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Risk aggregation in insurance

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Abstract

The topic of this thesis is aggregation of quantitative risks in the insurance business. The two major quantitative risk components are underwriting risk, i.e., risk entailed in writing an insurance policy, and market risk, i.e., risk of losses in positions arising from movements in market prices. The ultimate research goal is to develop a theoretically sound and practically useful model that can handle risks on the asset and liability sides of the balance sheet simultaneously. One feature of a practically useful model is that it can produce a reasonable solvency-capital estimate.

In the first paper, we aggregate underwriting risk for individual types of non-life insurance policies into an overall non-life underwriting risk. We develop a technique for constructing simulation models that could be used to get a better understanding of the stochastic nature of insurance claims payments, and to calculate solvency capital requirements (SCR) in view of the Solvency II framework. The modeling technique is illustrated with an analysis of motor insurance data from the Swedish insurer Folksam. The most important finding in this paper is that the uncertainty in prediction of the trend in ultimate claim amounts may affect the SCR substantially.

In the second paper, we aggregate interest-rate risk, interest-rate-spread risk, equity risk and exchange-rate risk into an overall market risk for an insurer. We investigate risks related to the common industry practice of engaging in interest-rate swaps to increase the duration of assets, and hence reduce the portfolio sensitivity to falling interest rates. The fundamental result in this paper is that engaging in swap contracts may reduce the standard deviation of changes in the insurer's net asset value, but it may at the same time significantly increase the exposure to tail risk; and when determining an appropriate level of solvency capital, the tail-risk exposure is of great importance.

Keywords: risk aggregation, solvency capital requirements, Solvency II, stochastic modeling, asset-liability management, extreme-value statistics

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List of Papers

The licentiate thesis includes the following papers.

I. J. Alm (2012). A simulation model for calculating solvency capital requirements for a non-life insurance company. *Submitted.*

II. J. Alm and F. Lindskog (2013). Foreign-currency interest-rate swaps in asset-liability management for insurers. *Submitted.*

Contents

Chapter 1

Introduction

Risk management is the art of identifying and assessing risks a business is exposed to, and making decisions whether to avoid, accept or mitigate each of these risks. Typically, the risk manager would like to avoid risks that could be avoided without changing the core business of the company. For example, a non-life insurer would ideally only take on risks that are due to the stochastic nature of claims payments.

Closely related to the concept of risk is the concept of uncertainty. In my view, uncertainty is to be interpreted as the state of having limited knowledge, while risks are potential bad outcomes due to this uncertainty. Uncertainty is not inherently bad though; profit is a potential good outcome due to uncertainty. To make the subject of mathematical statistics useful in risk management, the risk must, at least to some extent, be quantifiable. For an insurer this is true for some risks, most notably *underwriting risk*, i.e., risk entailed in writing an insurance policy, and *market risk*, i.e., risk of losses in positions arising from movements in market prices. These are the risks my work is concerned with. Other risks, for example the risk of fraud, are impossible to quantify in a meaningful way using methods from mathematical statistics, but may be mitigated using other techniques.

In an ideal world, both the owners and the policy holders are interested in the long-term survival of the insurance company. However, other interests, for example higher profits (for the owners) or lower premiums (for the policy holders), may at times overshadow the interest in long-term survival. This may be the case even in a mutual insurance company where the policy holders are the owners. To create a 'level playing field' the financial regulator designs and supervises rules that all insurers on the market must follow. In particular the regulator imposes a rule of what level of capital an insurer must hold to reduce the risk of insolvency. To calculate such a capital level, individual risks must somehow be aggregated to get an idea of the overall risk the insurer is exposed to, and in connection to this many questions arise. For example: How do we aggregate underwriting risk for two different lines of business? Or: How do we aggregate an insurer's underwriting risk and market risk?

The *Solvency II Directive* (European Parliament and the Council (2009)) is an EU directive that harmonizes the financial regulation so that the same set of rules applies to all insurers doing business within the European Union. These rules are supposed to be implemented in the not-very-far-away future, but I do not dare to predict a date; they have already been postponed more than once. The directive and its related papers (e.g., CEIOPS (2007) and European Commission (2010)) suggest a two-level model for risk aggregation in terms of a *standard formula* for calculation of *solvency capital requirements* or *SCR*. The standard formula has pre-defined correlations between different *risk modules* (there are, for example, a non-life underwriting risk module and a market risk module) on the top level, and between submodules within each risk module on the base level. My work does not concern the standard formula. The formula has an advantage in that it is easy to use, but the implicit assumptions are not easy to figure out. Moreover, correlation matrices do not capture dependencies in distributional tails in a realistic way unless the data is close to normally distributed; and for solvency purposes the tails are the interesting part of the distribution. Relevant critique of the standard formula is found in, e.g., Filipovic (2009), Ronkainen and Koskinen (2007), and Sandström (2007).

The directive also allows insurers to develop their own models, known as *internal models*, to calculate the SCR and to get a better understanding of the risk profile of the specific company. In Section 1.1 we start out from the balance sheet to create a top-down modeling framwork for risk aggregation. Valuation methods for both assets and liabilities are of great importance in risk aggregation, and such methods are presented in Section 1.2. We use the risk measure proposed in the directive, i.e., one-year value-at-risk at the level 0.005, but this can easily be altered in our modeling framwork. In particular, we are often interested in the entire distribution of one-year losses, and not just the value-at-risk which essentially is a high quantile of this distribution. Moreover, we use the Solvency II directive as a guideline when deciding how to value assets and liabilities.

1.1 Risk aggregation

In order to understand the role of the risk manager we need some basic accounting terminology. A *balance sheet* is a statement of a company's *assets*, *liabilities* and *owner's equity*. The assets are what the company owns, and the liabilities are what it owes. To get the numbers on the balance sheet we need valuation methods for both assets and liabilities. Given such valuation methods, the value of owner's equity is the *net asset value*, i.e, the difference between the value of assets and the value of liabilities.

The other important financial statement is the *income statement* which displays revenues recognized for a specified time period, and the cost and expenses charged against these revenues. Upper management is often concerned about the *net profit*, also known as the 'bottom line' of the income statement. One very real danger, especially in good times, is that too much focus is drawn to the income statement, and the balance sheet gets neglected. It is the risk manager's task to prevent this from happening.

From an accounting point of view, *insolvency*, or more precise *balance-sheet insolvency*, is the event that the net asset value becomes negative. If A_0 and L_0 are today's values of assets and liabilities, respectively, given some valuation methods. Then, today's net asset value V_0 is given by $V_0 = A_0 - L_0$. The risk manager is interested in what can happen to the net asset value over some time horizon (often a one-year horizon for an insurance company). Let A_1, L_1 and $V_1 = A_1 - L_1$ denote the asset value, liability value and net asset value, respectively, in one year. The change in net asset value over the coming year X is given by $X = V_1 - e^{r_0}V_0$, where r_0 is the zero rate of a non-defaultable bond with maturity in one year. The *discounted loss* Y over the same time period is given by $Y = -e^{-r_0}X = V_0 - e^{-r_0}V_1$. Large losses are what the risk manager fears, so estimation of the probability distribution of the discounted loss Y, especially the right tail of the distribution, is of great importance.

The financial regulator decides a minimum amount by which the asset value must exceed the liability value in order to consider the insurance company to be on good standing; this amount is known as the solvency capital requirements (SCR) in Solvency II. In Article 101 in the directive (European Parliament and the Council (2009)) it is stated that the SCR "shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period".

The *basic own funds* in Solvency II are identified with the net asset value. The *value-at-risk* (VaR) at level $p \in (0, 1)$ of a portfolio with value V in one year is a risk measure defined as

$$\mathrm{VaR}_p(V) := \min\{m : \mathrm{P}(me^{r_0} + V < 0) \leq p\},$$

and should be interpreted as the smallest amount of money that if added to the portfolio today and invested in a non-defaultable bond ensures that the probability of a strictly negative portfolio value in one year is not greater than p. In our notation the good-standing condition in Solvency II can be written $\mathrm{VaR}_{0.005}(V_1) \leq 0$, which is equivalent to $A_0 \geq L_0 + \mathrm{VaR}_{0.005}(X)$, and we get a natural definition of the solvency capital requirements

$$\mathrm{SCR} := \mathrm{VaR}_{0.005}(X) = \hat{F}_Y^{-1}(0.995), \tag{1.1}$$

where $\hat{F}_Y^{-1}$ is the estimated inverse distribution function (i.e., estimated quantile function) of the discounted loss Y (see Chapter 6 in Hult et al. (2012) for details).

Now, the fundamental question is: What is the distribution of the discounted loss Y? To answer this question we must understand how much the values of assets and liabilities may change over a one-year horizon given some valuation methods.

1.2 Valuation of assets and liabilities

To fill in the numbers on the balance sheet, assets and liabilities must be valued in some currency unit, for example Swedish krona or Euro. There is no unique way of doing this; there is always room for subjectivity when uncertainty is present. Article 75 in the Solvency II directive states that "assets shall be valued at the amount for which they could be exchanged between knowledgeable willing parties in an arm's length transaction" and "liabilities shall be valued at the amount for which they could be transferred, or settled, between knowledgeable willing parties in an arm's length transaction". The common interpretation of this is that market valuation, i.e., equating the value with the price paid in the latest market transaction, should be used when a deep and liquid market exists, and this is the case for most of the insurer's assets.

