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Two-scale Convergence and Homogenization of Some Partial Differential Equations

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Abstract

This thesis consists of four papers devoted to the homogenization of some partial differential equations by means of the two-scale convergence method and the Σ-convergence method. In the first two papers, I investigate the asymptotic behavior of second order self-adjoint elliptic eigenvalue problems with periodic rapidly oscillating coefficients and with indefinite (sign-changing) density function in periodically perforated domains.

In the first paper, we have a Steklov condition on the boundaries of the holes. I prove that the spectrum of the Steklov problem under consideration is discrete and consists of two sequences, one tending to $-\infty$ and another to $+\infty$, then I study the limiting behavior of positive and negative eigencouples which crucially depends on the sign of the average of the weight over the surface of the reference hole.

In the second paper, we have a homogeneous Neumann condition on the boundary of the holes. The spectrum is also discrete and consists of two sequences, one tending to $-\infty$ and another to $+\infty$. I study the asymptotic behavior of positive and negative eigencouples which critically depends on the sign of the average of the weight over the solid part of the reference cell. The third and fourth papers deal with deterministic homogenization.

In the third paper, we study the existence and almost periodic homogenization of some model of generalized Navier-Stokes equations. We establish an existence result for nonstationary Ladyzhenskaya equations with a given nonconstant density and an external force depending nonlinearly on the velocity. In the case of a nonconstant density of the fluid, we study the asymptotic behavior of the velocity field.

In the fourth paper, we first introduce a framework to study homogenization problems in general deterministic fissured media composed of blocks of an ordinary porous medium with fissures between them. Next, we study homogenization for nonstationary Navier-Stokes systems in a fissured medium of general deterministic type. Assuming that the blocks of the porous medium consist of deterministically distributed inclusions and the elasticity tensors satisfy general deterministic hypotheses, we prove that the macroscopic problem is a Navier-Stokes type equation for Newtonian fluid in a fixed domain. Our setting includes the classical periodic framework, the weakly almost periodic one and some others.

Keywords: Homogenization. Two-scale convergence. Perforated domains. Eigenvalue problem. Ergodic algebra. Algebra with mean value. Gelfand transform. Σ-convergence. Navier-Stokes equation. Deterministic fissured medium.

List of Papers

This thesis consists of an introduction and the following papers.

Paper A: H. Douanla, *Homogenization of Steklov spectral problems with indefinite density function in perforated domains*, Acta Applicandae Mathematicae, **123** (2013) 261-284.

Paper B: H. Douanla, *Two-scale convergence of periodic elliptic spectral problems with indefinite density function in perforated domains*, Asymptotic Analysis, **81** (2013) 251-272.

Paper C: H. Douanla and J.L. Woukeng, *Almost periodic homogenization of a generalized Ladyzhenskaya model for incompressible viscous flow*, Journal of Mathematical Sciences (N. Y.), **189** (2013) 431-458.

Paper D: H. Douanla, G. Nguetseng and J.L. Woukeng, *Incompressible viscous Newtonian flow in a fissured medium of general deterministic type*, Journal of Mathematical Sciences (N. Y.), **191** (2013) 214-242.

In addition to the above, there are three other papers by the author, which are not included in this thesis.

- H. Douanla and N. Svanstedt, *Reiterated homogenization of linear eigenvalue problems in multiscale perforated domains beyond the periodic setting*, Commun. Math. Anal., **11** (2011) 61-93.
- H. Douanla, *Two-Scale convergence of Stekloff eigenvalue problems in perforated domains*, Boundary Value Problems **(2010)**, Article ID 853717, 15 pages.
- H. Douanla and Nils Svanstedt, *Homogenization of a nonlinear elliptic problem with large nonlinear potential*, Applicable Analysis, **91** (2012) 1205-1218.

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Hermann Douanla
Gothenburg, April 2013

I dedicate this thesis to my wife

Mirene Douanla

Contents

Introduction

The homogenization theory aims at describing the effective (macroscopic) properties of composite materials by means of the characteristics of their micro-structures. Composite materials are widely used in engineering construction because of their properties that, in general, are better that those of their individual constituents. Well known examples of composites are the superconducting multifilamentary composites which are used in the composition of optical fibers. Under external loads, the behavior of a composite material on its microscale is often quite different from that of a uniform material with the same effective properties. Cracking, for instance, starts and develops in these materials in different ways [15]. The homogenization theory cover a wide range of applications ranging from the study of the characteristics of composites to optimal design. The homogenization theory made it possible to predict properties of a composite material even before it is engineered, and consequently, to consider the problem of developing a composite with predetermined or optimum characteristics. In mathematical terms, the homogenization theory study the asymptotic behavior of partial differential equations with varying coefficients depending on a small parameter $\varepsilon > 0$ (representing the size of the heterogeneities), as this parameter tends to zero.

The following classical example (see e.g., [15]) clarifies how studying the limiting behavior of partial differential equations with varying coefficients helps to describe the effective properties of composites.

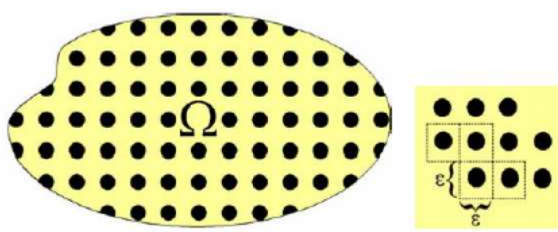

Figure 1: An ε-periodic composite.

We consider stationary heat conduction in a material with a periodic structure of period $\varepsilon > 0$ in all directions (see figure 1). This could, for example, be a cut across the section of a unidirectionally reinforced fibrous composite which is a system of unidirectional fibers of one compound and the matrix of another compound filling the space between the fibers. We assume that the thickness of fibers and that of the fibers spacing are of the same order $\varepsilon > 0$. Thus, if the temperature of the fibrous composite is constant along the orientation of the fibers, the two-dimensional thermal field in the composite is described by the Poisson equation

$$
\begin{cases} -\nabla \cdot (A_\varepsilon(x)\nabla u_\varepsilon) = f(x) \text{ in } \Omega \\ u_\varepsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}
$$

everywhere outside the interface of the two materials. Here we have assumed that the temperature on the boundary of the domain Ω is kept at 0, and $A_\varepsilon(x)$ is the conductivity at the point x. We have

$$
A_\varepsilon(x) = \begin{cases} A_f & \text{if } x \text{ is on a fiber,} \\ A_m & \text{if } x \text{ is on the matrix,} \end{cases}
$$

where A_f and A_m are the conductivity of the fibers and matrix, respectively. The function $f(x)$ is the density of the heat sources in the composite and $u_\varepsilon(x)$ is the temperature at the point x. Continuity conditions are satisfied on the interface for the temperature $[u_\varepsilon] = 0$ and for the heat flux density $\left[A_\varepsilon \frac{\partial u_\varepsilon}{\partial n}\right] = 0$, where $[h]$ stands for the jump of the function h at the matrix-fibers interface. The conductivity coefficient $A_\varepsilon(x)$ changes by $|A_f - A_m|$ when x changes by a value of order $\varepsilon << 1$. Thus the conductivity A_ε oscillates rapidly, making the numerical solvability of (1) (say by a finite difference or finite element method) very expensive or even impossible because of the discontinuities in the function A_ε at the fibers-matrix interface. In order to get reasonable results using a numerical scheme, the discretization step $\Delta(x)$ must satisfies $\Delta(x) << \varepsilon$. Otherwise, the microstructure is not taken into account and the computations are not realistic. Choosing for example $\varepsilon = 10^{-4}$ and $\Delta(x) = 10^{-1}\varepsilon$, and assuming that the ε-periodic bidimensional domain has order of magnitude 1, the order of degrees of freedom is $(10 \cdot 10^4)^2 = 10^{10}$! This is very expensive. Moreover, the variations of the temperature at that tiny microscale is of no interest, and composites are often heterogeneous on several scales, making the numerical approach tricky.

In the asymptotic process (as $\varepsilon \to 0$), we look for an equation

$$
\begin{cases} -\nabla \cdot (A(x)\nabla u) = f(x) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}
$$

whose coefficients are not oscillating, and whose solution is ‘close’ to that of the ε-problem (1). The new equation (2) is the averaged (or homogenized) equation and its coefficients describe the effective properties of the composite under consideration.

Let l and L denote the characteristic lengthes of the micro-cell and the whole domain, respectively ($l << L$). In the early work of Babuska on the homogenization approach in engineering [8], he pointed out that: ‘l is a given parameter, with physical meaning which cannot be changed, e.g., cannot be made “sufficiently” small’. Moreover, he mentioned in [7, 10] that ε in (1) is the ratio of the microscale to the macroscale: $\varepsilon = \frac{l}{L}$. Thus $\varepsilon \to 0$ is equivalent to $L \to \infty$, for this reason, the homogenization procedure is sometimes referred to as “upscaling” or “zooming out”. For an extensive description and further queries about the homogenization theory, we refer to leading books in this research area like, e.g., [15, 21, 29, 30, 46, 51, 83, 106, 128, 131, 143, 164]. We particularly recommend the introduction to homogenization by Cioranescu and Donato [46], which is the best start (at least to my opinion) to study the homogenization theory of partial differential equations.

1 Brief history

Following [15, 21, 46, 83], we give a short (possibly incomplete) historical development of homogenization methods. The problem of replacement of a heterogenous material by an "equivalent" homogeneous one dates back to at least the 19th century. This was raised in works by Poisson [135], Maxwell [105] and Rayleigh [138]. In 1881, Maxwell [105] studied the effective conductivity of media with small concentrations of randomly arranged inclusions, and Rayleigh [138] studied the same problem with periodically distributed inclusions in 1892. In 1906, Einstein [71] investigated the effective viscosity of suspensions with hard spherical particles in incompressible viscous fluids. A good survey of results on this question until 1926 can be found in [97].

Striking contributions were made in the 1930s. Voight [157] calculated effective parameters of polycrystal, such as, the stiffness tensor, by averaging the appropriate values over volume and orientation, while Reuss [139] used averaging of the component of the reverse tensor (compliance) for the same problem. Later on, Hill [80, 81] and Il'iushina [86] rigorously proved that Voight's and Reuss' methods give the upper bound and the lower bound, respectively, of those effective parameters. For results in the direction of the so-called Reuss-Voight inequality (Hill's fork), such as the Hashin-Shtrikman bounds, we refer to [164, Section 1.6 and Chapter 6] and [153, Chapter 25], and references therein. It should be noted that iterated homogenization type problems were considered for the first time by Bruggeman in 1935 [32].

The first asymptotically exact scheme for calculating effective parameters of laminated media was proposed in 1946 by Lifshits and Rozentsveig [98, 99]. Complex analysis based methods were used by Van Fo Fy [154], Grigoliuk and Filshtinskii [76] for exact solutions of two-dimensional problems of elasticity for composite materials and perforated plates and shells.

In 1964, Marchenko and Khruslov [103] introduced a general approach based on asymptotic tools which could handle numerous physical problems, including for example (for the first time), boundary value problems with fine-grained boundary [103, 104]. From the early 1970s, further development of the mathematical study of phenomena in heterogeneous media is done by averaging differential equations with rapidly oscillating coefficients, and the first results (according to e.g., [4, 15]) are in [8, 12, 14, 13, 20, 22, 54, 126, 142].

The name "homogenization" was introduced in 1974 by Babuska [9]. Several homogenization methods were developed in the 1970s, and homogenization became a subject in Mathematics. The methods introduced include:

- the multiple scales asymptotic expansion set out by engineers and mechanical scientists (see e.g., [26]), and systematically formalized to handle homogenization of boundary value problems with periodic rapidly oscillating coefficients by Bensoussan, Lions and Papanicolaou [21], see also Keller and Larsen [92, 93], and Sanchez-Palencia [143];

- the G-convergence of Spagnolo [146, 147] well adapted to problem involving Green kernel;

- the Γ-convergence of De Giorgi [55, 56] suitable for homogenization of optimization problem;

- the H-convergence of Tartar and Murat [107, 108, 109, 150, 151] (a generalization of the G-convergence method to non symmetric problems).

In 1989, Nguetseng [116, 117] introduced the two-scale convergence method to study homogenization of boundary value problems with periodic rapidly oscillating coefficients. The name "two-scale convergence" was later on introduced in 1992 by Allaire [1] who proposed an elegant proof of Nguetseng's compactness theorem and further studied properties of the two-scale convergence method. Nguetseng's breakthrough was "groundshaking and inspiring to periodic homogenization as was the fall of the Berlin wall in 1989 to world history", quoting Stelzig [149, Page 38]. The two-scale convergence method was extended to periodic surfaces in 1992 by Neuss-Radu [114], this extension has been further studied in [5, 115]. In 1994, Bourgeat, Mikelic, and Right [28] introduced the "stochastic two-scale convergence" to study random homogenization. The two-scale convergence has been developed in many other papers, among them, [1, 2, 3, 28, 100, 156, 162, 163].

Recently in 2002, Damlamian, Griso and Cioranescu [43, 44] introduced the "unfolding method" to study the homogenization of periodic heterogeneous composites while Nguetseng [119] extended the two-scale convergence method, under the label "Σ-convergence", to tackle deterministic homogenization beyond the periodic setting. In 2009, Wellander [159] introduced the two-scale Fourier transform (for periodic homogenization in Fourier spaces) which connects some existing techniques for periodic homogenization, namely: the two-scale convergence method, the periodic unfolding method and the Floquet-Bloch expansion approach to homogenization.

The homogenization methods utilized in this thesis are the two-scale convergence method and the Σ-convergence method. In the following section, we briefly present the asymptotic expansion method, the two-scale convergence method and the Σ-convergence method. But we will also recall the definition and main compactness theorems of the H-convergence method and the Γ-convergence method. As for the periodic unfolding method, its main ingredient, the unfolding operator, will be presented. For the sake of simplicity, we assume throughout (except otherwise specified) that all vector spaces are real vector spaces, and scalar functions are assumed to take real values.

2 Homogenization techniques

We start this section by giving a flavour of homogenization. We consider the following two-point boundary value problem (see e.g., [21, 46]):

$$\begin{cases} -\dfrac{d}{dx}\left(a(\dfrac{x}{\varepsilon})\dfrac{du_\varepsilon}{dx}\right) = f \quad \text{ in } (0,\,1) \\ u_\varepsilon(0) = u_\varepsilon(1) = 0, \end{cases} \tag{3}$$

where $f \in L^2(0,\,1)$, and where the coefficient $a \in L^\infty(0,\,1)$ is 1-periodic in y and satisfies

$$0 < \alpha \leq a(y) \leq \beta < \infty \quad \text{a.e. in } (0,\,1) \tag{4}$$

for some constants $\alpha, \beta \in \mathbb{R}$. Owing to (4), we have

$$\|u_\varepsilon\|_{H_0^1(0,\,1)} \leq \frac{1}{\alpha}\|f\|_{L^2(0,\,1)}. \tag{5}$$

Then, there exists a subsequence still denoted by ε such that

$$u_\varepsilon \to u_0 \qquad \text{weakly in } H_0^1(0,\,1) \tag{6}$$

as $\varepsilon \to 0$. We set

$$p^\varepsilon(x) = a(\frac{x}{\varepsilon})\frac{du_\varepsilon}{dx}(x) \qquad \text{a.e. in } x \in (0,\,1).$$

We have

$$-\frac{dp^\varepsilon}{dx} = f \quad \text{in } (0,\,1),$$

which together with (5) imply

$$\left\|\frac{dp^\varepsilon}{dx}\right\|_{L^2(0,\,1)} + \|p^\varepsilon\|_{L^2(0,\,1)} \leq \frac{\beta}{\alpha}\|f\|_{L^2(0,\,1)} + \|f\|_{L^2(0,\,1)}.$$

Therefore, up to a subsequence still denoted by ε we have by the Rellich-Kondrachov compactness theorem,

$$p^\varepsilon \to p^0 \qquad \text{strongly in } L^2(0,\,1)$$

as $\varepsilon \to 0$. Hence, passing to the limit, as $\varepsilon \to 0$, in the weak formulation of (3)

$$\int_0^1 a(\frac{x}{\varepsilon})\frac{du_\varepsilon}{dx}\frac{dv}{dx}dx = \int_0^1 fv\,dx \qquad \forall v \in H_0^1(0,\,1) \tag{7}$$

yields

$$-\frac{dp^0}{dx} = f \quad \text{in } (0,\,1). \tag{8}$$

We recall that, as the function a is 1-periodic and bounded from above and away from zero, the following convergence result hold

$$\frac{1}{a(\frac{\cdot}{\varepsilon})} \to \int_0^1 \frac{1}{a(x)}dx \equiv M(\frac{1}{a}) \qquad \text{weakly}^* \text{ in } L^\infty(0,\,1),$$

so that we can pass to the limit in the product

$$\frac{du_\varepsilon}{dx} = \frac{1}{a(\frac{\cdot}{\varepsilon})}p^\varepsilon$$

using the well-known weak-strong convergence result. We obtain

$$\frac{du_\varepsilon}{dx} \to M(\frac{1}{a})p^0 \quad \text{weakly in } L^2(0,\,1),$$

as $\varepsilon \to 0$. Consequently, from (6) we have

$$\frac{du_0}{dx} = M(\frac{1}{a})p^0,$$

which together with (8) yield

$$-\frac{d}{dx}\left(\frac{1}{M(\frac{1}{a})}\frac{du_0}{dx}\right) = f \quad \text{in } (0,\,1).$$

Hence, the limit u_0 is fixed by this equation and does not depend on any particular subsequence extracted in the process. Therefore, we have prove the following homogenization result.

Theorem 1. *Let $u_\varepsilon \in H_0^1(0,\,1)$ be the solution to (3) with hypothesis (4). Then, as $\varepsilon \to 0$, we have*

$$u_\varepsilon \to u_0 \quad \textit{weakly in } H_0^1(0,\,1),$$

where u_0 is the unique solution in $H_0^1(0,\,1)$ to the problem

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{M(\frac{1}{a})}\dfrac{du_0}{dx}\right) = f \quad \textit{in } (0,\,1) \\ u_0(0) = u_0(1) = 0. \end{cases}$$

Following the homogenization result in Theorem 1, we are tempted to claim that in the N-dimensional case $(N > 1)$ the homogenized coefficients can be obtained in terms of the mean value of the inverse matrix A^{-1} of A. This is not true as we will see in the following subsections where we briefly present popular homogenization techniques. In the sequel, the letter E stands for a sequence of strictly positive real numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ as $n \to \infty$.

2.1 The Γ-convergence method

Introduced by De Giorgi [55, 56] in the early 1970s, the Γ-convergence is an abstract notion of functional convergence aiming at describing the asymptotic behavior of families of minimum problems usually depending on some parameters whose nature may be geometric or constitutive, deriving from a discretization argument, an approximation procedure, etc [29]. The Γ-convergence has several applications in many research areas including for example the calculus of variation and the homogenization of partial differential equations. A good introduction to the Γ-convergence is [29] and several applications can be found in [30, 33, 51]. It should be noted that the *epiconvergence* introduced by Attouch [6] in 1984 is a functional convergence notion close to the Γ-convergence. We now give the definition of the Γ-convergence, its fundamental theorem and hint how it is utilized to handle the homogenization of partial differential equations.

