_q + A_q A_q^\dagger)A_r \\ &\quad + i\gamma(A_p A_q^t + A_q A_p^t)A_s^*, \end{aligned} \tag{32}$$

$$\begin{aligned} d_z A_s &= i\beta_s A_s + i\gamma(A_p^\dagger A_p + A_p A_p^\dagger + A_q^\dagger A_q + A_q A_q^\dagger)A_s \\ &\quad + i\gamma(A_p A_q^t + A_q A_p^t)A_r^*. \end{aligned} \tag{33}$$

The right sides of Eqs. (30)–(33) contain the scalar operators $A_p^\dagger A_p$ and $A_q^\dagger A_q$, which produce PM, and the tensor operators $A_p A_p^\dagger$ and $A_q A_q^\dagger$, which produce PR. Notice that in Eq. (30) one can replace $(A_p^\dagger A_p)A_p$ by $(A_p A_p^\dagger)A_p$ and in Eq. (31) one can replace $(A_q^\dagger A_q)A_q$ by $(A_q A_q^\dagger)A_q$. Notice also that in Eqs. (32) and (33) the self-coupling matrices are Hermitian, and the cross-coupling matrices satisfy the equation $A_p A_q^t + A_q A_p^t = (A_p A_q^t + A_q A_p^t)^t$, as required by Eqs. (1). Because the pump vectors A_p and A_q depend on z, so also do the coupling matrices.

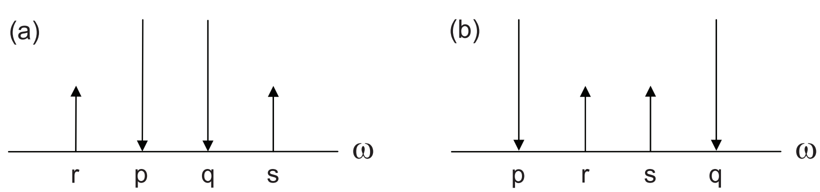

Fig. 2. Frequency diagrams for (a) outer-band and (b) inner-band phase conjugation. Long arrows denote pumps (p and q), whereas short arrows denote sidebands (r and s). Downward arrows denote modes that lose photons, whereas upward arrows denote modes that gain photons.

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It is convenient to define the operators O_p and O_q, which satisfy the evolution equations

$$d_z O_p = i[\beta_p + \gamma|A_q|^2 + \gamma(A_p A_p^\dagger + A_q A_q^\dagger)]O_p, \quad (34)$$
$$d_z O_q = i[\beta_q + \gamma|A_p|^2 + \gamma(A_p A_p^\dagger + A_q A_q^\dagger)]O_q, \quad (35)$$

together with the input conditions $O_p(0) = I$ and $O_q(0) = I$. Because the pump equations conserve the products $|A_p|^2$, $|A_q|^2$ and $A_p A_p^\dagger + A_q A_q^\dagger$, the operators

$$O_p(z) = \exp\{i[\beta_p + \gamma|A_q|^2 + \gamma(A_p A_p^\dagger + A_q A_q^\dagger)]z\}, \quad (36)$$
$$O_q(z) = \exp\{i[\beta_q + \gamma|A_p|^2 + \gamma(A_p A_p^\dagger + A_q A_q^\dagger)]z\}. \quad (37)$$

These unitary operators describe linear and nonlinear PM, and nonlinear PR, which in Stokes space is a rotation about the total Stokes vector of the pumps [9,21].

It is also convenient to define the transformed amplitude vectors

$$A_p(z) = O_p(z)B_p(z), \;\; A_q(z) = O_q(z)B_q(z), \quad (38)$$
$$A_r(z) = O_p(z)B_r(z), \;\; A_s(z) = O_q(z)B_s(z). \quad (39)$$