Valuation of liabilities is not as straightforward since no liquid market exists for insurance policies. For an insurer, the major part of the liability side of the balance sheet consists of obligations towards its policy holders. Such obligations are known as *technical provisions* in Solvency II. Article 77 in the directive states that "the value of technical provisions shall be equal to the sum of a best estimate and a risk margin", where the *best estimate* is the expected present value of future liability cash flows and the *risk margin* is commonly understood as the value of the non-hedgeable risks related to these cash flows.

Actuarial methods for calculating best estimates, known as *claims reserving methods*, have been around for quite some time. Two of the most famous are the *chain-ladder (CL) method* and the *Bornhuetter-Ferguson (BF) method* (Bornhuetter and Ferguson (1972)). These methods started off as purely algorithmic methods to calculate reserves, but later actuaries started to think about the stochastic models underlying these methods to assess the prediction uncertainty. There are several underlying assumptions that justify both the CL and the BF method. Assumptions underlying the CL method were first formulated in Mack (1993). These assumptions, as well as assumptions underlying the BF method, are found in Chapter 2 of Wüthrich and Merz (2008). Wüthrich and Merz (2008) also covers the mathematical theory of other stochastic claims reserving methods, e.g., *generalized linear models (GLM)*, and is a good start for the interested reader. For more information about GLM, see, e.g., England and Verrall (2002), and Björkwall et al. (2011).

There are several suggestions of how the risk margin should be calculated. One of the most common approaches is the *cost-of-capital method* (see, e.g., Keller (2006), Ohlsson and Lauzeningks (2009), and Salzmann and Wüthrich (2010)), where the value of the risk margin is supposed to be the cost of holding an amount equal to the SCR in own funds over the lifetime of the insurance obligations. An alternative approach, based on the risk aversion the financial agent providing the protection against adverse developments, is presented in Wüthrich et al. (2011).

Given valuation methods for assets and liabilities, and n years of insurance and financial data, we get a sample of net asset values $v_0, \ldots, v_n$. From these net asset values (and historical zero-rates) we can create a sample of discounted one-year losses $y_1, \ldots, y_n$, where $y_i = v_{i-1} - e^{-r_{i-1}} v_i$. This sample may then be used for making inferences on the SCR in view of (1.1).

Chapter 2

Introduction to papers

2.1 Introduction to Paper I

In the first paper, called *A simulation model for calculating solvency capital requirements for a non-life insurance company*, we develop a technique for constructing multidimensional simulation models that could be used to get a better understanding of the stochastic nature of insurance claims payments, and to calculate SCR, best estimates, risk margins and technical provisions. The only model input is assumptions about distributions of *payment patterns*, i.e., how fast claims are handled and closed, and *ultimate claim amounts*, i.e, the total amount paid to policyholders for accidents occuring in a specified time period. This kind of modeling works rather well on claims that are handled rather quickly, say in a few years. The assumptions made in the paper are based on an analysis of motor insurance data from the Swedish insurance company Folksam. Motor insurance is divided into the three subgroups *collision*, *major first party* and *third party property insurance*. The data analysis is interesting in itself and presented in detail in Chapter 3 of the paper.

Some of the more interesting findings of Paper I are that: the multivariate normal distribution fitted the motor insurance data rather well; modeling data for each subgroup individually, and the dependencies between the subgroups, yielded more or less the same SCR as modeling aggregated motor insurance data; uncertainty in prediction of trends in ultimate claim amounts affects the SCR substantially.

2.2 Introduction to Paper II

In the second paper, *Foreign-currency interest-rate swaps in asset-liability management for insurers*, co-authored with Filip Lindskog, we investigate risks related to the common industry practice of engaging in interest-rate swaps to increase the duration of assets. Our main focus is on foreign-currency swaps, but the same risks are present in domestic-currency swaps if there is a spread between the swap-zero-rate curve and the zero-rate curve used for discounting insurance liabilities.

We set up a stylized insurance company, where the size of the swap position can be varied, and conduct peaks-over-threshold analyses of the distribution of monthly changes in net asset value given historical changes in market values of bonds, swaps, stocks and the exchange rate. Moreover, we consider a 4-dimensional sample of risk-factor changes (domestic yield change, foreign-domestic yield-spread change, exchange-rate log return, and stock-index log return) and develop a structured approach to identifying sets of equally likely extreme scenarios using the assumption that the risk-factor changes are elliptically distributed. We define the *worst area* which is interpreted as the subset of a set of equally extreme scenarios that leads to the worst outcomes for the insurer.

The fundamental result of Paper II is that engaging in swap contracts may reduce the standard deviation of changes in net asset value, but it may at the same time significantly increase the exposure to tail risk; and tail risk is what matters for the solvency of the insurer.

Chapter 3

Future work

The first of the two papers in this thesis deals with risk aggregation on the liability side of the balance sheet, while the second deals with risk aggregation on the asset side. A natural continuation of this work is to develop a model that can handle quantitative risks on the asset and liability sides simultaneously. To develop such a model that is both theoretically sound and useful in practice is the ultimate research goal.

Before this complete-balance-sheet model can be constructed, we must extend the underwriting-risk model developed in the first paper so that it can incorporate lines of business (LoB) where claims may take many years to close, e.g., health or accident insurance. A first step is to find viable chain-ladder-factor parametrizations to reduce the number of model parameters that must estimated from data. Moreover, we need a better understanding of insurance data. In particular, what kind of tail behavior we may expect for each LoB, and whether or not data supports tail dependencies between different LoBs.

Finally, we need to further develop multivariate methods of threshold exceedances, based on a multivariate generalized Pareto distribution (see Ledford and Tawn (1996) and Rootzén and Tajvidi (2006)), that could be used with a limited number of observations.

Bibliography

Björkwall, S., Hössjer, O., Ohlsson, E., and Verrall, R. J. (2011). A generalized linear model with smoothing effects for claims reserving. *Insurance: Mathematics and Economics*, 49:27–37.

Bornhuetter, R. L. and Ferguson, R. E. (1972). The actuary and IBNR. *Proceedings of the Casualty Actuarial Society*, 59:181–195.

CEIOPS (2007). Consultation paper no. 20.

England, P. D. and Verrall, R. J. (2002). Stochastic claims reserving in general insurance. *British Actuarial Journal*, 8(3):443–518.

European Commission (2010). QIS5 Technical specifications.

European Parliament and the Council (2009). Directive of the European Parliament and of the Council on the taking-up and pursuit of the business of insurance and reinsurance (Solvency II).

Filipovic, D. (2009). Multi-level risk aggregation. *ASTIN Bulletin*, 39:565–575.

Hult, H., Lindskog, F., Hammarlid, O., and Rehn, C. J. (2012). *Risk and portfolio analysis*. Springer, New York, NY, USA.

Keller, P. (2006). A primer for calculating the Swiss Solvency Test "cost of capital" for a market value margin: Swiss Federal Office of Private Insurance.

Ledford, A. W. and Tawn, J. A. (1996). Statistics for near independence in multivariate extreme values. *Biometrika*, 83:169–187.

Mack, T. (1993). Distribution-free calculation of the standard error of chain ladder reserve estimates. *ASTIN Bulletin*, 23(2):214–225.

Ohlsson, E. and Lauzeningks, J. (2009). The one-year non-life insurance risk. *Insurance: Mathematics and Economics*, 45:203–208.

Ronkainen, V. and Koskinen, L. (2007). Topical modelling issues in Solvency II. *Scandinavian Actuarial Journal*, 2:135–146.

Rootzén, H. and Tajvidi, N. (2006). The multivariate generalized Pareto distribution. *Bernoulli*, 12:917–930.

Salzmann, R. and Wüthrich, M. V. (2010). Cost-of-capital margin for a general insurance liability runoff. *ASTIN Bulletin*, 40(2):415–451.

Sandström, A. (2007). Solvency II: Calibration for skewness. *Scandinavian Actuarial Journal*, 2:126–134.

Wüthrich, M. V., Embrechts, P., and Tsanakas, A. (2011). Risk margin for a non-life insurance run-off. *Statistics & Risk Modeling*, 28:299–317.

Wüthrich, M. V. and Merz, M. (2008). *Stochastic claims reserving methods in insurance*. Wiley, Chichester, England.

Paper I

A simulation model for calculating solvency capital requirements for a non-life insurance company

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May 3, 2012

Abstract

To stay solvent, an insurance company must have enough assets to cover its liabilities towards its policy holders. In this paper we develop a general technique for constructing a simulation model that is able to generate a solvency capital requirement (SCR) value for a non-life insurance company. The only input to the model is assumptions about the distributions of payment patterns and ultimate claim amounts. These assumptions should ideally be based on findings in empirical data studies.

We illustrate the modeling technique by considering a specific case with motor insurance data from the Swedish insurance company Folksam. The SCR values generated by the simulation model with different distributional assumptions in this specific case are analyzed and compared to the SCR value calculated using the Solvency II standard model. The most important finding was that the uncertainty in prediction of the trend in ultimate claim amounts affect the SCR substantially. Insurance companies and supervisory authorities should be aware of the effects of this trend prediction uncertainty when building and evaluating internal models in the Solvency II or other regulatory frameworks.