Definition 1. *Let W be a metric space endowed with a distance d. Let $(F_\varepsilon)_{\varepsilon\in E}$ be a sequence of real functions defined on W. The sequence $(F_\varepsilon)_{\varepsilon\in E}$ is said to Γ-converges to a limit function F_0 if for any $x \in W$, the following hold:*

1. *(*lim inf inequality*) any sequence $(x_\varepsilon)_{\varepsilon\in E}$ converging to x in W as $\varepsilon \to 0$ satisfies*

$$F_0(x) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon),$$

2. *(existence of a recovering sequence) there exists a sequence $(x_\varepsilon)_{\varepsilon\in E}$ converging to x as $\varepsilon \to 0$ and such that*

$$F_0(x) \geq \lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon).$$

Indeed, the Γ-convergence and the Γ-limit depend on the choice of the distance. A Γ-limit F_0 is a lower semi continuous function on W, viz, $F_0(x) \leq \liminf_{\varepsilon\to 0} F_0(x_\varepsilon)$ for any sequence $x_\varepsilon \to x$ as $\varepsilon \to 0$. The Γ-limit of the constant sequence F is the lower semicontinuous envelope (relaxation) of F.

To formulate the main (as regards the homogenization theory) theorems of the Γ-convergence, we recall that a sequence $(F_\varepsilon)_{\varepsilon\in E}$ of real functions defined on W is said to be *equi-mildly* coercive on W if there exists a compact set K (independent of ε) such that

$$\inf_{x\in W} F_\varepsilon(x) = \inf_{x\in K} F_\varepsilon(x).$$

Theorem 2. *Let $(F_\varepsilon)_{\varepsilon\in E}$ be an equi-mildly coercive sequence on W which Γ-converges to a limit F_0. Then,*

1. *the minima of F_ε converges to that of F_0, viz,*

$$\min_{x\in W} F_0(x) = \lim_{\varepsilon\to 0}\left(\inf_{x\in W} F_\varepsilon(x)\right),$$

2. *the minimizers of F_ε converge to those of F_0, viz, if $x_\varepsilon \to x$ in W and $\lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0}(\inf_{x\in W} F_\varepsilon(x))$, then, x is a minimizer of F_0.*

Theorem 3. *Assume that the metric space W endowed with the distance d is separable. For any sequence of functions $(F_\varepsilon)_{\varepsilon\in E}$ defined on W there exists a subsequence E' of E and a Γ-limit F_0 such that $(F_\varepsilon)_{\varepsilon\in E'}$ Γ-converges to F_0 as $E' \ni \varepsilon \to 0$.*

The proofs of the above theorems can be found in [51]. Loosely speaking, to utilize the Γ-convergence in homogenization, one usually transforms the partial differential equation into a minimization problem. To illustrate this, we consider the model problem

$$\begin{cases} -\boldsymbol{\nabla}\cdot\Big(A(\frac{x}{\varepsilon})\nabla u_\varepsilon\Big) = f \quad \text{in } \Omega \\ \qquad\qquad\qquad u_\varepsilon = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{9}$$

where Ω is a bounded domain in $\mathbb{R}^N$, and the matrix A is coercive, bounded and symmetric. For $\varepsilon \in E$, we set

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega A(\frac{x}{\varepsilon})\nabla u\cdot\nabla u\,dx - \int_\Omega f u\,dx \qquad u \in H_0^1(\Omega). \tag{10}$$

It is well known that, when the matrix A is symmetric, the problem (9) is equivalent to the following minimisation problem:

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^1(\Omega) \text{ such that} \\ F_\varepsilon(u_\varepsilon) \le F_\varepsilon(v) \\ \text{for all } v \in H_0^1(\Omega). \end{cases} \tag{11}$$

Hence, the Γ-convergence of the sequences of functionals $(F_\varepsilon)_{\varepsilon\in E}$ (defined by (10)) in $L^2(\Omega)$-strong is equivalent to the homogenization of the partial differential equation (9). The Γ-convergence method applied to homogenization theory is neither restricted to linear equation nor to periodic structure. Albeit the Γ-convergence method is one of the most utilized homogenization technique, it is sometimes blamed for being of limited interest to real-life problems of continuum physics. For example, Tartar [152, 153] believes that the energy minimization approaches to homogenization problems (or to continuum mechanics in general) is fake mechanics [153, Page 31], since from the first principle of thermodynamics, nature conserves energy rather than minimizing it. However, Benamou and Brenier [19] were able to formulate evolution problems of continuum physics as minimization problem by means of the optimal transportation theory.

2.2 The H-convergence and the G-convergence methods

As said earlier, the H-convergence of Tartar and Murat [107, 108, 109, 150, 151] is a generalization to non symmetric problems of the G-convergence of Spagnolo [146, 147].

The letters G and H stand for 'Green' and 'Homogenization', respectively. These convergence methods are equivalent to the convergence of the associated Green kernel. We briefly present the H-convergence method for specific simple example of operators. For an advanced description and several applications of these types of convergence, we refer to [128, 153].

Let Ω be a bounded open set in $\mathbb{R}^N$, and let $0 < \alpha \leq \beta$ be two positive constants. We define $\mathcal{M}(\alpha, \beta, \Omega)$ to be the set of all $N \times N$-matrices defined on Ω with uniform coercivity constant α and $L^\infty(\Omega)$-bound β.

$$\mathcal{M}(\alpha, \beta, \Omega) = \{A \in L^\infty(\Omega; \mathbb{R}^{N\times N}) : \alpha|\xi|^2 \leq A(x)\xi{\cdot}\xi \leq \beta|\xi|^2 \ \forall \xi \in \mathbb{R}^N \text{ and a.e } x \in \Omega\}.$$

We consider a sequence $(A_\varepsilon)_{\varepsilon\in E} \subset \mathcal{M}(\alpha, \beta, \Omega)$ without any periodicity assumption or any symmetric hypothesis either. Given $f \in L^2(\Omega)$, there exists, by the Lax-Milgram lemma, a unique solution $u_\varepsilon \in H^1_0(\Omega)$ to

$$\begin{cases} -\boldsymbol{\nabla} \cdot (A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \\ \qquad\qquad u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Definition 2. *The sequence $(A_\varepsilon)_{\varepsilon\in E}$ is said to H-convergence to a limit A^* as $E \ni \varepsilon \to 0$, if, for any right-hand side $f \in L^2(\Omega)$ in (12), we have*

$$\begin{array}{rcll} u_\varepsilon & \to & u_0 & \textit{weakly in } H^1_0(\Omega) \\ A_\varepsilon \nabla u_\varepsilon & \to & A^* \nabla u_0 & \textit{weakly in } L^2(\Omega)^N, \end{array}$$

as $E \ni \varepsilon \to 0$, where u_0 is the solution to the homogenized *equation associated with A:*

$$\begin{cases} -\boldsymbol{\nabla} \cdot (A^* \nabla u_0) = f & \textit{in } \Omega \\ \qquad\qquad u_0 = 0 & \textit{on } \partial\Omega. \end{cases} \tag{13}$$

Among useful properties of the H-convergence, we have the following:

1. if $(A_\varepsilon)_{\varepsilon\in E} \subset \mathcal{M}(\alpha, \beta, \Omega)$ H-converges, its H-limit is unique,

2. let $(A_\varepsilon)_{\varepsilon\in E}$ and $(B_\varepsilon)_{\varepsilon\in E}$ be two sequences in $\mathcal{M}(\alpha, \beta, \Omega)$ which H-converge to A^* and B^*, respectively. If $A_\varepsilon = B_\varepsilon$ in $\omega \subsetneq \Omega$, then $A^* = B^*$ in ω,

3. the H-limit does neither depend on the source term nor the boundary condition on $\partial\Omega$,

4. if $(A_\varepsilon)_{\varepsilon\in E} \subset \mathcal{M}(\alpha, \beta, \Omega)$ H-converges to A^*, then the associated density of energy also converges, viz, $A_\varepsilon \nabla u_\varepsilon {\cdot} \nabla u_\varepsilon$ converges to $A^* \nabla u {\cdot} \nabla u$ in the sense of distributions in Ω.

Indeed, the H-convergence is a local property. Note that the definition of the H-convergence differs from that of the G-convergence by requiring the convergence of the flux $(A_\varepsilon u_\varepsilon)_{\varepsilon\in E}$ in addition to that of the sequence $(u_\varepsilon)_{\varepsilon\in E}$. This additional requirement is essential when removing the symmetry hypothesis (say, when passing from G-convergence to H-convergence) since it ensures the uniqueness of the H-limit (see e.g., [83, Page 232]).

Without the following compactness result, the concept of H-convergence would be useless.

Theorem 4. *For any sequence $(A_\varepsilon)_{\varepsilon\in E} \subset \mathcal{M}(\alpha, \beta, \Omega)$, there exists a subsequence E' of E and a homogenized limit A^*, belonging to $\mathcal{M}(\alpha, \frac{\beta^2}{\alpha}, \Omega)$, such that A_ε H-converges to A^* as $E' \ni \varepsilon \to 0$.*

The G-convergence version of Theorem 4 was proved for the first time by Spagnolo [146] by means of the convergence of the Green functions associated with (12). It can also be proved using the Γ-convergence. Tartar [107, 151] proposed a simpler proof in the general framework of H-convergence. We present the idea (in the periodic setting) of tartar's proof which has been later called the *energy method*, and sometimes, the *oscillating test function method*. We assume that the matrix A_ε in (12) is given by $A_\varepsilon(x) = A(\frac{x}{\varepsilon})$, where A is 1-periodic in all coordinate directions. The variational formulation of (12) reads

$$\int_\Omega A(\frac{x}{\varepsilon})\nabla u_\varepsilon \cdot \nabla\varphi \, dx = \int_\Omega f(x)\varphi(x)\, dx \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{14}$$

The coercivity of the matrix A implies the boundedness of the sequence $(u_\varepsilon)_{\varepsilon\in E}$ in $H_0^1(\Omega)$. Hence, it weakly converges (up to a subsequence) to some $u_0 \in H_0^1(\Omega)$. We recall that, with the hypothesis on A, we have

$$A_\varepsilon \to \int_\Omega A(y)\, dy \ \text{ weakly in } L^2(\Omega),$$

so that the left-hand side of (14) involves a product of two weakly converging sequences in $L^2(\Omega)$: $(A(\frac{\cdot}{\varepsilon}))_{\varepsilon\in E}$ and $(\nabla u_\varepsilon)_{\varepsilon\in E}$. Therefore we cannot pass to the limit by means of classical arguments. Tartar's idea is to use the oscillating test function defined by

$$\varphi_\varepsilon(x) = \varphi(x) + \varepsilon \sum_{i=1}^N \frac{\partial\varphi}{\partial x_i}(x)\omega_i^*(\frac{x}{\varepsilon}) \qquad (\varepsilon \in E),$$

where $\varphi \in \mathcal{D}(\Omega)$, and where ω_i $(i = 1, \cdots, N)$ solve the so-called dual cell problem

$$\begin{cases} -\nabla_y \cdot (A^t(y)(e_i + \nabla_y\omega_i(y))) = 0 & \text{in } (0,\,1)^N \\ \qquad\qquad y \to \omega_i(y) & (0,\,1)^N\text{-periodic}, \end{cases}$$

with $e_i = (\delta_{ij})_{j=1}^N$. Utilizing this test function, Tartar was able to eliminate the 'bad terms' in (14), and to pass to the limit and get a macroscopic equation. However, constructing the oscillating test functions is not always as easy as in this simple periodic problem. To be convinced, see e.g., [109, 151, 153, 155] where this method is utilized to handle different non trivial homogenization problems. We conclude this subsection by mentioning that Donato et al. [31, 42, 53] have extended the H-convergence method (under the label H^0-convergence) to the case of perforated domains with a Neumann condition on the holes.

2.3 The asymptotic expansion method

The asymptotic expansion method in homogenization is a *formal procedure* to derive the limit problem. One usually makes use of further devices to justify the homogenization result. To sketch the method, we reconsider the model problem:

$$\begin{cases} -\boldsymbol{\nabla} \cdot \Big(A(\frac{x}{\varepsilon}) \nabla u_\varepsilon \Big) = f & \text{in } \Omega \\ \qquad\qquad\qquad u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and where we assume that the matrix A is symmetric, coercive, bounded, and Y-periodic, with $Y = (0,\, 1)^N$ (i.e., A is 1-periodic in all coordinate directions). To derive the limit problem for (15), we assume that the unknown function $u_\varepsilon \in H_0^1(\Omega)$ has a two-scale expansion (with respect to ε) of the form

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \cdots \tag{16}$$

where the coefficients functions $u_i(x, y)$ are Y-periodic with respect to the local variable $y = \frac{x}{\varepsilon}$. Next we plug (16) into (15) to get the formula

$$\begin{aligned} & \varepsilon^{-2} \nabla_y \cdot (A(y) \nabla_y u_0(x, y)) \\ + \quad & \varepsilon^{-1} (\nabla_y \cdot (A(y) \nabla_y u_1(x, y)) + \nabla_y \cdot (A(y) \nabla_x u_0(x, y)) \\ & \quad + A(y) \nabla_x \cdot \nabla_y u_0(x, y)) \\ + \quad & \varepsilon^0 (\nabla_y \cdot (A(y) \nabla_y u_2(x, y) + A(y) \nabla_x u_1(x, y)) + A(y) \nabla_x \cdot \nabla_y u_1(x, y) \\ & \quad + A(y) \nabla_x \cdot \nabla_x u_0(x, y)) \\ + \quad & \varepsilon^1 (\cdots) + \cdots = -f(x)|_{y = \frac{x}{\varepsilon}}. \end{aligned} \tag{17}$$

The next step is to compare the coefficients of different powers of ε in the above equation. The terms with ε^{-2} gives

$$\nabla_y \cdot (A(y) \nabla_y u_0(x, y)) = 0 \qquad \text{in } \in Y,$$

which admits only constant functions as Y-periodic solutions. Thus, $u_0(x, y) = u_0(x)$ (the leading term u_0 does not depend on the local variable). Therefore, the equation from the terms with ε^{-1} reduces to

$$\begin{cases} -\nabla_y \cdot (A(y) \nabla_y u_1(x, y)) = \nabla_y \cdot (A(y) \nabla_x u_0(x)) & \text{in } Y \\ \qquad\qquad\qquad u_1 \quad Y\text{-periodic in } y. \end{cases} \tag{18}$$

Since

$$\int_Y \nabla_y \cdot (A(y)\nabla_x u_0(x))\, dy = 0,$$

the equation (18) admits a unique (up to an additive constant) solution by the Fredholm alternative. Now, we rewrite the equation in (18) as follows

$$-\nabla_y \cdot (A(y)\nabla_y u_1(x,y)) = \sum_{j=1}^{N} \frac{\partial}{\partial y_j} A(y) \frac{\partial u_0}{\partial x_j}(x) \quad \text{for } y \in Y.$$

This suggests we introduce for $j = 1, \cdots, N$ the so-called *cell problems*

$$\begin{cases} -\nabla_y \cdot (A(y)\nabla_y \omega_j(y)) = \nabla_y \cdot (A(y)e_j) \quad \text{in } Y \\ \omega_j \quad Y\text{-periodic in } y, \end{cases} \tag{19}$$

so that comparing (18) and (19), and using the notation $\omega = (\omega_1, \cdots, \omega_N)$, we obtain

$$u_1(x,y) = \omega(y) \cdot \nabla_x u_0(x) + \widetilde{u}(x), \tag{20}$$

where $\widetilde{u}(x)$ is independent of y. As for the terms with ε^0, we integrate the resulting equation (from (17)) over Y,

$$\int_Y A(y)\nabla_x \cdot \nabla_y u_1(x,y)\, dy + \int_Y A(y)\nabla_x \cdot \nabla_x u_0(x)\, dy = -f(x) \ \text{ in } \ \Omega. \tag{21}$$

Note that some integrals have vanished after integrating by part therein and using the periodicity hypothesis of functions in the integrands. From (20), we have

$$\nabla_x \cdot \nabla_y u_1(x,y) = \sum_{i,j=1}^{N} \frac{\partial \omega_j}{\partial y_i}(y) \frac{\partial^2 u_0}{\partial x_i x_j},$$

so that (21) writes

$$\sum_{i,j=1}^{N} \int_Y A(y) \frac{\partial \omega_j}{\partial y_i}(y) \frac{\partial^2 u_0}{\partial x_i \partial x_j}\, dy + \int_Y A(y)\Delta_x u_0\, dy = -f(x) \ \text{ in } \ \Omega.$$

We introduce the following notation

$$\overline{a}_{ij} = \int_Y A(y)(\delta_{ij} + \frac{\partial \omega_j}{\partial y_i}(y))dy \qquad (1 \le i,j \le N),$$

with which, it appears that u_0 solves the homogenized equation

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \overline{a}_{ij} \frac{\partial^2 u_0}{\partial x_i \partial x_j} = f(x) \qquad \text{in } \ \Omega \\ \qquad u_0 = 0 \qquad \text{on } \ \partial\Omega. \end{cases} \tag{22}$$

The constant matrix $\overline{A} = (\overline{a}_{ij})_{1\leq i,j\leq N}$ is the homogenized tensor, and is symmetric and positive definite (this follows from the hypothesis on A). The asymptotic expansion method appears here to be simple but it is not that simple in practice since it is based on *establishing* the *ansatz* (16). However, it is useful *to guess* the homogenized equation. As said earlier, a further step is needed to prove the convergence of the sequence $(u_\varepsilon)_{\varepsilon\in E}$ to u_0. This step can be done by means of several methods, such as, the Γ or G-convergence, the oscillating test function methods, maximum principle, etc.

We present a justification of the formal expansion (16) by means of the maximum principle [21]. We further assume that the matrix A and the source term f, hence u_ε $(\varepsilon \in E)$ are smooth. To start, we take u_1 as in (20) with $\widetilde{u} = 0$ and put

$$v_\varepsilon = u_\varepsilon - (u_0 + \varepsilon u_1 + \varepsilon^2 u_2).$$

With this notation, we have

$$\begin{aligned}
&-\nabla \cdot (A(\frac{x}{\varepsilon})\nabla v_\varepsilon)\\
&\quad = \varepsilon \left\{ \nabla_y \cdot (A(\frac{x}{\varepsilon})\nabla_x u_2) + \nabla_x \cdot (A(\frac{x}{\varepsilon})\nabla_y u_2) + \nabla_x \cdot (A(\frac{x}{\varepsilon})\nabla_x (u_1 + \varepsilon u_2)) \right\}\\
&\quad \equiv \varepsilon \widetilde{v}_\varepsilon.
\end{aligned}$$

Since u_1 and u_2 are smooth we have

$$|\widetilde{v}_\varepsilon| \leq C \quad \text{in } \Omega.$$

On the boundary $\partial\Omega$ we have $v_\varepsilon = -(\varepsilon u_1 + \varepsilon^2 u_2)$. Hence

$$|v_\varepsilon| \leq C\varepsilon \quad \text{on } \partial\Omega$$

The maximum principle yields

$$|v_\varepsilon| \leq C\varepsilon \quad \text{in } \overline{\Omega}.$$

We have proved that

$$|u_\varepsilon - u_0| \leq C\varepsilon \quad \text{in } \overline{\Omega}.$$

Therefore, the sequence $(u_\varepsilon)_{\varepsilon\in E}$ uniformly converges to u_0 in $\overline{\Omega}$. Generally, it holds that

$$u_\varepsilon \to u_0 \text{ weakly in } H_0^1(\Omega)$$

as $E \ni \varepsilon \to 0$. As extensively discussed in [21], the previous convergence does not hold strongly in $H_0^1(\Omega)$, but with the correction εu_1 it holds (as $E \ni \varepsilon \to 0$) that

$$\left[u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon u_1(\frac{\cdot}{\varepsilon}) \right] \to 0 \quad \text{strongly in } H_0^1(\Omega).$$

Unlike the G and Γ-convergence methods, the asymptotic expansion method can only handle linear homogenization problems. Let now discuss the two-scale convergence method.