By substituting definitions (38) into Eqs. (30) and (31), and using Eqs. (36) and (37), one finds that $d_z B_p = 0$ and $d_z B_q = 0$: The transformed pump vectors are constant. By substituting definitions (38) and (39) in Eqs. (32) and (33), and using the facts that $O_p^\dagger O_q$, $O_p^t O_q^*$, $O_q^\dagger O_p$, and $O_q^t O_p^*$ are scalar operators, one obtains the transformed PC equations

$$d_z B_r = i(\beta_r - \beta_p + \gamma|B_p|^2)B_r + i\gamma(B_p B_q^t + B_q B_p^t)B_s^*, \quad (40)$$
$$d_z B_s = i(\beta_s - \beta_q + \gamma|B_q|^2)B_s + i\gamma(B_p B_q^t + B_q B_p^t)B_r^*. \quad (41)$$

Notice that the self-coupling matrices are still Hermitian and the (common) cross-coupling matrix is still symmetric, but all three matrices are now constant.

The transformed PC equations are similar to their MI counterparts. The self-coupling matrices are diagonal, with (repeated) entries $\delta_r = \beta_r - \beta_p + \gamma|B_p|^2$ and $\delta_s = \beta_s - \beta_q + \gamma|B_q|^2$, and the (common) cross-coupling matrix $\gamma(B_p B_q^t + B_q B_p^t)$ is symmetric. Hence, the polarization properties of PC are determined completely by the Schmidt vectors of the cross-coupling matrix. Specific formulas for these vectors are stated in terms of the pump components and Stokes vectors in [9] and [10], respectively. The latter formulas are more compact. Let $\vec{p}$ and $\vec{q}$ denote the (unit) Stokes vectors of pumps p and q, respectively. Then the Stokes representations of the idler and signal (unit) Schmidt vectors are $\pm\vec{r}$ and $\pm\vec{s}$, respectively, where

$$\vec{r} = \vec{s} = (\vec{p} + \vec{q})/(2 + 2\vec{p}\cdot\vec{q})^{1/2}. \quad (42)$$

For reference, if a Jones vector has the Stokes representation (v_1, v_2, v_3), the conjugate vector has the representation $(v_1, -v_2, -v_3)$. Pump vectors that are perpendicular in Jones space are anti-parallel in Stokes space [18]. This configuration, for which Eq. (42) is indeterminate, is discussed in [10]. The associated Schmidt coefficients (entries of D_γ) are

$$\gamma_\pm^2 = [3 + \vec{p}\cdot\vec{q} \pm 2(2 + 2\vec{p}\cdot\vec{q})^{1/2}]|B_p B_q|^2/2, \quad (43)$$

where $|B_p B_q| = (B_p^\dagger B_p B_q^\dagger B_q)^{1/2}$. The dependences of these coefficients (coupling strengths) on the polarization alignment of the pumps ($\vec{p}\cdot\vec{q}$) are illustrated in Fig. 3. Parallel pumps produce strong sideband-polarization-dependent coupling ($\gamma_+ = 2|B_p B_q|$ and $\gamma_- = 0$), whereas perpendicular pumps provide moderate polarization-independent coupling ($\gamma_+ = \gamma_- = |B_p B_q|$). Notice that $\gamma_+ + \gamma_- = 2|B_p B_q|$.

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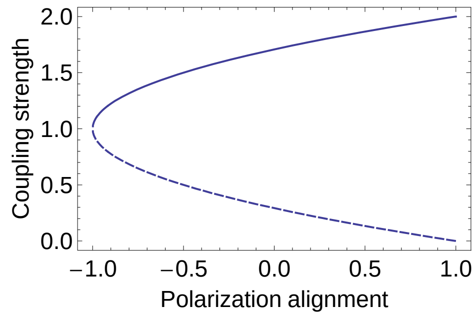

Fig. 3. Normalized Schmidt coefficients ($\gamma_\pm/|B_pB_q|$) plotted as functions of the pump-polarization alignment ($\vec{p}\cdot\vec{q}$). The solid and dashed curves represent γ_+ and γ_-, respectively.