Keywords: risk aggregation, stochastic modeling, SCR, Solvency II, premium and reserve risk

1 Introduction

The problem of how to aggregate risks of single insurance types, lines of business (LoBs) or risk classes (e.g., market risk and different non-life insurance risks)

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to come up with reasonable solvency capital requirements (SCR) for insurance companies have been hotly discussed the last few years. Actuaries, regulators and people in academia have been involved in these discussions as the Solvency II framework has been developed within the European Union. One of the main issues is how to take dependencies between insurance types, LoBs or risk classes into account. EIOPA[1] proposes a standard model using a two-level approach with pre-defined correlations between the different risk classes on the top level, and between the risk types within each risk class on the base level. The insurance companies also have the possibility to develop their own (full or partial) internal models as long as they comply with the general guidelines. One of the main features of the general guidelines is the use of value-at-risk at the level 0.005 as risk measure. For more details about the standard model and internal model guidelines, see, e.g., CEIOPS (2007), CEIOPS (2009), European Commission (2010) and Ronkainen and Koskinen (2007).

Several papers discussing the shortcomings of the standard model have been published: Filipovic (2009) studies the implications of using correlation matrices on two levels and shows that in general only parameters set at the base level lead to unequivocally comparable solvency capital requirements, Sandström (2007) proposes one way of calibrating the standard formula for for skewness, and Savelli and Clemente (2011) show limitations of Sandström's calibration method if the skewness of a single risk type is very high.

In this paper we start from scratch rather than trying to modify the Solvency II standard model. We develop a simulation model for calculating solvency capital requirements for a non-life insurance company which takes both the one-year reserve risk and the one-year premium risk into account.[2] The only input is assumptions about the distributions of payment patterns and ultimate claim amounts of the insurance types considered. These assumptions should ideally be based on findings in empirical data studies. Given the distributional assumptions, a simulation procedure generates values of best estimates, technical provisions, risk margins and solvency capital requirements. The risk measure used to calculate the SCR is value-at-risk at the level 0.005, as proposed in the Solvency II framework.

The technique used to develop the simulation model is very general: it can be applied to many different insurance types and to insurance companies of different sizes. We consider a specific set of motor insurance data from the mutual insurance company Folksam, one of the leading non-life insurance companies in Sweden, to illustrate the modeling technique. Even though we use Folksam data, the aim of this paper is not to build an internal model for Folksam but rather to show how different distributional assumptions may affect

[1]EIOPA (European Insurance and Occupational Pensions Authority) replaced CEIOPS (Committee of European Insurance and Occupational Pensions Supervisors) in January 2011.

[2]For a thoughrough discussion of the one-year reserve and premium risks, see Ohlsson and Lauzeningks (2009).

the solvency capital requirements.

In the special case with motor insurance data we find, as expected, that dependencies between insurance types and heaviness of tails of the ultimate claim amount distributions affect the SCR markedly. Perhaps more surprisingly, we find that the uncertainty in the prediction of trends in ultimate claim amounts affect the SCR considerably. This issue of prediction uncertainty has not been discussed to the same extent as dependencies and heavy tails in relation to the Solvency II framework, but due to its importance we suggest supervisory authorities to consider the effects of trend prediction uncertainty in detail when evaluating internal models.

Moreover, treating all policies as being of the same insurance type (the aggregate method) yielded more or less the same results as treating policies of the different insurance types individually (the individual method). A summary of how the distributional assumptions affect the SCR in this specific motor insurance case are shown in Table 14.

As mentioned above, exposure to risks caused by skewness or heavy tails is important when assessing the standing of an insurance company. We hence made standard GPD-analyses (cf. Coles (2001), pp. 74–91) of both the quarterly data and of the individual claims. All the estimated tails were light, and the data gave no indication of catastrophic risks. Since our simulations were based on the data, they could not either point to possibilities of catastrophes. Still, catastrophic risks exist even for motor insurance. For example, a very serious hail storm is not inconceivable even if it has not happened in Sweden up to now. However, as such risks were not present in the data they have to be handled by other, more ad hoc, methods, such as scenario analysis, which are outside the scope of this paper.

Now follows a short outline of the paper: In Section 2 we introduce the basic concepts and fix the notation, which mostly follows Wüthrich and Merz (2008). In Section 3 we present our findings from the analysis of Folksam's data. In Section 4 the simulation procedure is explained in detail. We motivate and state the assumptions of the different scenarios. In Section 5 we explain how the SCR is calculated using the Solvency II standard model if only non-life premium and reserve risk is taken into account, and discuss how the insurance types studied relate to the lines of business defined in the Solvency II framework. Results from the simulation study are presented in Section 6. The solvency capital requirements generated by the simulation model with different distributional assumptions are compared to the SCR calculated using the Solvency II standard model. A discussion of what impact the results may have on future insurance risk modeling follows in Section 7. The actuarial prediction methods are described in Appendix A, and data plots of payment patterns and claim amounts are shown in Appendix B.

2 Theory and model

In this section we introduce the notation that will be used throughout this paper. We discuss the general concept *solvency risk*, and the Solvency II concepts *solvency capital requirements*, *best estimate*, *risk margin* and *technical provisions*. Moreover, we outline the SCR calculation steps in our model as well as in the Solvency II standard model.

2.1 Notation

Consider a non-life insurance company with N different *insurance types*. We assume that all policies have a length of one year, and that they are written uniformly over the year. We divide each accident year into K *accident periods* of equal length, and denote the most recent accident period by I. The *ultimate development period* (i.e., the last period after a fixed accident period in which claim payments are made) is denoted by J.

The *incremental* and *cumulative claim amounts* for accident period i, development period j and insurance type n are denoted by $X_{ij}^{(n)}$ and $C_{ij}^{(n)}$, respectively. Each insurance type has a corresponding *claims development "triangle"*. The structure of these triangles is shown in Table 1.

Accident period	**Development period** 0	1	$\cdots$	J
1				
$\vdots$		observed $C_{ij}^{(n)}$'s		
$I-J+1$				
$\vdots$			$C_{ij}^{(n)}$'s to predict,	
I			incurred claims	
$I+1$				
$\vdots$		$C_{ij}^{(n)}$'s to predict,		
$I+K$		not yet incurred claims		

Table 1: The structure of the claims development triangle at time I for insurance type n.

Let $M_i^{(n)}$ denote the *number of policies* of type n at time i (i.e., the end of accident period i). Since the length of each accident period is $1/K$ years, and we assume that the policies are written uniformly over the year, the insurance company's *exposure* $E_i^{(n)}$ (i.e., the number of "one-year policy equivalents") to insurance type n in accident period i is given by

$$E_i^{(n)} := \frac{M_{i-1}^{(n)} + M_i^{(n)}}{2K}, \tag{1}$$

where we implicitly assume that the exposure is independent of season.

The *payment pattern* for accident period i and insurance type n is defined by

$$\boldsymbol{p}_i^{(n)} := \left(p_{i0}^{(n)}, \ldots, p_{iJ}^{(n)}\right)^T, \quad \begin{array}{l} i = 1, \ldots, I+K, \\ n = 1, \ldots, N, \end{array} \tag{2}$$

where $p_{ij}^{(n)}$ is the proportion of the ultimate claim amount paid in development period j for accident period i and insurance type n, i.e., $p_{ij}^{(n)} = X_{ij}^{(n)}/C_{iJ}^{(n)}$. The *normalized ultimate claim amounts* of accident period i are defined by

$$\boldsymbol{Y}_i := \left(Y_i^{(1)}, \ldots, Y_i^{(N)}\right)^T, \quad i = 1, \ldots, I+K, \tag{3}$$

where $Y_i^{(n)}$ is the individual ultimate claim amount for insurance type n and accident period i normalized by the exposure, i.e., $Y_i^{(n)} = C_{iJ}^{(n)}/E_i^{(n)}$.

Let $Q_{k\ell}$ denote the interest rate at time k of a zero-coupon bond maturing at time ℓ. In this paper discounting always means discounting by the zero-coupon rate. This implies that the discount factor $B_{k\ell}$ used at time k to discount a cash flow at time ℓ equals the price at time k of a zero-coupon bond with principal 1 maturing at time ℓ, i.e.,

$$B_{k\ell} = e^{-Q_{k\ell}t_{k\ell}}, \quad t_{k\ell} := \frac{\ell - k}{K}, \quad \ell \geq k.$$

Moreover, let $\mathcal{F}_t$ denote the information available at time t. This information includes the paid claim amounts (upper left corner of the development triangle in Table 1), the number of policies and the zero-coupon rate curves up to time t.

2.2 Solvency risk

The solvency risk of an insurance company is the risk that it will not have enough assets to cover its liabilities[3] at some future point in time. In this

[3]It is not completely clear what to mean by the "value" of an asset or a liability. For assets traded on liquid markets the asset value is often equated with the price paid for the asset in the latest market transaction (i.e., the "market value"). Insurance liability cash flows, however, are in general not traded on liquid markets, so there is no obvious answer to the question of how to value the liabilities in a "market consistent" way.

One way to think about this problem is to consider what amount of (publicly traded) assets an external investor would demand in order to be willing to take over the liabilities of paper we only consider what can happen to the values of assets and liabilities one year into the future, so we focus our attention on what we know today (i.e., at time I) and what we will know in one year (i.e., at time $I+K$).