2.4 The two-scale convergence method

In 1989, Nguetseng [116, 117] introduced a type of convergence which was later (1992) named "two-scale convergence" by Allaire [1]. Nguetseng's two-scale convergence method blends the formal asymptotic expansion and the test function method and give the homogenization result in one step: it constructs the homogenized problem and proves the convergence of solutions of ε-problems to the solution of the limit problem, simultaneously. Flagged in its early years as restricted to periodic homogenization, it was extended to the stochastic setting of homogenization in 1994 by Bourgeat, Mikelič, and Wright [28]. Recently in 2003, Nguetseng proposed a *direct* extension of the two-scale convergence method (under the label Σ-convergence), to a more general setting of deterministic homogenization, which can tackle almost periodic homogenization, weakly almost periodic homogenization, homogenization in Fourier-Stieltjes algebra, etc, *just like* the two-scale convergence method handles periodic homogenization problems. Unlike the asymptotic expansion method, the two-scale convergence method can handle nonlinear homogenization problems. We describe the two-scale convergence method below and present the Σ-convergence method in a further subsection. For a full exposition of the the two-scale convergence method, we recommend [1, 2, 100, 116, 162, 163].

We first introduce some notations. Ω is an open bounded set in $\mathbb{R}^N$, and $Y = (0,1)^N$. We write $\mathcal{C}^\infty_{per}(Y)$ and $\mathcal{C}_{per}(Y)$ for the space of functions respectively in $\mathcal{C}^\infty(\mathbb{R}^N)$ and $\mathcal{C}(\mathbb{R}^N)$ that are Y-periodic. As for Lebesgue spaces, for a given Lebesgue function space $F(\mathbb{R}^N)$, we denote by $F_{per}(Y)$ the space of functions in $F_{loc}(\mathbb{R}^N)$ (when it makes sense) that are Y-periodic, and by $F_{per}(Y)/\mathbb{R}$ the space of those functions $u \in F_{per}(Y)$ with $\int_Y u(y)dy = 0$. We will sometimes omit the subscript '*per*', and write for example $L^2(Y)$ in place of $L^2_{per}(Y)$. Now let $1 < p < \infty$ and $p' = \frac{p}{p-1}$.

Definition 3. *A sequence $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(\Omega)$ is said to:*

- *two-scale converge in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega; L^p_{per}(Y))$ $(\equiv L^p(\Omega \times Y))$ if*

$$\lim_{E\ni\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\varphi(x, \frac{x}{\varepsilon})\, dx = \iint_{\Omega\times Y} u_0(x,y)\varphi(x,y)\, dxdy, \tag{23}$$

  for all $\varphi \in L^{p'}(\Omega; \mathcal{C}_{per}(Y))$. This is expressed in the sequel by $u_\varepsilon \xrightarrow{2s} u_0$ in $L^p(\Omega)$.

- *strongly two-scale converge in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times Y)$ if it two-scale converges to u_0 in $L^p(\Omega)$ and*

$$\lim_{E\ni\varepsilon\to 0} \|u_\varepsilon\|_{L^p(\Omega)} = \|u_0\|_{L^p(\Omega\times Y)}.$$

Before formulating the compactness theorems that give sense to Definition 3, we give some examples of two scale convergent sequences.

1. Let $u_0 \in L^p(\Omega; \mathcal{C}_{per}(Y))$ and define $u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon})$ for $\varepsilon \in E$. Then, the sequence of traces $(u_\varepsilon)_{\varepsilon\in E}$ is well defined and two-scale converges to u_0 in $L^p(\Omega)$. However,

the sequence $(v_\varepsilon)_{\varepsilon\in E}$ defined by $v_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon^2})$ two-scale converges in $L^p(\Omega)$ to its weak $L^p(\Omega)$ limit: $\int_Y u_0(\cdot, y)\, dy$. The oscillations are lost in the limit since they are not in resonance with those of the test function.

2. Assume that $u_\varepsilon : \Omega \to \mathbb{R}$ $(\varepsilon \in E)$ admits the following asymptotic expansion

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \cdots ,$$

where the u_i belong to $L^p(\Omega; \mathcal{C}_{per}(Y))$. Then,

$$u_\varepsilon \xrightarrow{2s} u_0 \quad \text{in } L^p(\Omega)$$

and

$$\frac{u_\varepsilon(\cdot) - u_0(\cdot, \frac{\cdot}{\varepsilon})}{\varepsilon} \ \xrightarrow{2s} \ u_1 \quad \text{in } L^p(\Omega).$$

3. If the sequence $(u_\varepsilon)_{\varepsilon\in E}$ strongly converges to u_0 in $L^p(\Omega)$, then $u_\varepsilon \xrightarrow{2s} u_0$ in $L^p(\Omega)$.

The concept of two-scale convergence would be useless if the following compactness results were not available.

Theorem 5. *Let $1 < p < +\infty$. Any sequence $(u_\varepsilon)_{\varepsilon\in E}$, bounded in $L^p(\Omega)$, admits a subsequence which two-scale converges in $L^p(\Omega)$.*

Theorem 6. *Let $1 < p < +\infty$ and assume that the sequence $(u_\varepsilon)_{\varepsilon\in E}$ is bounded in $W^{1,p}(\Omega)$. Then, a subsequence E' can be extracted from E such that, as $E' \ni \varepsilon \to 0$,*

$$u_\varepsilon \quad \to \quad u_0 \quad \textit{weakly in } W^{1,p}(\Omega), \tag{24}$$

$$u_\varepsilon \quad \xrightarrow{2s} \quad u_0 \quad \textit{in } L^p(\Omega), \tag{25}$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \quad \xrightarrow{2s} \quad \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \quad \textit{in } L^p(\Omega) \quad (1 \le i \le N), \tag{26}$$

where $u_0 \in W^{1,p}(\Omega)$ and $u_1 \in L^p(\Omega; W^{1,p}_{per}(Y)/\mathbb{R})$.

The original proof of Theorem 5 is by Nguetseng [116] and uses advanced distribution theory and some density arguments. A simpler proof (similar to that of the existence of Young measures by Ball [16]) was proposed by Allaire [1] in 1992. Allaire [1] further developed the two-scale convergence method and applied it to various periodic homogenization problems. The following proposition (see e.g., [122]) is of capital interest when utilizing the two-scale convergence method.

Proposition 1. *If the sequence $(u_\varepsilon)_{\varepsilon\in E}$ two-scale converges to some $u_0 \in L^p(\Omega \times Y)$, then (23) holds for any $\varphi \in \mathcal{C}(\overline{\Omega}; L^\infty_{per}(Y))$.*

We apply the two-scale convergence method, by way of illustration, to the following locally periodic model problem

$$\begin{cases} -\boldsymbol{\nabla}\cdot\left(A(x,\frac{x}{\varepsilon})\nabla u_\varepsilon\right)=f \quad \text{in } \Omega \\ \qquad\qquad\qquad\qquad u_\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{27}$$

The matrix $A\in\mathcal{C}(\overline{\Omega};L^\infty_{per}(Y)^{N\times N})$ satisfies for almost every $y\in\mathbb{R}^N$ and for all $x\in\Omega$

$$\alpha|\xi|^2\leq(A(x,y)\xi,\xi)\leq\beta|\xi|^2 \qquad \text{for all} \quad \xi\in\mathbb{R}^N.$$

With the boundedness of the sequence $(u_\varepsilon)_{\varepsilon\in E}$ in $H^1_0(\Omega)$, Theorem 6 applies, and there is a subsequence E' of E such that

$$u_\varepsilon\to u_0 \text{ weakly in } H^1_0(\Omega), \;\text{ and }\; \nabla_x u_\varepsilon \xrightarrow{2s} \nabla_x u_0+\nabla_y u_1 \;\text{ in } L^2(\Omega\times Y)^N, \tag{28}$$

as $E'\ni\varepsilon\to0$, for some $(u_0,u_1)\in H^1_0(\Omega)\times L^2(\Omega;H^1_{per}(Y)/\mathbb{R})$. Let $(\varphi,\psi_1)\in\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)/\mathbb{R}$, and define

$$\psi_\varepsilon(x)=\varphi(x)+\varepsilon\psi_1(x,\frac{x}{\varepsilon}) \qquad (\varepsilon>0,\;\; x\in\Omega).$$

We have

$$\int_\Omega A(x,\frac{x}{\varepsilon})\nabla_x u_\varepsilon(x)\cdot\left(\nabla_x\varphi(x)+\nabla_y\psi_1(x,\frac{x}{\varepsilon})+\varepsilon\nabla_x\psi_1(x,\frac{x}{\varepsilon})\right)dx=\int_\Omega f(x)\psi_\varepsilon(x)dx.$$

A limit passage as $E'\ni\varepsilon\to0$ (use Proposition 1 in the left-hand side) yields

$$\iint_{\Omega\times Y}A(x,y)\left[\nabla_x u_0+\nabla_y u_1\right]\cdot\left[\nabla_x\varphi+\nabla_y\psi_1\right]dxdy=\int_\Omega f(x)\varphi(x)dx,$$

which, by density, still holds for (φ,ψ_1) in $H^1_0(\Omega)\times L^2(\Omega;H^1_{per}(Y)/\mathbb{R})$, a Hilbert space when endowed with the norm

$$\|(\varphi,\psi_1)\|=\left(\|\nabla_x\varphi\|^2_{L^2(\Omega)}+\|\nabla_y\psi_1\|^2_{L^2(\Omega\times Y)}\right)^{\frac{1}{2}}.$$

Since A is coercive, we claim by the Lax-Milgram lemma that (u_0,u_1) is the unique solution to

$$\begin{cases} (u_0,u_1)\in H^1_0(\Omega)\times L^2(\Omega;H^1_{per}(Y)/\mathbb{R}) \\ \displaystyle\iint_{\Omega\times Y}A(x,y)\left[\nabla_x u_0+\nabla_y u_1\right]\cdot\left[\nabla_x\varphi+\nabla_y\psi_1\right]dxdy=\int_\Omega f(x)\varphi(x)\,dx \\ \text{for all } (\varphi,\psi_1)\in H^1_0(\Omega)\times L^2(\Omega;H^1_{per}(Y)/\mathbb{R}). \end{cases} \tag{29}$$

Hence, (28) holds for the whole sequence E. We now derive the macroscopic problem. Taking (φ, ψ_1) in (29) such that $\varphi = 0$, yields

$$\begin{cases} -\nabla_y \cdot (A(x,y)\nabla_y u_1(x,y)) = \nabla_y \cdot (A(x,y)\nabla_x u_0(x)) & \text{in } \Omega \times Y \\ y \to u_1(x,y) & Y\text{-periodic}. \end{cases}$$

Therefore, we can represent u_1 in terms of u_0 as follows,

$$u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u_0}{\partial x_i}(x)\omega_i(x,y) \qquad \text{a.e. in } \Omega \times Y, \tag{30}$$

where ω_i $(i = 1, \cdots, N)$ is defined, at each point x, as the unique solution in $H^1_{per}(Y)/\mathbb{R}$ to the cell problem

$$\begin{cases} -\nabla_y \cdot (A(x,y)\nabla_y \omega_i(x,y)) = \nabla_y \cdot (A(x,y)e_i) & \text{in } Y \\ y \to \omega_i(x,y) & Y\text{-periodic}. \end{cases}$$

Now, we take take (φ, ψ_1) in (29), such that $\psi_1 = 0$ and use (30) to get, after easy calculations, the so-called macroscopic problem

$$\begin{cases} -\nabla \cdot (A^*(x)\nabla u_0(x)) = f(x) & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

where the homogenized diffusion tensor is given by

$$A^*_{i,j}(x) = \int_y A(x,y)[e_i + \nabla_y \omega_i(x,y)] \cdot [e_j + \nabla_y \omega_j(x,y)]\, dy \ \ (x \in \Omega,\ 1 \leq i,j \leq N).$$

Like in Subsection 2.3, corrector type results can be obtained by means of the two-scale convergence method, namely, as $E \ni \varepsilon \to 0$, we have

$$\left[u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon u_1(\cdot, \frac{\cdot}{\varepsilon}) \right] \to 0 \quad \text{strongly in } H^1_0(\Omega).$$

It should be noted that, unlike all the homogenization methods presented in previous subsections, the two-scale convergence exhibits the weak limit of the gradients, ∇u_ε, that is exactly the local behavior of u_ε, which is interesting from the physical point of view. However, with the two-scale convergence method, the homogenization procedure is usually straightforward that one cannot obtain convergence rates type results, which measure the magnitude of the error we commit when replacing the solution to the ε-problem by that of the homogenized equation. Other extensions of the two-scale convergence method can be found in [3, 37, 75, 77, 82, 129, 134, 156]

In 1990, Arbogast, Douglas, and Hornung [5] developed an idea similar to the two-scale convergence. They defined a so-called *dilation* operator to study homogenization problems in porous media. That dilation operator has been also used for example in [27, 36, 94, 95, 113]. Later on, Cioranescu, Damlamian, and Griso [43, 44] expanded and formalized that idea, and introduced the *periodic unfolding method* in homogenization, a dual formulation of the two-scale convergence method. Before we present Nguetseng's approach to general deterministic homogenization (the Σ-convergenve method), we briefly present the unfolding operator and formulate the dual definition of the two-scale convergence, which is the cornerstone of the periodic unfolding method.

2.5 The periodic unfolding method

For an extensive presentation and some applications of the unfolding method in periodic homogenization, we refer to e.g., [41, 43, 44, 47, 48, 52, 73, 74, 113]. Loosely speaking, the main ingredient of the unfolding method in periodic homogenization is the unfolding operator which transforms a function of one variable into a function of two variables, making it possible to reduce the concept of two scale-convergence to usual notions of weak and strong convergence in $L^p(\Omega \times Y)$.

Let Ω be an open set in $\mathbb{R}^N$ and let $Y = [0,\,1)^N$. More generally Y can be replaced by an N-dimensional parallelepiped

$$Y = \{\lambda_1 b_1 + \cdots + \lambda_N b_N : 0 \leq \lambda_i < 1,\ i = 1, \cdots, N\},$$

where $b_1, \cdots, b_N \in \mathbb{R}^N$ is an N-tuple of independent vectors. The cell Y has the *paving property*, i.e., the collection $\{Y_\xi = Y + \xi : \xi \in \Xi\}$ of cells Y shifted by a vector ξ from a set of shifts

$$\Xi = \{\xi = k_1 b_1 + \cdots + k_N b_N : k_1, \cdots, k_N \in \mathbb{Z}\}$$

is a partition of $\mathbb{R}^N$: the shifted cells Y_ξ are disjoint and cover $\mathbb{R}^N$. For $z \in \mathbb{R}^N$, let $[z]_Y$ denotes the unique integer combination $\sum_{j=1}^N k_j b_j$ such that $z - [z]_Y$ belongs to Y, and set $\{z\}_Y = z - [z]_Y$. In the case when $Y = [0,\,1)^N$, we have $\Xi = \mathbb{Z}^N$ and the decomposition $z = [z]_Y + \{z\}_Y$ is the usual decomposition into the integer and fractional parts. Then, for $x \in \mathbb{R}^N$ and $\varepsilon > 0$, we have

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right).$$

We introduce the following notations

$$\Xi_\varepsilon = \{\xi \in \mathbb{Z}^N : \varepsilon(Y+\xi) \subset \Omega\}, \quad \widehat{\Omega}_\varepsilon = \text{Interior}\left\{\cup_{\xi\in\Xi_\varepsilon} \varepsilon(\overline{Y} + \xi)\right\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon. \tag{31}$$

With theses notations we can now define the unfolding operator.

Definition 4. *For u Lebesgue measurable on Ω, the unfolding operator is defined a follows,*

$$\mathcal{T}_\varepsilon u(x,y) = \begin{cases} u\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x,y) \in \widehat{\Omega}_\varepsilon \times Y, \\ u(x) & \text{a.e. for } (x,y) \in \Lambda_\varepsilon \times Y, \end{cases} \tag{32}$$

Remark 1. *The unfolding operator $\mathcal{T}_\varepsilon$ defined by (32) for each $\varepsilon > 0$ has the following properties:*

1. $\mathcal{T}_\varepsilon(f)$ *is Lebesgue measurable on* $\Omega \times Y$,

$$\mathcal{T}_\varepsilon(\alpha f + \beta g) = \alpha \mathcal{T}_\varepsilon(f) + \beta \mathcal{T}_\varepsilon(g),$$

$$\mathcal{T}_\varepsilon(fg) = \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(g), \qquad \alpha, \beta \in \mathbb{R},\ f, g \in L^1(\Omega);$$

2. integral conservation

$$\frac{1}{|Y|}\iint_{\Omega\times Y} (\mathcal{T}_\varepsilon u)(x,y)\,dxdy = \int_\Omega f(x)\,dx, \qquad f \in L^1(\Omega). \tag{33}$$

The primary definition of the unfolding operator has been revisited by Francu [73, 74]. The primary definition by Cioranescu, Damlamian and Griso [43, 44] requires that $\mathcal{T}_\varepsilon(u)(x,y) = 0$ almost everywhere for $(x,y) \in \Lambda_\varepsilon \times Y$, which prevents the *integral conservation* property (33) to holds, making some proofs in the unfolding method in homogenization needlessly technical. By means of the unfolding operator we can now formulate the dual definition of the two scale convergence.

Definition 5. *Let E be as above.*

1. *The sequence $(u_\varepsilon)_{\varepsilon\in E}$ is said to two-scale converge to u_0 in $L^p(\Omega)$, if the sequence $(\mathcal{T}_\varepsilon u_\varepsilon)_{\varepsilon\in E}$ converges to u_0 weakly in $L^p(\Omega\times Y)$.*

2. *The sequence $(u_\varepsilon)_{\varepsilon\in E}$ is said to strongly two-scale converge to u_0 in $L^p(\Omega)$, if the sequence $(\mathcal{T}_\varepsilon u_\varepsilon)_{\varepsilon\in E}$ converges to u_0 strongly in $L^p(\Omega\times Y)$.*

Indeed, it is an easy exercise to check that this definition agrees with Definition 3, a key observation is that

$$\mathcal{T}_\varepsilon\left[\varphi(\cdot,\frac{\cdot}{\varepsilon})\right](x,y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, y\right) \qquad \text{for} \quad \varphi \in L^p(\Omega; \mathcal{C}_{per}(Y)) \ \ (1 < p < \infty).$$

2.6 The Σ-convergence method

Introduced in 2003 by Nguetseng [119], the Σ-convergence method is the generalization of the two scale convergence method which can handle general deterministic (beyond the periodic setting) homogenization problems in the same way the two-scale convergence tackle periodic homogenization problems. The main ingredient in this extension is the Gelfand representation theory of bounded uniformly continuous functions. We first give a short introduction to the concept of algebra with mean value which was introduced in 1983 by Zhikov and Krivenko [165] to study the problem of averaging of partial differential equations with almost periodic coefficients. Nguetseng [118, 119, 120, 121] has formalized and developed this concept of algebra with mean value which turns out to be the right candidate for replacement of $\mathcal{C}_{per}(Y)$ when moving from periodic homogenization to general deterministic homogenization. However, Nguetseng sometimes utilizes the slightly different concept of H-algebra. It is worth mentioning that in their paper [165], Zhikov and Krivenko already addressed the question of embedding the theory of deterministic homogenization into the theory of stochastic homogenization. We quote them without further comments: "... if the coefficients $a_{\alpha\beta}(x)$ are realizations of a homogenous stochastic field, then they are not necessarily almost-periodic in the sense of Besicovitch and have only rather simple and general properties of the type of existence of mean values".