Equations (17)–(28) also apply to PC (with the appropriate definitions of δ_r, δ_s and V), so no further analysis is required. Nonetheless, it is instructive to define the alternative amplitudes

$$B_r(z) = C_r(z)\exp(i\delta_d z), \;\; B_s(z) = C_s(z)\exp(-i\delta_d z), \tag{44}$$

where δ_d was defined after Eq. (23). By substituting these definitions in Eqs. (40) and (41), one obtains the alternative (symmetrized) PC equations

$$d_zC_r = i\delta_a C_r + i\gamma(B_pB_q^t + B_qB_p^t)C_s^*, \tag{45}$$

$$d_zC_s = i\delta_a C_s + i\gamma(B_pB_q^t + B_qB_p^t)C_r^*, \tag{46}$$

where δ_a also was defined after Eq. (23). In Eqs. (45) and (46) the mismatches are equal, so the phase factor $e(z)$ does not appear in the associated Schmidt-like decomposition (27) and only two unitary matrices (U and V) appear in the associated Schmidt decomposition (28), as stated previously.

In degenerate PC (inverse MI), $\omega_r = \omega_s$ and the pumps drive only a single sideband (s), subject to the frequency-matching condition $\omega_p + \omega_q = 2\omega_s$, which is illustrated in Fig. 1(b). For this degenerate process, the pump equations (30) and (31) are unchanged, and the signal equation is

$$\begin{aligned} d_zA_s = \; & i\beta_s A_s + i\gamma(A_p^\dagger A_p + A_pA_p^\dagger + A_q^\dagger A_q + A_qA_q^\dagger)A_s \\ & + i\gamma(A_pA_q^t + A_qA_p^t)A_s^*. \end{aligned} \tag{47}$$

It is only because the cross-coupling matrix is symmetric that Eqs. (32) and (33) have this common limit. It is convenient to define the unitary operator

$$\begin{aligned} O_s(z) = \; & \exp\{i[(\beta_p+\beta_q)/2 + \gamma(|A_p|^2 + |A_q|^2)/2 \\ & + \gamma(A_pA_p^\dagger + A_qA_q^\dagger)]z\}, \end{aligned} \tag{48}$$

which is a symmetric combination of the operators O_p and O_q. By using O_s in the second of Eqs. (39), one obtains the transformed signal equation

$$d_zB_s = i\delta_s B_s + i\gamma(B_pB_q^t + B_qB_p^t)B_s^*, \tag{49}$$

where the mismatch $\delta_s = \beta_s - (\beta_p+\beta_q)/2 + \gamma(|B_p|^2 + |B_q|^2)/2$ depends symmetrically on the pump wavenumbers and powers. Thus, the cross-coupling matrix for inverse MI is the same

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as that for PC, so it remains true that $K = VD_\gamma V^t$, where the Schmidt vectors (columns of V) and coefficients (entries of D_γ) were defined by Eqs. (42) and (43), respectively. The equations for the signal vector and its conjugate are similar to Eqs. (45) and (46), so the IO relations for these quantities can be written in the form of Eq. (27), but without the phase factor e (because $\delta_d = 0$).

For any pump alignment, there are two signal polarizations for which the signal experiences (one-mode) phase-sensitive amplification. The most useful configuration involves perpendicular pumps, for which the amplification strength is signal-polarization independent. If the pump vectors are used as basis vectors, the signal-polarization vectors are $[1, e^{i\phi}]^t/2^{1/2}$ and $[1, -e^{i\phi}]^t/2^{1/2}$, where ϕ is an arbitrary phase. For example, if the pumps are polarized linearly along reference axes, $\phi = 0$ corresponds to signals polarized linearly at $\pm 45^\circ$ to these axes, whereas $\phi = \pi/2$ corresponds to left- and right-circularly-polarized signals. If the pumps are circularly polarized, $\phi = 0$ corresponds to signals polarized linearly along the axes, whereas $\phi = \pi/2$ corresponds to signals polarized linearly at $\pm 45^\circ$ to the axes. The preceding results generalize those of [22, 23].

