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Transient Motions of Large Floating Structures

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SUMMARY

This study deals with transient motions of large-volume floating structures. If the structure/fluid system is regarded as linear, equations valid for transient motions can be obtained making use of frequency dependent hydrodynamic properties.

The theory associated with frequency-domain solutions is reviewed briefly and is subsequently applied to a vertical cylinder floating in water of finite depth. An analytical solution proposed by Yeung (1981) is used to solve the radiation problem, i.e. when the structure is moving in the absence of incident waves. Solving the radiation problem gives the added mass coefficients and the potential damping coefficients. The analytical solution has been extended to include determination of amplitudes and phases of the wave-exciting forces.

Relations between the frequency domain and the time domain are reviewed. The frequency dependent hydrodynamic coefficients associated with the vertical cylinder are transformed to corresponding time dependent functions using Fourier transforms.

The equations of motion obtained are formulated numerically, and the solving procedure has been implemented on a computer. A time simulation of the motions of the cylinder is performed. In the simulation the cylinder is tethered with pre-tensioned steel tendons and exposed to irregular waves.

PREFACE

During the 1970s, a project concerning wave energy utilization was performed at the Department of Hydraulics at Chalmers University of Technology. Part of the project concerned calculations of structure/fluid interaction.

When the North Sea offshore oilfields began to be exploited a new demand for hydrodynamic models for structure/fluid interaction arose. It was natural for the Dept. of Hydraulics to participate in the development of such models. A cooperation project, "Offshore Structures - Wave Forces and Motions", between the industry and the Universities of Technology in Gothenburg and Stockholm was initiated. The project has been sponsored by the National Swedish Board of Technical Development (STU).

This thesis is the result of a part of the project concerning motions of large-volume floating structures subjected to arbitrarily time-varying external forces. The work which has been carried out at the Dept. of Hydraulics is submitted in partial fulfillment of the requirements for the degree of Licentiate of Engineering.

I wish to express my thanks to Dr. Lars Bergdahl, my tutor, who has supported me throughout all phases of the study, Mrs. Ann-Marie Hellgren who patiently typed the manuscript, Mrs. Alicja Janiszewska who draw the figures and my colleagues at the Dept. of Hydraulics among whom it has been stimulating to work.

Göteborg, December 1986

Mickey Johansson

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1 INTRODUCTION

1.1 Motions of large-volume floating structures

The choice of methods for calculation of the motions of marine structures are greatly affected by the type of structure one intends to analyse. For slender structures, or structures with slender members, wave forces are ususally calculated considering the incident wave to be unaffected by the presence of the structure. On the other hand, if the structure is large in volume the presence of the structure will disturb the incident wave, and it is then more appropriate to use a diffraction theory which takes these effects into account. In the present report the interest is focused on motions of large-volume structures.

The methods are also affected by the type of loading considered. For example, if the structure is exposed to the force due to a regular wave and the structure is either free floating or attached to a linear mooring system, then, if a linear diffraction theory is used, the equation of motion simply becomes a linear differential equation with constant coefficients. However, the equation is only a frequency domain description since the coefficients of the equation are dependent on the frequency of the motion.

For some types of loading a frequency domain description is a poor reflection of the nature of the structure/fluid interaction. Consider for a moment a stone which is thrown into calm water. The stone will generate disturbances at the surface. The disturbances will gradually decrease in magnitude and eventually disappear. Obviously, the disturbances will remain a long time after the stone has left the water surface. In the same way, for loading cases when the motions become transient, it might be of importance to include what has happened to the structure/fluid system in the past, i.e. to include the time history of the system. In the field of hydrodynamics, such equations of motion valid for transient motions, was first developed by Cummins (1962).

An important disadvantage of time-domain models compared to frequency domain models is that they are far more time consuming when run in a computer. But the costs for computer resources have decreased and time-domain models have become more readily available, so they are thus more realistic tools in hydrodynamic analysis.

The equations of motion as they were formulated by Cummins have been used in various applications. Van Oortmerssen (1976) studied the behaviour of a moored ship subjected to long-crested waves. The mooring system was allowed to be non-linear and asymmetrical and effects from a quay could be considered. Greenhow (1982) analysed wave-energy devices with non-linear power take-off mechanisms. Sawaragi et al (1984) applied the same technique in order to search for improvements of mooring systems for ships subjected to storm waves. Mynett et al (1984) studied the behaviour of moored vessels inside a harbour configuration.

Other areas in which it might be of interest to apply Cummins equations are offshore structures with significantly non-linear mooring systems, analysis of cable failure in a mooring system, and analysis of the effects of collisions between, for example, icebergs and offshore structures.

The reference list of application areas is not intended to be complete.

1.2 Scope of the work

This work deals with the development of the equations of motion for a floating rigid structure. The interest is focused on the time-domain description first introduced by Cummins. In this technique the properties of the frequency-domain description are transformed and used in the time-domain. Consequently, the frequency-domain description is also of interest and is treated here. In Chapter 2 a short introduction to wave-loading and the dynamics of large-volume structures is given. The matrix form of the equation of motion in the frequency-domain is developed in Chapter 3. The relations between the frequency-domain and two mathematically different but physically equivalent time-domain descriptions are reviewed in Chapter 4.

The objective of this study is to establish and to solve necessary equations, but not to be complete in terms of loading. Hopefully, this will provide the basis for further studies as well as an idea of difficulties associated with numerical solutions in the time-domain.

In Chapter 5 the theory has been applied to a vertical cylinder floating in water of finite depth. In order to achieve stiffness also for motions in the water plane, the cylinder is tethered with pre-tensioned wire ropes. The formulation of the wave loading includes regular and irregular waves but not slowly varying drift forces.

2 LOADING OF LARGE-VOLUME FLOATING STRUCTURES

Marine structures are subjected to large dynamic forces from the environment. If the structures are free to move, the motions caused by the forces must remain limited. When analysing the motions of such structures use is made of Newton's second law, which for a system with a single degree of freedom (SDOF) is written:

$$m\ddot{x}(t) = f(t) \tag{2.1}$$

where

$x(t)$ = motion of the center of gravity of the structure

m = mass of the structure

$f(t)$ = external force acting at the center of gravity.

Generally the external force contains contributions from environmental forces due to waves, wind and current and from static and dynamic reaction forces from the surrounding fluid. The mooring system and, in polar regions, floating ice can also cause significant loads.

Wave forces on marine structures are traditionally calculated using one of two different methods. One of the methods, the Morison method, considers the force to be composed of the linear sum of a drag force and an inertia force. The drag force is formulated as if the structure were subjected to a uniform steady flow and is associated with flow separation. The inertia force is formulated as if the structure were subjected to a uniformly accelerated potential flow. Thus, for a fixed structure the Morison equation becomes

$$f(t) = \tfrac{1}{2}\rho A u|u| C_D + \rho \forall \dot{u} C_I \tag{2.2}$$

where A is the projected area of the structure, u is the fluid velocity and $\forall$ the displaced volume. Since for wave motions the flow is both unsteady and nonuniform, empirical evaluation of the drag coefficient, C_D, and the inertia coefficient, C_I, are required in order to use the Morison equation. Using a Morison approach for a floating structure changes the formulation of the force slightly since relative velocity and relative acceleration ought to be used. Therefore, the equation of motion of a floating structure becomes

$$m\ddot{x} = \tfrac{1}{2}\rho A(u-\dot{x})|u-\dot{x}|C_D + \rho \forall \dot{u} + C_m \rho \forall (\dot{u}-\ddot{x}) \qquad (2.3)$$

where the so called added mass coefficient C_m is given by $C_I=1+C_m$.

In the Morison approach velocities and accelerations of the incident wave are used. However, if the ratio structure dimension to wave length increases, the incoming wave will be significantly affected by the presence of the structure and the incoming wave will be diffracted. Hence, for large-volume structures methods that take these effects into account must be used. For such structures flow separation is usually unimportant and the viscous effects are located to the boundary layer adjacent to the structure. Consequently, the flow field can be determined independently of viscous effects and the wave forces can be calculated using a theoretical approach assuming potential flow and introducing boundary conditions associated with the presence of the structure. Usually viscous effects located to the boundary layer are negligible. However, for certain situations these effects may be of importance. One such situation occurs if the shape of the structure is such that inertial forces become small. For example the yaw moment of a structure with axisymmetry about the vertical axis is entirely a consequence of shear forces. Another situation occurs when a restrained structure reaches resonance. Then even a small amount of viscous damping has significant effects.

In order to discuss different regimes when diffraction and other effects become important, a dimensional analysis is performed. Following Sarpkaya and Isaacson (1981) a time-invariant force F on a fixed structure may be expressed in the form

$$\frac{F}{\rho g H D^2} = f\left(\frac{h}{L}, \frac{H}{L}, \frac{D}{L}, Re\right) \qquad (2.4)$$

where H is the wave height, D a characteristic dimension of the structure, h the water depth, L the wave length and Re a characteristic Reynolds number. The depth parameter h/L and the wave steepness parameter H/L both characterize the incident wave. If the steepness is assumed to be small, i.e. $H \ll L$, a linear formulation can be made such that the force becomes linearly dependent on the wave height. Consequently, assuming $H \ll L$ the steepness parameter may be omitted. Diffraction becomes important with an increasing diffraction parameter D/L. For values $D/L > 0.2$ diffraction should be taken into account. Since, in the diffraction region the flow separation effects become of less importance the Reynolds number may be omitted. Thus, for the linear diffraction problem one obtains

$$\frac{F}{\rho g H D^2} = f\left(\frac{h}{L}, \frac{D}{L}\right) \tag{2.5}$$

or alternatively

$$\frac{F}{\rho g H D^2} = f\left(\frac{D}{L}, \frac{D}{h}\right) \tag{2.6}$$

and consequently for a given structure at a given water depth the force parameter only varies with the diffraction parameter D/L.

In the flow separation region other physical phenomena occur. The Reynolds number may no longer be neglected and if diffraction is negligible the diffraction parameter no longer has any physical significance. Instead, in Eq. (2.4), it is preferable to use the Keulegan-Carpenter number defined by $K = \hat{u}T/D$ where $\hat{u}$ is the velocity amplitude of the flow and T the period of the flow.

In order to get an idea of when diffraction, flow separation and nonlinear effects become important Isaacson studied a fixed vertical cylinder. He presented a diagram, Figure 2.1, in terms of the Keulegan-Carpenter number and the diffraction parameter. It is seen that diffraction becomes increasingly important for increasing D/L and flow separation increasingly important for increasing K. For small values of both D/L and K the force becomes inertia dominated and for steep waves nonlinear effects become important.

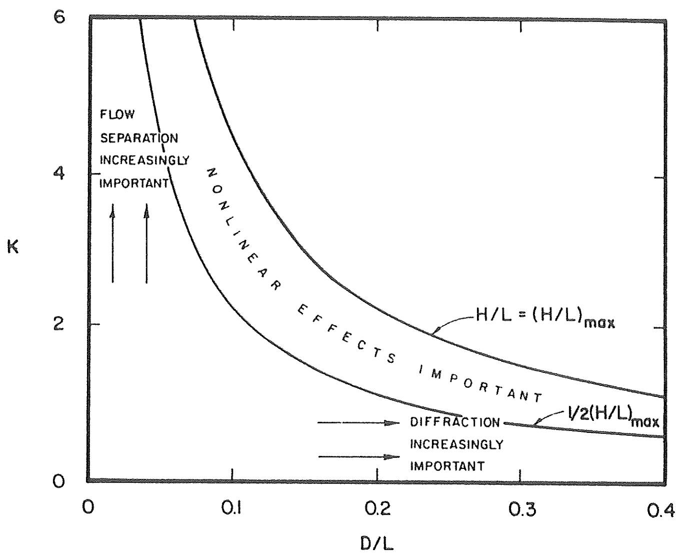

Figure 2.1 Wave force regimes for a vertical cylinder according to Isaacson. From Sarpkaya and Isaacson (1981).

In this work the formulation of the wave loading is based on the linear diffraction theory. In Chapter 3 a general formulation of the theory will be made and in Chapter 5 an analytical solution for a vertical cylinder in finite depth water is formulated. However, in order to make a qualitative discussion we may, at present, write the equation of motion in a schematic way analogous to a mechanical oscillator. Hence,

$$(m+a(\omega))\ddot{x}(t) + b(\omega)\dot{x}(t) + cx(t) = \mathrm{Re}\ \{\hat{F}e^{i\omega t}\} \qquad (2.7)$$

where $a(\omega)$ is the added mass coefficient, $b(\omega)$ the potential damping coefficient, c a buoyancy coefficient and $\hat{F}$ the complex amplitude of the wave exciting force.

However, Eq. (2.7) has to be treated with some caution. The equation is valid only if the structure oscillates sinusoidally in time with a given frequency, ω. This is a consequence of the frequency dependence of the hydrodynamic coefficients.

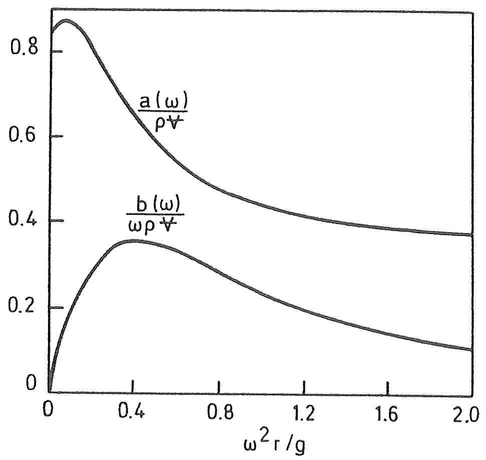

Figure 2.2 Example of frequency dependence of the hydrodynamic coefficients $a(\omega)$ and $b(\omega)$. The graphs are valid for a semi-immersed sphere with radius r heaving in deep water. From Havelock (1955).

If linear diffraction theory is used for an unrestrained structure subjected to regular waves, the wave exciting force as well as the motion become sinusoidal. Therefore it is appropriate to use Eq. (2.7). However, if any nonlinear force is included in the right hand side the motions do not become sinusoidal and strictly speaking Eq. (2.7) is no longer valid. The non-linear force may be a reaction force due to a non-linear mooring system.

If one wishes to solve the equation of motion for an arbitrarily time varying force a convolution integral has to be used. One approach is to use the impulse response function technique. This technique states that, if for any linear system, the response $r(t)$ to a unit impulse is known, then the response of the system to an arbitrarily time varying force $f(t)$ is

$$x(t) = \int_{-\infty}^{t} r(t-\tau)\, f(\tau)d\tau \qquad (2.8)$$

or if $f(\tau) = 0$ for $\tau<0$

$$x(t) = \int_{0}^{t} r(t-\tau)\, f(\tau)d\tau \qquad (2.9)$$

The idea of this convolution integral might be clearer if it is considered as a continuous summation of impulse responses, Figure

2.3. If the system is subjected to a unit impulse (Figure 2.3a) and the response r(t) (Figure 2.3b) to this impulse is known then an arbitrary force f(t) (Figure 2.3c) may be considered a sum of impulses $f(\tau)d\tau$ and thus the response x(t) (Figure 2.3d) as a sum of corresponding impulse responses.

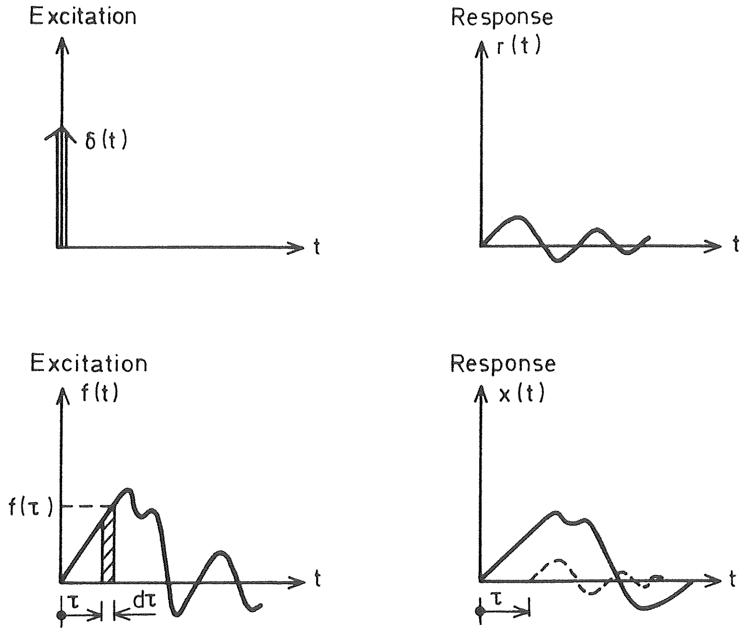

Figure 2.3 Illustration of the impulse response function technique.

As mentioned, Eq. (2.7) requires the system to be linear. This is not always true for a moored structure. However, if linear diffraction theory is used to calculate the hydrodynamic properties of the structure itself then the fluid/structure system can be regarded as linear. Non-linearities such as reaction forces from the mooring system can be excluded from the system and be thought of as external forces acting on the structure and can be included in f(t).

An alternative approach is to use the equations of motion in the time domain as they were developed by Cummins (1962). This formulation is more specifically developed for floating structures and makes more efficient use of the relations that exist between the hydrodynamic coefficients. The equation of motion according to Cummins becomes

$$(m + a_k)\ \ddot{x}(t) + \int_{-\infty}^{t} k(\tau)\ \dot{x}\ (t-\tau)\ d\tau + cx(t) = f(t) \quad (2.10)$$

where it is noted that a_k is not frequency dependent but a constant. The time dependent function $k(\tau)$ is referred to as the retardation function.

These two approaches are further discussed in Chapter 4 where the way in which the impulse response function $r(t)$ and the retardation function $k(t)$ are related to the hydrodynamic coefficients $a(\omega)$ and $b(\omega)$ is outlined.

3 FREQUENCY DOMAIN

A condition for the time domain analysis presented in the report is that the structure/fluid system can be regarded as linear. For such a system relations exist between the frequency domain and the time domain. The hydrodynamic properties in the frequency domain, represented by coefficients $a(\omega)$ and $b(\omega)$, can be transformed to corresponding properties in the time domain by means of Fourier transforms. Since the transforms are infinite integrals over the frequency, $a(\omega)$ and $b(\omega)$ must, in principle, be known for all frequencies. Moreover, if the structure is to be exposed to irregular sea the wave-exciting force $\underline{F}_e(\omega)$ must also be known in the frequency range corresponding to the sea state. In the present chapter the structure/fluid problem is formulated in the frequency domain, for the purpose of giving explicit expressions of the frequency dependent functions $a(\omega)$, $b(\omega)$ and $\underline{F}_e(\omega)$. Such a formulation is made according to potential theory and is developed in a second order partial differential equation, the Laplace equation, which, together with appropriate boundary conditions, gives the primary problem.

Unfortunately it is only possible to solve the Laplace equation analytically if the boundary conditions are sufficiently simple. For floating structures there are analytical solutions only for a few structures with simple geometries, such as spheres and cylinders. These solutions are usually based on a series expansion in some system of eigen functions. However, realistic floating structures usually have more complex geometries and must therefore be analysed using numerical techniques.

The most commonly used techniques seem to be the sink-source and the hybrid element techniques. The sink-source technique includes the formulation of Green's function and has extensively been studied, for example by Garrison (1974), Faltinsen and Michelsen (1975) and Eatock Taylor and Waite (1978). For floating structures the fluid domain is usually large and a pure finite element formulation requires an extremely large mesh. In a hybrid element method the fluid domain in a region close to the structure forms a mesh, and at the boundaries of this region analytical solutions are applied. Such methods have been studied, for example by Chen and Mei (1974), Bai and Yeung (1974) and Mei (1978).

3.1 Fundamentals

To justify use of a linear diffraction theory, both small structural motions and small wave amplitudes are assumed. The fluid is assumed to be ideal (non-viscous) and the flow to be irrotational. The assumption that the flow can be considered irrotational is important to the formulation of the problem. Generally, the flow is described by the velocity vector, but when the flow is irrotational it can be described by a single scalar, the velocity potential $\Phi(x,y,z,t)$. The space coordinates x, y and z refer to a Cartesian coordinate system.

For irrotational flow, also called potential flow, a velocity potential exists such that

$$\underset{\rightarrow}{u} = \nabla\Phi \qquad (3.1)$$

where the velocity vector $\underset{\rightarrow}{u}$ is given by

$$\underset{\rightarrow}{u} = u\underset{\rightarrow}{i} + v\underset{\rightarrow}{j} + w\underset{\rightarrow}{k}$$

and the operator, ∇, by

$$\nabla = \frac{\partial}{\partial x}\underset{\rightarrow}{i} + \frac{\partial}{\partial y}\underset{\rightarrow}{j} + \frac{\partial}{\partial z}\underset{\rightarrow}{k}$$

where $\underset{\rightarrow}{i}$, $\underset{\rightarrow}{j}$ and $\underset{\rightarrow}{k}$ are units vectors along the x, y and z axis respectively.

If the fluid is assumed to be incompressible the continuity equation may be written

$$\nabla\cdot\underset{\rightarrow}{u} = 0 \qquad (3.2)$$

and if Eq. (3.1) is substituted into Eq. (3.2) one obtains

$$\nabla^2\Phi = 0 \qquad (3.3)$$

Eq. (3.3) is the well-known Laplace equation, which expresses conservation of fluid mass for potential flow. The primary problem is to solve the Laplace equation with appropriate boundary conditions.

If we proceed from the Navier-Stokes equation for a viscous fluid the Bernoulli equation for irrotational flow can be derived. Expressed in terms of the velocity potential it becomes

$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}\,\nabla\Phi \cdot \nabla\Phi + gz + \frac{p}{\rho} = 0 \qquad (3.4)$$

and gives the pressure throughout the fluid domain. Since we are interested in a linear solution the second term is neglected. Then the Bernoulli equation becomes

$$p = -\rho\,\frac{\partial \Phi}{\partial t} - \rho g z \qquad (3.5)$$

where the first term represents the hydrodynamic pressure and the second term the hydrostatic pressure. For a review of the potential theory and a derivation of the Bernoulli equation see a textbook in hydraulics such as Daily and Harleman (1973).

From the Bernoulli equation the pressure distribution is known throughout the fluid, and by integrating the pressure over the wet surface of the structure, forces and moments are obtained. Expressed in terms of a generalised force vector one can write

$$\underline{F} = \iint_S p\,\underline{n}\,dS \qquad (3.6)$$

where $\underline{F}$ is the generalised force vector containing both forces and moments and $\underline{n}$ is the generalised normal vector defined by

$$\begin{aligned}
n_1 &= \cos(\vec{n}, \vec{i}) \\
n_2 &= \cos(\vec{n}, \vec{j}) \\
n_3 &= \cos(\vec{n}, \vec{k}) \\
n_4 &= (y - y_m)n_3 - (z - z_m)n_2 \\
n_5 &= (z - z_m)n_1 - (x - x_m)n_3 \\
n_6 &= (x - x_m)n_2 - (y - y_m)n_1
\end{aligned} \qquad (3.7)$$

where $\cos(\vec{n}, \vec{i})$ means the cosine for the angle from the x-axis to the normal $\vec{n}$, and $(x,y,z)_m$ is the point about which the moments are calculated. The normal $\vec{n}$ is assumed to be positive in the direction towards the structure.