Definition 6. *A closed subalgebra A of the $\mathcal{C}$*-algebra of bounded uniformly continuous functions $\mathcal{B}_{uc}(\mathbb{R}^N)$ is an* algebra with mean value *on $\mathbb{R}^N$ if it satisfies the following properties:*

- *A contains the constants,*
- *A is translation invariant, $u(\cdot + a) \in A$ for any $u \in A$ and each $a \in \mathbb{R}^N$,*
- *for any $u \in A$, the sequence $(u_\varepsilon)_{\varepsilon>0}$ $(u_\varepsilon(x) = u(\frac{x}{\varepsilon}), x \in \mathbb{R}^N)$ weakly $*$-converges in $L^\infty(\mathbb{R}^N)$ to some constant real number $M(u)$ as $\varepsilon \to 0$ (the real number $M(u)$ is called the mean value of $u \in A$).*

Endowed with the sup norm topology, A is a commutative $\mathcal{C}$*-algebra with identity (this is a classical result of the theory of commutative Banach algebras). We therefore denote by $\Delta(A)$ its spectrum and by $\mathcal{G}$ the Gelfand transform on A. We recall that $\Delta(A)$ (which is a subset of the topological dual A' of A) is the set of all nonzero multiplicative linear functionals on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u\rangle$ ($s \in \Delta(A)$), where $\langle\ ,\ \rangle$ denotes the duality pairing between A' and A. Equipped with the relative weak$*$ topology on A', $\Delta(A)$ is a compact topological space, and the Gelfand transform $\mathcal{G}$ is an isometric $*$-isomorphism identifying the $\mathcal{C}$*-algebras A and $\mathcal{C}(\Delta(A))$. In addition, the mean value M defined on A is a nonnegative continuous linear functional, and there exists a Radon measure β of total mass 1 such that the following integral representation holds [119]:

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta \qquad (u \in A).$$

We define Besicovitch spaces and some Sobolev type spaces. Let A be an algebra with mean value and let $m \in \mathbb{N}$. We define the following subspaces of A,

$$A^m = \{\psi \in \mathcal{C}^m(\mathbb{R}^N) :\ D_y^\alpha \psi \in A,\ \forall\, \alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N \text{ with } |\alpha| \leq m\},$$

where $D_y^\alpha \psi = \partial^{|\alpha|}\psi / \partial y_1^{\alpha_1} \cdots \partial y_N^{\alpha_N}$. Endowed with the norm

$$|||u|||_m = \sup_{|\alpha|\leq m} \left\| D_y^\alpha \psi \right\|_\infty \qquad (u \in A^m),$$

the space A^m is a Banach space. We also define

$$A^\infty = \{\psi \in \mathcal{C}^\infty(\mathbb{R}^N) :\ D_y^\alpha \psi \in A\, \forall\, \alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N\},$$

a Fréchet space when endowed with the locally convex topology defined by the family of norms $|||\cdot|||_m$, $m \in \mathbb{N}$. We consider the Besicovitch space $B_A^2(\mathbb{R}^N)$ defined as the closure of A with respect to the (Besicovitch) seminorm

$$\|u\|_2 = \left(\limsup_{r \to +\infty} \frac{1}{|B_r|} \int_{B_r} |u(y)|^2 \, dy \right)^{\frac{1}{2}},$$

where B_r is the open ball of $\mathbb{R}^N$ of radius r centered at the origin. It is well-known that $B_A^2(\mathbb{R}^N)$ is a complete seminormed vector space which, in general, is not separated since $u \neq 0$ does not entail $\|u\|_2 \neq 0$. However, the following properties hold [119, 123, 144].

(1) The Gelfand transform $\mathcal{G} : A \to \mathcal{C}(\Delta(A))$ extends by continuity to a unique continuous linear mapping (still denoted by $\mathcal{G}$) of $B_A^2(\mathbb{R}^N)$ into $L^2(\Delta(A))$, which induces an isometric isomorphism $\mathcal{G}_1$ of $B_A^2(\mathbb{R}^N)/\mathcal{N} = \mathcal{B}_A^2(\mathbb{R}^N)$ onto $L^2(\Delta(A))$ (where $\mathcal{N} = \{u \in B_A^2(\mathbb{R}^N) : \mathcal{G}(u) = 0\}$). Moreover, if $u \in B_A^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ then $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}^N)}$.

(2) The mean value M defined on A, extends by continuity to a positive continuous linear form (still denoted by M) on $B_A^2(\mathbb{R}^N)$ satisfying $M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta$ ($u \in B_A^2(\mathbb{R}^N)$), and $M(\tau_a u) = M(u)$ for each $u \in B_A^2(\mathbb{R}^N)$ and all $a \in \mathbb{R}^N$, where $\tau_a u(y) = u(y+a)$ for almost all $y \in \mathbb{R}^N$. Furthermore, for $u \in B_A^2(\mathbb{R}^N)$ we have $\|u\|_2 = \left[M(|u|^2) \right]^{1/2}$, and for $u + \mathcal{N} \in \mathcal{B}_A^2(\mathbb{R}^N)$, we define its mean value (which we still denoted by M) as expected: $M(u + \mathcal{N}) = M(u)$.

Let us justify the following important equality:

$$\|u\|_2 = \left[M(|u|^2) \right]^{1/2} \quad \text{for } \; u \in B_A^2(\mathbb{R}^N).$$

Let r be a positive real number. Set $\varepsilon = 1/r$. Then, $\varepsilon \to 0$ as $r \to +\infty$. Since $u^\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}^N)$-weak $*$, we have $\int u^{1/r} \chi_{B_1} dx \to M(u) \, |B_1|$ as $r \to +\infty$, where B_1 denotes the unit open ball in $\mathbb{R}^N$ and χ_{B_1} the characteristic function of B_1. But $\int u^{1/r} \chi_{B_1} dx = \int_{B_1} u(rx) dx$, and a change of variable $y = rx$ yields

$$\frac{1}{|B_1|} \int_{B_1} u(rx) dx = \frac{1}{r^N \, |B_1|} \int_{B_r} u(y) dy = \frac{1}{|B_r|} \int_{B_r} u(y) dy,$$

hence our claim is justified. We define Sobolev type spaces associated to the algebra A. To do this, we first consider the N-parameter group of isometries $\{T(y) : y \in \mathbb{R}^N\}$ defined by

$$\begin{aligned} T(y) : \mathcal{B}_A^2(\mathbb{R}^N) &\to \mathcal{B}_A^2(\mathbb{R}^N) \\ u + \mathcal{N} &\mapsto T(y)(u + \mathcal{N}) = \tau_y u + \mathcal{N} \qquad (u \in B_A^2(\mathbb{R}^N)). \end{aligned}$$

Since the elements of A are uniformly continuous, $\{T(y) : y \in \mathbb{R}^N\}$ is a strongly continuous group in $\mathcal{L}(\mathcal{B}^2_A(\mathbb{R}^N), \mathcal{B}^2_A(\mathbb{R}^N))$ (the Banach space of continuous linear functionals of $\mathcal{B}^2_A(\mathbb{R}^N)$ into $\mathcal{B}^2_A(\mathbb{R}^N)$), viz:

$$T(y)(u+\mathcal{N}) \to u+\mathcal{N} \text{ in } \mathcal{B}^2_A(\mathbb{R}^N) \quad \text{as } |y| \to 0.$$

Likewise, the N-parameter group $\{\overline{T}(y) : y \in \mathbb{R}^N\}$ defined by

$$\begin{array}{rcl} \overline{T}(y) : L^2(\Delta(A)) & \to & L^2(\Delta(A)) \\ \mathcal{G}_1(u+\mathcal{N}) & \mapsto & \overline{T}(y)\mathcal{G}_1(u+\mathcal{N}) = \mathcal{G}_1(T(y)(u+\mathcal{N})) \qquad (u \in B^2_A(\mathbb{R}^N)), \end{array}$$

is also strongly continuous. The infinitesimal generator of $T(y)$ (resp. $\overline{T}(y)$) along the i-th coordinate direction, denoted by $\overline{\partial}/\partial y_i$ (resp. ∂_i), is defined as follows

$$\frac{\overline{\partial} u}{\partial y_i} = \lim_{t\to 0}\left(\frac{T(te_i)u - u}{t}\right) \text{ in } \mathcal{B}^2_A(\mathbb{R}^N)$$

$$\left(\text{resp.} \quad \partial_i v = \lim_{t\to 0}\left(\frac{\overline{T}(te_i)v - v}{t}\right) \text{ in } L^2(\Delta(A))\right),$$

where, for the sake of simplicity, the equivalence class $u+\mathcal{N}$ of an element $u \in B^2_A(\mathbb{R}^N)$ is still denoted by u, and $e_i = (\delta_{ij})_{1\le j\le N}$ (δ_{ij} is the Kronecker δ). Indeed, this is justified by

$$\begin{aligned} \frac{\overline{\partial}(u+\mathcal{N})}{\partial y_i} &= \lim_{t\to 0}\left(\frac{T(te_i)(u+\mathcal{N}) - (u+\mathcal{N})}{t}\right) = \lim_{t\to 0}\left(\frac{(\tau_{te_i}u+\mathcal{N}) - (u+\mathcal{N})}{t}\right) \\ &= \lim_{t\to 0} \varrho\left(\frac{\tau_{te_i}u - u}{t}\right), \end{aligned}$$

where ϱ denotes the canonical mapping of $B^2_A(\mathbb{R}^N)$ onto $\mathcal{B}^2_A(\mathbb{R}^N)$, that is, $\varrho(u) = u+\mathcal{N}$ for $u \in B^2_A(\mathbb{R}^N)$. The domain of $\overline{\partial}/\partial y_i$ (resp. ∂_i) in $\mathcal{B}^2_A(\mathbb{R}^N)$ (resp. $L^2(\Delta(A))$) is denoted by $\mathcal{D}_i$ (resp. $\mathcal{W}_i$). The following result is just a consequence of the semigroup theory; see e.g., [68, Chap. VIII, Section 1].

Proposition 2. *$\mathcal{D}_i$ (resp. $\mathcal{W}_i$) is a vector subspace of $\mathcal{B}^2_A(\mathbb{R}^N)$ (resp. $L^2(\Delta(A))$), $\overline{\partial}/\partial y_i : \mathcal{D}_i \to \mathcal{B}^2_A(\mathbb{R}^N)$ (resp. $\partial_i : \mathcal{W}_i \to L^2(\Delta(A))$) is a linear operator, $\mathcal{D}_i$ (resp. $\mathcal{W}_i$) is dense in $\mathcal{B}^2_A(\mathbb{R}^N)$ (resp. $L^2(\Delta(A))$), and the graph of $\overline{\partial}/\partial y_i$ (resp. ∂_i) is closed in $\mathcal{B}^2_A(\mathbb{R}^N) \times \mathcal{B}^2_A(\mathbb{R}^N)$ (resp. $L^2(\Delta(A)) \times L^2(\Delta(A))$).*

For $1 \le i \le N$ and $u \in A^1$, we have $\varrho(u) \in \mathcal{D}_i$ and

$$\frac{\overline{\partial}}{\partial y_i}(\varrho(u)) = \varrho\left(\frac{\partial u}{\partial y_i}\right), \tag{34}$$

which shows that $\overline{\partial}/\partial y_i \circ \varrho = \varrho \circ \partial/\partial y_i$. We can also define higher order derivatives by setting $\overline{D}^\alpha_y = \frac{\overline{\partial}^{\alpha_1}}{\partial y_1^{\alpha_1}} \circ \cdots \circ \frac{\overline{\partial}^{\alpha_N}}{\partial y_N^{\alpha_N}}$ (resp. $\partial^\alpha = \partial_1^{\alpha_1} \circ \cdots \circ \partial_N^{\alpha_N}$) for $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$

with $\frac{\overline{\partial}^{\alpha_i}}{\partial y_i^{\alpha_i}} = \frac{\overline{\partial}}{\partial y_i} \circ \cdots \circ \frac{\overline{\partial}}{\partial y_i}$, α_i-times. The spaces of smooth functions on A and on $\Delta(A)$ are respectively defined by

$$\mathcal{D}_A(\mathbb{R}^N) = \varrho(A^\infty) \text{ and } \mathcal{D}(\Delta(A)) = \mathcal{G}_1(\mathcal{D}_A(\mathbb{R}^N)).$$

Since $\mathcal{G}_1 \circ \varrho = \mathcal{G}$, it holds that $\mathcal{D}(\Delta(A)) = \mathcal{G}(A^\infty)$. Likewise, for $u \in A^\infty$ we have $\overline{D}_y^\alpha \varrho(u) = \varrho(D_y^\alpha u)$ for any $\alpha \in \mathbb{N}^N$ (this is the generalization of (34)), hence $\partial^\alpha \mathcal{G}(u) = \mathcal{G}(D_y^\alpha u)$. Due to the density of A^∞ in $B_A^2(\mathbb{R}^N)$ (A^∞ is dense in A since the elements of A are uniformly continuous and A is translation invariant), the space $\mathcal{D}_A(\mathbb{R}^N)$ is dense in $\mathcal{B}_A^2(\mathbb{R}^N)$. For any $u \in \mathcal{D}_i$ we have $\mathcal{G}_1(u) \in \mathcal{W}_i$ and

$$\mathcal{G}_1(\overline{\partial} u/\partial y_i) = \partial_i \mathcal{G}_1(u),$$

which, together with (34) show that $\mathcal{D}_A(\mathbb{R}^N) \subset \mathcal{D}_i$ (hence $\mathcal{D}(\Delta(A)) \subset \mathcal{W}_i$) for any $1 \leq i \leq N$. From now on, we write $\widehat{u}$ either for $\mathcal{G}_1(u)$ if $u \in \mathcal{B}_A^2(\mathbb{R}^N)$ or for $\mathcal{G}(u)$ if $u \in B_A^2(\mathbb{R}^N)$. The following properties hold [160, Proposition 4]:

(i) $\int_{\Delta(A)} \partial^\alpha \widehat{u} d\beta = 0 \quad$ for all $u \in \mathcal{D}_A(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}^N$,

(ii) $\int_{\Delta(A)} \partial_i \widehat{u} d\beta = 0 \quad$ for all $u \in \mathcal{D}_i$ and $1 \leq i \leq N$,

(iii) $\frac{\overline{\partial}}{\partial y_i}(u\phi) = u\frac{\overline{\partial}\phi}{\partial y_i} + \phi\frac{\overline{\partial}u}{\partial y_i}$ for all $(\phi, u) \in \mathcal{D}_A(\mathbb{R}^N) \times \mathcal{D}_i$ and $1 \leq i \leq N$.

Formulae (i) and (iii), above imply

$$\int_{\Delta(A)} \widehat{\phi} \partial_i \widehat{u} d\beta = -\int_{\Delta(A)} \widehat{u} \partial_i \widehat{\phi} d\beta \text{ for all } (u, \phi) \in \mathcal{D}_i \times \mathcal{D}_A(\mathbb{R}^N),$$

which is a hint for the adaptation of the concepts of distribution and weak derivative to our functional setting. We endow $\mathcal{D}_A(\mathbb{R}^N) = \varrho(A^\infty)$ with its natural topology defined by the family of norms $N_m(u) = \sup_{|\alpha| \leq m} \sup_{y \in \mathbb{R}^N} \left|\overline{D}_y^\alpha u(y)\right|$, $m \in \mathbb{N}$, which makes $\mathcal{D}_A(\mathbb{R}^N)$ a Fréchet space. The topological dual of $\mathcal{D}_A(\mathbb{R}^N)$ is denoted in the sequel by $\mathcal{D}'_A(\mathbb{R}^N)$ and is endowed with the strong dual topology. The elements of $\mathcal{D}'_A(\mathbb{R}^N)$ are called *the distributions on* A. For $f \in \mathcal{D}'_A(\mathbb{R}^N)$, we define the weak derivative $D^\alpha f$ ($\alpha \in \mathbb{N}^N$) of f as expected, viz:

$$\langle D^\alpha f, \phi \rangle = (-1)^{|\alpha|} \left\langle f, \overline{D}_y^\alpha \phi \right\rangle \text{ for all } \phi \in \mathcal{D}_A(\mathbb{R}^N).$$

Since $\mathcal{D}_A(\mathbb{R}^N)$ is dense in $\mathcal{B}_A^2(\mathbb{R}^N)$, it is immediate that $\mathcal{B}_A^2(\mathbb{R}^N) \subset \mathcal{D}'_A(\mathbb{R}^N)$ with continuous embedding so that we can define the weak derivative of $f \in \mathcal{B}_A^2(\mathbb{R}^N)$. It satisfies the following functional equation:

$$\langle D^\alpha f, \phi \rangle = (-1)^{|\alpha|} \int_{\Delta(A)} \widehat{f} \partial^\alpha \widehat{\phi} d\beta \quad \text{for all} \quad \phi \in \mathcal{D}_A(\mathbb{R}^N).$$

In the case where $f \in \mathcal{D}_i$, we have

$$-\int_{\Delta(A)} \widehat{f}\partial_i\widehat{\phi}d\beta = \int_{\Delta(A)} \widehat{\phi}\partial_i\widehat{f}d\beta \qquad \text{for all } \phi \in \mathcal{D}_A(\mathbb{R}^N),$$

and $\overline{\partial} f/\partial y_i$ is identified with $D^{\alpha_i} f$, $\alpha_i = (\delta_{ij})_{1\leq j\leq N}$. Conversely, if $f \in \mathcal{B}_A^2(\mathbb{R}^N)$ is such that there exists $f_i \in \mathcal{B}_A^2(\mathbb{R}^N)$ with $\langle D^{\alpha_i} f, \phi\rangle = -\int_{\Delta(A)} \widehat{f_i}\widehat{\phi}d\beta$ for all $\phi \in \mathcal{D}_A(\mathbb{R}^N)$, then $f \in \mathcal{D}_i$ and $\overline{\partial} f/\partial y_i = f_i$. This being done, we define the following Sobolev type space,

$$\mathcal{B}_A^{1,2}(\mathbb{R}^N) = \cap_{i=1}^N \mathcal{D}_i \equiv \left\{ u \in \mathcal{B}_A^2(\mathbb{R}^N) : \frac{\overline{\partial} u}{\partial y_i} \in \mathcal{B}_A^2(\mathbb{R}^N) \text{ for all } 1 \leq i \leq N \right\},$$

a Hilbert space under the norm

$$\|u\|_{\mathcal{B}_A^{1,2}(\mathbb{R}^N)} = \left(\|u\|_2^2 + \sum_{i=1}^N \left\| \frac{\overline{\partial} u}{\partial y_i} \right\|_2^2 \right)^{\frac{1}{2}} \quad (u \in \mathcal{B}_A^{1,2}(\mathbb{R}^N)).$$

Likewise, we define $W^{1,2}(\Delta(A)) = \cap_{i=1}^N \mathcal{W}_i$ as expected and endow it with its natural norm. The space $\mathcal{D}_A(\mathbb{R}^N)$ (resp. $\mathcal{D}(\Delta(A))$) is obviously a dense subspace of $\mathcal{B}_A^{1,2}(\mathbb{R}^N)$ (resp. $W^{1,2}(\Delta(A))$).

With the Sobolev spaces defined, we proceed and recall some facts about *ergodic* algebras. A function $f \in \mathcal{B}_A^2(\mathbb{R}^N)$ is said to be *invariant* if for any $y \in \mathbb{R}^N$, $T(y)f = f$.