4. Bragg scattering

In the nondegenerate FWM process called Bragg scattering (BS), two strong pumps (p and q) drive weak sidebands (r and s), subject to the frequency-matching condition $\omega_p + \omega_s = \omega_q + \omega_r$, which is illustrated in Fig. 4. By substituting the four-frequency ansatz (29) in Eq. (7) and collecting terms of like frequency, one obtains the BS equations

$$\begin{aligned}
d_z A_p &= i\beta_p A_p + i\gamma(A_p^\dagger A_p + A_q^\dagger A_q + A_q A_q^\dagger)A_p, && (50)\\
d_z A_q &= i\beta_q A_q + i\gamma(A_q^\dagger A_q + A_p^\dagger A_p + A_p A_p^\dagger)A_q, && (51)\\
d_z A_r &= i\beta_r A_r + i\gamma(A_p^\dagger A_p + A_p A_p^\dagger + A_q^\dagger A_q + A_q A_q^\dagger)A_r \\
&\quad + i\gamma(A_p A_q^\dagger + A_q^\dagger A_p)A_s, && (52)\\
d_z A_s &= i\beta_s A_s + i\gamma(A_p^\dagger A_p + A_p A_p^\dagger + A_q^\dagger A_q + A_q A_q^\dagger)A_s \\
&\quad + i\gamma(A_p^\dagger A_q + A_q A_p^\dagger)A_r. && (53)
\end{aligned}$$

Equations (50) and (51) are identical to Eqs. (30) and (31), respectively. In Eqs. (52) and (53), the self-coupling matrices are Hermitian, and the coupling matrices satisfy the equation $A_p^\dagger A_q + A_q A_p^\dagger = (A_p A_q^\dagger + A_q^\dagger A_p)^\dagger$. Notice that A_r is coupled to A_s, rather than A_s^*. This type of coupling differentiates BS from MI and PC.

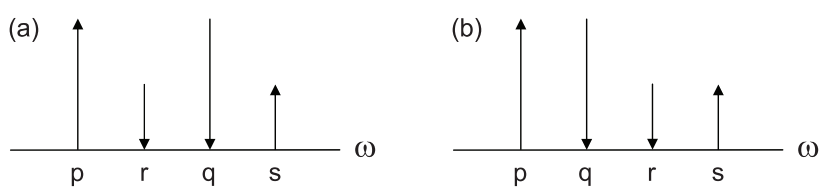

Fig. 4. Frequency diagrams for (a) distant and (b) nearby Bragg scattering. Long arrows denote pumps (p and q), whereas short arrows denote sidebands (r and s). Downward arrows denote modes that lose photons, whereas upward arrows denote modes that gain photons. The directions of the arrows are reversible.

The sideband equations can be written in the compact form

$$d_z X = iHX, \tag{54}$$

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where the mode vector and coefficient matrix are

$$X = \begin{bmatrix} A_r \\ A_s \end{bmatrix}, \ H = \begin{bmatrix} J_r & K \\ K^\dagger & J_s \end{bmatrix}, \tag{55}$$

respectively. J_r and J_s are the aforementioned self-coupling matrices and K is the (common) coupling matrix. Notice that H is Hermitian. Equation (54) is both a special case of Eq. (2), in which $J_1 = H$ and the other block matrices are absent, and an equation worthy of study in its own right.

The solution of Eq. (54) can be written in the form of Eq. (4) and the associated transfer matrix has the Schmidt decomposition

$$T(z) = \begin{bmatrix} V_1 D_\tau U_1^\dagger & V_1 D_\rho U_2^\dagger \\ -V_2 D_\rho U_1^\dagger & V_2 D_\tau U_2^\dagger \end{bmatrix}, \tag{56}$$

where U_j and V_j are unitary matrices, $D_\tau = \mathrm{diag}(\tau_j)$ and $D_\rho = \mathrm{diag}(\rho_j)$ are non-negative diagonal matrices whose entries satisfy the auxiliary equations $\tau_j^2 + \rho_j^2 = 1$, and $j = 1$ or 2 [24]. The physical significance of this result is that every BS process, no matter how complicated, can be decomposed into a collection of independent beam-splitter-like processes, about which much is known [25, 26].