Today's *outstanding loss liabilities* of the insurance company are described by the cash flow vector $\boldsymbol{X} := (X_{I+1}, \ldots, X_{I+J+K})^T$, where X_t is the amount to be paid by the company at time t, i.e.,

$$X_t = \sum_{n=1}^{N} \sum_{(i,j)\in S_t} X_{ij}^{(n)}, \quad t = I+1, \ldots, I+J+K,$$

with $S_t := \{(i,j) : \max(t-J,\, 1) \leq i \leq \min(t,\, I+K),\, j = t-i\}$. The so called *best estimate* (BE), which is an unbiased best estimate of the present value of the outstanding loss liability cash flows, is given by

$$\mathrm{BE} := \sum_{t=I+1}^{I+J+K} B_{It} \widehat{\mathrm{E}}\left[X_t | \mathcal{F}_I\right], \tag{4}$$

where B_{It} is the price today of a zero-coupon bond with principal 1 maturing at time t and $\widehat{\mathrm{E}}\left[X_t|\mathcal{F}_I\right]$ is to be interpreted as an unbiased prediction of X_t given the information $\mathcal{F}_I$ using some (pre-defined) actuarial method. (The actuarial prediction methods considered later in this paper are essentially versions of the chain ladder method combined with trend assumptions, see Appendix A for details.)

Let L_I and L_{I+K} denote the value today and in one year, respectively, of the insurance company's liabilities. Using the best estimate as a proxy for the liability value, we get

$$L_u = \sum_{t=u+1}^{I+J+K} B_{ut} \widehat{\mathrm{E}}\left[X_t | \mathcal{F}_u\right], \quad u = I, I+K. \tag{5}$$

Moreover, let A_I and A_{I+K} denote the value today and in one year, respectively, of the insurance company's assets. Assuming that all assets of the insurance company are zero-coupon bonds maturing in one year, and that some of these bonds are sold during the coming year to pay off maturing liabilities, we get the asset value in one year as the difference between the value (in one year)

the insurance company, and then equate the value of the liabilities with the market value of these assets. It is widely belived (although impossible to prove scientifically) that investors are risk averse, and since the future liability cash flow amounts are uncertain one could argue that an external investor would demand an amount of assets that is higher than an unbiased best estimate of the discounted liability cash flows. In the Solvency II framework the present value of the liabilities is estimated by the so called technical provisions, which are the sum of a best estimate and a risk margin, see CEIOPS (2007) and European Commission (2010) for details.

of today's assets and the value (in one year) of the coming year's maturing liabilities, i.e.,

$$A_{I+K} = \frac{A_I}{B_{I,I+K}} - \sum_{t=I+1}^{I+K} \frac{X_t}{B_{t,I+K}}. \tag{6}$$

Value-at-risk at the level 0.005 will be our risk measure, as proposed in the Solvency II framework. It is outside the scope of this paper to discuss whether this is a good choice of risk measure or not. However, it will serve the purpose of making comparisons of solvency capital requirements for different scenarios possible. If X is the value of a stochastic portfolio at time t, and $\alpha \in (0,1)$, then value-at-risk at the level α is defined by

$$\mathrm{VaR}_\alpha(X) := B_{It} F_{-X}^{-1}(1-\alpha), \quad t \geq I, \tag{7}$$

where F_{-X}^{-1} is the inverse of the distribution function (i.e., the quantile function) of $-X$ (note the minus sign), and B_{It} is the price today of a zero-coupon bond with principal 1 maturing at time t.

In one year the value of the insurance company's portfolio will be $A_{I+K} - L_{I+K}$. The regulator will conclude that the insurance company has enough assets to cover its liablities if $\mathrm{VaR}_{0.005}(A_{I+K} - L_{I+K}) \leq 0$ which is equivalent to

$$A_I \geq L_I + \mathrm{VaR}_{0.005}(\Delta), \tag{8}$$

where, using (5) and (6),

$$\begin{aligned} \Delta :=& A_{I+K} - \frac{A_I}{B_{I,I+K}} - \left(L_{I+K} - \frac{L_I}{B_{I,I+K}}\right) \\ =& \sum_{t=I+1}^{I+K} \left(\frac{B_{It}}{B_{I,I+K}} \widehat{\mathrm{E}}\left[X_t|\mathcal{F}_I\right] - \frac{X_t}{B_{t,I+K}}\right) \\ &+ \sum_{t=I+K+1}^{I+J+K} \left(\frac{B_{It}}{B_{I,I+K}} \widehat{\mathrm{E}}\left[X_t|\mathcal{F}_I\right]\right. \\ &\qquad \left. - B_{I+K,t} \widehat{\mathrm{E}}\left[X_t|\mathcal{F}_{I+K}\right]\right). \end{aligned} \tag{9}$$

The random variable Δ tells us how the balance (i.e., the difference between assets and liabilities) of the insurance company changes over the coming year. The value of Δ is affected by:

1. changes in bond prices over the coming year,
2. discrepancy between the actuary's expectations today about payments due within one year and the actual amounts paid, and,

3. discrepancy between the actuary's expectations today and their expectations in one year about payments due later than one year from today.

To set the change in balance (i.e., profit or loss) in relation to the size of the insurance company's liability portfolio, we construct the *normalized loss statistic* U,

$$U := \frac{-B_{I,I+K}\Delta}{\mathrm{BE}}. \tag{10}$$

We call U a loss statistic since a high value of U means a large (relative) loss for the insurance company.

The *solvency capital requirement* (SCR) is the minimum amount by which the present asset value must exceed the present liability value. From the condition in (8) we see that in our setup a natural definition of SCR is

$$\mathrm{SCR} := \mathrm{VaR}_{0.005}(\Delta) = \mathrm{BE} \cdot F_U^{-1}(0.995), \tag{11}$$

where we use (7) and (10) to get the last equality.

An external investor willing to take over the outstanding loss liabilities of the insurance company would demand an amount of assets to balance these liabilities. Assuming that the investor must hold capital equal to the calculated SCR for the duration of the liabilities, one could argue that they would demand an amount of assets that is $c \cdot T \cdot \mathrm{SCR}$ higher than the best estimate, where c is the investor's *cost-of-capital rate* (e.g., 6%, as proposed in European Commission (2010)) and T is the *estimated duration* of the liabilities,

$$T = \frac{\sum_{t=I+1}^{I+J+K} \frac{t-I}{K} \widehat{\mathrm{E}}\left[X_t|\mathcal{F}_I\right]}{\sum_{u=I+1}^{I+J+K} \widehat{\mathrm{E}}\left[X_u|\mathcal{F}_I\right]}. \tag{12}$$

With this reasoning we get the *risk margin* (RM) by

$$\mathrm{RM} := c \cdot T \cdot \mathrm{SCR}\,. \tag{13}$$

The *technical provisions* (TP), i.e., the estimated "market value" of the liabilites, are the sum of the best estimate and the risk margin,

$$\mathrm{TP} := \mathrm{BE} + \mathrm{RM}\,. \tag{14}$$

The approach used above, where we assume that the risk margin is proportional to the SCR value and leave it outside the SCR calculation, is known as the *simplified cost-of-capital method*, see Keller (2006) and Ohlsson and Lauzeningks (2009) for further details.[4]

[4] Other approaches to the calculation of the risk margin are found in Salzmann and Wüthrich (2010) and Wüthrich et al. (2011).

The construction with a best estimate and a risk margin is political rather than scientific. Which price an investor actually would pay depends on a range of factors. For example, an investor may be willing to accept a lower amount of assets if they believe that taking over the liabilities from the insurance company will strengthen their market position. However, we will stick to the BE plus RM approach in this paper.

The rationale for our model setup is that, since all best estimates are known today, the stochastic behavior of the loss of the insurance company over the coming year is captured in the single random variable U. If the distribution of U is known, then the solvency capital requirements, risk margin and technical provisions follows directly from (11), (13) and (14), respectively, assuming that the cost-of-capital rate c is known.

2.3 Splitting up the best estimate and the loss statistic

The best estimate BE can be split into two parts: a best estimate of incurred claims BE_R and a best estimate of future claims BE_P. The subscripts R and P indicate that these best estimates relate to the reserve risk and the premium risk, respectively. We get BE_R and BE_P by replacing X_t by R_t and P_t, respectively, in (4), where R_t is the part of the liability cash flow at time t arising from accidents before today and P_t is the part of the liability cash flow at time t arising from accidents after today, i.e.,

$$
\begin{aligned}
\mathrm{BE}_R &:= \sum_{t=I+1}^{I+J+K} B_{It} \widehat{\mathrm{E}}\left[R_t | \mathcal{F}_I\right] \quad \text{and} \\
\mathrm{BE}_P &:= \sum_{t=I+1}^{I+J+K} B_{It} \widehat{\mathrm{E}}\left[P_t | \mathcal{F}_I\right],
\end{aligned}
\tag{15}
$$

with

$$
\begin{aligned}
R_t &:= \sum_{n=1}^{N} \sum_{(i,j)\in S_t^R} X_{ij}^{(n)} \quad \text{and} \\
P_t &:= \sum_{n=1}^{N} \sum_{(i,j)\in S_t^P} X_{ij}^{(n)},
\end{aligned}
$$

where $S_t^R := S_t \cap \{(i,j) : i \leq I\}$ and $S_t^P := S_t \cap \{(i,j) : i > I\}$.