Definition 7. *An algebra with mean value A is said to be* ergodic *if every invariant function $f \in \mathcal{B}_A^2(\mathbb{R}^N)$ is constant in $\mathcal{B}_A^2(\mathbb{R}^N)$.*

The set of invariant functions $f \in \mathcal{B}_A^2(\mathbb{R}^N)$ is denoted in the sequel by I_A^2, and is a closed vector subspace of $\mathcal{B}_A^2(\mathbb{R}^N)$ satisfying the following property (see e.g., [28]):

$$f \in I_A^2 \quad \text{if and only if} \quad \frac{\overline{\partial} f}{\partial y_i} = 0 \quad \text{for all} \quad 1 \leq i \leq N. \tag{35}$$

Let $u \in \mathcal{B}_A^2(\mathbb{R}^N)$ (resp. $v = (v_1, ..., v_N) \in (\mathcal{B}_A^2(\mathbb{R}^N))^N$). We define the gradient operator $\overline{\nabla}_y u$ and the divergence operator $\overline{\operatorname{div}}_y v$ by

$$\overline{\nabla}_y u = \left(\frac{\overline{\partial} u}{\partial y_1}, ..., \frac{\overline{\partial} u}{\partial y_N} \right) \quad \text{and} \quad \overline{\operatorname{div}}_y v = \sum_{i=1}^N \frac{\overline{\partial} v_i}{\partial y_i}.$$

The Laplace operator is defined by $\overline{\Delta}_y u = \overline{\operatorname{div}}_y(\overline{\nabla}_y u)$, and we have $\overline{\Delta}_y \varrho(u) = \varrho(\Delta_y u)$ for all $u \in A^\infty$, where Δ_y denotes the usual Laplace operator on $\mathbb{R}_y^N$. The following properties are satisfied.

- The divergence operator $\overline{\operatorname{div}}_y$ sends the space $(\mathcal{B}_A^2(\mathbb{R}^N))^N$ continuously and linearly into the space $(\mathcal{B}_A^{1,2}(\mathbb{R}^N))'$, and

$$\left\langle \overline{\operatorname{div}}_y u, v \right\rangle = -\left\langle u, \overline{\nabla}_y v \right\rangle \text{ for } v \in \mathcal{B}_A^{1,2}(\mathbb{R}^N) \text{ and } u = (u_i) \in (\mathcal{B}_A^2(\mathbb{R}^N))^N, \quad (36)$$

where

$$\left\langle u, \overline{\nabla}_y v \right\rangle = \sum_{i=1}^N \int_{\Delta(A)} \widehat{u}_i \partial_i \widehat{v} d\beta.$$

- Taking in (36) $u = \overline{\nabla}_y w$ with $w \in \mathcal{B}_A^{1,2}(\mathbb{R}^N)$, we obtain

$$\left\langle \overline{\Delta}_y w, v \right\rangle = \left\langle \overline{\operatorname{div}}_y (\overline{\nabla}_y w), v \right\rangle = -\left\langle \overline{\nabla}_y w, \overline{\nabla}_y v \right\rangle = \left\langle w, \overline{\Delta}_y v \right\rangle$$

for all $v, w \in \mathcal{B}_A^{1,2}(\mathbb{R}^N)$.

The gradient operator $\partial = (\partial_1, ..., \partial_N)$, the divergence operator $\widehat{\operatorname{div}}$, and the Laplace operator $\widehat{\Delta}$ on $L^2(\Delta(A))$ are defined similarly by replacing $\mathcal{B}_A^2$ by $L^2(\Delta(A))$, $\overline{\nabla}_y$ by $\partial = (\partial_1, ..., \partial_N)$, $\overline{\operatorname{div}}_y$ by $\widehat{\operatorname{div}}$ and $\overline{\Delta}_y$ by $\widehat{\Delta}$ in the above definitions. Moreover, similar properties to the above ones hold, and we have the following relations between the just defined operators (when they make sense):

$$\partial = \mathcal{G}_1 \circ \overline{\nabla}_y, \ \widehat{\operatorname{div}} = \mathcal{G}_1 \circ \overline{\operatorname{div}}_y \text{ and } \widehat{\Delta} = \mathcal{G}_1 \circ \overline{\Delta}_y.$$

Now, we recall the definition and main compactness results of the Σ-convergence method. We eventually give some examples of ergodic algebras with mean value.

Definition 8. *A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^2(\Omega)$ is said to :*

- *Σ-converge in $L^2(\Omega)$ to some $u_0 \in L^2(\Omega; \mathcal{B}_A^2)$, if as $E \ni \varepsilon \to 0$ we have*

$$\int_\Omega u_\varepsilon(x) f\left(x, \frac{x}{\varepsilon}\right) dx \to \iint_{\Omega \times \Delta(A)} \widehat{u}_0(x,s) \widehat{f}(x,s)\, dx d\beta \quad (37)$$

for every $f \in L^2(\Omega; A)$, and where $\widehat{f} = \mathcal{G}_1 \circ (\varrho \circ f) = \mathcal{G} \circ f$. We express this by writing $u_\varepsilon \to u_0$ in $L^2(\Omega)$-weak Σ;

- *strongly Σ-converge in $L^2(\Omega)$ to some $u_0 \in L^2(\Omega; \mathcal{B}_A^2)$, if it Σ-converges to u_0 in $L^2(\Omega)$ and*

$$\lim_{E \ni \varepsilon \to 0} \|u_\varepsilon\|_{L^2(\Omega)} = \|\widehat{u}_0\|_{L^2(\Omega \times \Delta(A))},$$

where $\widehat{u}_0 = \mathcal{G}_1 \circ u_0$. We express this by writing $u_\varepsilon \to u_0$ in $L^2(\Omega)$-strong Σ

Remark 2. *Due to the equality $\mathcal{G}_1(\mathcal{B}_A^2) = L^2(\Delta(A))$, the right-hand side of (37) is equal to*

$$\int_\Omega M(u_0(x,\cdot)f(x,\cdot))\,dx.$$

Therefore, as expected, $u_\varepsilon \to u_0$ in $L^2(\Omega)$-weak Σ implies $u_\varepsilon \to M(u_0(x,\cdot))$ in $L^2(\Omega)$-weak. In particular, when $A = \mathcal{C}_{per}(Y)$, we have (up to a group isomorphism) $\Delta(A) = \mathbb{T}^N$, the N-dimensional torus. Thus, (37) reduces to

$$\int_\Omega u_\varepsilon(x) f\left(x, \frac{x}{\varepsilon}\right) dx \to \iint_{\Omega\times Y} u_0(x,y) f(x,y)\,dxdy$$

where $u_0 \in L^2(\Omega \times Y)$, which is the definition of the two-scale convergence.

The following compactness results are of great interest in practice.

Theorem 7. *Any bounded sequence $(u_\varepsilon)_{\varepsilon\in E}$ in $L^2(\Omega)$ admits a subsequence which Σ-converges in $L^2(\Omega)$.*

Theorem 8. *Let Ω be an open subset in $\mathbb{R}^N$. Let A be an ergodic algebra with mean value on $\mathbb{R}^N$, and let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $H^1(\Omega)$. There exist a subsequence E' of E and a pair $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega; \mathcal{B}_A^{1,2}(\mathbb{R}^N))$ such that*

$$u_\varepsilon \to u_0 \quad \text{in } H^1(\Omega)\text{-weak}, \tag{38}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\overline{\partial} u_1}{\partial y_j} \quad \text{in } L^2(\Omega)\text{-weak } \Sigma \quad (1 \le j \le N), \tag{39}$$

as $E' \ni \varepsilon \to 0$.

We end this subsection with some examples of algebras with mean value.

The periodic algebra with mean value. Let $A = \mathcal{C}_{per}(Y)$ be the algebra of Y-periodic continuous functions on $\mathbb{R}^N$. It is classical that A is an ergodic algebra with meean value. The mean of a function $u \in \mathcal{C}_{per}(Y))$ is given by

$$M(u) = \int_Y u(y)\,dy.$$

The almost periodic algebra with mean value. Let $AP(\mathbb{R}^N)$ be the algebra of Bohr continuous almost periodic functions $\mathbb{R}^N$. We recall that a function $u \in \mathcal{B}(\mathbb{R}^N)$ is in $AP(\mathbb{R}^N)$ if the set of translates $\{\tau_a u : a \in \mathbb{R}^N\}$ is relatively compact in $\mathcal{B}(\mathbb{R}^N)$ (the $\mathcal{C}^*$-algebra of bounded continuous functions on $\mathbb{R}^N$). Equivalently [24], $u \in AP(\mathbb{R}^N)$ if and only if u may be uniformly approximated by finite linear combinations of functions in the set $\{\gamma_k : k \in \mathbb{R}^N\}$, where $\gamma_k(y) = \exp(2i\pi k \cdot y)$ ($y \in \mathbb{R}^N$). It is also a classical result that A is an ergodic algebra with mean value ([165]). The mean value of a function $u \in AP(\mathbb{R}^N)$ is given by

$$M(u) = \int_{\mathcal{K}} \mathcal{G}(u) d\beta,$$

where $\mathcal{K}$ is the spectrum of $AP(\mathbb{R}^N)$, which more precisely, is the Bohr compactification of $\mathbb{R}^N$ and β is the Haar measure on the compact topological abelian group $\mathcal{K}$.

Let $\mathcal{R}$ be any subgroup of $\mathbb{R}^N$. We denote by $AP_{\mathcal{R}}(\mathbb{R}^N)$ the space of those functions in $AP(\mathbb{R}^N)$ that can be uniformly approximated by finite linear combinations in the set $\{\gamma_k : k \in \mathcal{R}\}$. Then $A = AP_{\mathcal{R}}(\mathbb{R}^N)$ is an ergodic algebra with mean value [144].

The algebra with mean value of convergence at infinity. Let $\mathcal{B}_\infty(\mathbb{R}^N)$ denote the space of all continuous functions on $\mathbb{R}^N$ that converge finitely at infinity, that is the space of all $u \in \mathcal{B}(\mathbb{R}^N)$ such that $\lim_{|y|\to\infty} u(y) \in \mathbb{R}$. The space $\mathcal{B}_\infty(\mathbb{R}^N)$ is an ergodic algebra with mean value [70]. The mean value of a function $u \in \mathcal{B}_\infty(\mathbb{R}^N)$ is given by $M(u) = \lim_{|y|\to\infty} u(y)$.

The weakly almost periodic algebra with mean value. The concept of weakly almost periodic function is due to Eberlein [70]. A continuous function u on $\mathbb{R}^N$ is weakly almost periodic if the set of translates $\{\tau_a u : a \in \mathbb{R}^N\}$ is relatively weakly compact in $\mathcal{C}(\mathbb{R}^N)$. Endowed with the sup norm topology, $WAP(\mathbb{R}^N)$ is a Banach algebra with the usual multiplication. As examples of Eberlein's functions we have the Bohr continuous almost periodic functions, the continuous functions vanishing at infinity, the positive definite functions (hence Fourier-Stieltjes transforms). $WAP(\mathbb{R}^N)$ is a translation invariant $\mathcal{C}^*$-subalgebra of $\mathcal{C}(\mathbb{R}^N)$ whose elements are uniformly continuous, bounded and possess a mean value

$$M(u) = \lim_{R\to+\infty} \frac{1}{|B_R|} \int_{B_R} u(y+a) dy \qquad (u \in WAP(\mathbb{R}^N)),$$

the convergence being uniform in $a \in \mathbb{R}^N$. Moreover, every $u \in WAP(\mathbb{R}^N)$ admits the unique decomposition $u = v + w$, v being a Bohr almost periodic function and w a continuous function with zero quadratic mean value: $M(|w|^2) = 0$. Hence denoting by $W_0(\mathbb{R}^N)$ the complete vector subspace of $WAP(\mathbb{R}^N)$ consisting of functions with zero quadratic mean value, we have

$$WAP(\mathbb{R}^N) = AP(\mathbb{R}^N) \oplus W_0(\mathbb{R}^N).$$

With this in mind, let $\mathcal{R}$ be a subgroup of $\mathbb{R}^N$ and set

$$WAP_{\mathcal{R}}(\mathbb{R}^N) = AP_{\mathcal{R}}(\mathbb{R}^N) \oplus W_0(\mathbb{R}^N) \tag{40}$$

(bear in mind that $WAP_{\mathcal{R}}(\mathbb{R}^N) = WAP(\mathbb{R}^N)$ when $\mathcal{R} = \mathbb{R}^N$). Then $WAP_{\mathcal{R}}(\mathbb{R}^N)$ is an ergodic algebra with mean value [144].

Remark 3. • *In practice, it is possible to sum (perturbed) algebras with mean value [119] to get the algebra with mean value that exactly meets the structural hypothesis of the homogenization problem under consideration.*

• *A generalisation of Theorem 8 without the ergodicity hypothesis has been proved [66, Theorem 3.9]. In this case, the "macroscopic" limit function u_0 belongs to $H^1(\Omega; I_A^2)$ and* `depend` *on the "local" variable. As already pointed out by Zhikov and Krivenko [165], the problem of averaging in nonergodic algebra is tricky and still open.*

3 Overview of Papers

During the first part of my doctoral studies, I investigated the asymptotic behavior of linear elliptic eigenvalue problems in perforated domains by means of the two-scale convergence method and the Σ-convergence method. The results obtained are published in [59, 60, 61, 62, 64]. It should be noted that many nonlinear problems lead, after linearization, to elliptic eigenvalue problems (see e.g., the survey paper by de Figueiredo [72] and the work of Hess and Kato [78, 79]). A vast literature in engineering, physics and applied mathematics deals with such problems arising, for instance, in the study of transport theory, reaction-diffusion equations and fluid dynamics. Generally speaking, spectral asymptotics is a two folded research area. On the one hand, it deals with asymptotic formulas (estimates) and asymptotic distribution of the eigenvalues, see e.g. [25, 140, 141] and the references therein. On the other hand, it is concerned with homogenization of eigenvalues of operators with oscillating coefficients on possibly varying domains such as perforated ones. My study falls within the second framework: the homogenization theory. The asymptotics of eigenvalue is a very important problem and has been widely explored (see e.g, [11, 23, 38, 87, 89, 90, 110, 111, 112, 127, 130, 155] and the references therein). In a fixed domain, the homogenization of spectral problems with point-wise positive density function goes back to Kesavan [89, 90]. In perforated domains, the eigenvalue asymptotics was first considered by Rauch and Taylor [136, 137] but the first homogenization result in that direction pertains to Vanninathan [155]. Before presenting my results, it is worth pointing out that I did not only address new problems in spectral asymptotics but I also utilized the two-scale convergence method (which dramatically simplified my proofs) for the first time for such problems, albeit I missed results of convergence of rate type.

3.1 Homogenization of Steklov spectral problems with indefinite density function in perforated domains, *Acta Applicandae Mathematicae, 123 (2013) 261-284*.

In this paper, I studied the asymptotic behavior of second order self-adjoint elliptic Steklov eigenvalue problems with periodic rapidly oscillating coefficients and with indefinite (sign-changing) density function in periodically perforated domains. To be more precise, let Ω be a bounded domain in $\mathbb{R}^N$, with integer $N \geq 2$ and let $T \subset Y = (0,1)^N$ be a compact subset of Y in $\mathbb{R}^N$. We assume that Ω and T have $\mathcal{C}^1$ boundaries $\partial\Omega$ and ∂T, respectively. For $\varepsilon > 0$, we define

$$t^\varepsilon = \{k \in \mathbb{Z}^N : \varepsilon(k+T) \subset \Omega\}, \ \ T^\varepsilon = \bigcup_{k \in t^\varepsilon} \varepsilon(k+T) \ \text{ and } \ \Omega^\varepsilon = \Omega \setminus T^\varepsilon.$$

We denote $Y \setminus T$ by Y^* and $n = (n_i)_{i=1}^N$ stands for the outer unit normal vector to ∂T with respect to Y^*. The ε-problem reads

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j}\right) = 0 \quad \text{in} \quad \Omega^\varepsilon \\ \displaystyle\sum_{i,j=1}^{N} a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j} n_i(\frac{x}{\varepsilon}) = \rho(\frac{x}{\varepsilon})\lambda_\varepsilon u_\varepsilon \quad \text{on} \quad \partial T^\varepsilon \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{41}$$

where $a = (a_{ij}) \in L^\infty(\mathbb{R}^N)^{N\times N}$ is symmetric, elliptic and Y-periodic. The density function $\rho \in \mathcal{C}_{per}(Y)$ changes sign on ∂T, that is, both the sets $\{y \in \partial T, \rho(y) < 0\}$ and $\{y \in \partial T, \rho(y) > 0\}$ are of positive surface measure. I proved that the spectrum of problem (4) is discrete and consists of two infinite sequences, one tending to $-\infty$ and another to $+\infty$, viz,

$$-\infty \leftarrow \lambda_\varepsilon^{n,-} \leq \cdots \leq \lambda_\varepsilon^{2,-} \leq \lambda_\varepsilon^{1,-} < 0 < \lambda_\varepsilon^{1,+} \leq \lambda_\varepsilon^{2,+} \leq \cdots \leq \lambda_\varepsilon^{n,+} \rightarrow +\infty.$$

The limiting behavior of positive and negative eigencouples crucially depends on whether the average of the weight over the surface of the reference hole

$$M_{\partial T}(\rho) = \int_{\partial T} \rho(y)d\sigma(y),$$

is positive, negative or equal to zero. To handle the case $M_{\partial T}(\rho) = 0$, I proved the following convergence result [59, Lemma 2.9].

Lemma 1. *Let $(u_\varepsilon)_{\varepsilon\in E} \subset H^1(\Omega)$ be such that, as $E \ni \varepsilon \to 0$,*

$$\begin{aligned} u_\varepsilon \quad &\xrightarrow{2s} \quad u_0 \quad \text{in } L^2(\Omega), \\ \frac{\partial u_\varepsilon}{\partial x_j} \quad &\xrightarrow{2s} \quad \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \quad \text{in } L^2(\Omega) \;\; (1 \leq j \leq N), \end{aligned}$$

for some $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega; H^1_{per}(Y))$. Then

$$\lim_{\varepsilon\to 0} \int_{\partial T^\varepsilon} u_\varepsilon(x)\varphi(x)\theta(\frac{x}{\varepsilon})d\sigma_\varepsilon(x) = \iint_{\Omega\times\partial T} u_1(x,y)\varphi(x)\theta(y)dxd\sigma(y)$$

for all $\varphi \in \mathcal{D}(\Omega)$ and $\theta \in \mathcal{C}_{per}(Y)$ with $\int_{\partial T} \theta(y)d\sigma(y) = 0$.

The results obtained in this paper generalize previous results by Pastukhova [130] and myself [61] where the same problem was considered (with $\mathcal{C}^\infty$ and L^∞ coefficients, respectively) but with a constant positive density, hence dealing only with one positive sequence of eigenvalues tending to $+\infty$. In short,

i) if $M_S(\rho) > 0$, then the positive eigencouples behave like in the case of point-wise positive density function, i.e., for $k \geq 1$, $\lambda_\varepsilon^{k,+}$ is of order ε and $\frac{1}{\varepsilon}\lambda_\varepsilon^{k,+}$ converges as $\varepsilon \to 0$ to the k^{th} eigenvalue of the limit Dirichlet spectral problem, corresponding extended eigenfunctions converge along subsequences.