The structure is assumed to be rigid, and therefore any motion of the structure can be described by motions in six modes, i.e. three translations and three rotations. A Cartesian coordinate system Oxyz is defined with its origin in the free mean water surface and its z-axis positive upwards and oriented through the center of gravity. The vector of motion is denoted by $\underline{x}$ and contains the following elements:

x_1 = translation in the x-direction = surge motion
x_2 = translation in the y-direction = sway motion
x_3 = translation in the z-direction = heave motion
x_4 = rotation about the x-axis = roll motion
x_5 = rotation about the y-axis = pitch motion
x_6 = rotation about the z-axis = yaw motion

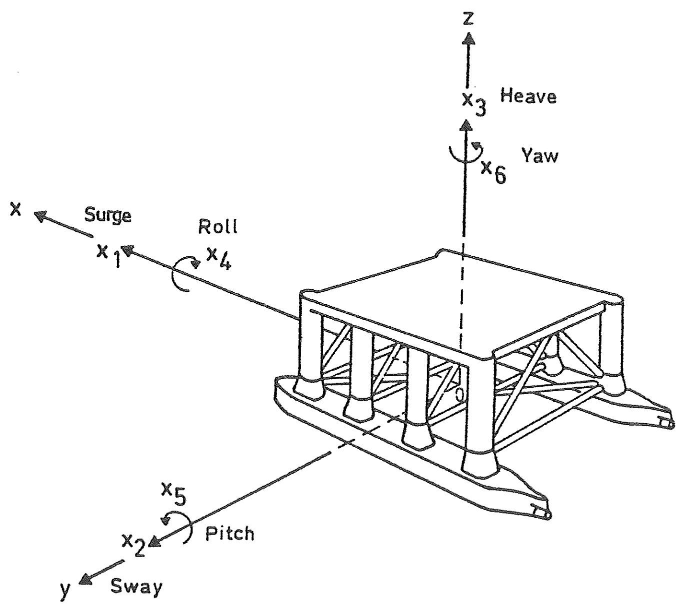

Figure 3.1 Definition of coordinate system and modes of motion.

Let us consider a regular wave propagating in a direction towards the structure. The incoming wave will be diffracted by the structure and will give rise to a scattered wave. In the linear theory the incoming wave and the scattered wave are superimposed linearly. Together they cause a change in the dynamic pressure which, integrated over the wet surface of the structure, gives the wave-exciting force $\underline{F}_e(\omega)$. Consistent with the linear theory, the force is calculated when the floating structure is fixed in its equilibrium position. The problem associated with the wave exciting force is referred to as the diffraction problem. The wave-exciting force (or any other sinusoidal force) will force the structure to oscillate. The oscillation creates a radiated wave, and consequently energy is lost from the system. This influence on the system is accounted for by the hydrodynamic reaction force $\underline{F}_h(\omega)$ which is determined when the structure is forced to oscillate in the absence of an incoming wave. The latter problem is referred to as the radiation problem. Thus, the total wave pattern is described by three physically and mathematically distinguishable waves, i.e. the incoming, scattered and radiated waves.

3.2 The free surface boundary condition

The free surface boundary condition is specific in as much as it causes difficulties associated with the solving procedure when its exact form is applied. The conventional way of treating the problem is to linearise the boundary condition and subsequently account for second order effects using an approximate method.

The linearised free surface boundary condition is formulated below. The condition is built up of two requirements, i.e. the derivative of the surface position must equal the velocity of the corresponding fluid particle (kinematic condition) and the pressure at the free surface has to be atmospheric (dynamic condition). Following Newman (1977), the exact kinematic condition is fulfilled if the substantial derivative of the quantity $z-\zeta$ vanishes at the free surface $z=\zeta(x,y,t)$. Consequently,

$$\frac{D(z-\zeta)}{Dt} = \frac{\partial\Phi}{\partial z} - \frac{\partial\zeta}{\partial t} - \frac{\partial\Phi}{\partial x}\frac{\partial\zeta}{\partial x} - \frac{\partial\Phi}{\partial y}\frac{\partial\zeta}{\partial y} = 0 \qquad (3.8)$$

The exact dynamic condition is given by the Bernoulli Eq. (3.4) which, if the atmospheric pressure is adopted as a reference pressure, becomes

$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} \nabla\Phi \cdot \nabla\Phi + gz = 0 \qquad (3.9)$$

Substituting ζ for z into Eq. (3.9) gives the associated free-surface elevation in terms of the velocity potential,

$$\zeta = -\frac{1}{g} \left(\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} \nabla\Phi \cdot \nabla\Phi \right) \qquad (3.10)$$

Eq. (3.8) and (3.9) are the exact form of the boundary condition and are seen to contain higher order terms. In the linear theory these terms are neglected and moreover the boundary condition is applied to z=0 instead of to the exact free surface $z=\zeta$. That this latter approximation is consistent with the linearisation can be seen by expanding Eq. (3.8) and Eq. (3.10) as Taylor series about z=0 and then neglecting higher order terms. Thus, the linear form of the equations become

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} \qquad (3.11)$$

and

$$\zeta = -\frac{1}{g} \frac{\partial \Phi}{\partial t} \qquad (3.12)$$

respectively. Finally, if the linear free-surface elevation is differentiated with respect to time and substituted into the kinematic condition, a single combined free-surface boundary condition is obtained,

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = 0 \qquad \text{at } z=0 \qquad (3.13)$$

3.3 The hydrodynamic reaction force

As mentioned in Section 3.1 the hydrodynamic reaction force is determined when the structure is forced to oscillate in the absence of an incoming wave. Since the radiated wave is a consequence of the motion of the structure, the velocity potential of the radiated wave is suitably divided into components on the basis of the modes of motion. Consequently, in the general case with motions in six modes, the velocity potential is

$$\Phi_R = \sum_{j=1}^{6} \Phi_j \tag{3.14}$$

where index j refers to the mode of motion considered. Furthermore, since the velocity varies harmonically with time it is convenient to separate the time and space dependence according to

$$\Phi_j = \dot{x}_j \, \phi_j \tag{3.15}$$

and if the motion of the structure is given by

$$x_j = \mathrm{Re}\,\{\hat{x}_j \, e^{i\omega t}\} \tag{3.16}$$

then

$$\Phi_j = \mathrm{Re}\,\{i\omega \, \hat{x}_j \, \phi_j \, e^{i\omega t}\} \tag{3.17}$$

By substituting Eq. (3.17) into the Bernoulli Eq. (3.5) and subsequently integrating the dynamic pressure over the surface of the structure, below the still water level, the force in mode i due to motion in mode j becomes

$$F_{h,ij} = \mathrm{Re}\,\{\omega^2 \hat{x}_j \, e^{i\omega t} \iint_S \rho \phi_j \, n_i \, dS\} \tag{3.18}$$

Conventionally, this force is written as one component in phase with the acceleration and one component in phase with the velocity. The velocity and the acceleration may be written

$$\dot{x}_j = \mathrm{Re}\,\{i\omega \hat{x}_j \, e^{i\omega t}\} = -\mathrm{Im}\,\{\omega \hat{x}_j \, e^{i\omega t}\} \tag{3.19}$$

and

$$\ddot{x}_j = \mathrm{Re}\,\{-\omega^2 \hat{x}_j \, e^{i\omega t}\} = -\omega \, \mathrm{Re}\,\{\omega \hat{x}_j \, e^{i\omega t}\} \tag{3.20}$$

respectively. If Eqs. (3.19) and (3.20) are substitued into Eq. (3.18), the hydrodynamic reaction force becomes

$$F_{h,ij} = -\mathrm{Re}\,\{(\ddot{x}_j + i\omega \dot{x}_j) \iint_S \rho \phi_j n_i \, dS\} \tag{3.21}$$

or alternatively

$$F_{h,ij} = - a_{ij} \ddot{x}_j - b_{ij} \dot{x}_j \tag{3.22}$$

if we define the added mass and damping coefficients, a_{ij} and b_{ij} as, respectively

$$a_{ij} = \mathrm{Re}\,\{\rho \iint_S \phi_j\, n_i\, dS\} \tag{3.23}$$

$$b_{ij} = \mathrm{Im}\,\{-\omega\rho \iint_S \phi_j\, n_i\, dS\} \tag{3.24}$$

In fact the hydrodynamic coefficients in Eqs. (3.23) and (3.24) are functions of the frequency, i.e. $a_{ij}=a_{ij}(\omega)$ and $b_{ij}=b_{ij}(\omega)$. This is important and implies that the corresponding equations of motion are only frequency domain descriptions. Moreover, the hydrodynamic coefficients a_{ij} and b_{ij} are related. If the damping coefficient is known for all frequencies the added mass coefficient (minus its value at the infinite frequency) can be determined for all frequencies, and vice versa, by an integral transform. These relations are called the Kramers-Kronig relations and were first pointed out in this context by Kotik & Mangulis (1962). The relations are as follows

$$a_{ij}(\omega) - a_{ij}(\infty) = \frac{2}{\pi}\, PV \int_0^\infty b_{ij}(\alpha)\, \frac{d\alpha}{\alpha^2 - \omega^2} \tag{3.25}$$

$$b_{ij}(\omega) = -\frac{2}{\pi}\, \omega^2 PV \int_0^\infty (a_{ij}(\alpha) - a_{ij}(\infty))\, \frac{d\alpha}{\alpha^2 - \omega^2} \tag{3.26}$$

where PV means the Cauchy principal value of the integral. Kotik & Lurye (1964) have also shown that

$$\int_0^\infty (a_{ij}(\omega) - a_{ij}(\infty))\, d\omega = 0 \tag{3.27}$$

These relations can sometimes be useful for simplifying the calculations or as controls for obtained results.

Now, let us return to the velocity potential of the radiated wave and express the boundary conditions it has to satisfy.

The linearised free surface boundary condition (3.13), discussed in Section 3.2, can be expressed in terms of the space dependent velocity potential. Thus

$$\frac{\partial \phi_j}{\partial z} - \frac{\omega^2}{g} \phi_j = 0 \qquad \text{on} \qquad z = 0 \tag{3.28}$$

Moreover, the sea bottom is usually assumed to be horizontal and impermeable. Therefore,

$$\frac{\partial \phi_j}{\partial z} = 0 \tag{3.29}$$

has to be satisfied for z=-d where d is the water depth. On the structure's wet surface the normal velocity of the structure must equal the normal velocity of the corresponding fluid particle. Expressed in terms of the space-dependent velocity potential this condition becomes

$$\frac{\partial \phi_j}{\partial n} = i\omega n_j \tag{3.30}$$

Finally, to give the problem a unique solution, the Sommerfeldt radiation condition must be applied. This condition states that the waves at an infinite distance from the structure behave like restricted outwards propagating waves. Mathematically, this is fulfilled if

$$\frac{\partial \phi_j}{\partial r} = (- \tfrac{1}{2}r + i\frac{\omega^2}{g})\, \phi_j \qquad \text{when} \qquad r \to \infty \tag{3.31}$$

To determine the velocity potential of the radiated wave for real offshore structures, these boundary conditions imply that a numerical method, such as the sink-source method must be used.

3.4 The wave exciting force

Assume a regular incoming wave with frequency ω. According to the discussion in Section 3.1 the wave-exciting force is determined when the floating structure is fixed in its equilibrium position. The velocity potential of the incoming wave and the velocity potential of the scattered wave are linearly superimposed and consequently

$$\Phi(x,y,z,t) = \Phi_I(x,y,z,t) + \Phi_S(x,y,z,t) \qquad (3.32)$$

Since the velocity of the fluid varies harmonically with time, the space and time dependence can be separated. Using complex notation the velocity potentials can be written

$$\Phi_I(x,y,z,t) = \mathrm{Re}\{\phi_I(x,y,z)\ e^{i\omega t}\} \qquad (3.33)$$

and

$$\Phi_S(x,y,z,t) = \mathrm{Re}\{\phi_S(x,y,z)\ e^{i\omega t}\} \qquad (3.34)$$

respectively. The wave exciting force is obtained by substituting the velocity potential Eq. (3.32) into the Bernoulli Eq. (3.5) and subsequently integrating the dynamic pressure over the wet surface of the structure. Thus

$$\underline{F}_e = \mathrm{Re}\ \{-i\omega \iint \rho\ (\phi_I+\phi_S)\ \underline{n}\ e^{i\omega t} dS\} \qquad (3.35)$$

where the velocity potential of the incoming wave is easy to determine and can, in principle, be considered as known. However, the velocity potential of the scattered wave has to satisfy more complicated boundary conditions. In analogy with the radiated wave, the free surface boundary condition, the sea bottom condition and the Sommerfeldt radiation condition have to be satisfied. Moreover, there is no flow across the surface of the structure and this implies that the boundary condition

$$\frac{\partial \phi_S}{\partial n} = - \frac{\partial \phi_I}{\partial n} \qquad (3.36)$$

has to be applied to the wet surface of the structure.

Although Φ_S and Φ_R represent different physical phenomena their mathematical formulations are seen to be quite similar. In fact it can be shown that there are relations between the wave exciting force and the velocity potential of the radiated wave. These relations are known as Haskind's relations. Haskind derived expressions for the wave exciting force on a fixed structure which do not require knowledge of the scattered wave, but depend instead on the velocity potential of the radiated wave.

Haskind's relation is simply an application of Green's theorem which states that if a volume $\underset{\sim}{V}$ is bounded by a closed surface $\underset{\sim}{S}$ and two potentials ϕ_1 and ϕ_2 satisfy Laplace's equation within the volume then

$$\iint_{\underset{\sim}{S}} (\phi_1 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial \phi_1}{\partial n}) \, d\underset{\sim}{S} = 0 \qquad (3.37)$$

Substituting the boundary condition Eq. (3.30) into Eq. (3.35) it follows that the wave exciting force can be written as

$$F_{ei} = \text{Re} \, \{-\rho \iint_S (\phi_I + \phi_S) \frac{\partial \phi_i}{\partial n} \, dS \, e^{i\omega t}\} \qquad (3.38)$$

where the integration is made over the wet surface of the structure S. For reasons that will become clear further on we would like to apply Green's theorem to the potential ϕ_S and ϕ_i. But, as mentioned above, the surface must be closed, and this is not fulfilled by S. Therefore Green's theorem must be applied not only for the wet surface of the structure S but for a surface also including the free surface, the bottom and a control surface at infinity. Fortunately, ϕ_S satisfies the same boundary condition as ϕ_j at these surfaces. Consequently, the integrals over the free surface, the bottom and the control surface vanish, and Green's theorem thus gives

$$\iint_S (\phi_i \frac{\partial \phi_S}{\partial n} - \phi_S \frac{\partial \phi_i}{\partial n}) \, dS = 0, \quad i=1,2,\ldots.,6 \qquad (3.39)$$

Substituting this relation in Eq. (3.38) gives

$$F_{ei} = \text{Re} \, \{-\rho \iint_S (\phi_i \frac{\partial \phi_S}{\partial n} + \phi_I \frac{\partial \phi_i}{\partial n}) \, dS \, e^{i\omega t}\} \qquad (3.40)$$

Finally if the boundary condition, Eq. (3.36), is used the wave exciting force can be written as

$$F_{ei} = \mathrm{Re}\,\{\rho \iint (\phi_i \frac{\partial \phi_I}{\partial n} - \phi_I \frac{\partial \phi_i}{\partial n})\, dS\, e^{i\omega t}\} \qquad (3.41)$$

This result is a form of Haskind's relation which, as mentioned above, expresses the wave exciting force without making use of the potential of the scattered wave.

Newman (1962) has further developed this idea and evaluated the wave exciting force on a control surface at infinity. Since the damping coefficients can be found from energy radiation at infinity Newman succeeded in deriving relations between the principal damping coefficients and the wave exciting force. Newman obtained

$$b_{ii}(\omega) = D \int_0^{2\pi} |F_{ei}\,(\omega,\beta)|^2 d\beta \qquad (3.42)$$

where

$$D = k/(\hat{\zeta}^2 8\pi\rho g V_g) \qquad (3.43a)$$

$$V_g = (\tfrac{1}{2} + \frac{kd}{\sinh 2kd})\,\frac{\omega}{k} \qquad (3.43b)$$

and $F_e(\omega,\beta)$ denotes the wave exciting force for waves at an angle of incidence β.

The discussion below follows the reasoning of Newman (1962). Eq. (3.42) can be used to calculate the damping coefficient if the wave exciting force is known for all angles of incidence in the interval $(0,2\pi)$. However, it is more likely that one would be interested in obtaining the wave exciting force when the damping coefficients are known. In general this is not possible since $F_e(\omega,\beta)$ is a function of the angle of wave incidence, with the exception of structures with a vertical axis of symmetry. For such structures the heave exciting force is independent of the angle of incidence and in the remaining modes the forces depend linearly on cos β or sin β, e.g. in surge the wave exciting force is given by

$$F_{e1}(\omega,\beta) = F_{e1}(\omega,\beta{=}0)\cos\beta \qquad (3.44)$$

Therefore, when evaluating the integral in Eq. (3.42) for structures with a vertical axis of symmetry, one finds

$$b_{11}(\omega) = \pi D |F_{e1}(\omega,\beta=0)|^2 \tag{3.45a}$$

$$b_{22}(\omega) = b_{11}(\omega) \tag{3.45b}$$

$$b_{33}(\omega) = 2\pi D |F_{e3}(\omega,\beta=0)|^2 \tag{3.45c}$$

$$b_{44}(\omega) = b_{55}(\omega) \tag{3.45d}$$

$$b_{55}(\omega) = \pi D |F_{e5}(\omega,\beta=0)|^2 \tag{3.45e}$$

$$b_{66}(\omega) = 0 \tag{3.45f}$$

Consequently for bodies with a vertical axis of symmetry, the principal damping coefficients can be found from the wave exciting forces and vice versa. However, it is important to bear in mind that information about the phases is lost in this kind of consideration.

3.5 The equations of motion

In the present section the equations of motion are established. Written in matrix form in the frequency domain they become

$$(\underline{\underline{m}} + \underline{\underline{a}}(\omega))\,\underline{\ddot{x}} + \underline{\underline{b}}(\omega)\,\underline{\dot{x}} + \underline{\underline{c}}\,\underline{x} = \underline{F} \tag{3.46}$$

The matrices should be regarded as generalized. For example, $\underline{x}$ includes both translations and rotations, $\underline{F}$ both forces and moments and $\underline{\underline{m}}$ both masses, moments of masses and moments of inertia.

The elements in the added mass matrix and the damping matrix have already been discussed and are defined by Eqs. (3.23) and (3.24) respectively. In Section 3.5.1 a linear transform between an earth fixed system Oxyz, already defined in Section 3.1, and a structure fixed system $\bar{O}\bar{x}\bar{y}\bar{z}$ is established. Subsequently, in Section 3.5.2 the elements of the mass matrix are developed and in

Section 3.5.3 the elements of the stiffness matrix are developed. In order to obtain a linear stiffness matrix the linear transform is used.

3.5.1 Coordinate system

When studying motions of floating rigid structures the geometrical configuration is most easily described in a structure-fixed coordinate system, whereas the motion of the fluid is most easily described in an earth-fixed system. When the hydrodynamic reaction force and the wave-exciting force were formulated, use was made of an earth-fixed coordinate system Oxyz with its origin in the free mean water surface and with its positive z-axis upwards.

Let us now introduce a structure-fixed coordinate system $\bar{O}\bar{x}\bar{y}\bar{z}$ which initially, when the structure is at rest, coincides with the earth-fixed system Oxyz.

Since the intention is to develop the equations of motion in the earth-fixed system, we need know how to pass from one system to the other.

Let the vector $\bar{\underset{\rightarrow}{r}}$ in $\bar{O}\bar{x}\bar{y}\bar{z}$ be given in Oxyz by successively rotating first about the x-axis with an angle α, then about the y-axis with an angle β and finally about the z-axis with an angle γ. Let the transformation matrices, corresponding to the three rotations mentioned above, be denoted $\underline{\underline{R}}_x$, $\underline{\underline{R}}_y$ and $\underline{\underline{R}}_z$ respectively and defined such that

$$\underset{\rightarrow}{r} = \underline{\underline{R}}_z \, \underline{\underline{R}}_y \, \underline{\underline{R}}_x \, \bar{\underset{\rightarrow}{r}} = \underline{\underline{R}} \, \bar{\underset{\rightarrow}{r}} \tag{3.47}$$

In addition to rotation, the structure-fixed system will also be translated. Consequently, let the translation be defined by the vector $\underset{\rightarrow}{r}_o$, so the total transformation is given by

$$\underset{\rightarrow}{r} = \underset{\rightarrow}{r}_o + \underline{\underline{R}} \, \bar{\underset{\rightarrow}{r}} \tag{3.48}$$

where

$$\underset{\rightarrow}{r} = (x,y,z) \tag{3.49}$$

$$\underset{\rightarrow}{r}_o = (x_o,y_o,z_o) \tag{3.50}$$

$$\underset{\rightarrow}{\bar{r}} = (\bar{x},\bar{y},\bar{z}) \tag{3.51}$$

and

$$\underset{=}{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{3.52}$$

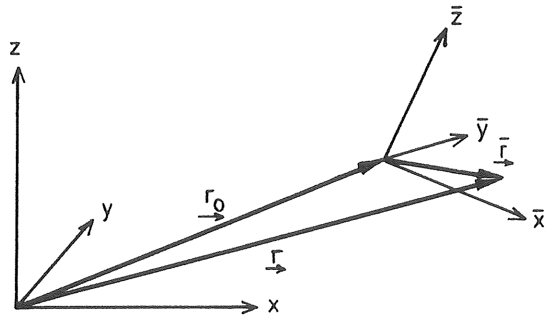

Figure 3.2 Structure-fixed system relative to the earth-fixed system.

The matrices corresponding to the rotations mentioned above are

$$\underset{=}{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \tag{3.53a}$$

$$\underset{=}{R}_y = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{3.53b}$$

$$\underset{=}{R}_z = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.53c}$$

By using Eq. (3.47) the following elements are obtained

$$
\begin{aligned}
R_{11} &= \cos\gamma\cos\beta \\
R_{12} &= -\sin\gamma\ \cos\alpha + \cos\gamma\sin\beta\sin\alpha \\
R_{13} &= \sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha \\
R_{21} &= \sin\gamma\cos\beta \\
R_{22} &= \cos\gamma\cos\alpha + \sin\gamma\sin\beta\sin\alpha \\
R_{23} &= -\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha \\
R_{31} &= -\sin\beta \\
R_{32} &= \cos\beta\sin\alpha \\
R_{33} &= \cos\alpha\cos\beta
\end{aligned}
$$

Further, assuming small rotations, we obtain

$$R = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \tag{3.54}$$

Consequently,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} \tag{3.55}$$

which is the linear transformation between the two systems when the rotations are assumed to be small.