We split Δ into Δ_R and Δ_P by doing the same replacement in (9). Since $X_t = R_t + P_t$ and all actuarial predictions are unbiased, we have

$$
\mathrm{BE} = \mathrm{BE}_R + \mathrm{BE}_P \quad \text{and} \quad \Delta = \Delta_R + \Delta_P.
$$

We get U_R by replacing Δ and BE by Δ_R and BE_R, respectively, in (10). Analogously, we get U_P by replacing Δ and BE by Δ_P and BE_P. In general, $U \neq U_R + U_P$, instead the relation

$$U = \frac{1}{\mathrm{BE}} \left(\mathrm{BE}_R \, U_R + \mathrm{BE}_P \, U_P \right) \tag{16}$$

holds.

Instead of considering the distribution of U directly, we could study the distribution of the 2-dimensional vector $\boldsymbol{U}_2 := (U_R, U_P)^T$, and use the relation (16) to get the distribution of U.

3 Data analysis

In this section we analyze historical payment patterns (in the sense of (2)) and normalized ultimate claim amounts (in the sense of (3)), and suggest distributions to be used in the simulation study. The data consist of Folksam's claim payments and exposures for three motor insurance types: *collision insurance* ($n = 1$), *major first party insurance* ($n = 2$) and *third party property insurance* ($n = 3$) for all accident quarters from 1998 to 2007 (40 accident periods, 9 development periods).

3.1 Payment patterns

By payment proportion ($p_{ij}^{(n)}$) we mean the proportion of the total claim amount (for a fixed accident period) paid in a specific development period. The payment pattern ($\boldsymbol{p}_i^{(n)}$) is the vector consisting of the payment proportions for all development periods. The entries of the payment pattern vector must sum to 1 by definition.

We make some remarks regarding the empirical payment proportions by considering the data plots in Figures 1, 2 and 3, and the sample means and sample standard deviations in Table 2.

- There is a rather distinct seasonal pattern, especially for collision insurance, where accidents in the first quarter of each year are handled more quickly than accidents occuring later in the year. In Figure 1 we see this pattern as peaks at accident quarters $1, 5, 9, \ldots$ in the diagram for development quarter 0 (upper left corner).

- There are negative proportions in development quarters 2 and higher for collision insurance (see Figure 1 and Table 2). This is due to the fact that Folksam pays their (collision) policy holders before it is clear who caused the accident. If it turns out that the counterpart to Folksam's policy holder caused the accident, then Folksam receives a payment from the counterpart's (third party property) insurance company. This is also one of the reasons why third party property insurance claims are not paid as quickly as collision insurance claims (see Figure 3).

- There is some variation between different accident years, but in general there are no clear trends over time except perhaps for a somewhat downward trend for development quarter 0, and a somewhat upward trend for development quarter 2, for collision insurance (see Figure 1).

Dev.	Collision		Major first party		Third party property	
quarter	Mean	SD	Mean	SD	Mean	SD
0	0.571	0.059	0.450	0.036	0.161	0.029
1	0.482	0.048	0.445	0.031	0.476	0.039
2	-0.010	0.026	0.069	0.014	0.207	0.038
3	-0.024	0.012	0.019	0.005	0.080	0.015
4	-0.011	0.008	0.008	0.002	0.037	0.009
5	-0.004	0.006	0.004	0.002	0.019	0.006
6	-0.002	0.004	0.002	0.001	0.011	0.004
7	-0.001	0.003	0.002	0.001	0.006	0.003
8	-0.001	0.002	0.001	0.001	0.004	0.002

Table 2: Sample means (m_{jn}'s) and sample standard deviations (s_{jn}'s) of the payment proportions ($p_{ij}^{(n)}$'s).

Despite the findings above regarding seasonal patterns and trends, we will assume that the payment patterns ($\boldsymbol{p}_i^{(n)}$'s, with n fixed) are i.i.d. random vectors in the simulations later in this paper. Moreover, we will assume that there is no dependence between payment patterns for different insurance types.

In particular, we we will use the Dirichlet distribution to simulate payment patterns. Since this distribution only handles non-negative payment proportions, we must modify the sample means in order to be able to use them as distribution parameters. Below follows some details regarding the Dirichlet distribution, and a description of the procedure used to get parameter estimates from data.

3.1.1 The Dirichlet distribution

The $J+1$-dimensional Dirichlet distribution[5] has a parameter vector $\lambda\boldsymbol{\pi}$, where $\lambda > 0$ is an inverse variability parameter and $\boldsymbol{\pi} := (\pi_0, \ldots, \pi_J)^T$ is a mean

[5] See, e.g., Kotz et al. (2000) for further details about the Dirichlet distribution.

vector with constraints $\pi_j \geq 0$ and $\sum_{j=0}^{J} \pi_j = 1$. If $p := (p_0, \ldots, p_J)^T \sim \mathrm{Dir}(\lambda\boldsymbol{\pi})$, then

$$\begin{aligned} \mathrm{E}[p_j] &= \pi_j, \quad \text{and} \\ \mathrm{Var}(p_j) &= \frac{\pi_j(1-\pi_j)}{\lambda+1}. \end{aligned}$$

We define the *modified sample mean* $\widetilde{m}_{jn}$ of the payment proportion in development quarter j for insurance type n by

$$\widetilde{m}_{jn} := \frac{\max(0, m_{jn})}{\sum_{k=0}^{J} \max(0, m_{kn})},$$

where the m_{jn}'s are the original sample means. Moreover, we define the *sample inverse variability* $\widetilde{\lambda}_n$ for insurance type n as the weighted average

$$\begin{aligned} \widetilde{\lambda}_n &:= \sum_{j=0}^{J} \widetilde{m}_{jn}\widetilde{\lambda}_{jn}, \\ \text{with} \quad \widetilde{\lambda}_{jn} &:= \frac{\widetilde{m}_{jn}(1-\widetilde{m}_{jn})}{s_{jn}^2} - 1, \end{aligned}$$

where s_{jn} is the sample standard deviation in development quarter j for insurance type n. The modified sample means and inverse variabilities are shown in Tables 3 and 4, respectively.

Dev. quarter	Collision	Major first party	Third party property
0	0.543	0.450	0.161
1	0.457	0.445	0.476
2	0	0.069	0.207
3	0	0.019	0.080
4	0	0.008	0.037
5	0	0.004	0.019
6	0	0.002	0.011
7	0	0.002	0.006
8	0	0.001	0.004

Table 3: Modified sample means ($\widetilde{m}_{jn}$'s) of the payment proportions ($p_{ij}^{(n)}$'s).

3.2 Normalized ultimate claim amounts

The individual normalized ultimate claim amount ($Y_i^{(n)}$) is the total amount paid (for claims in a fixed accident period) by the insurance company to its

	Collision	Major first party	Third party property
Inverse var.	86	259	195

Table 4: Sample inverse variabilities ($\widetilde{\lambda}_n$'s) of the payment proportions ($p_{ij}^{(n)}$'s).

policy holders divided by the exposure in terms of number of "one-year policy equivalents". Note that this is a measure of cost per policy, not cost per accident. For each accident period, we gather the individual amounts into a normalized ultimate claim amount vector ($\boldsymbol{Y}_i$). From now on, "amounts" will always mean "normalized ultimate claim amounts".

Data plots of the normalized ultimate claim amounts are shown in the left column of Figure 4. All amounts are in nominal SEK, i.e., the amounts have not been adjusted by some price index. These plots reveal some interesting properties of the data:

- For collision and third party property insurance the amounts are consistently higher in the winter quarters $(1, 4, 5, 8, \ldots)$ than in the summer quarters of the same accident year. This is due to the fact that slippery roads increase the number of car accidents in Sweden during the winter.

- For major first party insurance the amounts are slightly higher in the summer quarters than in the winter quarters. This may be due to that car thefts and fires are more common during the summer.

- For collision and third party property insurance there is an upward trend over time. This is very likely due to "claims inflation" (i.e., price increases for repair work and spare parts). Changes in deductibles may also affect the trend to some extent.

- For major first party insurance there is a rather sharpe downward trend in the amounts over the last few years. A probable explanation is that new cars are much more difficult to steal than older ones, so the number of car thefts decreases as people buy new cars.

The amounts are clearly non-stationary, so to be able to compare different accident quarters, we must adjust for both seasonal variation and trends over longer time frames. One quadratic function is fitted to the amounts of the winter accident quarters, and one quadratic function is fitted to the amounts of the summer accident quarters. We add the condition that the shape of the quadratic functions must be the same, i.e., the only parameter that may differ is the intercept. In this setting the difference between the two intercepts describes the seasonal variation and the curvature describes the trend from year to year.

The rationale for the choice of quadratic functions is the idea that the amounts increase (or decrease) linearly in some shorter time frame but that the slope may change (slowly) from year to year. One issue that should be emphasized at this point is that, even though the trends are clearly visible in hindsight, it is not clear if it is possible to predict the trend for the coming year. This issue will be discussed later in this paper.

The trend and seasonally adjusted amounts (i.e., the residuals from the quadratic fit plus the mean of the end values of the fitted summer and winter functions) are shown in the middle column of Figure 4. The adjusted amounts seem rather stationary over time, and the normal QQ plots in the right column indicate that the data are close to being normally distributed with the means and standard deviations shown in Table 5. Moreover, the scatter plots in Figure 5, and the sample correlation coefficients in Table 6, suggest a positive linear dependence between each pair of insurance types.