As for the "negative" eigencouples, $\lambda_\varepsilon^{k,-}$ converges to $-\infty$ at the rate $\frac{1}{\varepsilon}$ and the corresponding eigenfunctions oscillate rapidly. I utilized a factorization technique ([45, 155]) to prove that

$$\lambda_\varepsilon^{k,-} = \frac{1}{\varepsilon}\lambda_1^- + \xi_\varepsilon^{k,-} + o(1), \qquad k = 1, 2, 3, \dots$$

where $(\lambda_1^-, \theta_1^-)$ is the first negative eigencouple to the following local Steklov spectral problem

$$\begin{cases} -div(a(y)D_y\theta) = 0 \quad \text{in} \quad Y^* \\ a(y)D_y\theta \cdot n = \lambda\rho(y)\theta \quad \text{on} \quad \partial T \\ \theta \quad Y\text{-periodic}, \end{cases} \tag{42}$$

and $\{\xi_\varepsilon^{k,\pm}\}_{k=1}^{\infty}$ are eigenvalues of a Steklov eigenvalue problem similar to (41). With this trick, I proved that that $\{\frac{\lambda_\varepsilon^{k,-}}{\varepsilon} - \frac{\lambda_1^-}{\varepsilon^2}\}$ converges to the k^{th} eigenvalue of a limit Dirichlet spectral problem which is different from that obtained for positive eigenvalues. As for eigenfunctions, extensions of $\{\frac{u_\varepsilon^{k,-}}{(\theta_1^-)^\varepsilon}\}_{\varepsilon\in E}$ - where $(\theta_1^-)^\varepsilon(x) = \theta_1^-(\frac{x}{\varepsilon})$ - converge along subsequences to the k^{th} eigenfunctions of the limit problem.

ii) If $M_{\partial T}(\rho) = 0$, then the limit spectral problem is a quadratic operator pencil problem, and $\lambda_\varepsilon^{k,\pm}$ converges to the $(k,\pm)^{th}$ eigenvalue of the limit operator, extended eigenfunctions converge along subsequences as well. This case requires a new convergence result as regards the two-scale convergence theory, Lemma 1.

iii) The cases $M_{\partial T}(\rho) < 0$ and $M_{\partial T}(\rho) > 0$ are equivalent, just replace ρ with $-\rho$.

Remark 4. *Problems similar to the one study in this paper have been independently (and at the same time) investigated in [35, 40].*

In [35], Cao et al. studied the asymtotics of steklov eigenvalue for second order linear periodic operators with a zero lower order term, but, in a fixed domain, and with a constant density $\rho = 1$. The Steklov boundary condition is set on a fixed nonnegligeable part of the boundary $\partial\Omega$. Their work deals with one sequence of positive eigenvalue and has some numerical results.

In [40], Piatnitski et al. studied the problem considered in this paper, but for the Laplace operator ($a_{ij} = \delta_{ij}$). They combined the formal asymtotic expansion method with a justification procedure.

3.2 Two-scale convergence of elliptic spectral problems with indefinite density function in perforated domains, *Asymptotic Analysis, 81 (2013) 251-272*.

In this paper, I studied the spectral asymptotics of linear periodic elliptic operators with a sign-changing density function in perforated domains. More precisely, let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) and let $T \subset Y = (0,1)^N$ be a compact subset of Y in $\mathbb{R}^N$. We assume that Ω and T have $\mathcal{C}^1$ boundaries $\partial\Omega$ and ∂T, respectively. For $\varepsilon > 0$, we define

$$t^\varepsilon = \{k \in \mathbb{Z}^N : \varepsilon(k+T) \subset \Omega\}, \;\; T^\varepsilon = \bigcup_{k \in t^\varepsilon} \varepsilon(k+T) \quad \text{and} \quad \Omega^\varepsilon = \Omega \setminus T^\varepsilon.$$

We denote $Y \setminus T$ by Y^* and $n = (n_i)_{i=1}^N$ stands for the outer unit normal vector to ∂T with respect to Y^*. The ε-problem reads

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_i}\right) = \rho(\frac{x}{\varepsilon})\lambda_\varepsilon u_\varepsilon \quad \text{in} \quad \Omega^\varepsilon \\ \displaystyle\sum_{i,j=1}^{N} a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j} n_i(\frac{x}{\varepsilon}) = 0 \quad \text{on} \quad \partial T^\varepsilon \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{43}$$

where $a = (a_{ij}) \in L^\infty(\mathbb{R}^N)^{N\times N}$ is symmetric, elliptic and Y-periodic. The density function $\rho \in L^\infty(\mathbb{R}^N)$ is Y-periodic and changes sign on Y^*, that is, both the sets $\{y \in Y^*, \rho(y) < 0\}$ and $\{y \in Y^*, \rho(y) > 0\}$ are of positive Lebesgue measure. This hypothesis makes the problem nonstandard. The spectrum of (43) is discrete and consists of two infinite sequences

$$-\infty \leftarrow \lambda_\varepsilon^{n,-} \leq \cdots \leq \lambda_\varepsilon^{2,-} \leq \lambda_\varepsilon^{1,-} < 0 < \lambda_\varepsilon^{1,+} \leq \lambda_\varepsilon^{2,+} \leq \cdots \leq \lambda_\varepsilon^{n,+} \to +\infty.$$

The limiting behavior of positive and negative eigencouples crucially depends on whether the average of the weight over the solid part

$$M_{Y^*}(\rho) = \int_{Y^*} \rho(y)dy,$$

is positive, negative or equal to zero. I proved concise homogenization results in all three cases.

- In the case when $M_{Y^*}(\rho) = 0$, problem (43) generates in the limit (as $\varepsilon \to 0$) a quadratic operator pencil problem.

- In the case when $M_{Y^*}(\rho) > 0$ both parts of the spectrum (positive and negative) generate Dirichlet eigenvalue problems in the fixed domain Ω, though the negative part of the spectrum requires a non-obvious scaling trick involving the following local eigenvalue problem with sign-changing density function

$$\begin{cases} -div(a(y)D_y\theta) = \lambda\rho(y)\theta & \text{in } Y^* \\ a(y)D_y\theta \cdot n = 0 & \text{on } \partial T \\ \theta \quad Y\text{-periodic}. \end{cases} \tag{44}$$

More precisely, $\{\lambda_\varepsilon^{k,-} - \frac{1}{\varepsilon^2}\lambda_1^-\}$ converges to the k^{th} eigenvalue of the limit problem, a Dirichlet eigenvalue problem, where $(\lambda_1^-, \theta_1^-)$ is the first negative eigencouple to (44).

- The case $M_{Y^*}(\rho) < 0$ reduces to the case $M_{Y^*}(\rho) > 0$ by replacing ρ with $-\rho$.

My results in this paper simplify and generalize those in [112] where similar results where obtained in a fixed domain (no perforation !) via a combination of formal asymptotic expansion and a heavy justification procedure.

After working on spectral asymptotics for a while, I started broadening my interest in the homogenization theory of partial differential equations. This gave birth to [63, 65, 66].

3.3 Almost periodic homogenization of a generalized Ladyzhenskaya model for incompressible viscous flow, *Journal of Mathematical Sciences (N. Y.), 189 (2013) 431-458*. Translated from: *Problemy Matematicheskogo Analiza, 68 (2013) 89-112*.

The Navier-Stokes equations model the motion of Newtonian fluids. In order to understand the phenomenon of turbulence related to the motion of a fluid, several mathematical models have been developed and studied over the years. We refer to [18, 39, 69, 91], just to cite a few. In this paper, we proved an existence result for a nonstationary generalized Ladyzhenskaya equations with a given nonconstant density and an external force depending nonlinearly on the velocity, and then study the limiting behavior of the velocity field in an *almost periodic setting* by combining some compactness arguments with the Σ-convergence method. As for my contribution, I studied the whole problem under the supervision of my co-author who suggested the problem. Grigori Rozenblioum improved the presentation of the paper.

We consider N-dimensional problem, $N = 2, 3$ and assume that all the function spaces are real-valued spaces and scalar functions assume real values. Let $\varepsilon > 0$ be a small parameter representing the scale of the heterogeneities, and let $1 + \frac{2N}{N+2} \leq p < \infty$ and $T > 0$ be real numbers. Put $Q_T = Q \times (0, T)$ where Q is a bounded smooth domain in $\mathbb{R}^N$ and consider the following well-known spaces: $\mathcal{V} = \{\boldsymbol{\varphi} \in \mathcal{C}_0^\infty(Q)^N : \text{div}\, \boldsymbol{\varphi} = 0\}$; $\mathbb{V}$ = closure of $\mathcal{V}$ in $W^{1,p}(Q)^N$; $\mathbb{H}$ = closure of $\mathcal{V}$ in $L^2(Q)^N$. In view of the smoothness of Q, it is known that $\mathbb{V} = \{\boldsymbol{u} \in W_0^{1,p}(Q)^N : \text{div}\, \boldsymbol{u} = 0\}$ and $\mathbb{H} = \{\boldsymbol{u} \in L^2(Q)^N : \text{div}\, \boldsymbol{u} = 0$ and $\boldsymbol{u}|_{\partial Q} \cdot n = 0\}$, where $\boldsymbol{u}|_{\partial Q}$ denotes the trace of $\boldsymbol{u}$ on ∂Q and n is the outward unit vector normal to ∂Q. The space $\mathbb{V}$ is endowed with the gradient norm and $\mathbb{H}$ with the L^2-norm. With all this in mind, we introduce the functions a, b, ρ and f constrained as follows.

(A1) The matrix $a = (a_{ij})_{1\leq i,j\leq N} \in L^\infty(\mathbb{R}^{N+1})^{N\times N}$ satisfies $a_{ij} = a_{ji}$ and

$$(a(y,\tau)\lambda)\cdot\lambda \geq \nu_0 \left|\lambda\right|^2 \text{ for all } \lambda \in \mathbb{R}^N \text{ and a.e. } (y,\tau) \in \mathbb{R}^{N+1}$$

for some positive ν_0.

(A2) The function $b \in L^\infty(\mathbb{R}^{N+1})$ is such that $\nu_1 \leq b \leq \nu_2$ for some positive ν_1 and ν_2.

(A3) The density function ρ belongs to $\in L^\infty(\mathbb{R}^N)$ and satisfies $\Lambda^{-1} \leq \rho(y) \leq \Lambda$ for almost every $y \in \mathbb{R}^N$, for some positive Λ.

(A4) The mapping $f : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$, $(t,r) \mapsto f(t,r)$ is continuous with the following properties:

(i) f maps continuously $\mathbb{R}\times\mathbb{H}$ into $\mathbb{H}$ ($f(\tau,u) \in \mathbb{H}$ for any $u \in \mathbb{H}$)

(ii) There is a positive constant k such that

$$\begin{aligned}&\left|f(\tau,0)\right| \leq k \text{ for all } \tau \in \mathbb{R}\\&\left|f(\tau,r_1) - f(\tau,r_2)\right| \leq k\left|r_1 - r_2\right| \text{ for all } r_1, r_2 \in \mathbb{R}^N \text{ and } \tau \in \mathbb{R}.\end{aligned}$$

(A5) **Almost periodicity**. We assume that the functions a_{ij} and b lie in $B^2_{AP}(\mathbb{R}^{N+1}_{y,\tau}) \cap L^\infty(\mathbb{R}^{N+1}_{y,\tau})$ for all $1 \leq i,j \leq N$. We also assume that the function ρ belongs to $B^2_{AP}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $M_y(\rho) > 0$. Finally the function $\tau \mapsto f(\tau,r)$ lies in $(AP(\mathbb{R}_\tau))^N$ for any $r \in \mathbb{R}^N$.

Given $u^0 \in \mathbb{H}$, we are interested in the asymptotic behavior of the sequence of velocity field $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ of the following generalized Ladyzhenskaya equation

$$\begin{cases}\rho^\varepsilon \dfrac{\partial \boldsymbol{u}_\varepsilon}{\partial t} - \operatorname{div}\left(a^\varepsilon \nabla \boldsymbol{u}_\varepsilon + b^\varepsilon \left|\nabla \boldsymbol{u}_\varepsilon\right|^{p-2} \nabla \boldsymbol{u}_\varepsilon\right) + (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}_\varepsilon + \nabla q_\varepsilon = \rho^\varepsilon f^\varepsilon(\cdot,\boldsymbol{u}_\varepsilon) \text{ in } Q_T\\ \operatorname{div}\boldsymbol{u}_\varepsilon = 0 \text{ in } Q_T\\ \boldsymbol{u}_\varepsilon = 0 \text{ on } \partial Q\times(0,T)\\ \boldsymbol{u}_\varepsilon(x,0) = \boldsymbol{u}^0(x) \text{ in } Q,\end{cases} \tag{45}$$

where the scaled functions $\rho^\varepsilon \in L^\infty(Q)$, $a^\varepsilon = (a^\varepsilon_{ij})_{1\leq i,j\leq N} \in L^\infty(Q_T)^{N\times N}$, $f^\varepsilon(\cdot,r) \in \mathcal{C}(0,T)$ (for any $r \in \mathbb{R}^N$) and $b^\varepsilon \in L^\infty(Q_T)$ are defined as follows:

$$\begin{aligned}\rho^\varepsilon(x) &= \rho\left(\frac{x}{\varepsilon}\right),\ a^\varepsilon_{ij}(x,t) = a_{ij}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right),\ f^\varepsilon(\cdot,r)(t) = f\left(\frac{t}{\varepsilon^2},r\right) \text{ and}\\ b^\varepsilon(x,t) &= b\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) \text{ for } (x,t) \in Q_T.\end{aligned}$$

The equation (45) models various types of motion of non-Newtonian fluids. We cite a few examples. If $a = (\nu_0 \delta_{ij})_{1\leq i,j\leq N}$ (δ_{ij} is the Kronecker delta) and $b = \nu_1$ where ν_0 and ν_1 are positive constants, then we get the Ladyzhenskaya equations. In that case, the analysis conducted in [69] reveals that one may either let $\nu_1 \to 0$ (and get the usual Navier-Stokes equations) or let $\nu_0 \to 0$, and get the power-law fluids equations. In particular when $p = 3$, we get the Smagorinski's model of turbulence [145] with ν_0 being the molecular viscosity and ν_1 the turbulent viscosity. Another model included in (45) is the equation of incompressible bipolar fluids [18]. It should be noted that if in equation (45) we replace the gradient $\nabla \boldsymbol{u}$ by its symmetric part

$$\frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}),$$

then thanks to the Korn's inequality, the mathematical analysis does not change, although the model becomes in this case, physical. The following existence theorem is the first result of this paper.

Theorem 9. *Let $1 + \frac{2N}{N+2} \leq p < \infty$. Suppose $\boldsymbol{u}^0 \in \mathbb{H}$. Under assumptions **(A1)-(A4)**, there exists (for each fixed $\varepsilon > 0$) a couple $(\boldsymbol{u}_\varepsilon, q_\varepsilon) \in L^p(0,T;\mathbb{V}) \cap L^\infty(0,T;\mathbb{H}) \times W^{-1,\infty}(0,T;L^{p'}(Q))$ solution to* (45)*. The function $\boldsymbol{u}_\varepsilon$ also belongs to $\mathcal{C}([0,T];\mathbb{H})$ and q_ε is unique up to a constant function of x:*

$$\int_Q q_\varepsilon dx = 0.$$

The second result is the homogenization result.

Theorem 10. *Assume that **(A1)-(A5)** hold. Let $1 + \frac{2N}{N+2} \leq p < \infty$. For each $\varepsilon > 0$ let $(\boldsymbol{u}_\varepsilon, q_\varepsilon)$ be a solution of* (45)*. Then there exists a subsequence of $(\boldsymbol{u}_\varepsilon, q_\varepsilon)_{\varepsilon>0}$ which converges strongly in $L^2(Q_T)^N$ (with respect to the first component $\boldsymbol{u}_\varepsilon$) and weakly in $L^{p'}(Q_T)$ (with respect to q_ε) to the solution to*

$$\begin{cases} \mathfrak{M}_y(\rho)\frac{\partial \boldsymbol{u}_0}{\partial t} - \operatorname{div}\left(\mathsf{m}D\boldsymbol{u}_0) + M(D\boldsymbol{u}_0)\right) + B(\boldsymbol{u}_0) + \nabla q_0 = \mathfrak{M}(\rho f(\cdot, \boldsymbol{u}_0)) \\ \operatorname{div} \boldsymbol{u}_0 = 0 \text{ in } Q_T \\ \boldsymbol{u}_0 = 0 \text{ on } \partial Q \times (0,T) \\ \boldsymbol{u}_0(x,0) = \boldsymbol{u}^0(x) \text{ in } Q, \end{cases} \tag{46}$$

where we refer to the appended version of the paper for the definition of the limit operators. Moreover, any limit point in $L^2(Q_T)^N \times L^{p'}(Q_T)$ (in the above sense) of $(\boldsymbol{u}_\varepsilon, q_\varepsilon)_{\varepsilon>0}$ is a solution to (46)*.*

3.4 Incompressible viscous Newtonian flow in a fissured medium of general deterministic type, *Journal of Mathematical Sciences (N. Y.), 191 (2013) 214-242*. Translated from: *Problemy Matematicheskogo Analiza, 70 (2013) 105-128*.

Jointly with Gabriel Nguetseng and Jean Louis Woukeng, I generalized the usual approach for upscaling flow in periodic porous media and proposed a framework which can handle general deterministically fissured media. However, I should stress that my contribution in this paper does not include the new developments in the theory of Σ-convergence herein, but is the a priori estimates, the compactness results and the homogenization procedure. Grigori Rozenblioum improved the presentation of the paper.

A fissured porous medium is a structure made up of blocks of an usual porous medium with fissures between them. The blocks form the *porous matrix* while the fissures are characterized by substantially higher flow rates and lower relative volume. In order to study the flow of a fluid in a fractured porous medium, several mathematical models have been suggested and upscaled by homogenization. The classical and most studied *double diffusion model* for fissured porous rock domain was introduced by Barenblatt, Zheltov and Kochina in 1960 [17], and further developed by Warren and Root in 1963 [158], Coats and Smith in 1964 [50], and some other researchers, including [5, 85, 88, 102, 101, 125, 132, 133, 148, 161].

All the previous models are characterized by the common fact that they have been investigated under the classical *equidistribution* (periodicity) assumption on the geometry of the medium under consideration. In this paper we propose an approach to handle the general case where the medium is fissured in a deterministic manner including the special case of equidistribution commonly known in the literature as the periodic fissured medium. We then apply the approach to a double diffusion type problem.

To be precise, let us describe the geometry of the problem. Let $N = 2$ or 3 and let $Y = (0,1)^N$ be the reference cell. Let Y_1 and Y_2 be two open disjoint subsets of Y representing the local structure of the porous matrix and the local structure of fissures, respectively. We assume that $\overline{Y}_1 \subset Y$, $Y = \overline{Y}_1 \cup Y_2$, Y_2 is connected and that the boundary ∂Y_1 of Y_1 is Lipschitz continuous. Moreover, we assume that Y_1 and Y_2 have positive Lebesgue measure. Let $S \subset \mathbb{Z}^N$ be an infinite subset of $\mathbb{Z}^N$ and put

$$G_j = \cup_{k\in S}(k + Y_j) \quad (j = 1, 2).$$

Then G_j is an open subset of $\mathbb{R}^N$ and moreover, G_2 is connected. Since the cells $(k+Y)_{k\in S}$ are pairwise disjoint, the characteristic function χ_j of the set G_j in $\mathbb{R}^N$ satisfies

$$\chi_j = \sum_{k\in S} \chi_{k+Y_j},$$

or more precisely

$$\chi_j = \sum_{k\in\mathbb{Z}^N} \theta(k)\chi_{k+Y_j},$$

where θ is the characteristic function of S in $\mathbb{Z}^N$. The function θ is the *distribution function of the fissured cells*. The special case $\theta(k) = 1$ for all $k \in \mathbb{Z}^N$ (that is when $S = \mathbb{Z}^N$) leads to the equidistribution of fissured cells, commonly known as the *periodic fissured medium*. The function θ may also be assumed to be periodic, that is, there exists a discrete subgroup of $\mathbb{R}^N$ of rank N, $\mathcal{R}$ (such an $\mathcal{R}$ is called a network in $\mathbb{R}^N$), with $\mathcal{R} \subset \mathbb{Z}^N$ and $\theta(k + r) = \theta(k)$ for all $k \in \mathbb{Z}^N$ and all $r \in \mathcal{R}$. We get in that case a periodic distribution of fissured cells. One can consider the case when θ is almost periodic (which corresponds to the almost periodic distribution of fissured cells). Some other assumptions may also be considered.