Finally, if we adopt the notation mentioned in Section 3.1 for rigid body motions we obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & -x_6 & x_5 \\ x_6 & 1 & -x_4 \\ -x_5 & x_4 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} \tag{3.56}$$

3.5.2 The mass matrix

Consider an intertial system xyz fixed in space and a particle of mass m at a distance $\underset{\sim}{r}$ from the origin (Figure 3.3).

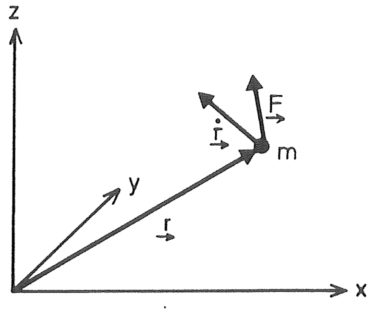

Figure 3.3 Particle of mass, m, in an inertial system xyz.

According to the second law of Newton, a particle acted upon by a force moves so that the force vector is equal to the time rate of change of the linear momentum. The linear momentum is defined as the product of mass and velocity, $\vec{p}=m\dot{\vec{r}}$, and consequently

$$\vec{F} = \frac{d}{dt}(m\dot{\vec{r}}) = \frac{d\vec{p}}{dt} \tag{3.57}$$

Furthermore, the moment of momentum, or angular momentum, of the mass m with respect to the origin of the inertial system is defined as the vector product of the space vector $\vec{r}$ and the linear momentum $\vec{p}$. If the angular momentum is denoted by $\vec{L}$ we obtain

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\dot{\vec{r}} \tag{3.58}$$

The rate of the change of angular momentum is found from

$$\dot{\vec{L}} = \dot{\vec{r}} \times m\dot{\vec{r}} + \vec{r} \times \frac{d}{dt}(m\dot{\vec{r}}) = \vec{r} \times \frac{d}{dt}(m\dot{\vec{r}}) \tag{3.59}$$

But, we also know that the moment of the force $\vec{F}$ with respect to the origin of the inertial system is defined by the vector product of the space vector $\vec{r}$ and the force $\vec{F}$ and hence if the moment is denoted by $\vec{M}$ we obtain

$$\underset{\rightarrow}{M} = \underset{\rightarrow}{r} \times \underset{\rightarrow}{F} = \underset{\rightarrow}{r} \times \frac{d}{dt}(m\dot{\underset{\rightarrow}{r}}) = \dot{\underset{\rightarrow}{L}} \qquad (3.60)$$

Thus the moment with respect to the inertial system is equal to the time rate of change of angular momentum.

The intention in this section is not to formulate the equation of motion for a particle of mass but for a rigid structure. Therefore, use is again made of the inertial system Oxyz and the structure fixed system $\bar{O}\bar{x}\bar{y}\bar{z}$ defined in Section 3.5.1. The latter is obtained from the former by means of three translations of the origin and three rotations about the displaced origin (Figure 3.4).

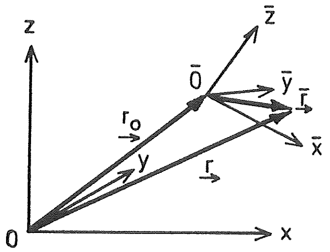

Figure 3.4 Inertial and structure fixed systems.

Since, in a rigid structure, there is no motion of mass with respect to the structure fixed system, the velocity with respect to the inertial system of a mass point is

$$\dot{\underset{\rightarrow}{r}} = \dot{\underset{\rightarrow}{r}}_0 + \underset{\rightarrow}{\omega} \times \underset{\rightarrow}{\bar{r}} \qquad (3.61)$$

where $\dot{\underset{\rightarrow}{r}}_0$ is the velocity of the origin of the structure fixed system, $\underset{\rightarrow}{\bar{r}}$ is the coordinate of the mass point with respect to the structure fixed system and $\underset{\rightarrow}{\omega}$ is the angular velocity vector. The linear momentum of the rigid body can then be found by integrating the product of mass and velocity of the structure,

$$\underset{\rightarrow}{p} = \iiint (\dot{\underset{\rightarrow}{r}}_o + \underset{\rightarrow}{\omega} \times \bar{\underset{\rightarrow}{r}})\, dm = \dot{\underset{\rightarrow}{r}}_o \iiint dm + \underset{\rightarrow}{\omega} \times \iiint \bar{\underset{\rightarrow}{r}}\, dm$$

$$= m(\dot{\underset{\rightarrow}{r}}_o + \underset{\rightarrow}{\omega} \times \bar{\underset{\rightarrow}{r}}_G) \qquad (3.62)$$

where $\bar{\underset{\rightarrow}{r}}_G$ is the vector to the center of gravity. The angular momentum of a rigid body about the origin $\bar{O}$ is defined by

$$\underset{\rightarrow}{L} = \iiint \bar{\underset{\rightarrow}{r}} \times \dot{\underset{\rightarrow}{r}}\, dm = \iiint \bar{\underset{\rightarrow}{r}} \times (\dot{\underset{\rightarrow}{r}}_o + \underset{\rightarrow}{\omega} \times \bar{\underset{\rightarrow}{r}})\, dm$$

$$= -\dot{\underset{\rightarrow}{r}}_o \times \iiint \bar{\underset{\rightarrow}{r}}\, dm + \iiint \bar{\underset{\rightarrow}{r}} \times (\underset{\rightarrow}{\omega} \times \bar{\underset{\rightarrow}{r}})\, dm$$

$$= -\dot{\underset{\rightarrow}{r}}_o \times m\bar{\underset{\rightarrow}{r}}_G + \iiint (\underset{\rightarrow}{\omega}(\bar{\underset{\rightarrow}{r}} \cdot \bar{\underset{\rightarrow}{r}}) - \bar{\underset{\rightarrow}{r}}(\underset{\rightarrow}{\omega} \cdot \bar{\underset{\rightarrow}{r}}))\, dm \qquad (3.63)$$

where use has been made of relations known from vector algebra. Since we know that the force and the moment acting upon the structure is equal to the time rate of change of linear and angular momentum respectively we can, by using Eqs. 3.62 and 3.63, establish equations of motion for a rigid body. It is convenient to leave the concept of physical vectors and instead express these equations by matrices. Thus the components of the force are obtained as,

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \dot{x}_o \\ \dot{y}_o \\ \dot{z}_o \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} 0 & m\bar{z}_G & -m\bar{y}_G \\ -m\bar{z}_G & 0 & m\bar{x}_G \\ m\bar{y}_G & -m\bar{x}_G & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad (3.64)$$

and the components of the moments as

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = -\frac{d}{dt} \begin{bmatrix} 0 & m\bar{z}_G & -m\bar{y}_G \\ -m\bar{z}_G & 0 & m\bar{x}_G \\ m\bar{y}_G & -m\bar{x}_G & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_o \\ \dot{y}_o \\ \dot{z}_o \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_o \end{bmatrix} \qquad (3.65)$$

where the three elements

$$I_{xx} = \iiint (\bar{r}^2 - \bar{x}^2)\, dm,\ I_{yy} = \iiint (\bar{r}^2 - \bar{y}^2)\, dm,$$

$$I_{zz} = \iiint (\bar{r}^2 - \bar{z}^2)\, dm$$

are called moments of inertia and the six elements

$$I_{xy}=I_{yx}= \iiint \bar{x}\bar{y}dm,\ I_{xz}=I_{zx}= \iiint \bar{x}\bar{z}dm,\ I_{yz}=I_{zy}= \iiint \bar{y}\bar{z}dm$$

are called products of inertia. Observe that all inertia terms are defined in the structure fixed coordinate system and therefore remain constant while the structure is in motion.

In Section 3.1 the motions of the rigid structure were described in a Cartesian coordinate system Oxyz. In that formulation the motion vector was not a physical vector but a generalised matrix of motions consisting of both translations and rotations. If we formulate the equations above using generalised matrices of forces and moments and also adopt the coordinate systems defined in Section 3.1 such that $(\dot{x}_0, \dot{y}_0, \dot{z}_0)=(\dot{x}_1, \dot{x}_2, \dot{x}_3)$ and $(\omega_x, \omega_y, \omega_z)=(\dot{x}_4, \dot{x}_5, \dot{x}_6)$ we obtain

$$\underline{F} = \underline{\underline{m}}\, \ddot{\underline{x}}$$

where the generalised mass matrix is given by

$$\underline{\underline{m}} = \begin{bmatrix} m & 0 & 0 & 0 & m\bar{z}_G & -m\bar{y}_G \\ 0 & m & 0 & -m\bar{z}_G & 0 & m\bar{x}_G \\ 0 & 0 & m & m\bar{y}_G & -m\bar{x}_G & 0 \\ 0 & -m\bar{z}_G & m\bar{y}_G & I_{xx} & -I_{xy} & -I_{xz} \\ m\bar{z}_G & 0 & -m\bar{x}_G & -I_{xy} & I_{yy} & -I_{yz} \\ -m\bar{y}_G & m\bar{x}_G & 0 & -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \tag{3.66}$$

Relations and equations used in the derivation of the generalised mass matrix can, for example, be found in the book of analytical dynamics of Meirovitch (1970).

3.5.3 Hydrostatics

The object of this section is to formulate the hydrostatic reaction force that arises when the structure is forced to move from its equilibrium position. In the formulation use is made of the transformation between the structure-fixed and the earth-fixed coordinates.

For floating structures the only reaction forces due to buoyancy occur in the heave, roll and pitch motions. In heave the reaction force is simply the buoyancy force. In roll and pitch the reaction moments are moments due to the buoyancy force. Since the buoyancy force is proportional to the displaced volume some volume integrals have to be evaluated. These integrals are evaluated after decomposing the displaced volume into a static volume $\forall$ beneath the plane $\bar{z}=0$ and a volume between the planes $\bar{z}=0$ and $z=0$.

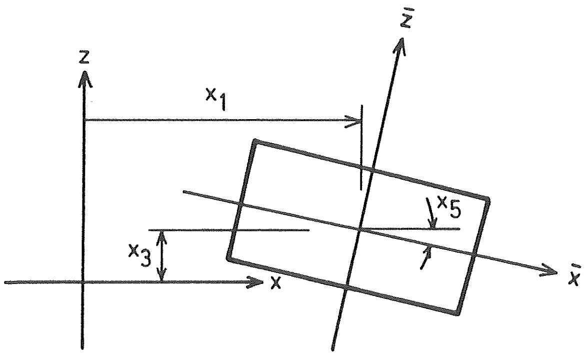

Figure 3.5 Schematic figure showing the earth-fixed and the structure-fixed coordinate system.

If the forces are evaluated with respect to the origin of the structure-fixed system and second order quantities are neglected then the following hydrostatic forces are found

$$F_3 = \rho g(\forall - \iint_{A_w} z(\bar{z}=0)\, d\bar{x}d\bar{y} \tag{3.67a}$$

$$F_4 = \rho g(\iiint_{\forall} (y-x_2)dV - \iint_{A_w} z(\bar{z}=0)\bar{y}d\bar{x}d\bar{y} \tag{3.67b}$$

$$F_5 = \rho g(-\iiint_{\forall} (x-x_1)dV - \iint_{A_w} z(\bar{z}=0)\bar{x}d\bar{x}d\bar{y} \tag{3.67c}$$

where $\forall$ and A_w are the displaced volume and the waterplane area in equilibrium respectively.

Furthermore, if use is made of the transforms Eq. (3.56) derived in Section 3.5.1 the hydrostatic forces become

$$F_3 = \rho g(\forall - \iint_{A_w} \{x_3 - x_5\bar{x} + x_4\bar{y}\}\, d\bar{x}d\bar{y}) \qquad (3.68a)$$

$$F_4 = \rho g(\iiint_{\forall} \{x_6\bar{x}+\bar{y}-x_4\bar{z}\}dV - \iint_{A_w} \{x_3-x_5\bar{x}+x_4\bar{y}\}\bar{y}d\bar{x}d\bar{y}) \qquad (3.68b)$$

$$F_5 = \rho g(-\iiint_{\forall} \{\bar{x}-x_6\bar{y}+x_5\bar{z}\}dV + \iint_{A_w} \{x_3-x_5\bar{x}+x_4\bar{y}\}\bar{x}d\bar{x}d\bar{y}) \qquad (3.68c)$$

Before these expressions are simplified some definitions will be given.

The moments of inertia of the water plane area:

$$J_{xx} = \iint_{A_w} \bar{x}^2 d\bar{x}d\bar{y} \qquad (3.69a)$$

$$J_{yy} = \iint_{A_w} \bar{y}^2 d\bar{x}d\bar{y} \qquad (3.69b)$$

$$J_{xy} = J_{yx} = \iint_{A_w} \bar{x}\bar{y}d\bar{x}d\bar{y} \qquad (3.69c)$$

The center of gravity of the waterplane area:

$$x_c = \frac{1}{A_w} \iint_{A_w} \bar{x}d\bar{x}d\bar{y} \qquad (3.70a)$$

$$y_c = \frac{1}{A_w} \iint_{A_w} \bar{y}d\bar{x}d\bar{y} \qquad (3.70b)$$

The center of buoyancy:

$$x_B = \frac{1}{\forall} \iiint_{\forall} \bar{x}dV \qquad (3.71a)$$

$$y_B = \frac{1}{\forall} \iiint_{\forall} \bar{y}dV \quad (3.71b)$$

$$z_B = \frac{1}{\forall} \iiint_{\forall} \bar{z}dV \quad (3.71c)$$

If these definitions are used the hydrostatic forces become

$$F_3 = \rho g(\forall - A_w x_3 - y_c A_w x_4 + x_c A_w x_5) \quad (3.72a)$$

$$F_4 = \rho g(y_B \forall - y_c A_w x_3 - (z_B \forall + J_{yy})x_4 + J_{xy} x_5 + x_B \forall x_4) \quad (3.72b)$$

and

$$F_5 = \rho g(-x_B \forall + x_c A_w x_3 + J_{yx} x_4 - (z_B \forall + J_{xx})x_5 + y_B \forall x_6) \quad (3.72c)$$

Equilibrium requires that the center of gravity and the center of buoyancy must lie on the same vertical line. Since the z-axis is oriented through the center of gravity the horizontal coordinates of the center of buoyancy are given by

$$x_B = y_B = 0$$

We are not interested in the hydrostatic forces but in the net reaction forces that arise when the structure is forced to move from its equilibrium position. Therefore the gravitational force is included in the expressions above which, if it is taken into account that $x_B = y_B = 0$, become

$$\Delta F_3 = \rho g \forall - mg + \rho g(-A_w x_3 - y_c A_w x_4 + x_c A_w x_5) \quad (3.73a)$$

$$\Delta F_4 = (mgz_G - \rho g \forall z_B - \rho g J_{yy})x_4 + \rho g(-y_c A_w x_3 + J_{xy} x_5) \quad (3.73b)$$

and

$$\Delta F_5 = (mgz_G - \rho g \forall z_B - \rho g J_{xx})x_5 + \rho g(x_c A_w x_3 + J_{yx} x_4) \quad (3.73.c)$$

If we define a hydrostatic stiffness matrix such that

$$\underline{\Delta F} = - \underline{c}\underline{x} \quad (3.74)$$

then the stiffness coefficients become

$$
\begin{aligned}
c_{33} &= \rho g A_w \\
c_{34} &= \rho g y_c A_w \\
c_{35} &= -\rho g x_c A_w \\
c_{43} &= \rho g y_c A_w \\
c_{44} &= \rho g \forall z_B - mgz_G + \rho g J_{yy} \\
c_{45} &= -\rho g J_{xy} \\
c_{53} &= -\rho g x_c A_w \\
c_{54} &= -\rho g J_{xy} \\
c_{55} &= \rho g \forall z_B - mgz_G + \rho g J_{xx}
\end{aligned}
\qquad (3.75)
$$

and all other coefficients c_{ij} equal zero. If the structure is symmetrical about the xz-plane then y_c equals zero and consequently c_{34} and c_{43} also equal zero. Analogously, if the yz-plane is a plane of symmetry then c_{35} and c_{53} equal zero.

4 TIME-DOMAIN

In the frequency domain the motions of the structure are described by coupled ordinary differential equations Eq. (3.46). Since the equations are valid only for harmonic excitations they cannot be considered as genuine differential equations. Therefore, when analysing non-linear or transient responses, a relevant time-domain description must be used. As mentioned in Section 2.1, for a linear structure/fluid system, such a time domain analysis contains a convolution integral. Such an integral takes into account what has happened to the system in the past, i.e. includes the time-history of the dynamic system. In the present chapter two time domain models are discussed and their relations to the frequency domain are pointed out.

4.1 Fundamentals

It is probably most straightforward to use a convolution integral over the excitation. Using matrices the motion is then given by

$$\underline{x}(t) = \int_{-\infty}^{t} \underline{\underline{r}}(t-\tau)\ \underline{f}(\tau)d\tau \qquad (4.1a)$$

$$\underline{x}(t) = \int_{0}^{\infty} \underline{\underline{r}}(\tau)\ \underline{f}(t-\tau)d\tau \qquad (4.1b)$$

where the impulse response function $r_{ij}(t)$ is the response (motion) in mode j due to a unit impulse, a Dirach function, at time t=0 in mode i. The impulse response function is a real function of time which depends not only on the geometry of the structure but also on the fluid domain. Therefore the impulse response function is not a pure structure property but is also affected by the water depth or the distance to a vertical boundary such as a dock, for example.

On the basis of the definition of the impulse response function it holds that

$$r_{ij}(t) = 0 \text{ for } t<0 \qquad (4.2)$$

This is called the principle of causality and for such a linear system the response is given by the value of the excitation at present and in the past. As an initial condition it is stated that

$$r_{ij}(0) = 0 \tag{4.3}$$

The only assumption required, aside from linearity of the structure/fluid system, is convergence of the convolution integral. A sufficient (but not necessary) condition for convergence is that the absolute value of the functions $f_i(t)$ and $r_{ij}(t)$ are integrable, i.e.

$$\int_0^\infty |f_i(t)|\ dt < \infty \tag{4.4}$$

$$\int_0^\infty |r_{ij}(t)|\ dt < \infty \tag{4.5}$$

where the latter condition states that the system is stable. Consequently, assuming the excitation to be restricted and integrable, then the remaining question is with regard to the behaviour of the impulse response function $r_{ij}(t)$. Fontijn (1978), who has studied the berthing ship problem, discussed convergence of the convolution integral and the asymptotic behaviour of the impulse response function. The discussion below follows the reasoning of Fontijn.

If the convolution integral is formulated according to Eq. (4.1) with the motion as the output signal then, in modes where the stiffness coefficient c_{ij} is zero (surge, sway and yaw if only stiffness due to buoyancy is considered), the asymptotic value of the impulse response function $r_{ij}(t)$ will increase indefinitely. In the remaining modes (heave, roll and pitch) the impulse response function will approach zero asymptotically and therefore it will be integrable. Consequently, the requirements are only fulfilled for motions out of the plane of the water surface.

Alternatively, the convolution integral can be formulated with the velocity of the structure as output signal.

Thus, the velocity can be written

$$\dot{\underline{x}}(t) = \int_{-\infty}^{t} \underline{\underline{p}}(t-\tau)\, \underline{f}(\tau)\, d\tau \qquad (4.6)$$

where the impulse response function $p_{ij}(t)$ is the velocity in mode j due to a unit impulse in mode i.

It can be shown that, for modes with zero stiffness coefficients, the impulse response function, $p_{ij}(t)$, asymptotically reaches a constant non-zero value. In fact such a case can be treated since

$$\int_{0}^{\infty} |p_{ij}(t) - p_{ij}(\infty)|\, dt \quad < \infty \qquad (4.7)$$

However, in this case the meaning of the transforms will be somewhat different. For further details see Cummins (1962) or Fontijn (1978).

Although there is no hydrostatic restoring force in the horizontal plane, in many applications the structure is connected to a mooring system that supplies stiffness in these modes as well. If the restoring force from the mooring system is linear it can be directly included in the stiffness matrix $\underline{\underline{c}}$. If the mooring system is non-linear then some representative linear stiffness matrix $\underline{\underline{c}}$ can be used when evaluating the impulse response function $\underline{\underline{r}}(t)$, while the non-linear contribution can be considered as an external force and included in the force $\underline{f}(t)$ when evaluating the convolution integral, Eq. (4.1). Thus, for a moored structure, stable behaviour can be obtained in all modes of motion.

If the formulation with retardation function, Eq. (2.10) is used instead of Eq. 4.1 the problems associated with non-existing transforms are avoided. Therefore, when motions in coupled modes both in and out of the plane of the free water surface are analysed, this latter formulation is probably a more practical tool for solving for non-linear motions.

In the next sections relations between the frequency-domain and the time-domain are described for both the impulse response function and the retardation function.

4.2 The impulse response function with the motions regarded as output signal

Generally, the response in the frequency domain is characterised by the frequency response function matrix $\underline{\underline{R}}(i\omega)$. If the elements in this complex matrix are known they can be transformed to corresponding elements in the impulse response function matrix $\underline{\underline{r}}(t)$.

In order to develop relations between the frequency domain and the time domain, assume the dynamic system to be exposed to a harmonic force

$$\underline{F}(i\omega) = \underline{\hat{f}}e^{i\omega t} \tag{4.8}$$

By substituting Eq. (4.8) into the convolution integral Eq. (4.1b) one obtains

$$\begin{aligned}\underline{x}(t) &= \int_0^\infty r\ (\underline{\underline{\tau}})\ \underline{\hat{f}}\ e^{i(\omega t-\omega\tau)}d\tau \\ &= \int_0^\infty \underline{\underline{r}}(\tau)e^{-i\omega\tau}d\tau\ \underline{\hat{f}}\ e^{i\omega t} \\ &= \underline{\underline{R}}(i\omega)\ \underline{\hat{f}}\ e^{i\omega t}\end{aligned} \tag{4.9}$$

if the complex frequency response function $\underline{\underline{R}}(i\omega)$ is defined

$$\underline{\underline{R}}\ (i\omega) = \int_0^\infty \underline{\underline{r}}(\tau)e^{-i\omega\tau}d\tau \tag{4.10a}$$

from which it is seen that the frequency response function is the Fourier transform of the impulse response function. From the theory of Fourier transforms it is known that there exists an inverse Fourier transform

$$\underline{\underline{r}}(t) = \frac{2}{\pi} \int_0^\infty \underline{\underline{R}}(i\omega)\, e^{i\omega t} d\omega \qquad (4.10b)$$

if $\underline{\underline{r}}(t)$ is piecewise differentiable and the absolute value of $\underline{\underline{r}}(t)$ is integrable. Thus, $\underline{\underline{r}}$ and $\underline{\underline{R}}$ together form a Fourier transform pair that makes it possible to shift from the frequency domain to the time domain or vice versa.