	Collision	Major first party	Third party property
Mean	887.3	432.7	492.8
SD	71.6	25.1	23.6

Table 5: Sample means and sample standard deviations of the trend and seasonally adjusted (normalized ultimate claim) amounts ($Y_i^{(n)}$'s). Values in SEK.

	Collision	Major first party	Third party property
Collision	1	0.67	0.61
M.F.P.	0.67	1	0.40
T.P.P.	0.61	0.40	1

Table 6: Sample correlations of the trend and seasonally adjusted (normalized ultimate claim) amounts ($Y_i^{(n)}$'s).

Even though the adjusted amounts seem normally distributed, we will use two other distributions (in addition to the normal distribution) when simulating the amounts in our model. One of the reasons for this is that the Folksam data sample may not be representative of data from the insurance types considered. The calculated SCR value could be misleading if we rely only on what we see in one specific data sample.

The two distributions are Student's t-distribution and the log-normal distribution. Below we give some details about these distributions, and show what parameter choices our data suggest.

3.2.1 The multivariate Student's t-distribution

The multivariate Student's t-distribution[6] is a generalization of the one-dimensional Student's t-distribution.

Let $\boldsymbol{G}$ be a random vector from an N-dimensional normal distribution with mean $\mathbf{0}$ and covariance matrix Σ, and let χ^2_ν be a random variable from a chi-squared distribution with ν degrees of freedom. If $\boldsymbol{G}$ and χ^2_ν are independent and $\boldsymbol{\mu}$ is an N-dimensional vector, then $\boldsymbol{Z}$, defined by

$$\boldsymbol{Z} := \boldsymbol{\mu} + \boldsymbol{G}\sqrt{\frac{\nu}{\chi^2_\nu}},$$

is from an *N-dimensional Student's t-distribution* with parameters μ (location vector), Σ (dispersion matrix) and ν (degrees of freedom). The mean vector and covariance matrix of $\boldsymbol{Z}$ are given by

$$\mathrm{E}\left[\boldsymbol{Z}\right] = \begin{cases} \boldsymbol{\mu}, & \text{if } \nu > 1, \\ \text{undefined}, & \text{otherwise}, \end{cases} \qquad \text{and}$$

$$\mathrm{Var}\left(\boldsymbol{Z}\right) = \begin{cases} \frac{\nu}{\nu-2}\Sigma, & \text{if } \nu > 2, \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

respectively.

Using the sample means and sample standard deviations in Table 5, and the sample correlations in Table 6, a sample location vector $\boldsymbol{\mu}$ and a sample covariance matrix W can be constructed. Taking this location vector and covariance matrix as given, we maximize the likelihood function of the N-dimensional Student's t-distribution with respect to the degrees of freedom parameter ν. We add the condition $\nu > 2$, to make sure that the covariance matrix is properly defined. Note that the dispersion matrix Σ is not equal to the covariance matrix W. For a given ν, we get the dispersion matrix as $\Sigma = \frac{\nu-2}{\nu}W$.

The maximization procedure suggests $\nu = 59$ as degrees of freedom parameter for the multivariate Student's t-distribution. This indicates that our data are almost normally distributed.

3.2.2 The multivariate log-normal distribution

Now, instead of studying the original values, we look at the logarithms of the amounts. The normal QQ plots of the trend and seasonally adjusted "log amounts" in the right column of Figure 6 indicate that the logaritmized data are close to being normally distributed. Sample means and sample standard deviations of the log amounts are shown in Table 7. The scatter plots in

[6] See, e.g., McNeil et al. (2005) for further details about the multivariate Student's t-distribution.

Figure 7, together with the sample correlation coefficients in Table 8, show a positive linear dependence between each pair of insurance types.

	Collision	Major first party	Third party property
Mean	6.79	6.07	6.20
SD	0.08	0.06	0.05

Table 7: Sample means and sample standard deviations of the trend and seasonally adjusted **log** amounts ($\log Y_i^{(n)}$'s).

	Collision	Major first party	Third party property
Collision	1	0.67	0.62
M.F.P.	0.67	1	0.40
T.P.P.	0.62	0.40	1

Table 8: Sample correlations of the trend and seasonally adjusted **log** amounts ($\log Y_i^{(n)}$'s).

4 Simulation setup

In this section we set up five different scenarios. For each scenario we specify distributions of the payment patterns ($\boldsymbol{p}_i^{(n)}$'s, see (2)), and the normalized ultimate claim amounts ($\boldsymbol{Y}_i$'s, see (3)). The distribution and parameter choices are based on the findings in Section 3. Throughout this paper we assume that the (normalized ultimate claim) amounts are independent of the payment patterns.

4.1 Simplifications and general assumptions

In this paper we are only interested in the non-life insurance risks. Since the claims of all our insurance types are handled quickly, we simplify by setting all zero-coupon rates to zero in all scenarios, i.e., $Q_{k\ell} = 0$ for all k and ℓ, ignoring the market and interest rate risks. Moreover, we set the cost-of-capital rate c to 0.06 as suggested in the Solvency II framework (see, e.g., European Commission (2010)).

We divide each year into quarters ($K = 4$), and simulate data for the latest 40 quarters ($I = 40$) and the coming four quarters. We assume that no payments are made later than 8 quarters after the quarter in which the accident happens ($J = 8$). The insurance types in the simulations are the same as the three we studied in Section 3 ($N = 3$). For each insurance type (n fixed), we let the number of policies be constant up to time I and then decline linearly to zero from time I to time $I+K$, i.e.,

$$M_i^{(n)} = \begin{cases} a_n M, & i \leq I, \\ a_n M \left(1 - \frac{i-I}{K}\right), & i > I, \end{cases}$$

where $M = 430{,}000$, $a_1 = 0.26$, $a_2 = 0.35$ and $a_3 = 0.39$. Using (1), we get the exposures

$$E_i^{(n)} = \begin{cases} \frac{a_n M}{K}, & i \leq I, \\ \frac{a_n M}{K}\left(1 - \frac{2(i-I)-1}{2K}\right), & i > I. \end{cases}$$

In the equations above we implicitly assume that the contracts are written uniformly over the year, that no new business is written after time I and that the exposure is independent of season. The exposure is in general not independent of season, as we have seen in the data analysis. However, this will not matter in the simulations since we only consider seasonally adjusted amounts.

In all scenarios we assume that the $\boldsymbol{p}_i^{(n)}$'s (payment patterns) are independent in n, and that $\boldsymbol{p}_1^{(n)}, \ldots, \boldsymbol{p}_{I+K}^{(n)}$ (n fixed) are i.i.d. random vectors from a Dirichlet distribution. The expected proportions and inverse variability parameters in the Dirichlet distributions are set to the empirical values shown in Tables 3 and 4, respectively.

4.2 Scenario specific assumptions

Below follows an outline of the scenario specific distribution and parameter choices.

Multivariate normal distribution, no correlation and no trend

In this scenario we assume that $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{I+K}$ are i.i.d. random vectors from an N-dimensional normal distribution with diagonal covariance matrix. (The diagonal covariance matrix implies that there is no dependence between claim amounts of different insurance types.) The means and standard deviations are set to the sample means and sample standard deviations seen in Table 5.

Multivariate normal distribution, correlation but no trend

In this scenario we assume that $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{I+K}$ are i.i.d. random vectors from an N-dimensional normal distribution. The means and standard deviations are the same as in the previous scenario. The covariance matrix is chosen to get the correlations between insurance types suggested by the data analysis, see Table 6.

Multivariate normal distribution, correlation and trend

In this scenario we assume that $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{I+K}$ are independent random vectors. All vectors are assumed to be normally distributed with the same coefficients of variation (i.e., ratio between the standard deviations and the means) as in the two previous scenarios. The means will be same as in the two previous scenarios up to time I, but will thereafter increase by 2% per quarter. The correlations between insurance types are the same as in the previous scenario, i.e., the correlations seen in Table 6.

Multivariate Student's t-distribution, correlation but no trend

In this scenario we assume that $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{I+K}$ are i.i.d. random vectors from an N-dimensional Student's t-distribution. The location vector and covariance matrix are the same as the mean vector and covariance matrix, respectively, in the multivariate normal scenario with correlation but no trend. The degrees of freedom parameter ν is set to 3.

Note that the data analyis in Section 3 suggests a degrees of freedom parameter ν equal to 59. However, with such a high ν, it is essentially impossible to distinguish Student's t-distribution from the normal distribution. We set $\nu = 3$ in order to examine what happens to the SCR if the distribution is more heavy-tailed than our data sample indicates.

Multivariate log-normal distribution, correlation but no trend

In this scenario we assume that $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{I+K}$ are i.i.d. random vectors from an N-dimensional log-normal distribution. The means and standard deviations of the logarithms are set to the sample means and sample standard deviations seen in Table 7. The logarithm covariance matrix is chosen to get the correlations between insurance types suggested by the data analysis, see Table 8.

4.3 Simulation procedure and actuarial calculations

For each scenario we make 10,000 simulations. Consider a specific simulation, say simulation ℓ, for a fixed scenario. This simulation has the following steps:

1. Amounts and payment patterns,

$$\begin{array}{lll} \boldsymbol{Y}_i^{[\ell]}, & i = 1, \ldots, I+K, & \text{and,} \\ \boldsymbol{p}_i^{(n)\,[\ell]}, & i = 1, \ldots, I+K, & n = 1, \ldots, N, \end{array}$$

   are simulated given the scenario specific distributions.