Bearing this in mind, let Ω be an open bounded Lipschitz domain in $\mathbb{R}^N$ and let $\varepsilon > 0$ be a small parameter. We define our ε-fissured medium as follows: $\Omega_j^\varepsilon = \Omega \cap \varepsilon G_j$ ($j = 1, 2$). The set Ω_1^ε (resp. Ω_2^ε) is the portion of Ω occupied by the matrix (resp. fissures) and we have $\Omega = \Omega_1^\varepsilon \cup \Gamma_{12}^\varepsilon \cup \Omega_2^\varepsilon$ (disjoint union) where $\Gamma_{12}^\varepsilon = \partial\Omega_1^\varepsilon \cap \partial\Omega_2^\varepsilon \cap \Omega$ is the part of the interface of Ω_1^ε with Ω_2^ε lying in Ω. Let ν_j denote the outward unit normal vector to $\partial\Omega_j^\varepsilon$. It is clear that $\nu_1 = -\nu_2$ on Γ_{12}^ε. With the geometry of the medium described, let

$$\mathcal{Q} = (a_{ij})_{1\leq i,j\leq N} \quad \text{and} \quad B = (b_{ij})_{1\leq i,j\leq N}$$

be two real symmetric matrices with entries in $L^\infty(\mathbb{R}^N)$, satisfying

$$\mathcal{Q}(y)\xi \cdot \xi \geq \alpha \left|\xi\right|^2, \quad B(y)\xi \cdot \xi \geq \alpha \left|\xi\right|^2 \tag{47}$$

for almost every $y \in \mathbb{R}^N$, and for all $\xi \in \mathbb{R}^N$, where α is a given positive constant independent of y and ξ. For $j = 1, 2$, let $\rho_j : \mathbb{R}^N \to \mathbb{R}$ be a bounded continuous function satisfying

$$\Lambda^{-1} \leq \rho_j(y) \leq \Lambda \text{ for all } y \in \mathbb{R}^N, \tag{48}$$

where Λ is a positive constant independent of y. It is well known that the function $x \mapsto \mathcal{Q}(x/\varepsilon)$ (resp. $x \mapsto B(x/\varepsilon)$, $x \mapsto \rho_j(x/\varepsilon)$) defined from Ω into $\mathbb{R}^{N\times N}$ (resp. $\mathbb{R}^{N\times N}$, $\mathbb{R}$) is well-defined as an element of $L^\infty(\Omega)^{N^2}$ (resp. $L^\infty(\Omega)^{N^2}$, $\mathcal{C}(\Omega)$) denoted by $\mathcal{Q}^\varepsilon$ (resp. B^ε, ρ_j^ε). Next, let $T > 0$ be freely fixed and put $Q = \Omega \times (0, T)$. Assume that $f, g \in L^\infty(0, T; L^2(\Omega)^N)$ and $\boldsymbol{u}^0, \boldsymbol{v}^0 \in L^2(\Omega)^N$ with $\operatorname{div} \boldsymbol{u}^0 = \operatorname{div} \boldsymbol{v}^0 = 0$ in Ω.

A nonstationary flow of an incompressible viscous Newtonian fluid is governed by the Navier-Stokes system. We study the evolution state of the medium Ω which is saturated with viscous Newtonian fluids satisfying a slip condition on the rough part of the boundary as well as on the fluid-solid interface. The Navier-Stokes equations in Ω_j^ε ($j = 1, 2$) and the continuity equations of the normal stress and velocity at the solid-fluid interface Γ_{12}^ε are:

$$\rho_1^\varepsilon \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} - \operatorname{div}(\mathcal{Q}^\varepsilon \nabla \boldsymbol{u}_\varepsilon) + (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon = \rho_1^\varepsilon f \quad \text{in} \quad \Omega_1^\varepsilon \times (0, T) \tag{49}$$

$$\operatorname{div} \boldsymbol{u}_\varepsilon = 0 \quad \text{in} \quad \Omega_1^\varepsilon \times (0, T) \tag{50}$$

$$\rho_2^\varepsilon \frac{\partial \boldsymbol{v}_\varepsilon}{\partial t} - \operatorname{div}(B^\varepsilon \nabla \boldsymbol{v}_\varepsilon) + (\boldsymbol{v}_\varepsilon \cdot \nabla)\boldsymbol{v}_\varepsilon + \nabla q_\varepsilon = \rho_2^\varepsilon g \quad \text{in} \quad \Omega_2^\varepsilon \times (0, T) \tag{51}$$

$$\operatorname{div} \boldsymbol{v}_\varepsilon = 0 \quad \text{in} \quad \Omega_2^\varepsilon \times (0, T) \tag{52}$$

$$\boldsymbol{u}_\varepsilon = \boldsymbol{v}_\varepsilon \quad \text{on} \quad \Gamma_{12}^\varepsilon \times (0, T) \tag{53}$$

$$(\mathcal{Q}^\varepsilon \nabla \boldsymbol{u}_\varepsilon - p_\varepsilon I) \cdot \nu_1 = (B^\varepsilon \nabla \boldsymbol{v}_\varepsilon - q_\varepsilon I) \cdot \nu_1 \quad \text{on} \quad \Gamma_{12}^\varepsilon \times (0,T) \tag{54}$$

$$(\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) \cdot \nu_1 = (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) \cdot \nu_1 \quad \text{on} \quad \Gamma_{12}^\varepsilon \times (0,T) \tag{55}$$

$$\boldsymbol{u}_\varepsilon = 0 \quad \text{on} \quad (\partial\Omega_1^\varepsilon \cap \partial\Omega) \times (0,T), \quad \boldsymbol{v}_\varepsilon = 0 \quad \text{on} \quad (\partial\Omega_2^\varepsilon \cap \partial\Omega) \times (0,T) \tag{56}$$

$$\boldsymbol{u}_\varepsilon(x,0) = \boldsymbol{u}^0(x) \quad \text{for} \quad x \in \Omega_1^\varepsilon, \quad \boldsymbol{v}_\varepsilon(x,0) = \boldsymbol{v}^0(x) \quad \text{for} \quad x \in \Omega_2^\varepsilon \tag{57}$$

where on the matrix (or solid part) the fluid has the stress tensor $\sigma_1^\varepsilon = \mathcal{Q}^\varepsilon \nabla \boldsymbol{u}_\varepsilon - p_\varepsilon I$, density ρ_1^ε, velocity $\boldsymbol{u}_\varepsilon$, pressure p_ε, and the fluid part Ω_2^ε has the stress tensor $\sigma_2^\varepsilon = B^\varepsilon \nabla \boldsymbol{v}_\varepsilon - q_\varepsilon I$, velocity $\boldsymbol{v}_\varepsilon$ and pressure q_ε; I is the identity tensor, $\rho_1^\varepsilon f$ (or $\rho_2^\varepsilon g$) is the external body force per volume. The above system consists of a nonstationary incompressible Navier-Stokes equations in Ω_1^ε coupled across the interface Γ_{12}^ε to another nonstationary incompressible Navier-Stokes system in Ω_2^ε. The structure of our fissured medium produces very high frequency spatial variations of pressures in the matrix Ω_1^ε, hence leads to corresponding variations of velocity fields. The positive definite symmetric elasticity tensors $\mathcal{Q} = (a_{ij})_{1\leq i,j\leq N}$ and $B = (b_{ij})_{1\leq i,j\leq N}$ provide a model for general anisotropic materials. Conditions (53)-(55) are the interface conditions stating the continuity of the fluid flow at the interface Γ_{12}^ε while condition (56) is the homogeneous Dirichlet boundary conditions on the outer boundary, and (57) is the initial condition.

The limit passage $\varepsilon \to 0$ in the above system is done under suitable hypotheses on the elasticity tensors and on the density of the fluids when the medium is subjected to a general deterministic hypothesis. Each of these hypotheses covers a great set of patterns of deterministic behavior such as the periodicity, the almost periodicity, the convergence at infinity, the weak almost periodicity, and many more. It should be noted that such a mathematical analysis has never been done previously for our model, even in the so-called periodic setting. By means of the Σ-convergence method we prove the following theorem.

Theorem 11. *Let A be an ergodic algebra with mean value and assume that*

$$\chi_j \in B_A^2(\mathbb{R}^N) \quad \textit{with} \quad M(\chi_j) > 0 \ , j = 1,2,$$

$$\mathcal{Q}, B \in (B_A^2(\mathbb{R}^N))^{N^2} \quad \textit{and} \quad \rho_j \in A \quad \textit{for all} \quad j = 1,2.$$

For each $\varepsilon > 0$ let $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ and $(\boldsymbol{v}_\varepsilon, q_\varepsilon)$ be the sequence of solutions to (49)-(57). *Let $\boldsymbol{u}^\varepsilon$ and p^ε denote the global velocity and global pressure, respectively defined by*

$$\boldsymbol{u}^\varepsilon = \chi_1^\varepsilon \boldsymbol{u}_\varepsilon + \chi_2^\varepsilon \boldsymbol{v}_\varepsilon \quad \textit{and} \quad p^\varepsilon = \chi_1^\varepsilon p_\varepsilon + \chi_2^\varepsilon q_\varepsilon.$$

There exists a subsequence of $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ still denoted here by $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ such that, as $\varepsilon \to 0$, we have

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u}_0 \quad \textit{in} \quad L^2(Q)^N\textit{-strong}$$

$$p^{\varepsilon} \to p \quad \textit{in} \quad L^2(Q)\textit{-weak}$$

where $(\boldsymbol{u}_0, p)$ *is a solution to the following Navier-Stokes system*

$$\begin{cases} \rho \dfrac{\partial \boldsymbol{u}_0}{\partial t} - \mathit{div}\,(\mathcal{C}\nabla \boldsymbol{u}_0) + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 + \nabla p = \boldsymbol{F} \text{ in } Q \\ \mathit{div}\,\boldsymbol{u}_0 = 0 \text{ in } Q \\ \boldsymbol{u}_0 = 0 \text{ on } \partial\Omega \times (0,T) \\ \boldsymbol{u}_0(x,0) = \chi_1 \boldsymbol{u}^0(x) + \chi_2 \boldsymbol{v}^0(x),\ x \in \Omega. \end{cases}$$

Moreover, any limit point of $(\boldsymbol{u}^{\varepsilon}, p^{\varepsilon})$ *in* $L^2(Q)^N \times L^2(Q)$ *(in the above sense) is a solution to the above system. The matrix* $\mathcal{C} = (c_{ij})_{1\leq i,j\leq N}$ *is the effective homogenized elasticity tensor, and has constant entries.*

References

[1] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., **23** (1992) 1482-1518.

[2] G. Allaire and M. Briane, Multi-scale convergence and reiterated homogenization, Proc. Roy. Soc. Edinb., **126 A** (1996) 297-342.

[3] M. Amar, Two-scale convergence and homogenization on $BV(\Omega)$, Asymptot. Anal., **65** (1998) 65-84.

[4] A.Y. Amenadze, Elasticity theory, Vysshaia Shkola, Moscow (1976).

[5] T. Arbogast, J. Douglas and U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, SIAM J. Math. Anal., **21** (1990) 823-836.

[6] H. Attouch, Variational convergence for functions and operators, Pitman, Boston, 1984.

[7] I. Babuska, Homogenization and its application. Mathematical and computational problems, Numerical solution of partial differential equations III, Academic Press (1976) 89-116.

[8] I. Babuska, Homogenization approach in engineering, Lecture notes in economics and mathematical systems, M Beckman and H.P. Kunzi (eds.), Springer-Verglag (1975) 137-153.

[9] I. Babuska, Solution of problem with interfaces and singularities, in Mathematical aspects of finite elements in partial differential equations, C. de Boor ed., Academic Press, New York (1974) 213-277.

[10] I. Babuska, The computational aspects of the homogenization problem, Computing methods in applied sciences and engineering I, Lecture notes in mathematics, Springer-Verlag, Berlin, Heidelberg, New York (1977) 309-316.

[11] L. Baffico, C. Conca and M. Rajesh, Homogenization of a class of nonlinear eigenvalue problems. Proc. Roy. Soc. Edinb., **136A** (2006) 7-22.

[12] N.S. Bakhvalov, Averaged characteristics of bodies with periodic structure, (Russian) Dokl. Akad. Nauk SSSR, **218** (1974) 1046-1048.

[13] N.S. Bakhvalov, Averaging of nonlinear partial differential equations with rapidly oscillating coefficients, (Russian) Dokl. Akad. Nauk SSSR **225** (1975) 249-252.

[14] N.S. Bakhvalov, Averaging of partial differential equations with rapidly oscillating coefficients, (Russian) Dokl. Akad. Nauk SSSR, **221** (1975) 516-519.

[15] N.S. Bakhvalov and G.P. Panasenko, Homogenization: Averaged Processes in Periodic Media, Kluwer, Dordrecht, 1989.

[16] J. Ball, A version of the fundamental theorem for Young measures, in Pde's and continuum models of phase transitions , M. Rascle, D. Serre, and M. Slemrod, eds., Lecture Notes in Phys., **344**, Springer-Verlag, New York, 1989.

[17] G.I. Barenblatt, I.P. Zheltov and I.N. Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks (strata), J. Appl. Math. Mech., **24** (1960) 1286-1303.

[18] H. Bellout, F. Bloom and J. Nečas, Young measure-valued solutions for non-Newtonian incompressible fluids, Commun. Partial Diff. Equ., **19** (1994) 1763-1803.

[19] J.D. Benamou and Y. Brenier, A computational fluid mechanics solution to the monge-kantorovich mass transfer problem. Numerische Mathematik, **84** (2000) 375- 393.

[20] A. Bensoussan, J.L. Lions and G. Papanicolaou, Sur quelques phénomènes asymptotiques stationaires, C. R. Acad. Sci. Paris **281** (1975) 89-94.

[21] A. Bensoussan, J.L. Lions and G. Papanicolaou, Asymptotic analysis for periodic structures, North-Holland, Amsterdam, 1978.

[22] A.L. Berdichevskii, Spatial averaging of periodic structures, DAN SSSR, **222** (1975) 565-567.

[23] H. Berestycki, F. Hamel and N. Nadirashvili, Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena. Commun. Math. Phys., **253** (2005) 451-480.

[24] A.S. Besicovitch, Almost periodic functions, Dover Publications, Cambridge, 1955.

[25] M.S. Birman and M. Solomyak, Estimates for the number of negative eigenvalues of the Schrdinger operator and its generalizations, Adv. Soviet Math. **7** (1991) 1-55.

[26] N.N. Bogoliubov and Y. Mitropolsky, Asymptotic methods in nonlinear mechanics, Gordon and Breach, New York, 1961.

[27] A. Bourgeat, S. Luckhaus and A. Mikelic, Convergence of the homogenization process for a double-porosity model of immiscible two-phase flow, SIAM. J. Math. Anal., **27** (1996) 15201543.

[28] A. Bourgeat, A. Mikelić and S. Wright, Stochastic two-scale convergence in the mean and applications, J. Reine Angew. Math., **456** (1994) 19-51.

[29] A. Braides, Γ-convergence for Beginners, Oxford University Press, 2002.

[30] A. Braides and A. Defranceschi, Homogenization of multiple integrals, Oxford Lecture Series in Mathematics and its applications **12**, Oxford University Press, 1998.

[31] M. Briane, A. Damlamian and P. Donato, H-convergence for perforated domains. Nonlinear partial differential equations and their applications, Collge de France Seminar, Vol. XIII (Paris, 1994/1996), 62100, Pitman Res. Notes Math. Ser., **391**, Longman, Harlow, 1998.

[32] D.A.G. Bruggeman, Berechnung verschiedener physikalisher konstanten von heterogenen substanzen, Ann. Physik., **24** (1935) 636-664.

[33] G. Buttazzo. Semicontinuity, relaxation and integral representation in the calculus of variations, Pitman, London, 1989.

[34] R. Bunoiu and J. Saint Jean Paulin, Linear flow in porous media with double periodicity, Portugaliae Mathematica, **56** (1999).

[35] L. Cao, L. Zhang, W. Allegretto and Y. Lin, Multiscale asymptotic method for Steklov eigenvalue equations in composite media, SIAM J. Numer. Anal., **51** (2013) 273-296.

[36] J. Casado-Díaz, Two-scale convergence for nonlinear Dirichlet problems in perforated domains, Proc. Roy. Soc. Edinb., **130A** (2000) 249-276.

[37] J. Casado-Díaz and I. Gayte, The two-scale convergence method applied to generalized Besicovitch spaces, Proc. R. Soc. Lond. **458A** (2002) 2925-2946.

[38] T. Champion and L. De Pascale, Asymptotic behaviour of nonlinear eigenvalue problems involving p-Laplacian-type operators, Proc. R. Soc. Lond., **137A** (2007) 1179-1195.

[39] H. Cheng, Resonance and generalized Navier-Stokes equations, Syst. Sci. Math. Sci., **13** (2000) 250-259.

[40] V. Chiado Piat, S.S. Nazarov and A.L. Piatnitski, Steklov problems in perforated domains with a coefficient of indefinite sign, Netw. Heterog Media, **7** (2012), 151-178.

[41] D. Cioranescu, A. Damlamian and R. De Arcangelis, Homogenization of quasiconvex integrals via the periodic unfolding method, SIAM J. Math. Anal., **37** (2006) 1435-1453.

[42] D. Cioranescu, A. Damlamian, P. Donato and L. Mascarenhas, H^0-convergence as a limit-case of H-convergence, Adv. Math. Sci. Appl., **9** (1999) 319-331.

[43] D. Cioranescu, A. Damlamian and G. Griso, Periodic unfolding and homogenization, C.R. Acad. Sci. Paris, Ser. 1, **335** (2002) 99104.

[44] D. Cioranescu, A. Damlamian and G. Griso, Periodic unfolding method in homogenization, Siam J. Math. Anal., **40** (2008) 1585-1620.

[45] D. Cioranescu and P. Donato, Homogénéisation du problème de Neumann non homogène dans des ouverts perforés, Asymptotic Anal., **1** (1988) 115-138.

[46] D. Cioranescu and P. Donato, An introduction to Homogenization, Oxford Lecture Series in Mathematics and its applications **17**, Oxford University Press, 1999.

[47] D. Cioranescu, P. Donato and R. Zaki, The periodic unfolding method in perforated domains, Port. Math., **63** (2006) 467-496.

[48] D. Cioranescu, P. Donato and R. Zaki, Asymptotic behavior of elliptic problems in perforated domains with nonlinear boundary conditions, Asymptot. Anal., **53** (2007) 209-235.