Alternatively, if the complex frequency response function is separated into real and imaginary parts according to

$$\underline{\underline{R}}(i\omega) = \underline{\underline{R}}^c(\omega) - i\underline{\underline{R}}^s(\omega) \qquad (4.11)$$

the following real Fourier transform pairs obtained are

$$\underline{\underline{R}}^c(\omega) = \int_0^\infty \underline{\underline{r}}(\tau) \cos(\omega\tau)\, d\tau \qquad (4.12a)$$

$$\underline{\underline{r}}(\tau) = \frac{2}{\pi} \int_0^\infty \underline{\underline{R}}^c(\omega) \cos(\omega t)\, d\omega \qquad (4.12b)$$

and

$$\underline{\underline{R}}^s(\omega) = \int_0^\infty \underline{\underline{r}}(\tau) \sin(\omega\tau)\, d\tau \qquad (4.13a)$$

$$\underline{\underline{r}}(t) = \frac{2}{\pi} \int_0^\infty \underline{\underline{R}}^s(\omega) \sin(\omega t)\, d\omega \qquad (4.13b)$$

Furthermore, in the frequency domain the equations of motion may be written as

$$(\underline{\underline{m}} + \underline{\underline{a}}(\omega))\, \ddot{\underline{x}}(t) + \underline{\underline{b}}(\omega)\dot{x}(t) + \underline{\underline{c}}x(t) = \underline{\hat{f}}\, e^{i\omega t} \qquad (4.14)$$

where $\underline{\underline{c}}$ is assumed to contain non-zero diagonal elements. If, the motion, Eq. (4.9), and its first and second derviatives are substituted into Eq. (4.14) one obtains

$$(\{\underline{\underline{c}} - \omega^2(\underline{\underline{m}} + \underline{\underline{a}}(\omega))\} + i\omega\underline{\underline{b}}(\omega))\, R(i\omega) = \underline{\underline{E}} \qquad (4.15)$$

where $\underline{\underline{E}}$ is the unit matrix defined by having diagonal elements equal to one and the other elements equal to zero. Solving for the frequency response function gives

$$\underline{\underline{R}}(i\omega) = (\{\underline{\underline{c}} - \omega^2(\underline{\underline{m}} + \underline{\underline{a}}(\omega))\} + i\omega\underline{\underline{b}}(\omega))^{-1} \qquad (4.16)$$

Alternatively, the real frequency response function matrices may be found if Eq. (4.11) is substituted into Eq. (4.15) and then real and imaginary parts are identified. This results in two coupled matrix equations,

$$\{\underline{\underline{c}} - \omega^2(\underline{\underline{m}} + \underline{\underline{a}}(\omega))\}\ \underline{\underline{R}}^c(\omega) + \omega\underline{\underline{b}}(\omega)\ \underline{\underline{R}}^s(\omega) = \underline{\underline{E}} \qquad (4.17a)$$

and

$$\{\underline{\underline{c}} - \omega^2(\underline{\underline{m}} + \underline{\underline{a}}(\omega))\}\ \underline{\underline{R}}^s(\omega) + \omega\underline{\underline{b}}(\omega)\ \underline{\underline{R}}^c(\omega) = \underline{\underline{0}} \qquad (4.17b)$$

respectively. In order to solve for the two real matrices let

$$\underline{\underline{A}}(\omega) = \underline{\underline{c}} - \omega^2(\underline{\underline{m}} + \underline{\underline{a}}(\omega)) \qquad (4.18)$$

$$\underline{\underline{B}}(\omega) = \omega\underline{\underline{b}}(\omega) \qquad (4.19)$$

The coupled matrix equations can then be written

$$\underline{\underline{A}}(\omega)\underline{\underline{R}}^c(\omega) + \underline{\underline{B}}(\omega)\underline{\underline{R}}^s(\omega) = \underline{\underline{E}} \qquad (4.20)$$

$$\underline{\underline{A}}(\omega)\ \underline{\underline{R}}^s(\omega) - \underline{\underline{B}}(\omega)\underline{\underline{R}}^c(\omega) = 0 \qquad (4.21)$$

and by solving these one obtains

$$\underline{\underline{R}}^c(\omega) = (\underline{\underline{A}}(\omega) + \underline{\underline{B}}(\omega)\underline{\underline{A}}^{-1}(\omega)\underline{\underline{B}}(\omega))^{-1} \qquad (4.22)$$

$$\underline{\underline{R}}^s(\omega) = \underline{\underline{A}}^{-1}(\omega)\underline{\underline{B}}(\omega)\underline{\underline{R}}^c(\omega) \qquad (4.23)$$

From Eq. (4.22) and (4.23) it is seen that the real frequency response matrices can be solved explicitly. In order to evaluate these equations two matrices have to be inverted. When either one of the two frequency response functions is known then the impulse response function can be determined by using the corresponding inverse Fourier transforms element by element, i.e. either Eq. (4.12b) or (4.13b).

For the particular case of uncoupled equations the frequency response function becomes

$$R_{ii}(i\omega) = \frac{1}{c_{ii} - \omega^2(m_{ii}+a_{ii}(\omega))+i\omega b_{ii}(\omega)} \tag{4.24}$$

or expressed as real frequency response functions

$$R_{ii}^{c}(\omega) = \frac{c_{ii}-\omega^2(m_{ii}+a_{ii}(\omega))}{(\omega b_{ii}(\omega))^2+(c_{ii}-\omega^2(m_{ii}+a_{ii}(\omega)))^2} \tag{4.25}$$

$$R_{ii}^{s}(\omega) = \frac{\omega\, b_{ii}(\omega)}{(\omega b_{ii}(\omega))^2+(c_{ii}-\omega^2(m_{ii}+a_{ii}(\omega)))^2} \tag{4.26}$$

4.3 The retardation function formulation

In order to obtain the equations of motion in the time domain the approach of Cummins (1962) is used in the applications in this report. The main problem is to find the reaction force of the fluid to arbitrarily time varying motions of the structure. Cummins assumed that the motion of the structure could be regarded as a sum of impulsive displacements.

Use an earth fixed system Oxyz as defined in Section 2.1. Suppose that the structure is given an impulsive displacement in the j-th mode. This displacement is achieved by moving the structure at the instantaneous velocity $\dot{x}(\tau)$ for an infinitessimal time interval $d\tau$. Then the impulsive displacement is given by

$$dx_j = \dot{x}_j(\tau)d\tau \tag{4.27}$$

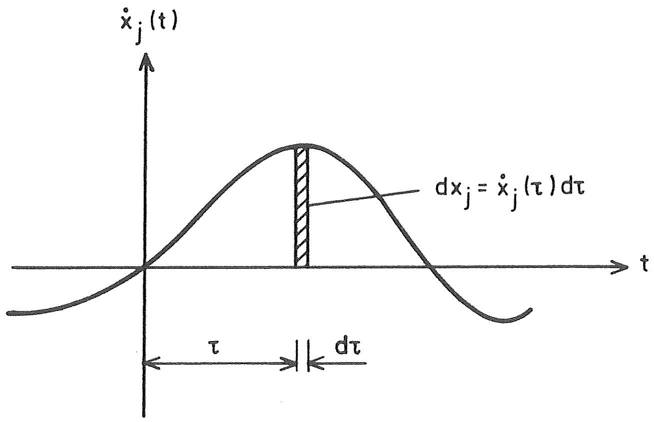

Figure 4.1 The instantaneous velocity gives the impulsive displacements.

During the impulse the flow is characterised by a velocity potential proportional to the instantaneous velocity of the structure:

$$\Phi = \sum_{j=1}^{6} \dot{x}_j(\tau)\ \psi_j \tag{4.28}$$

During the impulse, the free surface will be elevated. However, after the impulse this elevation also dissipates in a radiating disturbance of the surface. In a linear formulation the velocity potential of this decaying wave will be proportional to the impulsive displacement. Thus, the velocity potential of the decaying wave can be written

$$\Phi = \sum_{j=1}^{6} \phi_j(\tau)dx_j = \sum_{j=1}^{6} \phi_j(\tau)\dot{x}_j(\tau)d\tau \tag{4.29}$$

In order to find the total velocity potential for a certain time t Eq. (4.28) and the time history integral of Eq. (4.29) are added linearly. The time history integration is performed considering appropriate time lags from the instant of corresponding impulsive displacements. Thus, the total velocity potential becomes

$$\Phi = \sum_{j=1}^{6} (\dot{x}_j\psi_j + \int_{-\infty}^{t} \phi_j(t-\tau)\dot{x}_j(\tau)d\tau) \tag{4.30}$$

Of course this velocity potential has to satisfy boundary conditions at the free surface, at the surface of the structure and at the bottom.

As before the linearized dynamic pressure of the fluid is given by the Bernoulli equation:

$$p = -\frac{\partial\Phi}{\partial t}$$

$$= \rho \sum_{j=1}^{6} (\ddot{x}_j \psi_j + \int_{-\infty}^{t} \frac{\partial}{\partial t} \Phi_j \; (t-\tau) \; \dot{x}_j(\tau) d\tau) \qquad (4.31)$$

The hydrodynamic reaction force in the i-th direction is obtained if the pressure is integrated over the wet surface of the structure:

$$F_i = \iint p n_i ds$$

$$= -\sum_{j=1}^{6} (\rho \ddot{x}_j \iint_S \psi_j n_i dS + \rho \iint_S n_i dS \int_{-\infty}^{t} \frac{\partial \phi_j(t-\tau)}{\partial t} \dot{x}_j(\tau) d\tau) \qquad (4.32)$$

Finally, using matrices, the equations of motion can be written in the form

$$(\underline{\underline{m}} + \underline{\underline{a}}_k) \; \underline{\ddot{x}}(t) + \int_{-\infty}^{t} \underline{\underline{k}}(t-\tau) \; \underline{\dot{x}}(\tau) d\tau + \underline{\underline{c}} \; \underline{x}(t) = \underline{f}(t) \qquad (4.33)$$

if the following respective definitions are used

$$a_{k,ij} = \rho \iint_S \psi_j n_i dS \qquad (4.34)$$

$$k_{ij} = \rho \iint \frac{\partial \phi_j(t)}{\partial t} n_i dS \qquad (4.35)$$

From Eq. (4.33) it is obvious that $a_{k,ij}$ has the dimension of mass and the function $k_{ij}(t)$ may be interpreted physically as the hydrodynamic reaction force due to a unit impulsive displacement.

So far nothing has been said about the solution of the velocity potentials ψ_j and ϕ_j. However, Ogilvie (1964) has shown how the equations of motion in the time domain Eq. (4.33) are related to those in the frequency domain Eq. (3.47). By using these relations the unknowns in Eq. (4.33), i.e. $\underline{\underline{a}}_k$ and $\underline{\underline{k}}(t)$, may be expressed in terms of the frequency dependent hydrodynamic coefficients $\underline{\underline{a}}(\omega)$ and $\underline{\underline{b}}(\omega)$. These relations can be found if the Fourier transform of Eq. (4.33) is compared with the frequency domain description. If following notation is used for the Fouriers transform,

$$G(i\omega) = \int_0^\infty g(t)\, e^{-i\omega t} dt$$

the Fourier transform of Eq. (4.33) becomes

$$(-\omega^2(\underline{\underline{m}} + \underline{\underline{a}}_k) + i\omega\underline{\underline{K}}(i\omega) + \underline{\underline{c}})\ \underline{X}(i\omega) = F(i\omega) \qquad (4.36)$$

Separate the complex function $\underline{\underline{K}}(i\omega)$ in real and imaginary parts so that

$$\begin{aligned}
\underline{\underline{K}}(i\omega) &= \int_0^\infty \underline{\underline{k}}(t) e^{-i\omega t} dt \\
&= \int_0^\infty \underline{\underline{k}}(t)\cos(\omega t) - i \int_0^\infty \underline{\underline{k}}(t) \sin(\omega t) dt \\
&= \underline{\underline{K}}^c(\omega) - i\ \underline{\underline{K}}^s(\omega) \qquad (4.37)
\end{aligned}$$

If Eq. (4.37) is substituted into Eq. (4.36) one obtains

$$(\underline{\underline{c}} - \omega^2\underline{\underline{m}} + (\omega\underline{\underline{K}}^s(\omega) - \omega^2\underline{\underline{a}}_k) + i\omega\underline{\underline{K}}^c(\omega))\ \underline{X}(i\omega) = F(i\omega) \qquad (4.38)$$

but the equations of motion in the frequency domain can be written

$$(c - \omega^2\underline{\underline{m}} + (-\omega^2\underline{\underline{a}}(\omega)) + i\omega\underline{\underline{b}}(\omega))\ \underline{X}(i\omega) = \underline{F}(i\omega) \qquad (4.39)$$

and if these are compared one finds the two following relations

$$K^s(\omega) = -\omega(\underline{\underline{a}}(\omega) - \underline{\underline{a}}_k) \tag{4.40a}$$

$$K^c(\omega) = b(\omega) \tag{4.40b}$$

In Eq. (4.37) we have already defined real Fourier transforms. The respective corresponding inverse Fourier transforms are

$$\underline{\underline{k}}(t) = \frac{2}{\pi} \int_0^\infty \underline{\underline{K}}^s(\omega) \sin(\omega t)\, d\omega \tag{4.41a}$$

$$\underline{\underline{k}}(t) = \frac{2}{\pi} \int_0^\infty \underline{\underline{K}}^c(\omega) \cos(\omega t)\, d\omega \tag{4.41b}$$

Thus, if we substitute Eq. (4.40b) into Eq. (4.41b) the retardation function becomes

$$\underline{\underline{k}}(t) = \frac{2}{\pi} \int_0^\infty b(\omega) \cos(\omega t)\, d\omega \tag{4.42}$$

and if we identify Eq. (4.40a) with the imaginary part of Eq. (4.37) the mass coefficient becomes

$$\underline{\underline{a}}_k = a(\omega') + \frac{1}{\omega'} \int_0^\infty \underline{\underline{k}}(t) \sin(\omega' t)\, dt \tag{4.43}$$

where ω' is an arbitrarily chosen value of the frequency. The meaning of Eq. (4.43) may be further examined. Since the retardation function reaches zero asymptotically for high frequencies and $(\sin(\omega t))/\omega$ also reaches zero asymptotically it follows that

$$\underline{\underline{a}}_k = \underline{\underline{a}}(\infty) \tag{4.44}$$

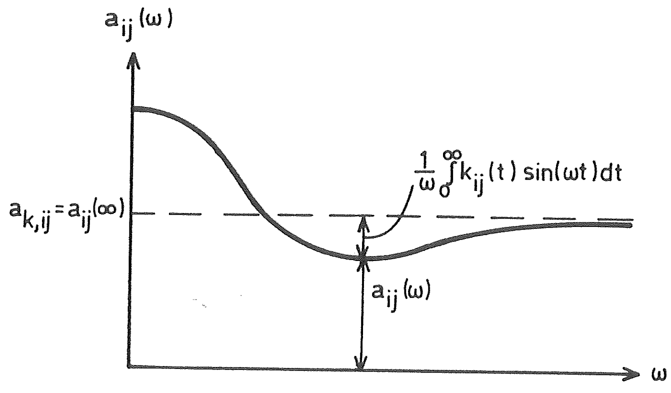

Figure 4.2 Schematical graph of Eq. (4.43).

Consequently, Eq. (4.42) and (4.43) make it possible to pass from the frequency domain to the time domain if the damping coefficients are known throughout the frequency domain as well as the values of the added mass coefficients for a single frequency ω'. A similar formulation of the equations of motion in the time domain is given by Wehausen (1971).

4.4 Experimental possibilities

Having formulated the relations between oscillatory motions in the frequency domain and transient responses to an impulse in the time domain we have a theoretical tool for solving motions to arbitrarily time-varying forces. The formulation of the motions in the time domain also suggests experimental methods for determining the frequency response function (or the added mass and potential damping). Traditionally, the frequency response function is calculated from a series of experiments at different frequencies. However, using relations between the frequency domain and the time domain, one properly performed experiment should give the same information. Such experiments have been made by Smith & Cummins (1965) who studied the heave and pitch motions of a ship. The basic ideas of such experiments are, in principle, simple.

As has been shown in Section 4.2 the impulse response function can be used without being directly measured, as it is simply the Fourier transform of the frequency response function. If the latter is known for all frequencies, the impulse response function can be computed. But to determine the frequency response function directly, one must measure the response to a set of frequencies at suitably close intervals over the whole range of frequencies in which there is a significant response. The alternative approach is to apply a known excitation and observe the response. By analysing the respective frequency contents of the excitation and the response, the frequency response can be computed. Thus, one single experiment can replace a series of experiments at different frequencies.

Although in principle the experiment is very simple, in practice there are many difficulties to be overcome. These are discussed by Smith & Cummins (1965).

5 APPLICATIONS OF THE RETARDATION FUNCTION FORMULATION ON A VERTICAL CYLINDER

5.1 General

In the preceding chapters the theory associated with the hydrodynamics of large-volume floating structures has been reviewed. In the present chapter the theory is applied to a specific structure. We have seen that if the structure is forced to move in non-harmonic oscillations the hydrodynamic reaction force can be obtained from the hydrodynamic coefficients in the frequency domain using a Fourier transform. In order to do this the added mass and the potential damping must be known for a suitable number of frequencies. For a structure with a complex geometry it is necessary to use a numerical method to solve for the added mass and the potential damping. Such methods require large amount of computer resources and are thus expensive to run. It is also difficult to estimate errors occurring in a numerical solution. In order to obtain a fast solution that does not demand large of computer resources, but still has high accuracy, an analytical solution has been preferred. There exist only analytical solutions for structures with simple geometries.

In the application in the present chapter a vertical cylinder floating in finite depth water is chosen. Such a structure has been extensively investigated. Garret (1970) studied the scattering problem when the cylinder was subjected to a plane incident wave and solved for the near field solution of the problem and thus obtained the wave exciting force. With a near field solution it is as well possible to solve for wave disturbances caused by the structure. Furthermore, Yeung (1981) studied the radiation problem of a cylinder in the absence of incident waves and obtained the added mass and the damping coefficients. Yeung divided the fluid domain in an interior region beneath the cylinder and an exterior region outside the cylinder. He treated the conditions at the common boundary as if these were known and developed expression of the interior and exterior problems. By matching these expressions of eigenfunctions at the common boundary Yeung established an infinite system of equations. The system showed to have excellent truncation characteristics and therefore only a small number of coupled equations had to be solved. In this report the method and formulation suggested by Yeung are adopted and used in order to calculate the added mass and damping coefficients. The principal equations are given in Section 5.3.

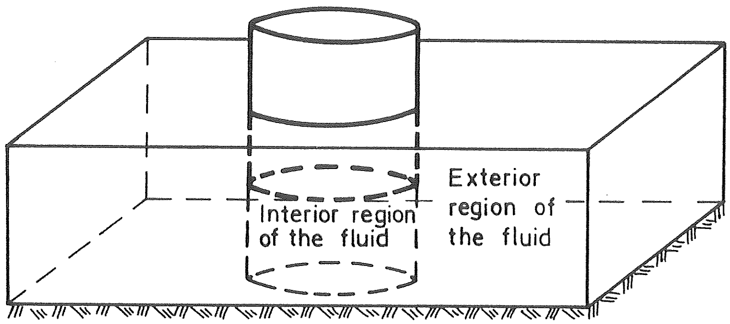

Figure 5.1 Definition of interior region and exterior region of the fluid according to Yeung (1981).

The wave exciting force can, as mentioned, be found by solving the diffraction problem according to Garret using a technique similar to that of Yeung. However, Haskind has used Green's theorem to derive expressions for the wave exciting force which do not include the solution of the diffraction problem, but instead make use of the solution of the radiation problem. Moreover, it can be shown that the integration does not necessarily have to be performed over the wet surface of the structure but can, instead be performed over a control surface at an infinite distance from the structure. This is a consequence of Green's theorem. Thus, for a given incident wave, the exciting force can be calculated if the asymptotic behaviour of the radiation problem is known. This way of treating the problem has been discussed by Newman (1962), who derived expressions for the wave exciting force on a submerged ellipsoid. In Section 5.4 this technique is applied and expressions are derived for the wave exciting force on a cylinder.

Wave loading due to irregular waves is discussed in Section 5.5. The force is given as a series of sinusoidal waves linearly superimposed. The amplitude of each sinusoidal term is obtained from a wave energy spectrum, and the phases are chosen arbitrarily.

In order to obtain more realistic motions and to supply stiffness in the horizontal plane as well, the cylinder is tethered with pre-tensioned wire ropes. The equations for such a system are given in Section 5.6.

The numerical evaluation of the retardation function, Eq. (4.42) and a numerical formulation of the equations of motion are presented in Section 5.7.

Finally, in Section 5.8 the results from the calculations are presented and discussed. The results include the added mass and the damping coefficients, the wave exciting forces and the phases between the incident wave and the forces, and the retardation functions. A simulation of the motions due to a realistic time dependent force is outside the scope of this application, but the equations of motion are solved for a simple type of loading in order to check the numerical formulation.

In Figure 5.2 a principle scheme for the steps included in the calculation is given.

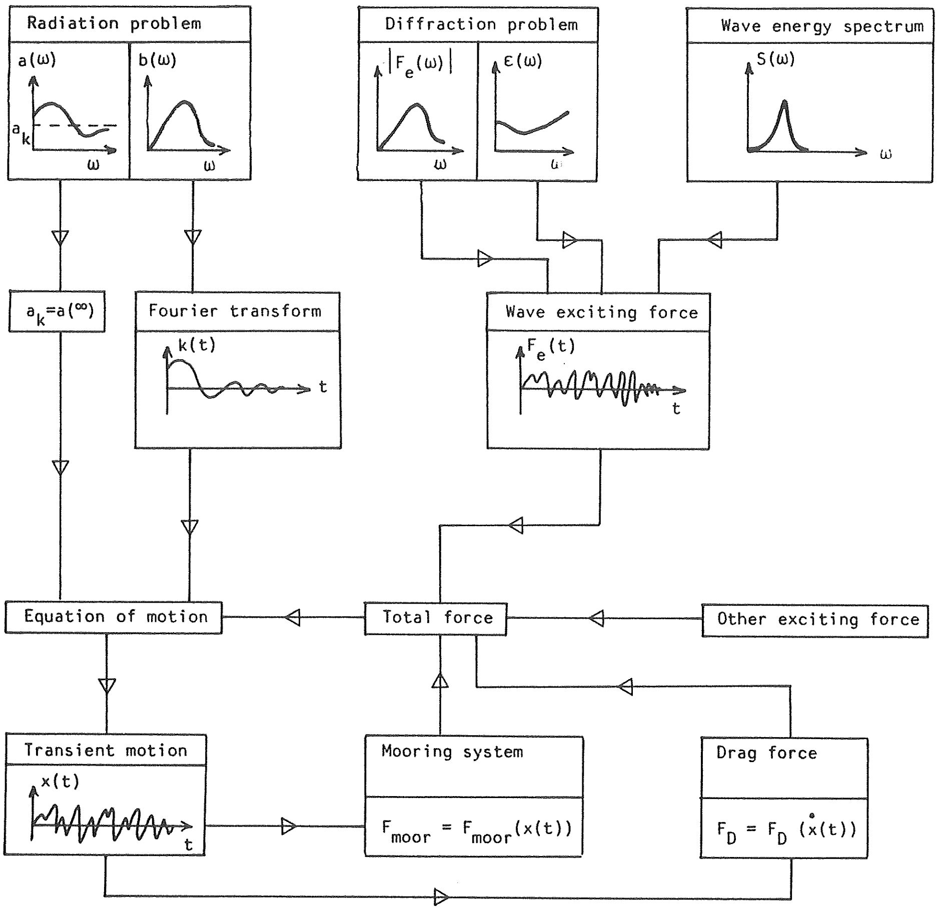

Figure 5.2 A principal scheme for calculating transient motions using the retardation function technique.