2. Using the relation $X_{ij}^{(n)} = p_{ij}^{(n)} E_i^{(n)} Y_i^{(n)}$, development triangles are created given the information available at times I (today) and $I+K$ (in one year), respectively.

3. A "fictive actuary" calculates $\widehat{\mathrm{E}}\left[X_t|\mathcal{F}_I\right]$ and $\widehat{\mathrm{E}}\left[X_t|\mathcal{F}_{I+K}\right]$ using the triangles in the previous step and the prediction methods defined in Appendix A.

4. The actuary calculates best estimates ($\mathrm{BE}^{[\ell]}$, $\mathrm{BE}_R^{[\ell]}$, $\mathrm{BE}_P^{[\ell]}$) using (4), balance changes ($\Delta^{[\ell]}$, $\Delta_R^{[\ell]}$, $\Delta_P^{[\ell]}$) using (9), and loss statistics ($U^{[\ell]}$, $U_R^{[\ell]}$, $U_P^{[\ell]}$) using (10). The estimated liability duration ($T^{[\ell]}$) is calculated using (12).

The best estimate BE is calculated as the mean of the $\mathrm{BE}^{[\ell]}$'s, and the duration T as the mean of the $T^{[\ell]}$'s. Analoguosly, BE_R and BE_P are calculated as the means of the $\mathrm{BE}_R^{[\ell]}$'s and $\mathrm{BE}_P^{[\ell]}$'s, respectively.

The $U^{[\ell]}$'s make up the simulated distribution of U. In an analog way, the $U_R^{[\ell]}$'s and $U_P^{[\ell]}$'s make up the simulated distributions of U_R and U_P, respectively. Using the simulated 0.995 quantile of U we are able to calculate the SCR using (11), and then, RM and TP using (13) and (14), respectively.

Note that for an actuary looking at the data at time I the two multivariate normal scenarios with correlation will be indistinguishable since the trend in the mean value for one the scenarios is only present for future accident quarters. The chain ladder development factors and trend assumptions will therefore be the same for these two scenarios. However, the losses compared to the best estimates (U's) will be larger in the scenario with trend due to the incorrect trend assumptions.

5 The Solvency II standard model

In this section we explain how solvency capital requirements are calculated using the *Solvency II standard model*, and discuss how the insurance types studied in this paper relate to the lines of business (LoBs) defined in the Solvency II framework. Moreover, we make an interpretation of the standard model in terms of a normalized loss statistic ($\widetilde{U}$) to illustrate the implicit assumptions of this model.

5.1 Standard model calculations

The capital requirements for the non-life underwriting risk module $\mathrm{SCR}_{\mathrm{NL}}$ in the Solvency II standard model is derived by combining the capital requirements for the three submodules: premium and reserve risk, lapse risk, and catastrophe risk. In this paper we only consider premium and reserve risk, so we set the non-life underwriting risk equal to the premium and reserve risk. Using the standard model formula for the capital requirement for the premium and reserve risk[7] we get

$$\begin{aligned} \mathrm{SCR}_{\mathrm{NL}} &= V \cdot g(\sigma), \\ \text{with } g(\sigma) &:= \left(\frac{e^{N_{0.995}\sqrt{\log(\sigma^2+1)}}}{\sqrt{\sigma^2+1}} - 1 \right), \end{aligned} \tag{17}$$

where $N_{0.995}$ is the 0.995 quantile of the standard normal distribution ($N_{0.995} \approx 2.58$), V is a volume measure and σ is the combined standard deviation per volume unit of the non-life LoBs.

One interpretation of the standard model is that the *loss per volume unit* $\widetilde{U}$ is a random variable with mean zero and variance σ^2, but unknown distribution, and the function $g(\sigma)$ is the 0.995 quantile of $\widetilde{U}$. If $\widetilde{U}$ is normally distributed, then $g(\sigma) = N_{0.995}\sigma \approx 2.58\sigma$. However, in the standard model, with g defined as in (17), we have $g(\sigma)$ between 2.7σ and 3.1σ for standard deviations in the appropriate range. So, the assumption in the standard model is that insurance data have heavier tails than the normal distribution.

With this interpretation we can rewrite (17) as

$$\mathrm{SCR}_{\mathrm{NL}} = V \cdot F_{\widetilde{U}}^{-1}(0.995),$$

which is very simliar to the SCR of our simulation model seen in (11),

$$\mathrm{SCR} = \mathrm{BE} \cdot F_{U}^{-1}(0.995).$$

The volume measure V is the sum of the volume measures of the individual LoBs. For an individual LoB, say ℓ, the volume measure $V^{(\ell)}$ is the sum of the volume of outstanding incurred claims $V_R^{(\ell)}$ and the volume of claims expected to arise in the future $V_P^{(\ell)}$. Typically, the volume of outstanding incurred claims is set to the best estimate of outstanding incurred claims, i.e., $V_R^{(\ell)} = \mathrm{BE}_R^{(\ell)}$, while the volume of claims expected to arise in the future is set to the expected premium volume of the coming year. In this paper we do not work with premium volume data, instead we use the best estimate of future claims multiplied by the estimated *total cost*[8] *to claim cost ratio* $\gamma^{(\ell)}$ as a proxy for the volume of claims expected to arise in the future, i.e., $V_P^{(\ell)} = \gamma^{(\ell)}\,\mathrm{BE}_P^{(\ell)}$.

The combined standard deviation (per volume unit) σ is given by

$$\sigma = \frac{1}{V}\left(\sum_{\ell}\sum_{m} \rho_{\ell m}\sigma^{(\ell)}\sigma^{(m)}V^{(\ell)}V^{(m)} \right)^{1/2}, \tag{18}$$

[7]For details about the Solvency II standard model calculations for the non-life premium and reserve risk submodule, see pp. 196–203 in European Commission (2010).

[8]The total cost is the claim cost plus the operating cost less the interest income due to the time lag between premium payments and claim payments.

where $\sigma^{(\ell)}$ is the standard deviation of LoB ℓ, and $\rho_{\ell,m}$ is the correlation between LoBs ℓ and m. The correlations between the LoBs are given to the insurance company by the regulator. The standard deviation of LoB ℓ is given by

$$\sigma^{(\ell)} = \frac{1}{V^{(\ell)}} \left(\left(\sigma_R^{(\ell)} V_R^{(\ell)} \right)^2 + 2\rho \sigma_R^{(\ell)} \sigma_P^{(\ell)} V_R^{(\ell)} V_P^{(\ell)} + \left(\sigma_P^{(\ell)} V_P^{(\ell)} \right)^2 \right)^{1/2}, \tag{19}$$

where $\sigma_R^{(\ell)}$ and $\sigma_P^{(\ell)}$ are the standard deviations for reserve risk and premium risk, respectively, for LoB ℓ, and ρ is the correlation between the reserve risk and the premium risk (note that ρ is independent of ℓ).

5.2 Standard model parameters

For each LoB, the Solvency II standard model specifies values of standard deviations per volume unit of the reserve risk ($\sigma_R^{(\ell)}$ in (19)) and premium risk ($\sigma_P^{(\ell)}$ in (19)), respectively. Moreover, it specifies the correlation coefficient (ρ in (19)) beween reserve risk and premium risk, as well as the correlations between each pair of LoBs ($\rho_{\ell m}$ in (18)).

In terms of the standard model, the insurance types collision and major first party belong to the LoB *other motor* (OM), while the insurance type third party property belongs to the LoB *motor vehicle liability* (MVL).

The standard deviations per volume unit of the reserve risk and premium risk, respectively, for these LoBs are shown in Table 9. The correlation (ρ) between the reserve risk and premium risk is 0.5, and the correlation ($\rho_{\mathrm{OM,MVL}}$) between other motor and motor vehicle liability is also 0.5.

	SD, reserve risk	SD, premium risk
Other motor	0.100	0.070
Motor vehicle liability	0.095	0.100

Table 9: Standard deviations per volume unit of the reserve risk ($\sigma_R^{(\ell)}$) and premium risk ($\sigma_P^{(\ell)}$), respectively.

According to Folksam's annual report 2010 (see Folksam General Insurance (2011), p. 76), the total cost to claim cost ratio ($\gamma^{(\ell)}$) for the LoBs other motor and motor vehicle liability are 1.25 and 1.26, respectively, assuming that the interest income due to the time lag between premium payments and claim payments is zero. So, the premium risk volumes are estimated by $V_P^{(\mathrm{OM})} = 1.25 \cdot \mathrm{BE}_P^{(\mathrm{OM})}$ and $V_P^{(\mathrm{MVL})} = 1.26 \cdot \mathrm{BE}_P^{(\mathrm{MVL})}$, respectively.

Now, given best estimates $\mathrm{BE}_R^{(\mathrm{OM})}$, $\mathrm{BE}_P^{(\mathrm{OM})}$, $\mathrm{BE}_R^{(\mathrm{MVL})}$ and $\mathrm{BE}_P^{(\mathrm{MVL})}$, the calculation of $\mathrm{SCR}_{\mathrm{NL}}$ using (17) becomes straightforward.

6 Results

In this section we compare solvency capital requirements, best estimates, durations, risk margins and technical provisions for the different scenarios defined in Section 4 and the two actuarial methods defined in Appendix A. Moreover, we compare the SCR calculated using the Solvency II standard model with the SCR values our simulation model generates. The main results are summarized in Table 14.