Because the pump equations for BS are identical to those for PC, the pump evolution (linear and nonlinear PM, and nonlinear PR) is described by Eqs. (34)–(38). By substituting definitions (38) and (39) in Eqs. (52) and (53), and using the facts that $O_p^\dagger O_q$ and $O_q^\dagger O_p$ are scalar operators, one obtains the transformed BS equations

$$d_z B_r = i(\beta_r - \beta_p + \gamma |B_p|^2) B_r + i\gamma(B_p B_q^\dagger + B_q^\dagger B_p) B_s, \tag{57}$$

$$d_z B_s = i(\beta_s - \beta_q + \gamma |B_q|^2) B_s + i\gamma(B_p^\dagger B_q + B_q B_p^\dagger) B_r. \tag{58}$$

Notice that the self-coupling matrices are still Hermitian and cross-coupling is still described by a single matrix (and its Hermitian conjugate), but all three matrices are now constant. The cross-coupling matrix has the Schmidt decomposition $K = U D_\gamma V^\dagger$, whereas the self-coupling matrices are proportional to the identity matrix, which has the unitary decompositions $I = UU^\dagger = VV^\dagger$. Hence, the polarization properties of BS are determined completely by the Schmidt vectors of the coupling matrix. Specific formulas for these vectors are stated in terms of the pump components and Stokes vectors in [9] and [10], respectively. The Stokes representation of the idler and signal Schmidt vectors are $\pm\vec{r}$ and $\pm\vec{s}$, respectively, where

$$\vec{r} = (2\vec{p} + \vec{q})/(5 + 4\vec{p}\cdot\vec{q})^{1/2}, \ \vec{s} = (\vec{p} + 2\vec{q})/(5 + 4\vec{p}\cdot\vec{q})^{1/2}, \tag{59}$$

and the associated Schmidt coefficients are

$$\gamma_\pm^2 = [3 + 2\vec{p}\cdot\vec{q} \pm (5 + 4\vec{p}\cdot\vec{q})^{1/2}] |B_p B_q|^2/2. \tag{60}$$

The dependences of these coefficients on the pump-polarization alignment is illustrated in Fig. 5. For any pump alignment, there are strongly- and weakly-coupled sideband polarizations: The coupling is always sideband-polarization dependent. Notice that $\gamma_+ - \gamma_- = |B_p B_q|$.

By substituting the decompositions

$$B_r = \textstyle\sum_j b_{rj} U_j, \ B_s = \sum_j b_{sj} V_j, \tag{61}$$

which are based on different Schmidt vectors, in Eqs. (57) and (58), one obtains the scalar equations

$$d_z b_{rj} = i\delta_r b_{rj} + i\gamma_j b_{sj}, \ d_z b_{sj} = i\delta_s b_{sj} + i\gamma_j b_{rj}, \tag{62}$$

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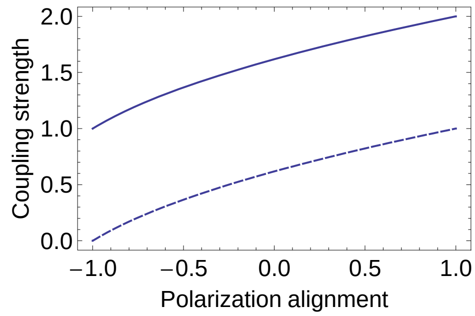

Fig. 5. Normalized Schmidt coefficients ($\gamma_\pm/|B_pB_q|$) plotted as functions of the pump-polarization alignment ($\vec{p}\cdot\vec{q}$). The solid and dashed curves represent γ_+ and γ_-, respectively.

where the mismatches $\delta_r = \beta_r - \beta_p + \gamma|B_p|^2$ and $\delta_s = \beta_s - \beta_q + \gamma|B_q|^2$, and $j = +$ or $-$. Equations (62) describe two-mode beam splitting (frequency conversion). Their solutions, which are well known, can be written in the IO forms

$$b_{rj}(z) = e(z)\tau_j(z)b_{rj}(0) + e(z)\rho_j(z)b_{sj}(0), \tag{63}$$

$$b_{sj}(z) = -e(z)\rho_j^*(z)b_{rj}(0) + e(z)\tau_j^*(z)b_{sj}(0), \tag{64}$$

where the transfer functions and phase factor are

$$\tau_j(z) = \cos(k_jz) + i\delta_d\sin(k_jz)/k_j, \tag{65}$$

$$\rho_j(z) = i\gamma_j\sin(k_jz)/k_j, \tag{66}$$

$$e(z) = \exp(i\delta_a z), \tag{67}$$

respectively. In these formulas, the mismatches $\delta_a = (\delta_r + \delta_s)/2$ and $\delta_d = (\delta_r - \delta_s)/2$, and the BS wavenumbers $k_j = (\delta_d^2 + \gamma_j^2)^{1/2}$. Notice that τ_j and ρ_j depend on δ_d, rather than δ_a, and the k_j are real, so BS is always stable.