5.2 The mass matrix and the hydrostatic stiffness matrix

For a vertical cylinder, the elements of the matrices of the equations of motion can be somewhat simplified. Below, all non-zero elements of the mass matrix and the hydrostatic stiffness matrix are given. They become

$$m_{11} = m_{22} = m_{33} = m$$
$$m_{15} = m_{51} = mz_G$$
$$m_{24} = m_{42} = -mz_G$$
$$m_{44} = I_{xx} = mr_r^2$$
$$m_{55} = I_{yy} = mr_p^2$$
$$m_{66} = I_{zz} = mr_y^2$$

$$c_{33} = \rho g \pi a^2$$
$$c_{44} = \rho g \forall Z_B - mgZ_G + \rho g(\pi a^4)/4$$
$$c_{55} = c_{44}$$

where r_r, r_p and r_y are the respective radii of gyration associated with roll, pitch and yaw motions.

The elements of the added mass matrix and the potential damping matrix are given in Section 5.3.

5.3 The hydrodynamic coefficients as determined by Yeung

As mentioned previously, when solving the radiation problem, the formulation suggested by Yeung (1981) is adopted. His work includes the determination of the added mass and damping coefficients for the heave, sway and roll motions as well as the coupling coefficients between sway and roll. Throughout this section the notations and conventions of Yeung are followed. Consequently, a Cartesian coordinate system Oxyz as well as polar coordinates (r,θ) are defined. The Oxy-plane is located at the sea bottom and the z-axis is positive upwards.

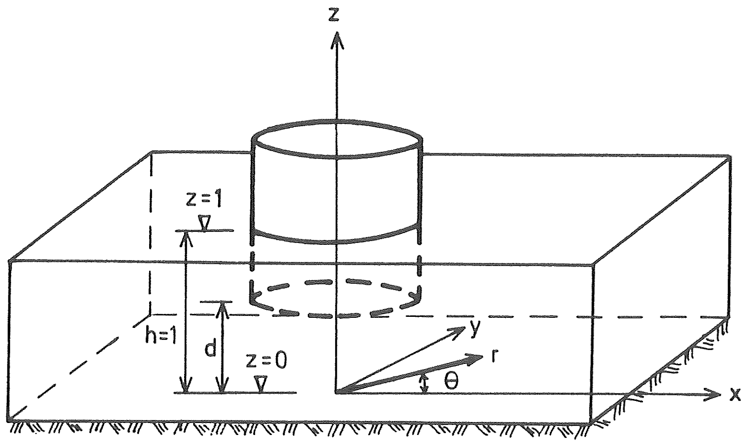

Figure 5.3 Definition of coordinate system and geometric variables.

Geometric variables are non-dimensionalized by the water depth $\bar{h}$, e.g. the non-dimensional radius is given by $a=\bar{a}/\bar{h}$ where $\bar{a}$ is the dimensional radius.

In the present section only the equations included in the boundary value problem is given, together with the necessary equations for calculating the hydrodynamic coefficients.

5.3.1 The heave radiation problem

Consider a vertical cylinder which is assumed to heave harmonically in finite depth water. The velocity potential for radiation is conveniently separated into space and time dependence as

$$\Phi = \bar{h} \text{ Re } \{-i\omega\hat{x}_3\phi e^{-i\omega t}\} \tag{5.1}$$

If the Laplace equation and the boundary conditions are expressed in terms of the spatial velocity potential, the following equations are obtained

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (5.2a)$$

$$\frac{\partial\phi}{\partial z} - \frac{\omega^2}{g}\bar{h}\,\phi = 0 \qquad z=h \qquad (5.2b)$$

$$\frac{\partial\phi}{\partial z} = 1 \qquad z=d \text{ and } 0\leq r\leq a \qquad (5.2c)$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad z=0 \qquad (5.2d)$$

$$\frac{\partial\phi}{\partial r} = 0 \qquad r=a \text{ and } d\leq z\leq h \qquad (5.2e)$$

Yeung divided the fluid domain into an interior and an exterior region. The velocity potentials of these regions are denoted $\phi^{(i)}$ and $\phi^{(e)}$ respectively. The velocity potential of the interior region becomes

$$\phi^{(i)}(r,z) = \frac{1}{2d}\left(z^2 - \frac{r^2}{2}\right) + \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \frac{I_0(\lambda_n r)}{I_0(\lambda_n a)} \cos(\lambda_n z) \qquad (5.3)$$

where

$$\lambda_n = \frac{n\pi}{d} \qquad (5.4)$$

$$\alpha_n = \frac{2}{d}\int_0^d \phi^{(e)}(a,z)\cos(\lambda_n z)\,dz - \alpha_n^* \qquad (5.5)$$

$$\alpha_n^* = \begin{cases} \frac{1}{6}\left(2d - \frac{3a^2}{d}\right) & n=0 \\ \frac{2d(-1)^n}{(n\pi)^2} & n\geq 1 \end{cases} \qquad (5.6)$$

In Eq. (5.3) I_0 is the modified Bessel function of first kind and zero order.

The velocity potential of the exterior region becomes

$$\phi^{(e)}(r,z) = \sum_{k=0}^{\infty} A_k R_k(m_k r)\, Z_k(m_k z) \qquad (5.7)$$

where A_k are unknown coefficients. The radial function R_k is given by

$$R_k(m_k r) = \begin{cases} H_0^{(1)}(m_0 r) & k=0 \\ K_0\ (m_k r) & k\geq 1 \end{cases} \qquad (5.8)$$

where $H_0^{(1)}$ is the Hankel function of first kind and zero order and K_0 is the modified Bessel function of second kind and zero order. The vertical function Z is given by

$$Z_k(m_k z) = \begin{cases} \cosh\ (m_0 z)/N_0^{\frac{1}{2}} & k=0 \\ \cos\ (m_k z)/N_k^{\frac{1}{2}} & k\geq 1 \end{cases} \qquad (5.9)$$

where the normalisation factor N_k is defined by

$$N_k = \begin{cases} \frac{1}{2}(1 + \sinh\ (2m_0)/(2m_0)) & k=0 \\ \frac{1}{2}(1 + \sin\ (2m_k)/(2m_k)) & k\geq 1 \end{cases} \qquad (5.10)$$

The eigenvalues m_k are defined by

$$\begin{cases} m_0 \tanh m_0 = \frac{\omega^2}{g}\,\bar{h} \\ m_k \tan m_k = -\frac{\omega^2}{g}\,\bar{h} \end{cases} \qquad (5.11)$$

and are drawn schematically in Figure 5.4.

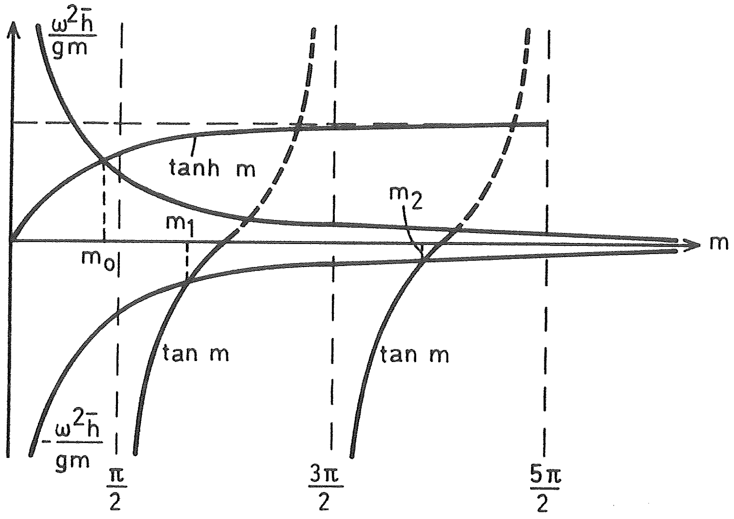

Figure 5.4 The solution of Eq. (5.11) drawn schematically.

By matching the derivatives of Eqs. (5.3) and (5.7) at the common boundary r=a, Yeung derived a system of coupled equations. The system becomes

$$\alpha_n - \sum_{j=0}^{\infty} e_{nj}\alpha_j = g_n \qquad n=0,\ 1,\ \ldots. \qquad (5.12)$$

where

$$e_{nj} = \left(\sum_{k=0}^{\infty} \frac{c_{nk}\ c_{jk}}{m_k} \left(\frac{R_k}{R_k'}\right) \right) S_j \qquad (5.13)$$

$$g_n = \sum_{k=0}^{\infty} \frac{c_{nk}}{m_k} \left(\frac{R_k}{R_k'}\right) A_k^* - \alpha_n^*$$

$$c_{nk} = \begin{cases} \dfrac{2(-1)^n \sinh(m_o d)/(m_o d)}{\left(1 + \left(\frac{n\pi}{m_o d}\right)^2\right) N_o^{\frac{1}{2}}} & k=0 \qquad (5.14) \\[2ex] \dfrac{2\sin(m_k d - n\pi)}{(m_k d - n\pi)} \ \dfrac{m_k d}{(m_k d + n\pi) N_k^{\frac{1}{2}}} & k \geq 1 \end{cases}$$

$$S_n = \frac{n\pi}{2} I_o'(\lambda_n a)/I_o(\lambda_n a) \qquad (5.15)$$

The primes indicate differentiation with respect to the argument. Using matrices Eq. (5.12) becomes

$$\begin{bmatrix} 1-e_{oo} & -e_{o1} & \cdots & -e_{on} \\ -e_{1o} & 1-e_{11} & \cdots & -e_{1n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -e_{no} & -e_{n1} & \cdots & 1-e_{nn} \end{bmatrix} \begin{bmatrix} \alpha_o \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} = \begin{bmatrix} g_o \\ g_1 \\ \cdot \\ \cdot \\ \cdot \\ g_n \end{bmatrix} \tag{5.16}$$

Once Eq. (5.16) is solved the Fourier coefficients α_n are known and the velocity potential of the inerior region is obtained from Eq. (5.3). Furthermore, the unknown coefficients A_k are related to α_n by

$$A_k = \sum_{n=0}^{\infty} \frac{S_n c_{nk} \alpha_n + A_k^*}{m_k R'_k(m_k a)} \tag{5.17}$$

where

$$A_k^* = \begin{cases} -\dfrac{a}{2N_o^{\frac{1}{2}}} \dfrac{\sinh(m_o d)}{m_o d} & k=0 \\ -\dfrac{a}{2N_k^{\frac{1}{2}}} \dfrac{\sin(m_k d)}{m_k d} & k \geq 1 \end{cases} \tag{5.18}$$

The velocity potential of the exterior region then is given by Eq. (5.7).

Finally, the hydrodynamic reaction force is obtained from the Bernoulli equation by integrating the pressure over the bottom of the cylinder. Expressed in terms of the added mass and the potential damping, defined in Section 3.3, this yields

$$a_{33} + ib_{33} = \frac{\bar{a}_{33} + i\bar{b}_{33}/\omega}{\pi\rho\bar{a}^3} = 2\left(\frac{1}{4}\left(\frac{d}{a}\right) - \frac{1}{16}\left(\frac{a}{d}\right) + \frac{1}{a}\left(\frac{\alpha_o}{4} + 2\left(\frac{d}{a}\right)\sum_{n=1}^{\infty}(-1)^n \frac{\alpha_n S_n}{(n\pi)^2}\right)\right) \tag{5.19}$$

5.3.2 The sway (surge) radiation problem

For sway motion the velocity potential is conveniently written

$$\Phi = \bar{h}\ \mathrm{Re}\ \{-i\omega\hat{x}_1\phi\ e^{-i\omega t}\}\ \cos\theta \qquad (5.20)$$

The Laplace equation and the boundary conditions become

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} - \frac{\phi}{r^2} = 0 \qquad (5.21a)$$

$$\frac{\partial\phi}{\partial z} - \frac{\omega^2}{g}\bar{h}\ \phi = 0 \qquad z=h \qquad (5.21b)$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad z=d \text{ and } 0 \leqq r \leqq a \qquad (5.21c)$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad z=0 \qquad (5.21d)$$

$$\frac{\partial\phi}{\partial r} = 1 \qquad r=a \text{ and } d \leqq z \leqq h \qquad (5.21e)$$

The only modifications compared to the solution of the heave radiation problem are

$$\phi^{(i)}(r,z) = \frac{\alpha_0^*}{2}\left(\frac{r}{a}\right) + \sum_{n=1}^{\infty}\alpha_n\frac{I_1(\lambda_n r)}{I_1(\lambda_n a)}\cos\lambda_n z \qquad (5.22)$$

$$\alpha_n^* = 0 \qquad n \geqq 0 \qquad (5.23)$$

$$A_k^* = \begin{cases} (\sinh m_0 - \sinh(m_0 d))/(N_0^{\frac{1}{2}} m_0) & k=0 \\ (\sin m_k - \sin(m_k d))/(N_k^{\frac{1}{2}} m_k) & k \geqq 1 \end{cases} \qquad (5.24)$$

$$R_k/R'_k = \begin{cases} H_1^{(1)}(m_0 a)/H_1^{(1)'}(m_0 a) & k=0 \\ K_1(m_k a)/K_1'(m_k a) & k \geqq 1 \end{cases} \qquad (5.25)$$

$$S_n = \begin{cases} d/4a & n=0 \\ \dfrac{n\pi}{2} \dfrac{I_1'(\lambda_n a)}{I_1(\lambda_n a)} & n\geq 1 \end{cases} \tag{5.26}$$

The hydrodynamic coefficients for the sway motion are given by

$$a_{11} + ib_{11} = \frac{\bar{a}_{11} + i\bar{b}_{11}/\omega}{\pi\rho\bar{a}^{\,2}(\bar{h}-\bar{d})} = \frac{1}{a(d-1)} \sum_{k=0}^{\infty} A_k A_k^* R_k \tag{5.27}$$

5.3.3 The roll (pitch) radiation problem

For roll motion the velocity potential can be written

$$\Phi = \bar{h}^{\,2} \, \mathrm{Re}\,\{-i\,\omega\,\hat{x}_5 \phi\, e^{-i\omega t}\} \cos\theta$$

If the roll motion occurs about the point (0,0,1) the boundary value problem is now as follows:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} - \frac{\phi}{r^2} = 0 \tag{5.28a}$$

$$\frac{\partial\phi}{\partial z} - \frac{\omega^2}{g}\bar{h}\,\phi = 0 \qquad z=h \tag{5.28b}$$

$$\frac{\partial\phi}{\partial z} = -r \qquad z=d \text{ and } 0\leq r\leq a \tag{5.28c}$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad z=0 \tag{5.28d}$$

$$\frac{\partial\phi}{\partial r} = -(h-z) \qquad r=a \text{ and } d\leq z\leq h \tag{5.28e}$$

The only modifications compared to the solution of the sway radiation problem are

$$\phi^{(i)} = -\frac{1}{2d}\left(z^2 r - \frac{r^3}{4}\right) + \frac{\alpha_o r}{2a} + \sum_{n=1}^{\infty} \frac{\alpha_n I_1(\lambda_n r)}{I_1(\lambda_n a)} \cos\lambda_n z \tag{5.29}$$

$$\alpha_n^* = \begin{cases} -ad\left(\frac{1}{3} - \frac{1}{4}\left(\frac{a}{d}\right)^2\right) & n=0 \\ 2ad\,(-1)^{n+1}/(n\pi)^2 & n\geq 1 \end{cases} \tag{5.30}$$

$$A_k^* = N_k^{-\frac{1}{2}} \begin{cases} \dfrac{\sinh(m_o d)}{m_o d}\left(\dfrac{3}{8}a^2 - \dfrac{3}{2}d^2 + d\right) - \\ -\dfrac{1}{m_o^2}\left(\cosh m_o + \dfrac{\sinh(m_o d)}{m_o d} - 2\cosh(m_o d)\right) & k=0 \\ \dfrac{\sin(m_k d)}{m_k d}\left(\dfrac{3}{8}a^2 - \dfrac{3}{2}d^2 + d\right) - \\ -\dfrac{1}{m_k^2}\left(2\cos(m_k d) - \cos m_k - \dfrac{\sin(m_k d)}{m_k d}\right) & k\geq 1 \end{cases} \tag{5.31}$$

The hydrodynamic coefficients for the roll motion become

$$a_{55} + ib_{55} = \frac{\bar{a}_{55} + i\bar{b}_{55}/\omega}{\pi\rho\bar{a}^3\bar{h}^2} = \tag{5.32}$$

$$= \frac{1}{8}\left(ad - \frac{a^3}{6d} - \alpha_o\right) - \frac{d}{a}\sum_{n=1}^{\infty}(-1)^n\left(\frac{I_o(\lambda_n a)/I_1(\lambda_n a)}{n\pi} - \frac{2d}{a(n\pi)^2}\right)\alpha_n +$$

$$+ \left(\frac{1}{a}\right)^2 \sum_{k=0}^{\infty} A_k R_k(m_k a)\, E_k$$

where

$$E_k = N_k^{-\frac{1}{2}} \begin{cases} (d-1)\dfrac{\sinh(m_o d)}{m_o} + \dfrac{\cosh m_o - \cosh(m_o d)}{m_o^2} & k=0 \\ (d-1)\dfrac{\sin(m_k d)}{m_k} - \dfrac{\cos m_k - \cos(m_k d)}{m_k^2} & k\geq 1 \end{cases} \tag{5.33}$$

Finally, the coupling coefficients between sway and roll become

$$a_{51} + ib_{51} = \frac{\bar{a}_{51} + i\bar{b}_{51}/\omega}{\pi\rho\bar{a}^2\bar{h}^2} = (-1)\frac{1}{a}\sum_{k=0}^{\infty} A_k R_k A_k^{*(s)} \tag{5.34}$$

where $A_k^{*(s)}$ is given by Eq. (5.24).

Observe that Eq. (5.31) differs from that of Yeung (1981) in terms of a positive sign. However, the added mass and the damping coefficients for the roll motion are in very good agreement with the results of Yeung and therefore it is probably only misprint that the positive sign is not included in his calculations.

5.4 The wave exciting force

In this section the analytical solution proposed by Yeung has been extended also to include amplitudes and phases of the wave exciting forces.

As mentioned previously, the wave exciting force, for a given indicent wave, may be calculated without knowledge of the velocity potential of the scattered wave. According to Newman (1962) the wave exciting force can be expressed in terms of the velocity potential of the radiated wave. Thus,

$$F_{ei} = \text{Re}\ \{-\hbar^{\ 2} i\omega\rho\ e^{i\omega t} \iint_S (\phi_I \frac{\partial\phi_i}{\partial n} - \phi_i \frac{\partial\phi_I}{\partial n})\ dS\} \quad (5.35)$$

if, respectively,

$$\Phi_I = \text{Re}\ \{\hbar\phi_I e^{i\omega t}\} \quad (5.36)$$

and

$$\Phi_i = \text{Re}\ \{\hbar\ i\omega\phi_i \hat{x}_i e^{i\omega t}\} \quad (5.37)$$

If the incident wave is propagating in the positive x-direction the wave elevation can be written

$$\zeta(t) = \hbar\ \hat{\xi}\ \cos\ (m_o x - \omega t) \quad (5.38a)$$

or if using complex notation

$$\zeta(t) = \text{Re}\ \{\hbar\ \hat{\xi}\ e^{-i(m_o x - \omega t)}\} \quad (5.38b)$$

The spatial potential of the incident wave is given by

$$\phi_I = \frac{ig\hat{\xi}}{\omega}\ \frac{\cosh m_o(z+h)}{\cosh m_o h}\ e^{-im_o x} \quad (5.39a)$$

where the spatial coordinates are referred to a coordinate system with its origin in the free surface and the z-axis positive upwards. Using cylindrical coordinates, Eq. (5.39a) becomes

$$\phi_I = \frac{ig\hat{\zeta}}{\omega} \frac{\cosh m_o(z+h)}{\cosh m_o h} e^{-im_o r \cos\theta} \tag{5.39b}$$

In Section 5.3.1 the heave radiation problem was treated and the velocity potential of the exterior region was given by

$$\phi_3 = \sum_{k=0}^{\infty} A_k R_k(m_k r)\, Z_k(m_k z) \tag{5.40a}$$

where the coefficients A_k are defined by Eq. (5.17), the radial function by Eq. (5.8) and the vertical function by Eq. (5.9). Observe that in Section 5.3 the z-coordinate is referred to a coordinate system with its origin at the sea bottom and the time dependence of the velocity potential is separated using $e^{-i\omega t}$. Referred to a system with its origin in the free surface and separating the time dependence using $e^{i\omega t}$ the velocity potential of the exterior region can be written

$$\phi_3 = \sum_{k=0}^{\infty} \bar{A}_k R_k(m_k r)\, Z_k(m_k z) \tag{5.40b}$$

where $\bar{A}_k$ is the complex conjugate of A_k and the vertical function is defined by

$$Z_k(m_k z) = \begin{cases} \cosh m_o\ (z+h)/N_o^{\frac{1}{2}} & k=0 \\ \cos m_k\ (z+h)/N_k^{\frac{1}{2}} & k\geq 1 \end{cases} \tag{5.41}$$

Before the wave exciting force is developed Eq. (5.35) can be further simplified. Since both ϕ_I and ϕ_i satisfy the free surface condition and the bottom condition, it is a consequence of Green's theorem that

$$\iint_{S+S_\infty} (\phi_I \frac{\partial\phi_3}{\partial n} - \phi_3 \frac{\partial\phi_I}{\partial n})\, dS = 0 \qquad (5.42)$$

where S_∞ is a control surface at some distance from the structure. Hence, by substituting Eq. (5.42) into Eq. (5.35) one obtains

$$F_{e3} = \text{Re}\,\{\, \bar{h}\,^2 i\omega\rho e^{i\omega t} \iint_{S_\infty} (\phi_I \frac{\partial\phi_i}{\partial n} - \phi_3 \frac{\partial\phi_I}{\partial n})\, dS \,\} \qquad (5.43)$$

In Eq. (5.43) the pressure is integrated over the control surface which is conveniently chosen as a cylinder at an infinite distance from the structure.