[49] G.W. Clark and R.E. Showalter, Two-scale convergence of a model for flow in a partially fissured medium, Electron. J. Differ. Equ., **1999** (1999) 1-20.

[50] K.H. Coats and B.D. Smith, Dead-end pore volume and dipersion in porous media, Trans. Soc. Petr. Engin., **231** (1964) 73-84.

[51] G. Dal Maso, Introduction to Γ-convergence, Birkhäuser, Berlin, 1993.

[52] A. Damlamian, An elementary introduction to periodic unfolding, in Proceedings of the Narvik Conference 2004, GAKUTO Internat. Ser. Math. Sci. Appl. **24**, A. Damlamian, D. Lukkassen, A. Meidell, and A. Piatnitski, eds., Gakkotosho, Tokyo, (2006) 119-136.

[53] A. Damlamian and P. Donato, H^0-convergence and iterated homogenization, Asymptot. Anal., **39** (2004) 45-60.

[54] E. De Giorgi and S. Spagnolo, Sulla convergenza degli integrali dell'energia per operatori ellitici del secondo ordine, Boll. Un. Mat. Ital., **8** (1973) 391-411.

[55] E. De Giorgi, Sulla convergenca di alcune successioni di integrali del tipo dell'area, Rendi Conti di Mat., **8**(1975) 277-294.

[56] E. De Giorgi, G-operators and Γ-convergence. In Proc. Int. Congr. Math., (Warszawa, August 1983) 1175-1191. PWN Polish Scientific Publishing and North-Holland, 1984.

[57] E. De Giorgi and T. Franzoni, Su un tipo di convergenza variazionale, Atti Acad. Naz. Lincei Rend., Cl. Sci. Fis. Mat. Natur., **58** (1975) 842-850.

[58] P. Donato and J. Saint Jean Paulin, Homogenization of Laplace equation in a porous medium with double periodicity. Japan J. Indust. Appl. Math., **10** (1993) 333-349.

[59] H. Douanla, Homogenization of Steklov spectral problems with indefinite density function in perforated domains, Acta Appl. Math., **123** (2013) 261-284.

[60] H. Douanla, Two-scale convergence of elliptic spectral problems with indefinite density function in perforated domains, Asymptot. Anal., **81** (2013) 251-272.

[61] H. Douanla, Two-scale convergence of Stekloff eigenvalue problems in perforated domains, Bound. Value Probl., **2010**, 853717 (2010), 15 pages.

[62] H. Douanla, Spectral asymptotics in porous media, Licentiate Thesis, Chalmers Publication Library, 2010 (Available at : http://publications.lib.chalmers.se/records/fulltext/127455.pdf).

[63] H. Douanla and N. Svanstedt, Homogenization of a nonlinear elliptic problem with large nonlinear potential, Appl. Anal., **91** (2012) 1205-1218.

[64] H. Douanla and N. Svanstedt, Reiterated homogenization of linear eigenvalue problems in multiscale perforated domains beyond the periodic setting, Commun. Math. Anal., **11** (2011) 61-93.

[65] H. Douanla and J.L. Woukeng, Almost periodic homogenization of a generalized Ladyzhenskaya Model for incompressible viscous flow, J. Math. Sci. (N. Y.) **189** (2013) 431-458.

[66] H. Douanla, G. Nguetseng and J.L. Woukeng, Incompressible viscous Newtonian flow in a fissured medium of general deterministic type, J. Math. Sci., (N. Y.) **191** (2013) 214-242.

[67] J. Douglas, M. Peszyńska and R.E. Showalter, Single phase flow in partially fissured media, Transp. Porous Media **28** (1997) 285-306.

[68] N. Dunford and J.T. Schwartz, Linear operators, Parts I and II, Interscience Publishers, Inc., New York, 1958, 1963.

[69] Q. Du and M.D. Gunzburger, Analysis of a Ladyzhenskaya model for incompressible viscous flow, J. Math. Anal. Appl., **155** (1991) 21-45.

[70] W.F. Eberlein, Abstract ergodic theorems and weak almost periodic functions, Trans. Amer. Math. Soc., **67** (1949) 217-240.

[71] A. Einstein, Ann. Phys., 19, **289** (1906).

[72] D. G. Figueiredo, Positive solutions of semilinear elliptic problems, Lecture Notes in Maths., **954**, Springer-Verlag, Berlin (1982) 34-87.

[73] J. Franců, Modification of unfolding approach to two scale convergence and periodic unfolding, Math. Bohem., **135** (2010) 403-412.

[74] J. Franců, On two-scale convergence and periodic unfolding, Tatra Mt. Math. Publ., **48** (2011) 73-81.

[75] E. Frénod, P.A. Raviart and E. Sonnendrcker. Asymptotic expansion of the Vlasov equation in a large external magnetic field, J. Math. Pures et Appl., **80** (2001) 815-843. 2001.

[76] E.I. Grigoliuk and L.A.Filshtinskii, Perforated plates and shells, Nauka, Moscow, 1970.

[77] M. Heida, An extension of the stochastic two-scale convergence method and application, Asymptot. Anal., **72** (2011) 130.

[78] P. Hess, On bifurcation from infinity for positive solutions of second order elliptic eigenvalue problems, Nonlinear Phenomena in Mathematical Sciences. Academic Press, New York, 1982.

[79] P. Hess and T. Kato, On some linear and nonlinear eigenvalue problems with an indefinite weight function, Comm. Partial. Differential Equations. **5** (1980) 999-1030.

[80] R. Hill, Elastic properties of composite media, some theoritical principles, Mechanics, **5** (1964), Collection of translations.

[81] R. Hill, Theory of mechanical properties of fibers-strengthened materials, Journ. Mech. Phys. Solids **12** (1964) 199-218.

[82] A. Holmbom, J. Silfver, N. Svanstedt and N. Wellander, On two-scale convergence and related sequential compactness topics, Appl. Math. **51** (2006) 247-262.

[83] U. Hornung, Homogenization and porous media, Interdisciplinary Applied Mathematics, **6** Springer, New-York, 1997.

[84] U. Hornung and R.E. Showalter, Diffusion models for fractured media, J. Math. Anal. Appl., **147** (1990) 68-90.

[85] P.S. Huyakorn and G.F. Pinder, Computational methods in subsurface flow, Academic Press, New York, 1983.

[86] E.A. Il'iushina, A version of moment plasticity theory for a one-dimensionnal continuous medium of inhomogeneous periodic structure, P.P.M., **36** (1972) 1086-1093.

[87] S. Kaizu, Homogenization of eigenvalue problems for the laplace operators with nonlinear terms in domains in many tiny holes. Nonlin. Anal. TMA., **28** (1997) 377-391.

[88] H. Kazemi, Pressure transient analysis of naturally fractured reservoirs with uniform fracture distribution, Soc. Pet. Eng. J., **9** (1969) 451-462.

[89] S. Kesavan, Homogenization of elliptic eigenvalue problems. I. Appl. Math. Optim., **5** (1979) 153-167.

[90] S. Kesavan, Homogenization of elliptic eigenvalue problems. II. Appl. Math. Optim., **5** (1979) 197-216.

[91] O. Ladyzhenskaya, New equations for the description of the viscous incompressible fluids and solvability in the large of the boundary value problems for them, *in* ”Boundary value problems of mathematical physics V”, Amer. Math. Soc., Providence, RI, 1970.

[92] E.W. Larsen, Neutron transport and diffusion in inhomogeneous media. I., J. Mathematical Phys., **16** (1975) 1421-1427.

[93] E.W. Larsen and J.B. Keller, Asymptotic solution of neutron transport problems for small mean free paths, J. Mathematical Phys., **15** (1974) 75-81.

[94] M. Lenczner, Homogénéisation dun circuit électrique, C.R. Acad. Sci. Paris, Ser. 2, **324** (1997) 537-542.

[95] M. Lenczner and D. Mercier, Homogenization of periodic electrical networks including voltage to current amplifiers, SIAM Multiscale Model. Simul., **2** (2004) 359-397.

[96] B. Li, F. Habbal and M. Ortiz, Optimal transportation meshfree approximation schemes for fluid and plastic flows. International Journal for Numerical Methods in Engineering, **83** (2010) 1541-1579.

[97] K. Lichtenecker, Die dielektrizit ätskonstante natürlicher und künstli cher mischkörper, Phys. Zeitschr., **XXVII** (1926) 115-158.

[98] I.M. Lifshits and L.N. Rozentsveig, On the theory of elastic properties of polycrystals, ZhETF, **16** (1946).

[99] I.M. Lifshits and L.N. Rozentsveig, On the construction of Green’s tensor for the basic equation of elasticity theory in the case of an unbounded elastic anisotropic medium, ZhETF, **17** (1947).

[100] D. Lukkassen, G. Nguetseng and P. Wall, Two-scale convergence, Int. J. Pure Appl. Math., **2** (2002) 35-86.

[101] M. Mabrouk and H. Samadi, Homogenization of a heat transfer problem in a highly heterogeneous periodic medium, Int. J. Engng Sci., **40** (2002) 1233-1250.

[102] M. Mabrouk and A. Boughammoura, Homogenization of a strongly heterogeneous periodic elastic medium, Int. J. Engng Sci., **41** (2002) 817-843.

[103] V.A. Marchenko and E.Y. Khruslov, Boundary-value problems with fine-grained boundary, Mat. Sb. (N.S.), **65(107)** (1964) 458-472.

[104] V.A. Marchenko and E.Y. Khruslov, Boundary value problems in domains with a fine-grained boundary, Naukova Dumka, Kiev, 1974.

[105] J.C. Maxwell, A treatise on electricity and magnetism, 3rd Ed., Clarendon Press, Oxford, 1881.

[106] G. Milton, The theory of composites, Cambridge Monographs on Applied and Computational Mathematics, **6**. Cambridge University Press, Cambridge, 2002.

[107] F. Murat and L. Tartar, H-convergence, In R. V. Kohn, editor, Séminaire d’Analyse Fonctionnelle et Numérique de l’Université d’Alger (1977).

[108] F. Murat and L. Tartar, Calcul des variations et homegénéisation, In R. V. Kohn, editor, Cours de l’Ecole d’Eté d’Analyse Numérique CEA-EDF-INRIA (1983). English translation in “Topics in the Mathematical Modeling of Composite Materials”, Progress in Nonlinear Differential Equations and their Applications, Birkhuser, Boston, 1995.

[109] F. Murat and L. Tartar, H-convergence, In R. V. Kohn, editor, Topics in the Mathematical Modeling of Composite Materials", Progress in Nonlinear Differential Equations and their Applications, Birkhuser, Boston, 1995.

[110] S.A. Nazarov, Asymptotics of negative eigenvalues of the Dirichlet problem with the density changing sign, J. Math. Sci., **163** (2009) 151-175.

[111] S.A. Nazarov and A.L. Piatnitski, Homogenization of the spectral Dirichlet problem for a system of differential equations with rapidly oscillating coefficients and changing sing density, J. Math. Sci., **169** (2010) 212-248.

[112] S.A. Nazarov, A.L. Piatnitski and I.L. Pankratova, Homogenization of the spectral problem for periodic elliptic operators with sign-changing density function. Arch. Ration. Mech. Anal., 200 (2011), 747788.

[113] L. Nechvátal, Alternative approaches to the two-scale convergence, Appl. Math., **49** (2004) 97-110.

[114] M. Neuss-Radu, Homogenization techniques, Diplomarbeit, University of Heidelberg: Faculty of Mathematics, July 1992.

[115] M. Neuss-Radu, Some extensions of two-scale convergence, C. R. Acad. Sci. Paris Sér. I Math., **9** (1996) 899-904.

[116] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal., **20** (1989) 608-623.

[117] G. Nguetseng, Asymptotic analysis of a stiff variational problem arising in mechanics, SIAM J. Math. Anal., **21** (1990) 1394-1414.

[118] G. Nguetseng, Homogenization in perforated domains beyond the periodic setting, J. Math. Anal. Appl., **289** (2004) 608-628.

[119] G. Nguetseng, Homogenization structures and applications I, Z. Anal. Anwen., **22** (2003) 73-107.

[120] G Nguetseng, Homogenization structures and applications. II. Z. Anal. Anwendungen, **23** (2004) 483-508.

[121] G. Nguetseng, Mean value on locally compact abelian groups, Acta Sci. Math., (Szeged) **69** (2003) 203-221.

[122] G. Nguetseng, Σ-convergence of parabolic differential operators. Multiple scales problems in biomathematics, mechanics, physics and numerics, 93-132, GAKUTO Internat. Ser. Math. Sci. Appl., **31**, Gakk$\bar{\text{o}}$tosho, Tokyo, 2009.

[123] G. Nguetseng, M. Sango and J.L. Woukeng, Reiterated ergodic algebras and applications, Commun. Math. Phys., **300** (2010) 835-876.

[124] G. Nguetseng and N. Svanstedt, Σ-convergence, Banach J. Math. Anal., **5**(2011) 101-135.

[125] A.S. Odeh, Unsteady-state behavior of naturally fractured reservoirs, Soc. Pet. Eng. J., **5** (1965) 60-66.

[126] O.A. Oleinik, On the convergence of solutions of elliptic and parapolic equations with weak convergence of the coefficients, UMN, **30** (1975) 257-258.

[127] O.A. Oleinik, A.S. Shamaev and G.A. Yosifan, Mathematical problems in elasticity and homogenization, North-Holland, Amsterdam, 1992.

[128] A. Pankov, G-convergence and homogenization of nonlinear partial differential operators, Kluwer academic publ., Dordrecht, 1997.

[129] H.E. Pak, Geometric two-scale convergence on forms and its applications to maxwell's equations, Proc. Roy. Soc. Edinburgh Sect. **A 135** (2005) 133-147.

[130] S.E., Pastukhova, Averaging error for the Steklov problem in a perforated domain. Differ. Equ., **31** (1995) 975-986.

[131] G.A. Pavliotis and A.M. Stuart, Multiscale methods. Averaging and homogenization. Texts in Applied Mathematics, **53**. Springer, New York, 2008.

[132] M. Peszyńska and R.E. Showalter, Multiscale elliptic-parabolic systems for flow and transport, Electron. J. Differ. Equ., **2007** (2007) 1-30.

[133] M. Peszyńska, R.E. Showalter and S.Y. Yi, Homogenization of a pseudoparabolic system, Appl. Anal., **88** (2009) 1265-1282.

[134] A.L Piatnitski and V.V. Zhikov, Homogenization of random singular structures and random measures, Izv. Math., **70** (2006) 19-67.

[135] S. Poisson, Second mémoire sur la théorie du magnétisme, Mem. Acad. France, **5** (1822).

[136] J. Rauch, The mathematical theory of crushed ice, Partial Differential Equations and related topics, Springer Verlag, Lectures Notes in mathematics, **446** (1975).

[137] J. Rauch and M. Taylor, Potential and scattering theory on wildly pertubed domains, J. Funct. Analysis, **18** (1975) 27-59.

[138] J.W. Rayleigh, On the influence of obstacles arranged in rectangular order upon the properties of a medium, Phil. Mag., (1892) 481-491.

[139] A. Reuss, Berechnung der fliehgrenze von mischkristallen auf grund der plastizitätsbedingung für einkristalle, Z. Angew. Math. und Mech., **9** (1929) 49-58.

[140] G. Rozenblioum and M. Solomyak, Spectral estimates for Schrödinger operators with sparse potentials on graphs, J. Math. Sci. (N. Y.), **176** (2011) 458-474.

[141] G. Rozenblioum and A.V. Sobolev, Discrete spectrum distribution of the Landau operator perturbed by an expanding electric potential. Spectral theory of differential operators, 169190, Amer. Math. Soc. Transl. Ser. 2, 225, Amer. Math. Soc., Providence, RI, 2008.

[142] E. Sanchez-Palencia, Comportements local et macroscopique d'un type de milieux physiques hétérogènes, Intern. Jour. Engin. Sci., **12** (1974) 331-351.

[143] E. Sanchez-Palencia, Non homogeneous media and vibration theory, Springer, Heidelberg (1980).

[144] M. Sango, N. Svanstedt and J.L. Woukeng, Generalized Besicovitch spaces and application to deterministic homogenization, Nonlin. Anal. TMA, **74** (2011) 351-379.

[145] J.S. Smagorinsky, General circulation model of the atmosphere, Mon. Weather Rev., **91** (1963) 99-164.

[146] S. Spagnolo, Sulla convergenza delle soluzioni di equazioni paraboliche ed ellittiche, Ann. Scu. Norm. Super. Pisa, Cl. Sci., **22** (1968) 571-597.

[147] S. Spagnolo, Convergence in energy for elliptic operators, Numerical solution of partial differential equations, III (Proc. Third Sympos. (SYNSPADE), Univ. Maryland, College Park, Md., (1975) 469-498. Academic Press, New York, 1976.

[148] R.E. Showalter and D.B. Visarraga, Double-diffusion models from a highly-heterogeneous medium, J. Math. Anal. Appl., **295** (2004) 191-201.

[149] P.E. Stelzig, On problems in homogenization and two-scale convergence, Ph. D. Thesis, Universit Degli Studi di Trento (2012) (Available at: http://eprints-phd.biblio.unitn.it/780/).

[150] L. Tartar, Cours peccot, Collège de france, 1977.

[151] L. Tartar, Quelques remarques sur l'homogénéisation. In H. Fujita, editor, Proc. of the Japan-France Seminar 1976 "Functional Analysis and Numerical Analysis" (1978) 469-481.

[152] L. Tartar, Qu'est-ce que l'homogénéisation?, Portugaliae Mathematica **4** (2007) 389-444.

[153] L. Tartar, The general theory of homogenization. A personalized introduction, Lecture Notes of the Unione Matematica Italiana (**7**), Springer-Verlag, Berlin (2009).

[154] G.A. Van Fo Fy, Theory of reinforced materials, Nauka, Moscow, 1971.

[155] M. Vanninathan, Homogenization of eigenvalue problems in perforated domains. Proc. Indian Acad. Sci. Math. Sci., **90** (1981) 239-271.

[156] A. Visintin, Some properties of two-scale convergence, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **15** (2004) 93-107.

[157] W. Voight, Lehrbuch der kristallpfysik. Berlin, Teubner, 1928.

[158] J.E. Warren and P.J. Root, The behavior of naturally fractured reservoirs, Soc. Petro. Eng. J., **3** (1963) 245-255.

[159] N. Wellander, The two-scale Fourier transform approach to homogenization; periodic homogenization in Fourier space. Asymptot. Anal., **62** (2009) 1-40.

[160] J.L. Woukeng, Homogenization in algebras with mean value, arXiv: 1207.5397v1, 2012 (Submitted).

[161] S.Y. Yi, M. Peszyńska and R.E. Showalter, Numerical upscaled model of transport with non-separable scale, *in* XVIII Intern. Conf. on Water Resources CMWR 2010, J. Carrera (Ed), CIMNE, Barcelona, (2010) 1-8.

[162] V.V. Zhikov, On an extension and an application of the two-scale convergence method. (Russian) Mat. Sb. **191** (2000) 31-72; translation in Sb. Math. **191** (2000) 973-1014.

[163] V.V. Zhikov, On two-scale convergence, J. Math. Sci. (N. Y.), **120** (2004) 1328-1352.

[164] V.V. Zhikov, S.M. Kozlov and O.A. Oleinik, Homogenization of differential operators and integral functionals, Springer-Verlag, Berlin, 1994.

[165] V.V. Zhikov and E.V. Krivenko, Homogenization of singularly perturbed elliptic operators, Math. Notes, **33** (1983) 294-300.