By combining Eqs. (63) and (64) with Eqs. (61) and their inverses

$$b_{rj} = U_j^\dagger B_r, \;\; b_{sj} = V_j^\dagger B_s, \tag{68}$$

one can write the solutions of Eqs. (57) and (58) in the vector IO forms

$$B_r(z) = \textstyle\sum_j U_je(z)\tau_j(z)U_j^\dagger B_r(0) + \sum_j U_je(z)\rho_j(z)V_j^\dagger B_s(0), \tag{69}$$

$$B_s(z) = -\textstyle\sum_j V_je(z)\rho_j^*(z)U_j^\dagger B_r(0) + \sum_j V_je(z)\tau_j^*(z)V_j^\dagger B_s(0). \tag{70}$$

Equations (69) and (70) can be rewritten in the compact form

$$\begin{bmatrix} B_r(z) \\ B_s(z) \end{bmatrix} = \begin{bmatrix} UeD_\tau U^\dagger & UeD_\rho V^\dagger \\ -VeD_\rho^* U^\dagger & VeD_\tau^* V^\dagger \end{bmatrix} \begin{bmatrix} B_r(0) \\ B_s(0) \end{bmatrix}. \tag{71}$$

The transfer matrix in Eq. (71) is in Schmidt-like form, because the diagonal matrices $eD_\tau^{(*)}$ and $eD_\rho^{(*)}$ are complex. Nonetheless, Eq. (71) shows that the polarization properties of BS are determined by only two unitary matrices (U and V), rather than the four matrices allowed by Eq. (56). Let $\phi_\tau = \arg(\tau)$ and $\phi_\rho = \arg(\rho)$, and define the phase average $\phi_a = (\phi_\tau + \phi_\rho)/2$ and phase difference $\phi_d = (\phi_\rho - \phi_\tau)/2$, which depend implicitly on j. Furthermore, define

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the column vectors $U_{rj} = U_j \exp(i\phi_d)$, $V_{rj} = U_j \exp[i(\phi_e + \phi_a)]$, $U_{sj} = V_j \exp(-i\phi_d)$ and $V_{sj} = V_j \exp[i(\phi_e - \phi_a)]$. Then, by using this notation, one can rewrite Eq. (71) in the Schmidt form

$$\begin{bmatrix} B_r(z) \\ B_s(z) \end{bmatrix} = \begin{bmatrix} V_r|D_\tau|U_r^\dagger & V_r|D_\rho|U_s^\dagger \\ -V_s|D_\rho|U_r^\dagger & V_s|D_\tau|U_s^\dagger \end{bmatrix} \begin{bmatrix} B_r(0) \\ B_s(0) \end{bmatrix}, \tag{72}$$

where the diagonal matrices $|D_\tau|$ and $|D_\rho|$ are non-negative.

5. Summary

In this paper, vector four-wave mixing in a randomly-birefringent fiber was studied for arbitrary pump polarizations. The coupled-mode equations for (inverse) modulation interaction, phase conjugation and Bragg scattering were derived from the Manakov equation (7) and solved analytically. For each process, one can reduce a complicated system of four coupled equations to two simple systems of two coupled equations by using the Schmidt vectors of the cross-coupling matrix as basis vectors. Not only do these Schmidt vectors facilitate the solution of the coupled-mode equations and the Schmidt decomposition of the associated transfer matrix, they also determine completely the polarization properties of each process. This simplification is not required by the Schmidt decomposition theorem. It is a consequence of the facts that the dispersion term in the Manakov equation does not depend on the wave polarizations and the nonlinearity term depends on the polarizations in a relatively simple way.

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