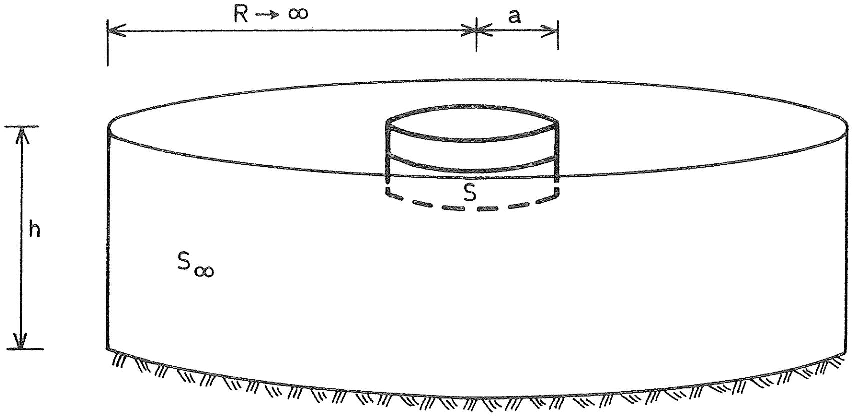

Figure 5.5 Control surface.

For a cylinder the wave exciting force in heave becomes

$$F_{e3} = \text{Re}\,\{\, \bar{h}\,^2 i\omega\rho e^{i\omega t} \int_0^{2\pi}\int_{-d}^{0} (\phi_I \frac{\partial\phi_3}{\partial R} - \phi_3 \frac{\partial\phi_I}{\partial R})\, R\, dz d\theta \} \qquad (5.44)$$

Since we only need know the asymptotic behaviour of the velocity potentials the radial funcion may be simplified. The radial function for large arguments is given by Abramovitz and Stegun (1972) as

$$R_k(m_kR) = \begin{cases} H_o^{(1)}(m_oR) \cong (\frac{2}{\pi m_oR})^{\frac{1}{2}} e^{i(m_oR-(\pi/4))} \\ K_o(m_kR) \cong (\frac{\pi}{2m_kR})^{\frac{1}{2}} e^{-m_kR} \end{cases} \tag{5.45}$$

Since for large arguments the modified Bessel function of the second kind, $K_o(m_kR)$, reaches zero the spatial potential of the radiated wave is given by

$$\phi_3 = \bar{A}_o(\frac{2}{\pi m_oR})^{\frac{1}{2}} e^{i(m_oR-(\pi/4))} \frac{\cosh m_o(z+h)}{N_o^{\frac{1}{2}}} \tag{5.47}$$

By substituting Eqs. (5.39) and (5.47) into Eq. (5.44) the wave exciting force becomes

$$F_{e3} = \mathrm{Re}\ \{\ \bar{h}\ ^2 i\rho g\xi\bar{A}_o e^{i\omega t}\ (\frac{2Rm_o}{\pi})^{\frac{1}{2}}\ e^{i(m_oR-(\pi/4))} P_\theta Q_z\} \tag{5.48}$$

where

$$Q_z = \int_{-h}^{0} \frac{\cosh^2 m_o(z+h)}{\cosh\ (m_oh)N_o^{\frac{1}{2}}}\ dz \tag{5.49}$$

$$P_\theta = \int_0^{2\pi} (1 + \cos\ \theta)\ e^{-im_oR\ \cos\theta}\ d\theta \tag{5.50}$$

The first integral is easily solved and since h=1 one obtains

$$Q_z = \frac{N_o^{\frac{1}{2}}}{\cosh\ m_o} \tag{5.51}$$

The second integral may be solved using the method of stationary phase proposed by Newman (1962). Following Newman (1977) one obtains

$$P_\theta = \left(\frac{8\pi}{Rm_0} \right)^{\frac{1}{2}} e^{-i(m_0 R-(\pi/4))} \tag{5.52}$$

Finally, by substituting Eqs. (5.51) and (5.52) into Eq. (5.48) the wave exciting force in heave becomes

$$F_{e3} = \mathrm{Re}\, \{ \hbar^{\,2} i\, 4\rho g \xi\, \bar{A}_0 \frac{N_0^{\frac{1}{2}}}{\cosh m_0} e^{i\omega t} \} \tag{5.53a}$$

Analogously for the sway force and the roll moment one obtains, respectively,

$$F_{e1} = \mathrm{Re}\, \{ \hbar^{\,2} i\, 4\rho g \xi\, \bar{A}_0^{(s)} \frac{N_0^{\frac{1}{2}}}{\cosh m_0} e^{i(\omega t-(\pi/2))} \} \tag{5.53b}$$

and

$$F_{e5} = \mathrm{Re}\, \{ \hbar^{\,3} i\, 4\rho g \xi\, \bar{A}_0^{(r)} \frac{N_0^{\frac{1}{2}}}{\cosh m_0} e^{i(\omega t-(\pi/2))} \} \tag{5.53c}$$

where $\bar{A}_0^{(s)}$ indicates coefficients from the solution of the sway radiation problem and $\bar{A}_0^{(r)}$ coefficients from the roll radiation problem.

Observe that Eqs. (5.53a-c) are complex quantities from which the amplitude and the phase can be determined. It is convenient to write the wave exciting force in the form

$$F_i = \xi\, |X_i| \cos(\omega t + \varepsilon_i) \tag{5.54}$$

where $|X_i|$ is the amplitude of the force due to a wave with unit amplitude. The amplitude can be checked by using the Newman relations, Eq. (3.45). Expressed in agreement with the notations of Yeung they become

$$|X_1| = \pi\rho g\bar{a}^2 \left(\frac{4D(1-d)}{\pi m_o a^2} b_{11} \right)^{\frac{1}{2}} \tag{5.55a}$$

$$|X_3| = \pi\rho g\bar{a}^2 \left(\frac{2D}{\pi m_o a} b_{33} \right)^{\frac{1}{2}} \tag{5.55b}$$

$$|X_3| = \pi\rho g\bar{a}^3 \left(\frac{4D}{\pi m_o a^3} b_{55} \right)^{\frac{1}{2}} \tag{5.55c}$$

where $D = \tanh m_o - m_o(1-\tanh^2 m_o)$.

5.5 Description of the sea

The sea state is usually described using one of two different methods, the design wave method or the spectrum method. In the design wave method the sea state is considered as characterised by a single, regular wave with a fixed amplitude and frequency. This method can be used when the natural frequency of the structure is far from the frequency content of the original sea state. Then the response of the structure is received as a deterministic quantity in the frequency domain if the system is linear or in the time domain if significant non-linearities exist. However, this regular wave does not give a realistic description of the sea state. For most applications a stochastic description of the sea state is more appropriate. A stochastic description of the sea state is usually characterised by a continuous wave energy spectrum. Again if the system is considered linear the analysis is suitably performed in the frequency domain but if significant non-linearities exist a time domain analysis has to be used.

Conventionally the wave elevation is considered relative to the mean level of the free surface as a stationary Gaussian distributed stochastic process. Such a process has the advantage that if a linear operator acts on the process the response will also be a Gaussian distributed process. Thus, in a frequency domain analysis the response spectrum is determined, via a linear transfer function, from the wave energy spectrum. Subsequently, maximum values for the Gaussian distributed response can be statistically evaluated, according to Cartwright and Longuet-Higgins (1956). If, however, significant non-linearities exist in the dynamic system the analysis ought to be performed in the time domain. The spectral description of the sea state, i.e. the wave energy spectrum, then has to be converted to the time domain. In order to describe the wave elevation a quasi-stochastic time domain realisation can be created from the wave energy spectrum as a series of sinusoidal waves. Thus the wave elevation can be written

$$\zeta(t) = \sum_{n=1}^{N} c_n \cos(\omega_n t + \theta_n) \tag{5.56}$$

where

$c_n = (2S(\omega_n)\,\Delta\omega)^{\frac{1}{2}}$

$S(\omega)$ = wave energy spectrum

$\Delta\omega$ = frequency step

ω_n = $n\Delta\omega$

θ_n = phase angles randomly chosen in the interval $(0,2\pi)$

Since the original wave elevation that gave the wave energy spectrum is considered as a stationary Gaussian distributed process the recreated wave elevation must also possess these properties. This requires N, in principle, to be set infinitely large. This difficulty and other problems associated with time realisations from wave energy spectra are beyond the scope of this work but have been discussed, for example, by Tucker et al (1984) and Elgar et al (1985).

The time realisation of the wave exciting force corresponding to the wave elevation given by Eq. (5.56) is obtained from

$$F_e(t) = \sum_{n=1}^{N} c_n \, |X_n| \, \cos(\omega_n t + \theta_n + \varepsilon_n) \tag{5.57}$$

where $|X_n|$ is the amplitude of the wave exciting force due to a wave with frequency ω_n and unit amplitude. The phase ε_n is the time-lag between the incident wave and the wave exciting force.

Since the frequency spacing $\Delta\omega$ in Eq. (5.56) is kept constant both the wave elevation $\zeta(t)$ and the wave exciting force $F_e(t)$ will repeat with the period $T=2\pi/\Delta\omega$.

5.6 The mooring system

In the application in this report the structure is assumed to be tethered with pretensioned wire ropes. Initially, when there is no external force acting on the cylinder, the wire ropes are assumed to be vertical and the bottom of the cylinder is assumed to be parallel with the free surface. Below, reaction forces of the mooring system are developed as functions of the motions of the structure.

Again, use is made of the earth fixed system Oxyz located with its origin in the free surface with the positive z-axis pointing upwards. $\bar{O}\bar{x}\bar{y}\bar{z}$ is a structure-fixed coordinate system which initially coincides with Oxyz. $(\tilde{O}\tilde{x}\tilde{y}\tilde{z})_i$ is an earth-fixed system with its origin located to the point where the i-th wire rope is connected to the sea bottom.

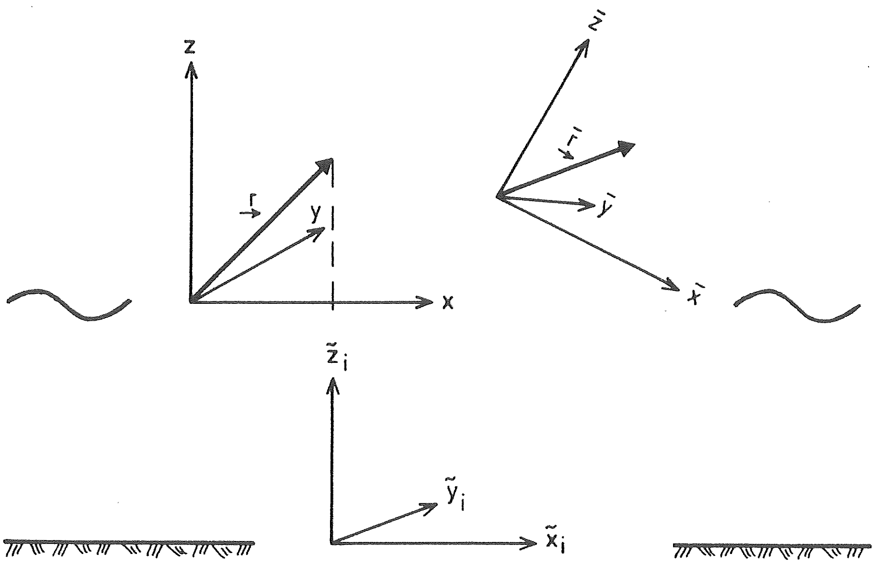

Figure 5.6 Definition of coordinate systems associated with the mooring system.

Since the cylinder is regarded as rigid, the body motions are described by three translations (x_1,x_2,x_3) and three rotations (x_4,x_5,x_6). As has been discussed in Section 3.5.1 relations between Oxyz and $\bar{O}\bar{x}\bar{y}\bar{z}$ can be linearised if the rotations are assumed to be small. Thus, if the cylinder is translated and rotated the coordinates of the fairlead of the i-th wire rope, expressed in xyz-coordinates, become

$$\vec{r}_i = \vec{r}_o + R\vec{\bar{r}}_i \tag{5.58}$$

where

$$\vec{r}_o = x_1\vec{i} + x_2\vec{j} + x_3\vec{k} \tag{5.59}$$

$$R = \begin{bmatrix} 1 & -x_6 & x_5 \\ x_6 & 1 & -x_4 \\ -x_5 & x_4 & 1 \end{bmatrix} \tag{5.60}$$

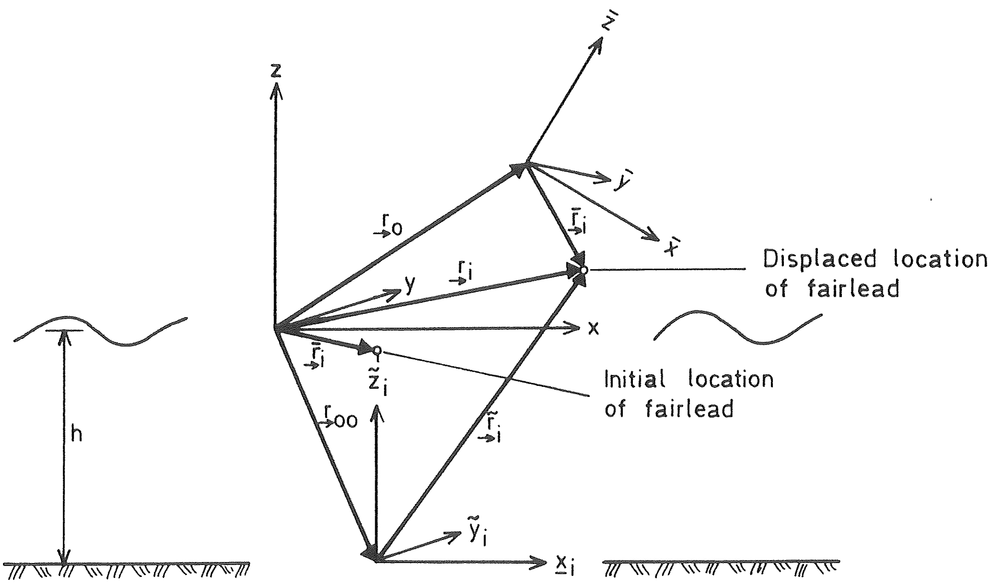

Figure 5.7 Initial location and displaced location of the fairlead of the i-th wire rope.

Expressed in $\tilde{x}\tilde{y}\tilde{z}$-coordinates the coordinates of the fairlead become

$$\vec{\tilde{r}}_i = \vec{r}_i - \vec{r}_{oo} \tag{5.61}$$

where

$$\underset{\rightarrow}{r}_{oo} = \bar{x}_i \underset{\rightarrow}{i} + \bar{y}_i \underset{\rightarrow}{j} - h\underset{\rightarrow}{k} \tag{5.62}$$

By substituting Eq. (5.58) into Eq. (5.61) one obtains

$$\underset{\rightarrow}{\tilde{r}}_i = \underset{\rightarrow}{r}_o - \underset{\rightarrow}{r}_{oo} + R\underset{\rightarrow}{\bar{r}}_i \tag{5.63a}$$

or, using matrices,

$$\begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \\ \tilde{z}_i \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ -h \end{bmatrix} + \begin{bmatrix} 1 & -x_6 & x_5 \\ x_6 & 1 & -x_4 \\ -x_5 & x_4 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{z}_i \end{bmatrix} \tag{5.63b}$$

Hence, by using Eq. (5.63) for a given water depth and given structure-fixed coordinates of the fairlead, we can calculate the coordinates $(\tilde{x}\tilde{y}\tilde{z})_i$ of the fairlead when the structure is moving rigidly.

Furthermore, the magnitude of the reaction force of the i-th wire rope is denoted by T_i and is assumed always to act through the origin $\tilde{O}$.

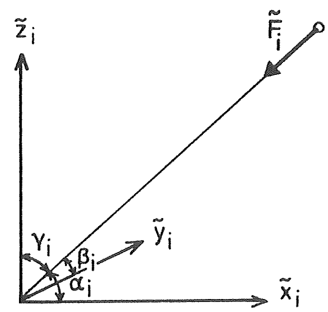

Figure 5.8 Definition of angles.

Then we can express the reaction force as a vector quantity $\tilde{F}_i$, defined in Figure 5.8, as

$$\begin{bmatrix} \tilde{F}_{i,x} \\ \tilde{F}_{i,y} \\ \tilde{F}_{i,z} \end{bmatrix} = - | T_i | \begin{bmatrix} \cos \alpha_i \\ \cos \beta_i \\ \cos \gamma_i \end{bmatrix} \tag{5.64a}$$

where

$$\cos \alpha_i = \frac{\tilde{x}_i}{(\tilde{x}_i^2 + \tilde{y}_i^2 + \tilde{z}_i^2)^{\frac{1}{2}}} \tag{5.64b}$$

$$\cos \beta_i = \frac{\tilde{y}_i}{(\tilde{x}_i^2 + \tilde{y}_i^2 + \tilde{z}_i^2)^{\frac{1}{2}}} \tag{5.64c}$$

$$\cos \gamma_i = \frac{\tilde{z}_i}{(\tilde{x}_i^2 + \tilde{y}_i^2 + \tilde{z}_i^2)^{\frac{1}{2}}} \tag{5.64d}$$

It should be noted that, when establishing the mass matrix, the angular momentum was referred to the translated origin and therefore the external moment should also be referred to this point. Consequently, the moments associated with the i-th wire rope are obtained from

$$\underset{\rightarrow}{M}_i = (R\underset{\rightarrow}{\bar{r}}_i) \times \underset{\rightarrow}{\tilde{F}} \tag{5.64e}$$

By adding the contributions from all wire ropes, forces and moment of the complete mooring system become

$$F_x = \sum_i \tilde{F}_{i,x} \tag{5.65a}$$

$$F_y = \sum_i \tilde{F}_{i,y} \tag{5.65b}$$

$$F_z = \sum_i \tilde{F}_{i,z} \tag{5.65c}$$

$$M_x = \sum_i (R\underset{\rightarrow}{\bar{r}}_i)_y \tilde{F}_{i,z} - (R\underset{\rightarrow}{\bar{r}}_i)_z \tilde{F}_{i,y} \tag{5.65d}$$

$$M_y = \sum_i - (R\underset{\rightarrow}{\bar{r}}_i)_x \tilde{F}_{i,z} + (R\underset{\rightarrow}{\bar{r}}_i)_z \tilde{F}_{i,x} \tag{5.65e}$$

$$M_z = \sum_i (R\underset{\rightarrow}{\bar{r}}_i)_x \tilde{F}_{i,y} - (R\underset{\rightarrow}{\bar{r}}_i)_y \tilde{F}_{i,x} \tag{5.65f}$$

Finally, we need to calculate the reaction force T_i. This may be done by using some simple relations:

Initial length of wire rope: $L_{i,o} = h + \bar{z}_i$ (5.66a)

Length of wire rope: $L_i = (\tilde{x}_i^2 + \tilde{y}_i^2 + \tilde{z}_i^2)^{\frac{1}{2}}$ (5.66b)

Extension of wire rope: $\Delta L_i = (L_i - L_{i,o})$ (5.66c)

Pretension: $T_{i,o}$ (5.66d)

Additional force: $\Delta T_i = (\Delta L_i / L_{i,o}) EA$ (5.66e)

Reaction force of wire rope: $T_i = T_{i,o} + \Delta T_i$ (5.66f)

The axial stiffness of the wire ropes, EA, is the product of the modulus of elasticity and the cross sectional area of the wire rope.

5.7 Numerical formulation

5.7.1 The retardation function

The retardation function $k_{ij}(t)$ is obtained from the inverse Fourier transform

$$k_{ij}(t) = \frac{2}{\pi} \int_0^\infty b_{ij}(\omega) \cos \omega t \, d\omega \qquad (5.67)$$

In a numerical evaluation the infinite integral has to be truncated and a proper frequency spacing has to be chosen. Since it is quite time consuming to evaluate the damping coefficients it is important not to keep the frequency spacing too large, but for large values of t, cos ωt will vary rapidly with frequency, which implies that the frequency spacing should be kept not too large.

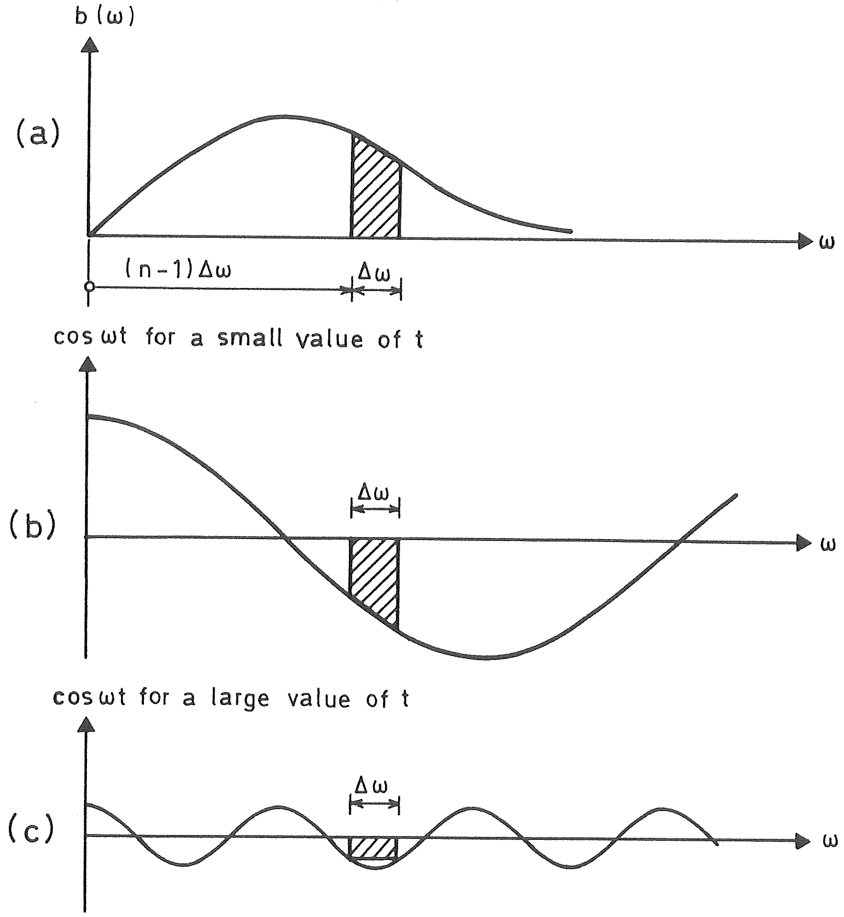

Figure 5.9 Numerical evaluation of the retardation function. The frequency spacing $\Delta\omega$ is set small enough to evaluate $b(\omega)$ with good accuracy, Figure 5.8a. When t is small $\Delta\omega$ will also be small enough to evaluate cos ωt properly, Figure 5.8b, but for large values of t, $\Delta\omega$ is too large to give a proper value of the integral over cos ωt.

One practical way of overcoming the problems associated with frequency spacing is to use different values of the spacing for the damping coefficient and for cos ωt. The frequency spacing $\Delta\omega$ for the damping coefficient is chosen so that the retardation function, for t=0, is evaluated with the desired accuracy. If the trapezoidal method of numerical integration is used, $k_{ij}(0)$ becomes

$$k_{ij}(0) = \sum_{n=1}^{N} \tfrac{1}{2} \left(b_{ij}((n-1)\Delta\omega) + b_{ij}(n\Delta\omega)\right)\Delta\omega \qquad (5.68)$$

where N is chosen large enough not to give a significant truncation error.

When evaluating the retardation function for $t>0$ it is important that a cycle of $\cos \omega t$ is described at a certain number of points. If the required number of points is denoted by N_c, then the associated spacing of $\cos \omega t$ becomes

$$\Delta\omega_2 = \frac{2\pi}{N_c t} \tag{5.69}$$

If $\Delta\omega_2$ is chosen such that $M=\Delta\omega/\Delta\omega_2$ is an integer and, say, $N_c \geqq 10$ then the retardation function is given by

$$k_{ij}(t) = \sum_{n=1}^{N} \{\tfrac{1}{2}(b_{ij}((n-1)\Delta\omega) + b_{ij}(n\Delta\omega)) \sum_{m=1}^{M} \tfrac{1}{2}(\cos\{((n-1)\Delta\omega+(m-1)\Delta\omega_2)t\} +$$

$$+ \cos\{((n-1)\Delta\omega + m\Delta\omega_2)t\})\Delta\omega_2\} \tag{5.70}$$

5.7.2 The equations of motion

The equations of motion expressed in terms of the retardation function become

$$\underline{\underline{M}}\,\ddot{\underline{x}}\,(t) + \int_0^t \underline{\underline{k}}\,(t-\tau)\,\dot{\underline{x}}\,(\tau)\,d\tau + \underline{\underline{c}}x(t) = \underline{f}(t) \tag{5.71}$$

where

$$\underline{\underline{M}} = \underline{\underline{m}} + \underline{\underline{a}}_k$$

In order to solve the equations a numerical procedure must be chosen. In the application in this report a central difference method is used, in which the derivatives of the unknown variable are approximated with expressions based on a quadratic approximation.

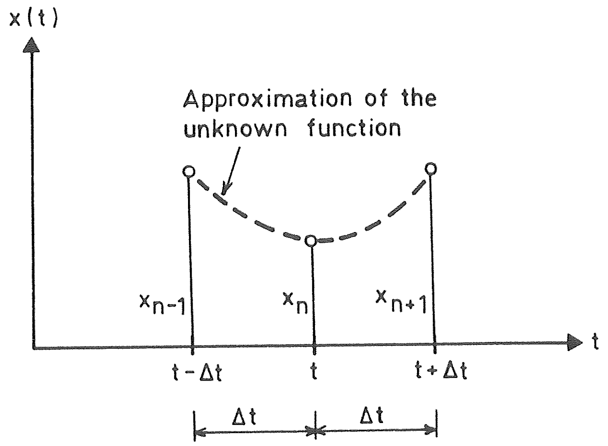

Figure 5.10 The central difference method based on a quadratic approximation of the unknown variable.

If the unknown variable is approximated locally as a parabola and is running through the points x_{n-1}, x_n and x_{n+1} then the approximations of the first and second derivatives with respect to time become

$$\dot{\underline{x}}_n \cong \frac{\underline{x}_{n+1} - \underline{x}_{n-1}}{2\Delta t} \tag{5.72a}$$

$$\ddot{\underline{x}}_n \cong \frac{\underline{x}_{n+1} - 2\,\underline{x}_n + \underline{x}_{n-1}}{\Delta t^2} \tag{5.72b}$$

As an initial condition we state that $\underline{x}_0 = \dot{\underline{x}}_0 = 0$. Before the unknown variable can be solved, the convolution integral in Eq. (5.71) must be approximated as a series. Using a trapezoidal method and the same time step as before we can write

$$\int_0^t \underline{\underline{k}}\,(t-\tau)\;\dot{\underline{x}}\,(\tau)\;d\tau \cong \sum_{k=0}^{n} c_k\;\underline{\underline{k}}_{n-k}\dot{\underline{x}}_k\Delta t \tag{5.73a}$$

where

$$c_k = \begin{cases} \frac{1}{2} \text{ for } k=0 \text{ and } k=n \\ 1 \text{ for } 1 \leq k \leq n-1 \end{cases}$$

Using the initial condition $\dot{x}_0=0$ and substituting Eq. (5.72a) into (5.73a) gives

$$\int_0^t \underline{\underline{k}}(t-\tau)\, \underline{\dot{x}}(\tau)d\tau \cong c_0\, \tfrac{1}{2}\, \underline{\underline{k}}_0(\underline{x}_{n+1} - \underline{x}_{n-1}) + \tfrac{1}{2} \sum_{k=1}^{n-1} \underline{\underline{k}}_{n-k}(\underline{x}_{k+1} - \underline{x}_{k-1}) \qquad (5.73b)$$

Finally, substituting Eqs. (5.72) and (5.73) into (5.71) and solving for $\underline{x}_{n+1}$ yields

$$\underline{x}_{n+1} = \left(\frac{1}{\Delta t^2}\underline{\underline{M}} + c_0\tfrac{1}{2}\underline{\underline{k}}_0\right)^{-1} \Big(f_n - \tfrac{1}{2}\sum_{k=1}^{n-1}\underline{\underline{k}}_{n-k}(\underline{x}_{k+1} - \underline{x}_{k-1}) -$$

$$- \left(\underline{\underline{c}} - \frac{2}{\Delta t^2}\underline{\underline{M}}\right)\underline{x}_n - \left(\frac{1}{\Delta t^2}\underline{\underline{M}} - c_0\,\tfrac{1}{2}\,\underline{\underline{k}}_0\right)\underline{x}_{n-1}\Big) \qquad (5.74)$$

Eq. (5.74) is the numerical formulation of the equations of motion in the time domain. As a starting procedure the following relations can be used

$$\underline{x}_0 = 0 \qquad (5.75a)$$

$$\underline{x}_1 = \tfrac{1}{2}\Delta t^2\, \underline{\underline{M}}^{-1}\underline{f}_0 \qquad (5.75b)$$

Then $\underline{x}_2$, $\underline{x}_3$, $\underline{x}_4$, ... can be computed successively by using Eq. (5.74).

Of course a drag force, as well as any other force can be included in the right hand side of the equations of motion, as long as a time relisation of the force can be found. The drag force is usually calculated using the relative velocity between the fluid and the structure. Usually the drag force has no significance on the motions of the structure, but when the structure reaches resonance even a small amount of damping can be of importance. In such cases the velocity of the structure becomes much higher than the fluid velocity and therefore it is a reasonable simplification to neglect the fluid velocity. Then the non-linear drag force simply becomes a function of the velocity of the structure and can easily be included in the calculation. The problem of finding a proper drag coefficient does, of course remain.

The explicit scheme described above turns out to solve the motions of the structure with sufficient accuracy to give a proper evaluation of the hydrodynamic reaction force. However, the system is very stiff in the axial direction of the wire ropes which causes a significant local error in the evaluation of the reaction force of the mooring system. This problem can be overcome without too much increase in processing time since all forces except the reaction force from the mooring system are sufficiently evaluated using the explicit scheme. An implicit correction based on an estimate of the local error can be applied to the reaction force from the mooring system. Thus the motion of structure becomes

$$\underline{x}_{n+1}^{(i)} = \underline{x}_{n+1}^{(1)} + \frac{1}{12}\,\underline{\underline{M}}^{-1}(\underline{F}_{moor,n+1}^{(i-1)} - 2\,\underline{F}_{moor,n} + \underline{F}_{moor,n-1}) \qquad (5.76)$$

where $\underline{x}_{n+1}^{(1)}$, the starting value in the implicit correction scheme, is given from Eq. (5.74).

5.8 Numerical calculations

In the present section numerical results associated with the solution of the motions of a vertical cylinder are given. The calculations include a frequency domain solution as well as a time simulation. All results, except those given in Tables 5.1-5.3, refer to the cylinder defined in Figure 5.11.

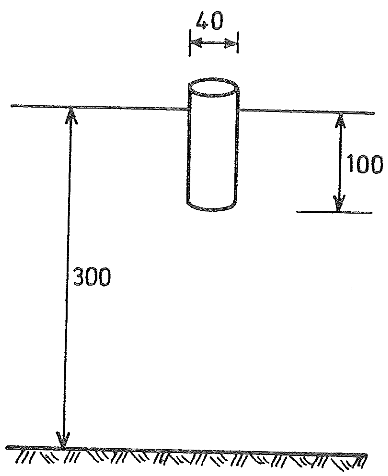

Figure 5.11 Geometry of the cylinder.

5.8.1 The frequency domain

The theory associated with the frequency domain solution was reviewed in Sections 5.3 and 5.4. The results obtained for the hydrodynamic coefficients have been compared with the results obtained by Yeung (1981) and, as far as the resolution of the graphs of Yeung allow comparison, they are in full agreement. The hydrodynamic coefficients associated with the cylinder defined by Figure 5.11 are presented below, non-dimensionalised according to Section 5.3.

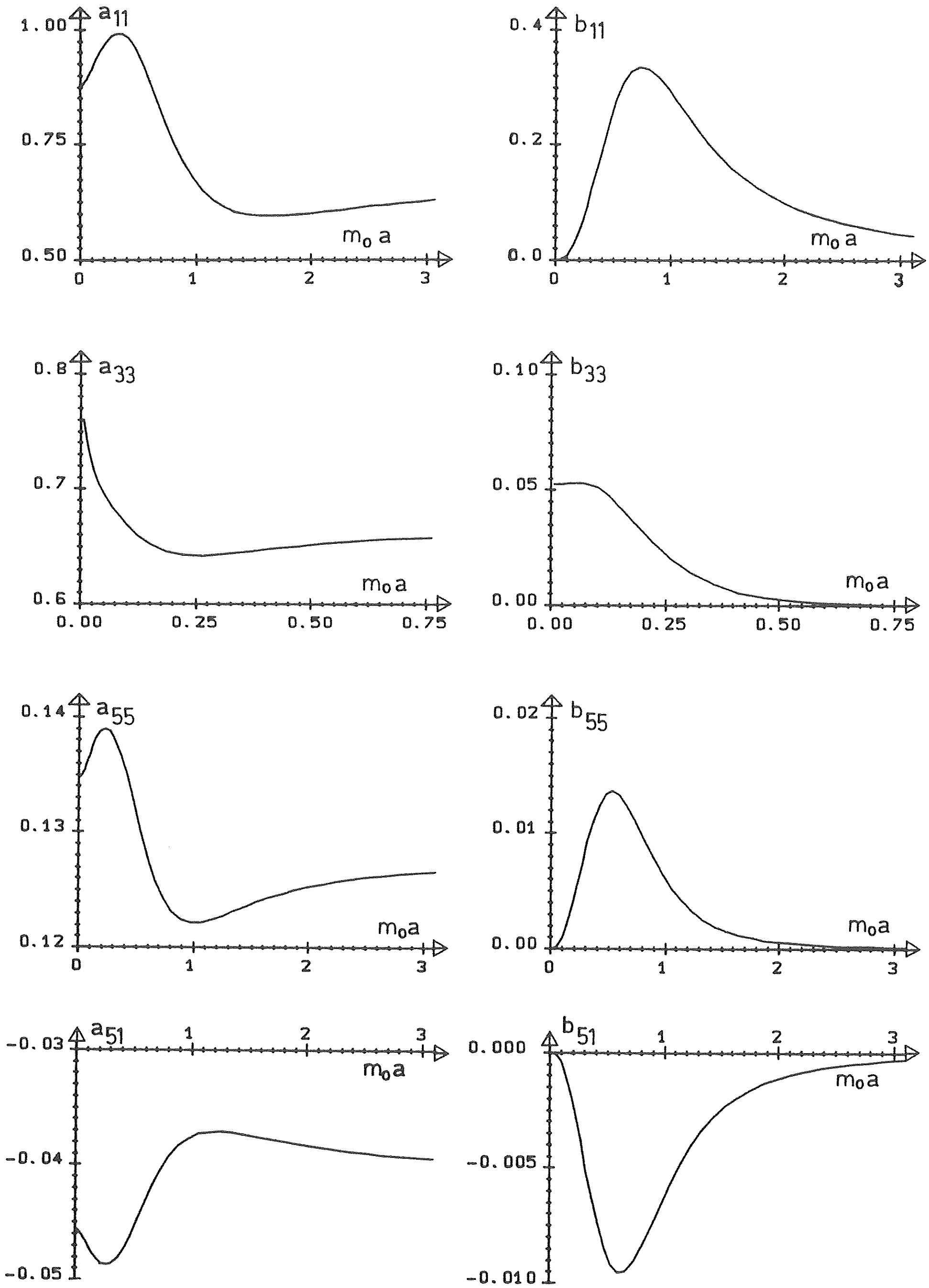

Figure 5.12 The hydrodynamic coefficients for the cylinder defined in Figure 5.11. The coefficients have been non-dimensionalised according to Section 5.3.

The wave exciting force obtained from Eqs (5.53a-c) has been compared with the results of Garret (1970) who solved the diffraction problem. Again, as far as the resolution of Garret's graphs allow comparison, the results are in full agreement. The amplitudes and the phases associated with the cylinder defined by Figure 5.11 are presented below.

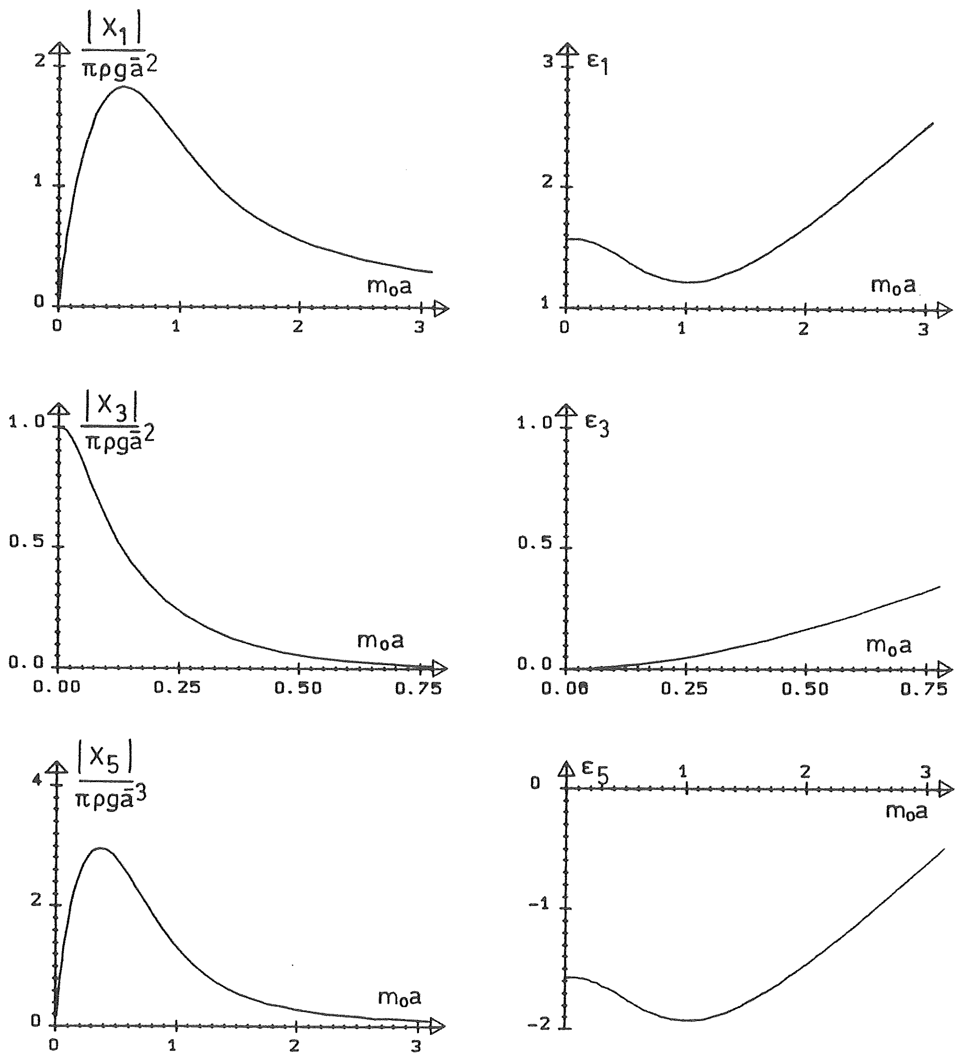

Figure 5.13 The wave exciting force for the cylinder defined by Figure 5.11 expressed in terms of amplitudes and phases.

It should be emphazised that Garret took the option of plotting the phases as continuous curves rather than by plotting them in the interval $(-\pi,\pi)$. Then the information of half a period get lost. For example the phases associated with the horizontal force and the torque are not always identical but could differ by 180 degrees.

As an additional reliability check, the frequency domain results have been compared with corresponding results from the computer program WADIF (NV1459), a program based on a Greens function formulation and described by Faltinsen and Michelsen (1975). The results, supplied by Götaverken Arendal AB, have been calculated for a cylinder approximated by rectangular elements, where the diagonal D of each element fulfilled the requirement $D<L_{min}/7$, where L_{min} is the shortest wave-length of the incident waves. The radius of the cylinder is 50 m, the depth of submergence 25 m, and the water depth 100 m. The comparison is shown below in Tables 5.2-5.3. The dimensionalisation factors, presented in Table 5.1, are those of WADIF with a characteristic length of 100 m. When multiplied by the non-dimensional values in Tables 5.2 and 5.3 they give the corresponding values expressed in SI-units.

Table 5.1 Dimensionalisation factors associated with the Tables 5.2-5.3.

i	j	a_{ij}	b_{ij}	$\lvert x_i \rvert$
1	1	$0.1964\cdot10^{9}$	$0.6150\cdot10^{8}$	$0.1926\cdot10^{8}$
3	3	$0.1964\cdot10^{9}$	$0.6150\cdot10^{8}$	$0.1926\cdot10^{8}$
5	5	$0.1964\cdot10^{13}$	$0.6150\cdot10^{12}$	$0.1926\cdot10^{10}$
5	1	$0.1964\cdot10^{11}$	$0.6150\cdot10^{10}$	-

The frequency range covered in the tables below corresponds to wave lengths in the interval 100-700 meters. The comparison shows satisfactory agreement. In Table 5.3 it can be noted that the phase angles associated with the horizontal force and the torque are either equal or they differ by 180 degrees.

Table 5.2 Added mass coefficients and damping coefficients computed according to Section 5.3 and compared with results from the Greens function program WADIF.

Angular frequency ω	Own calculations				WADIF			
	a_{11}	a_{33}	a_{55}	a_{51}	a_{11}	a_{33}	a_{55}	a_{51}
0.251	0.508	1.23	0.0342	-0.0063	0.516	1.25	0.0342	-0.0063
0.302	0.538	1.13	0.0341	-0.0071	0.546	1.15	0.0341	-0.0072
0.348	0.559	1.06	0.0341	-0.0089	0.568	1.08	0.0340	-0.0090
0.401	0.557	1.00	0.0339	-0.0125	0.565	1.02	0.0339	-0.0126
0.447	0.515	0.97	0.0339	-0.0162	0.522	0.98	0.0339	-0.0164
0.483	0.452	0.96	0.0340	-0.0188	0.458	0.97	0.0339	-0.0190
0.528	0.358	0.96	0.0341	-0.0203	0.363	0.97	0.0340	-0.0205
0.554	0.303	0.96	0.0341	-0.0202	0.307	0.97	0.0341	-0.0204
0.584	0.246	0.97	0.0342	-0.0195	0.249	0.98	0.0342	-0.0196
0.620	0.191	0.99	0.0342	-0.0180	0.194	1.00	0.0342	-0.0181
0.663	0.141	1.01	0.0342	-0.0157	0.144	1.01	0.0342	-0.0157
0.785	0.070	1.06	0.0341	-0.0095	0.072	1.10	0.0340	-0.0094

Angular frequency ω	Own calculations				WADIF			
	b_{11}	b_{33}	b_{55}	b_{51}	b_{11}	b_{33}	b_{55}	b_{51}
0.251	0.043	0.469	0.0002	0.0027	0.044	0.474	0.0002	0.0027
0.302	0.093	0.502	0.0003	0.0049	0.095	0.507	0.0003	0.0049
0.348	0.173	0.507	0.0003	0.0073	0.176	0.511	0.0003	0.0074
0.401	0.323	0.482	0.0003	0.0097	0.327	0.485	0.0003	0.0096
0.447	0.484	0.436	0.0002	0.0092	0.490	0.438	0.0002	0.0091
0.483	0.609	0.386	0.0001	0.0063	0.616	0.387	0.0001	0.0060
0.528	0.722	0.318	0.0000	0.0002	0.729	0.318	0.0000	-0.0003
0.554	0.761	0.277	0.0000	-0.0041	0.768	0.277	0.0000	-0.0047
0.584	0.783	0.232	0.0001	-0.0090	0.790	0.231	0.0001	-0.0097
0.620	0.783	0.185	0.0003	-0.0142	0.790	0.182	0.0003	-0.0150
0.663	0.760	0.136	0.0005	-0.0193	0.766	0.132	0.0005	-0.0202
0.785	0.636	0.050	0.0011	-0.0263	0.641	0.057	0.0011	-0.0276

Table 5.3 Amplitudes and phase angles of the wave exciting force calculated according to Section 5.4 and compared with results from the Greens function program WADIF.

Angular frequency ω	Own calculations			WADIF			Own calculations			WADIF		
	$\lvert X_1\rvert$	$\lvert X_3\rvert$	$\lvert X_5\rvert$	$\lvert X_1\rvert$	$\lvert X_3\rvert$	$\lvert X_5\rvert$	ε_1	ε_3	ε_5	ε_1	ε_3	ε_5
0.251	1.19	2.77	0.0730	1.19	2.78	0.0726	88	8	88	88	8	88
0.302	1.44	2.36	0.0748	1.44	2.36	0.0743	87	12	87	87	12	87
0.348	1.64	1.99	0.0696	1.65	1.99	0.0690	85	17	85	85	17	85
0.401	1.83	1.58	0.0548	1.83	1.58	0.0539	83	25	83	83	25	83
0.447	1.90	1.27	0.0362	1.90	1.27	0.0352	82	33	82	82	33	82
0.483	1.88	1.06	0.0195	1.88	1.06	0.0184	82	41	82	82	41	82
0.528	1.78	0.83	0.0004	1.77	0.83	0.0007	85	52	85	85	52	-95
0.554	1.69	0.72	0.0091	1.69	0.72	0.0103	88	59	-92	88	60	-92
0.584	1.57	0.61	0.0181	1.57	0.60	0.0193	92	68	-88	93	69	-87
0.620	1.44	0.49	0.0261	1.43	0.49	0.0273	100	79	-80	100	80	-80
0.663	1.28	0.38	0.0324	1.27	0.37	0.0337	111	94	-69	111	95	-69
0.785	0.91	0.18	0.0374	0.90	0.20	0.0389	153	142	-27	153	143	-27

5.8.2 The time domain

In the present section a time simulation of the dynamic behaviour of the cylinder defined in Figure 5.11 is presented. The cylinder is tethered with a group of steel tendons running from the center of the bottom of the cylinder to the sea bed. In the simulation the cylinder is exposed to a constant horisontal force and a force due to irregular waves. The constant force roughly simulates contributions from wind, currents and drift forces. A time realisation of the wave exciting force is calculated using Eq. (5.57) in which a Pierson-Moskowitz wave energy spectrum has been chosen. The Pierson-Moskowitz spectrum describes the sea as fully developed and is determined by one parameter, the wind speed, while the fetch and the duration are assumed not to limit the development of the waves. As an alterantive to the wind speed the significant wave height, H_s, can be used as a parameter, see for example Chakrabarti (1980).

The retardation functions associated with the cylinder in Figure 5.11 were calculated using the numerical formulation described in Section 5.7.1. The results are shown in Figure 5.14.

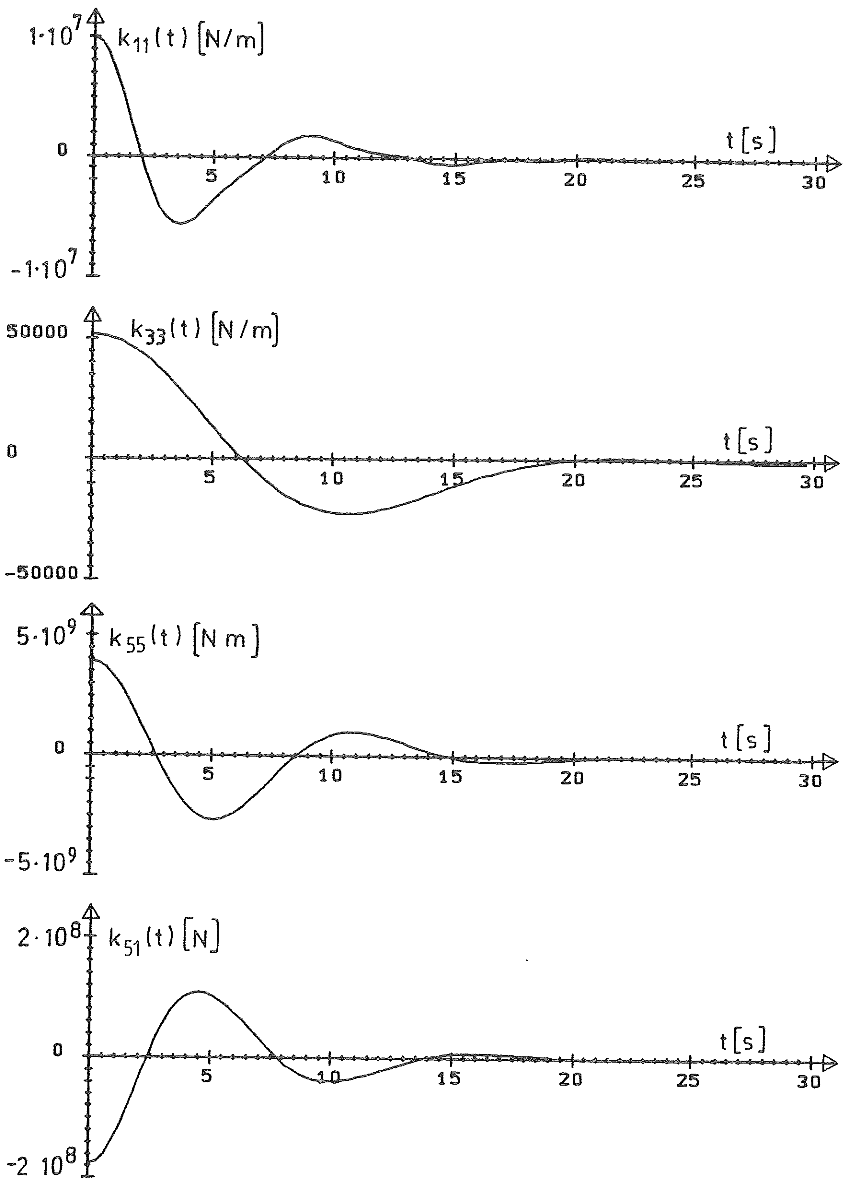

Figure 5.14 The retardation functions for the cylinder defined in Figure 5.11.

Below follow input data and results for a time simulation. The results include environmental forces, motions and tension in the steel tendons. The environmental force in surge consists of both the force due to irregular waves and the constant horizontal force. Analogously, in pitch the environmental force includes the force due to irregular waves and the moment caused by the constant horizontal force. The environmental forces were applied linearly increasing from zero to full value in the interval t=0 to t=50 seconds. The numerical scheme defined in Section 5.7.2 was used in order to solve the equations of motion in the time domain.

Input data for time simulation:

Mass of cylinder	m	$= 110.3 \cdot 10^6$ kg
Displacement	$\forall$	$= 125.6 \cdot 10^3$ m^3
z-coordinate for the center of gravity	z_G	= -60.0 m
Pitch radius of gyration	r_p	= 80.0 m
Location of fairlead for the group of tendons	(x,y,z)	= (0.,0.,-100)
Cross sectional area of group of tendons	A_{wr}	= 1.4 m^2
Modulus of elasticity	E	= 210 GPa
Initial length of tendons	L_o	= 200 m
Total pre-tension	T_o	= 150.0 MN
Pierson-Moskowitz wave energy spectrum		
Significant wave height	H_s	= 20.0 m
Frequency spacing used in spectrum	$\Delta\omega$	= 0.02 rad/s
Constant horizontal force	F_H	= 5.0 MN
z-coordinate of the point where F_H is applied	z_H	= -10.0 m
Time step	Δt	= 0.3 s

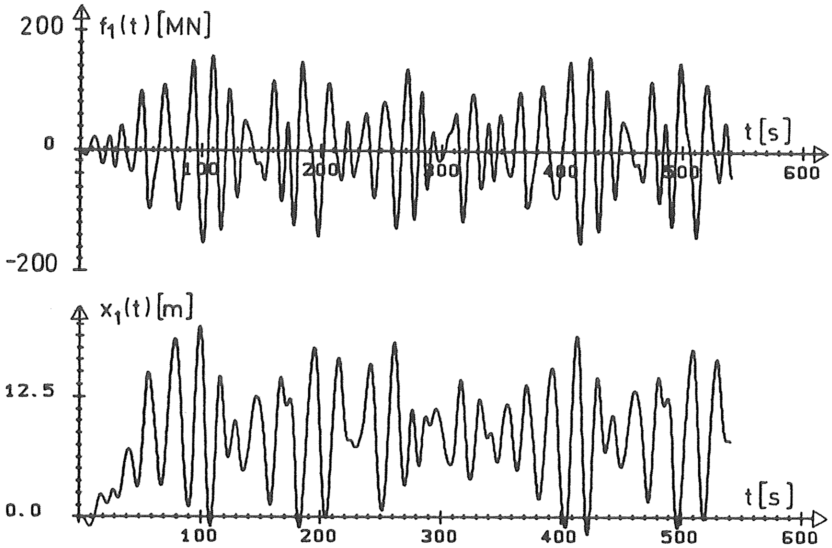

Figure 5.15 Horizontal environmental force, $f_1(t)$, and motion, $x_1(t)$.

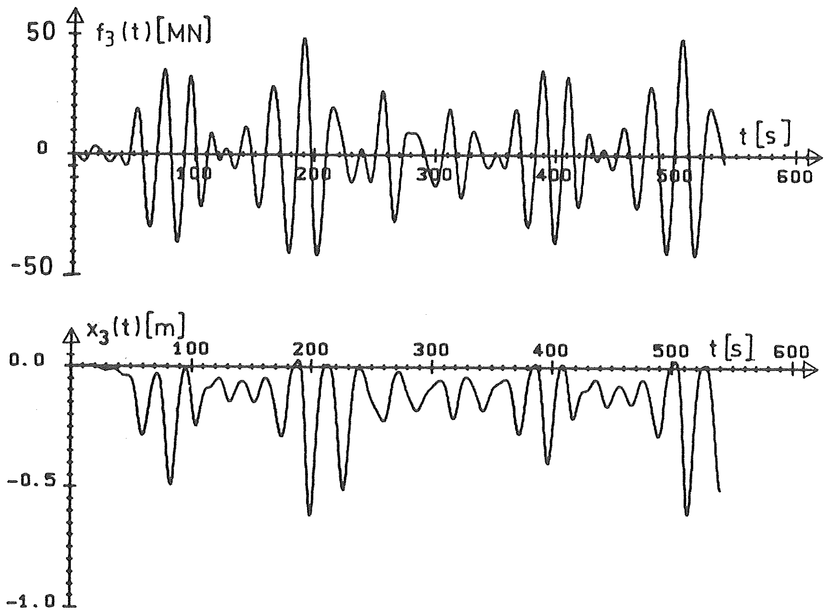

Figure 5.16 Vertical environmental force, $f_3(t)$, and motion, $x_3(t)$.

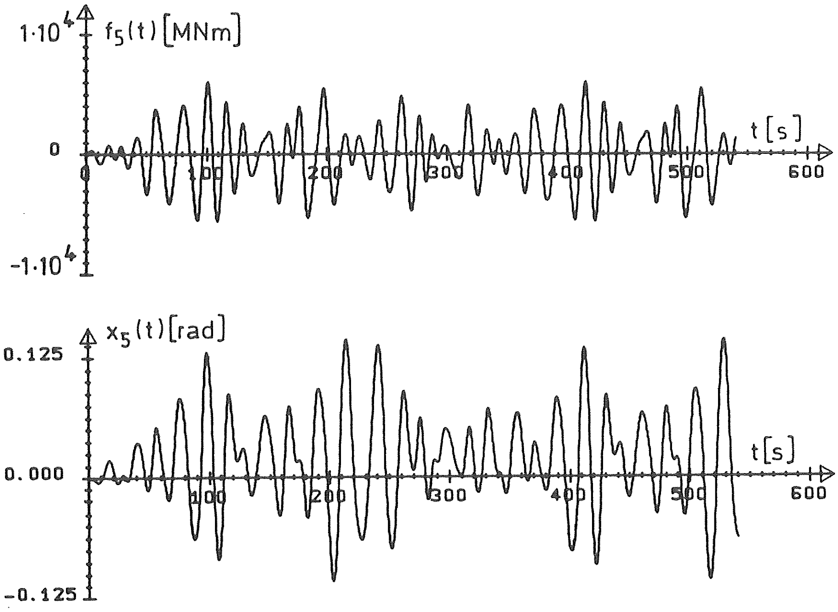

Figure 5.17 Environmental moment, $f_5(t)$, and motion, $x_5(t)$.

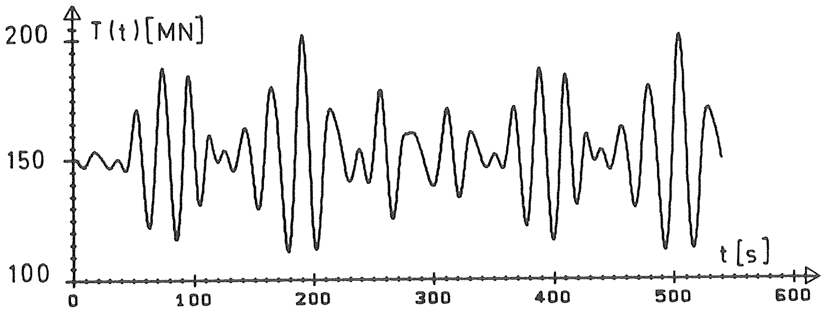

Figure 5.18 Total tension of the steel tendons.

From the simulations above it can be seen that the tethered cylinder is in static equilibrium with the constant horizontal force after it has translated in the positive x-direction and negative z-direction and rotated in a positive angle about the y-axis. A return period of the time series can be identified. The return period is equal to the longest period of the terms in Eq. (5.57) which is given by $T=2\pi/\Delta\omega=2\pi/0.02\cong314$ seconds.

5.9 Comments

In Chapter 5 the solution associated with a vertical cylinder floating in water of finite depth was presented. In order to solve the retardation function the potential damping had to be known for a range of frequencies in which the potential damping had a significant value. Since in Section 5.3 a fast analytical method was used the potential damping could easily be determined for higher frequencies. When solving the problem for a structure with more complex geometry a numerical technique such as the sink-source technique has to be applied. Then the surface of the structure is divided into a mesh of elements, where the number of elements determines the required amount of processing time. For higher frequencies the size of the elements has to be small in order to maintain the accuracy. Consequently, when using a numerical technique it is important either to find a simplified expression valid in the high frequency region or to use a proper extrapolation technique. It can be noted that van Oortmerssen (1976) has given asymptotic expressions for the potential damping for a ship and Jefferys (1983) has discussed a technique for interpolation and extrapolation of the hydrodynamic coefficients.

In Section 5.8.2 the time simulations were performed using some rough estimates of environmental forces. The equations of motion in the time domain allow analysis of responses caused by arbitrarily time varying forces. Thus, for example, slowly varying drift forces, dynamic effects from the mooring system or any other force can be included. This, of course, assumes that a reliable time realisation of the force can be obtained.

LIST OF SYMBOLS

SI-units are used throughout the text. It has been my aim to use notations and symbols as consistently as possible. In Section 5.3, however, the notation of Yeung (1981) has been used for the more major parameters.

An arrow underneath the symbol is used to indicate a vector quantity. Similarly, a line is used to indicate column matrices and two lines to indicate quadratic matrices. A circumflex placed over a symbol indicates an amplitude value, e.g. $\hat{u}$ is the amplitude of the fluid velocity.

a	$= \bar{a}/\bar{h}$ non-dimensional radius of cylinder
$\underline{\underline{a}}$	added mass matrix
a_k	constant added mass coefficient defined by (4.43)
a_{ij}	element of $\underline{\underline{a}}$
a_{ij}	non-dimensionalised added mass coefficient (Chapter 5)
$a(\omega)$	added mass coefficient
A_w	waterline area
A_k	defined by (5.17)
$\bar{A}_k$	complex conjugate of A_k
A_k^*	defined by (5.18), (5.24) and (5.31)
$\underline{\underline{A}}(\omega)$	$= \underline{\underline{c}} - \omega^2(\underline{\underline{m}} + \underline{\underline{a}}(\omega))$
$\underline{\underline{b}}$	damping matrix
b_{ij}	element of $\underline{\underline{b}}$
$\bar{b}_{ij}$	dimensional value of the damping coefficient (Chapter 5)
$b(\omega)$	damping coefficient
$B(\omega)$	$= \omega\underline{\underline{b}}(\omega)$
c	hydrostatic coefficient
$\underline{\underline{c}}$	hydrostatic matrix
c_n	Fourier coefficients associated with the time realisation of a seastate from a wave energy spectrum
c_k	defined by (5.73)
c_{nk}	defined by (5.14)
C_D	drag coefficient
C_I	inertia coefficient
C_m	added mass coefficient
d	$= \bar{d}/\bar{h}$

$\bar{d}$	distance from the sea bottom to the bottom of the cylinder
D	characteristic dimension of structure
D	defined by (5.55)
D/Dt	substantial derivative
e_{nj}	defined by (5.13)
$\underline{\underline{E}}$	unit matrix defined by $\underline{\underline{A}}\,\underline{\underline{E}}=\underline{\underline{A}}$
E_k	defined by (5.33)
f(t)	time dependent force
$\underline{f}_n$	external force at the n-th time step
$\underline{F}$	generalised force vector
$\underline{F}_h$	hydrodynamic reaction force
$\underline{F}_e$	wave exciting force
$F(i\omega)$	Fourier transform of f(t)
F_x	x-component of the reaction force due to the mooring system
F_y	y-component of the reaction force due to the mooring system
F_z	z-component of the reaction force due to the mooring system
$\lvert F_{ei} \rvert$	amplitude of the wave exciting force in the i-th mode
g	acceleration due to gravity
g_n	defined by (5.13)
h	water depth
h	=1 non-dimensional water depth (Chapter 5)
$\bar{h}$	water depth (Chapter 5)
H	wave height
$H_o^{(1)}$	Hankel function of first kind and zero order
$H_1^{(1)}$	Hankel function of first kind and first order
i	$=(-1)^{\frac{1}{2}}$
$\vec{i}$	unit vector along the x-axis
I_o	modified Bessel function of first kind and zero order
I_1	modified Bessel function of first kind and first order
I_{xx}	moment of inertia of the structure
I_{yy}	moment of inertia of the structure
I_{xy}	products of inertia of the structure
Im{ }	imaginary part of the complex quantity inside the braces
$\vec{j}$	unit vector along the y-axis
J_{xx}	moment of inertia of A_w
J_{xy}	product of inertia of A_w

$\underline{\underline{k}}$	retardation function matrix
$\underline{\underline{k}}_n$	retardation function of the n-th time step
$\vec{k}$	unit vector along the z-axis
K	$=\hat{u}T/D$ the Keulegan-Carpenter number
K_o	modified Bessel function of second kind and zero order
K_1	modified Bessel function of second kind and first order
$K(i\omega)$	Fourier transform of k(t)
L	wave length
L_i	length of the i-th wire rope
$L_{i,o}$	initial length of the i-th wire rope
$\vec{L}$	angular momentum
m	mass of the structure
$\underline{\underline{m}}$	generalised mass matrix
m_k	eigenvalues defined by (5.11)
m_o	wave number
M_x	reaction moment about the x-axis due to the mooring system
M_y	reaction moment about the y-axis due to the mooring system
M_z	reaction moment about the z-axis due to the mooring system
$\underline{\underline{M}}$	$= \underline{\underline{m}} + \underline{\underline{a}}_k$
$\vec{M}$	moment vector
$\underline{n}$	generalised normal vector
$\vec{n}$	physical normal vector
N_c	number of points per cycle of the function cos ωt
N_k	normalisation factor associated with eigenfunctions
Oxyz	earth-fixed coordinate system
$\bar{O}\bar{x}\bar{y}\bar{z}$	structure-fixed coordinate system
$(\tilde{O}\tilde{x}\tilde{y}\tilde{z})_i$	coordinate system associated with the i-th wire rope
p	pressure of the fluid
p(t)	impulse response function (velocity)
$\vec{p}$	linear momentum
PV	principle value
P_θ	defined by (5.50) and evaluated in (5.52)
Q_z	defined by (5.51) and evaluated in (5.53)
r	polar coordinate
r_p	pitch radius of gyration
r_r	roll radius of gyration
r_y	yaw radius of gyration

$r(t)$	impulse response function (motion)
$\underset{\rightarrow}{r}$	vector in the Oxyz system
$\underset{\rightarrow}{\bar{r}}$	vector in the $\bar{O}\bar{x}\bar{y}\bar{z}$ system
$\underset{\rightarrow}{r}_o$	translation of the structure fixed coordinate system
Re	the Reynolds number
Re{ }	real part of the complex quantity inside the braces
R_k	k-th term of the radial function associated with an eigenvalue problem
$\underline{\underline{R}}$	$= \underline{\underline{R}}_z\underline{\underline{R}}_y\underline{\underline{R}}_x$ transformation matrix
$\underline{\underline{R}}_x$	transformation matrix associated with a rotation about the x-axis
$\underline{\underline{R}}_y$	transformation matrix associated with a rotation about the y-axis
$\underline{\underline{R}}_z$	transformation matrix associated with a rotation about the z-axis
$R(i\omega)$	complex frequency response function
$R^c(\omega)$	cosine frequency response function
$R^s(\omega)$	sine frequency response function
S	surface of the structure below the still water level
S_∞	control surface at infinite distance from the structure
$S(\omega)$	wave energy spectrum
S_n	defined by (5.15)
$\underset{\sim}{S}$	surface enclosing the volume $\underset{\sim}{V}$
t	time
T	period
T_i	magnitude of the reaction force of the i-th wire rope
$T_{i,o}$	pretension of the i-th wire rope
ΔT_i	additional reaction force of the i-th wire rope
u	x-component of the fluid velocity
v	y-component of the fluid velocity
w	z-component of the fluid velocity
$\underset{\sim}{V}$	volume enclosed by $\underset{\sim}{S}$
V_g	group velocity of waves
$\forall$	displaced volume
$x(t)$	motion of structure
$\underline{x}$	generalised motion vector
x_B	x-coordinate of the center of buoyancy
x_C	x-coorcinate of the center of gravity of A_W
$\underline{x}_n$	motion vector at the n-th time step
x_m	x-coordinate of the point to which the moments are referred

$X(i\omega)$	Fourier transform of $x(t)$
X_i	complex amplitude of the wave exciting force due to a wave of unit amplitude
$\|X_i\|$	real amplitude of X_i
y_B	y-coordinate of the center of buoyancy
y_c	y-coordinate of the center of gravity of A_w
y_m	y-coordinate of the point to which the moment is referred
z_B	z-coordinate at the center of buoyancy
z_m	z-coordinate of the point to which the moment is referred
Z_k	k-th term of the vertical function associated with an eigenvalue problem
α	angle of rotation about the x-axis
α_i	angle defined by (5.64)
α_n	Fourier coefficients
α_n^*	defined by (5.6), (5.23) and (5.30)
β	angle of rotation about the y-axis
β	angle of the incident wave
β_i	angle defined by (5.64)
γ	angle of rotation about the z-axis
γ_i	angle defined by (5.64)
Δt	time step
$\Delta \underline{F}$	hydrostatic reaction force
ΔL_i	extension of the i-th wire rope
ΔT_i	additional reaction force of the i-th wire rope
$\Delta\omega$	frequency step
$\Delta\omega_2$	frequency step associated with the evaluation of cos ωt
ε_i	phase angle between the incident wave and the wave exciting force
ζ	elevation of the free water surface
λ_n	eigenvalues
θ	polar coordinate
θ_n	phase angle associated with creation of an irregular sea state from a wave energy spectrum
τ	variable of integration of time
ρ	fluid density
ϕ	space dependent velocity potential
ϕ_i	velocity potential of the radiated wave due to motion in mode i

ϕ_I	velocity potential of the incident wave
ϕ_S	velocity potential of the diffracted and scattered wave
$\phi^{(i)}$	velocity potential of the interior region
$\phi^{(e)}$	velocity potential of the exterior region
Φ	space and time dependent velocity potential
Φ_i	velocity potential of the radiated wave due to motion in mode i
Φ_I	velocity potential of the incident wave
Φ_R	velocity potential of the radiated wave
Φ_S	velocity potential of the diffracted and scattered wave
ψ_j	space dependent velocity potential associated with an impulsive displacement
ω	angular frequency
$\vec{\omega}$	angular velocity vector
ω'	arbitrarily chosen angular frequency
∇	del-operator
$\int$	one dimensional integral
$\iint$	two dimensional integral
$\iiint$	three dimensional integral

LIST OF FIGURES

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