6.1 Simulation model results

There are some interesting observations to be made about the simulated values for the different scenarios shown in Tables 10 and 11:

- We got more or less the same best estimates, durations, solvency capital requirements, risk margins and technical provisions for the aggregate method as for the individual method.
- The SCR was markedly affected by the correlation, it increased about 10% when correlation was added in the normally distributed scenarios.
- The SCR was even more affected by not being able to predict a trend of 2% per quarter, it increased about 20% when the trend was added in the normally distributed scenarios.
- Assuming log-normal amounts instead of normal did not change the values a lot.
- However, assuming Student's t-distributed amounts did increase the SCR, and hence increased the RM and TP.
- The risk margin was small compared to the solvency capital requirements, so leaving it outside the SCR calculation did not affect the model much.

For a generic random variable X, let $\widehat{\mu}_X$, $\widehat{\sigma}_X$ and $\widehat{F}_X^{-1}(0.995)$ denote the simulated mean, standard deviation and 0.995 quantile, respectively. If Y is another generic random variable, then the simulated correlation between X and Y is denoted by $\widehat{\rho}_{X,Y}$.

By considering the simulated values for the individual method in Table 12 (the values are very similar in the aggregate case), we notice the following regarding the normalized loss statistics U, U_R and U_P:

Scenario	Normal No cor. No trend	Normal **Cor.** No trend	Normal **Cor.** **Trend**	Student's t **Cor.** No trend	Log-normal **Cor.** No trend
BE	177.9	177.9	177.9	178.0	178.0
T	0.6	0.6	0.6	0.6	0.6
SCR	25.7	27.8	33.3	29.7	27.9
RM	0.9	1.0	1.2	1.1	1.0
TP	178.9	178.9	179.1	179.0	178.9

Table 10: **Individual method:** Simulated best estimates, durations, solvency capital requirements, risk margins and technical provisions (assuming the cost-of-capital rate $c = 0.06$). BE, SCR, RM and TP in million SEK, T in years.

Scenario	Normal No cor. No trend	Normal **Cor.** No trend	Normal **Cor.** **Trend**	Student's t **Cor.** No trend	Log-normal **Cor.** No trend
BE	177.8	177.8	177.8	177.8	177.8
T	0.6	0.6	0.6	0.6	0.6
SCR	25.2	28.1	33.6	30.1	28.2
RM	0.9	1.0	1.2	1.1	1.0
TP	178.7	178.8	179.0	178.9	178.8

Table 11: **Aggregate method:** Simulated best estimates, durations, solvency capital requirements, risk margins and technical provisions (assuming the cost-of-capital rate $c = 0.06$). BE, SCR, RM and TP in million SEK, T in years.

- The estimates of parameters related to accident periods up to time I (i.e., parameters with subscript U_R) were more or less the same in all scenarios.
- Adding correlation between insurance types made the standard deviation of U_P increase in value.
- Adding a trend after time I made the mean of U_P increase in value.
- Assuming log-normally distributed amounts instead of normally distributed did not affect the values much.
- Assuming Student's t-distributed amounts instead of normally distributed increased the 0.995 quantile of U_P, but not the mean and standard deviation.
- The correlation between U_R and U_P decreased when we added correlation between insurance types.

Scenario	Normal No cor. No trend	Normal **Cor.** No trend	Normal **Cor.** **Trend**	Student's t **Cor.** No trend	Log-normal **Cor.** No trend
$\widehat{\mu}_U$	0.002	0.003	0.030	0.003	0.003
$\widehat{\sigma}_U$	0.052	0.057	0.059	0.058	0.057
$\widehat{F}_U^{-1}(0.995)$	0.144	0.156	0.187	0.167	0.157
$\widehat{\mu}_{U_R}$	0.005	0.005	0.005	0.005	0.005
$\widehat{\sigma}_{U_R}$	0.088	0.088	0.088	0.088	0.088
$\widehat{F}_{U_R}^{-1}(0.995)$	0.253	0.254	0.254	0.254	0.254
$\widehat{\mu}_{U_P}$	0.002	**0.003**	**0.041**	0.003	0.003
$\widehat{\sigma}_{U_P}$	**0.045**	**0.057**	0.059	0.057	0.057
$\widehat{F}_{U_P}^{-1}(0.995)$	0.126	**0.159**	0.204	**0.179**	0.159
$\widehat{\rho}_{U_C,U_P}$	0.597	0.470	0.470	0.467	0.470

Table 12: **Individual method:** Simulated means, standard deviations, 0.995 quantiles of U, U_R and U_P, and correlation between U_R and U_P for all scenarios.

Some of the observations made in Tables 10 and 11 can be understood by considering the above items. For example, an increase in the mean, standard deviation or 0.995 quantile of U_P implies an increase in the 0.995 quantile of U, and hence an increase in SCR.

6.2 Comparison to the Solvency II standard model

In the Solvency II standard model, the motor insurance types collision and major first party ($n = 1, 2$) belong to the LoB other motor, and the insurance type third party property ($n = 3$) belongs to the LoB motor vehicle liability. Using the best estimate (in terms of (15)) as a proxy of the volume measure of reserve risk ($V_R^{(\ell)}$), and the best estimate multiplied by the total cost to claim cost ratio as a proxy of the volume measure of premium risk ($V_P^{(\ell)}$), we get the values shown in Table 13. Combining these volume measures with the standard

Volume measure	Value (million SEK)
$V_R^{(\mathrm{OM})}$	23.2
$V_P^{(\mathrm{OM})}$	102.7
$V_R^{(\mathrm{MVL})}$	31.3
$V_P^{(\mathrm{MVL})}$	52.1

Table 13: Volume measures of the reserve risk ($V_R^{(\ell)}$'s) and premium risk ($V_P^{(\ell)}$'s) for the LoBs other motor (OM) and motor vehicle liability (MVL).

deviations and correlations of the Solvency II standard model (see Section 5.2), we get the solvency capital requirement $\mathrm{SCR}_{\mathrm{NL}}$ using (17). The SCR values for all scenarios and both methods, as well as for the Solvency II standard model are shown in Table 14.

Scenario	Normal No cor. No trend	Normal **Cor.** No trend	Normal **Cor.** **Trend**	Student's t **Cor.** No trend	Log-normal **Cor.** No trend	Solvency II standard model
Individual	25.7	27.8	33.3	29.7	27.9	37.8
Aggregate	25.2	28.1	33.6	30.1	28.2	-

Table 14: Simulated solvency capital requirements for all scenarios and both methods (assuming the cost-of-capital rate $c = 0.06$), as well as the calculated SCR for the Solvency II standard model. Values in million SEK.

An interesting remark is that if we use the best estimate ($\mathrm{BE}_P^{(\ell)}$) instead of the estimated premium volume ($\gamma^{(\ell)}\,\mathrm{BE}_P^{(\ell)}$) as volume measure ($V_P^{(\ell)}$), and let $g(\sigma)$ in (17) equal $N_{0.995}\sigma\,(\approx 2.58\sigma)$, then the SCR calculated using the Solvency II standard model becomes very close to the values generated by the simulation model.

7 Discussion

In this section we discuss pros and cons of the simulation model for calculating solvency capital requirements developed in this paper. Moreover, we pinpoint some of the results of the previous section and discuss what impact they may have on future insurance risk modeling.

The most important characteristics of the modeling technique are summarized below.

Data driven. No a priori assumptions about the distributions are needed. The data decide which distributions and parameters to use.

Universal across insurance types. Applicable on a wide range of insurance types (not only motor), as long as there are data available.

Universal in space. The technique can be used regardless of the insurance company's geographical region or jurisdiction. However, the distribution and parameter choices may differ between regions due to, e.g., climate or local laws.

Scalable. The technique can be used by all non-life insurance companies, regardless of their sizes.

Even though the model is data driven, there may be ambiguity in the choice of distribution. In the specific case with three motor insurance types it is, due to the small data sample, more or less impossible to determine from data if one should choose the multivariate normal, Student's t or log-normal distribution. However, the simulated SCR values for the different assumptions are in a quite narrow range (see Table 14) so the choice did not matter too much.

The SCR values generated by the simulation model are markedly lower than the value calculated using the Solvency II standard model. One reason for this may be that the LoB motor vehicle liability contains not only third party property claims but also *third party personal injury* claims which are often greater in size and take longer time to handle. Another reason may be that the standard model assumptions about the data distribution of the insurance types studied in this paper are different from the distribution of Folksam's data.

The big advantage of using the simulation model instead of the standard model is that the distributional assumptions are very explicit in the former. We do not only get a numerical SCR value, we also formulate our beliefs about the data. These beliefs may, of course, be wrong but without formulating them there is no possibility to discuss them.

Risks, in particular catastrophic risks, which are not present at all in the data cannot be assessed by statistical methods. Since our simulation model is statistically based it cannot account for such risks either. Instead these risks have to be handled by ad hoc methods, such as scenario analysis. This is however not a disadvantage in comparison to the Solvency II standard model where ad hoc methods are used for catastrophe risk assessment.

The simulation model clearly shows that dependencies (correlations in the linear case) between insurance types must be taken into account when calculating the SCR level. This issue has been discussed a lot the last few years as the Solvency II framework has been developed. Another issue that has not been discussed as frequently as dependencies is the importance of trend assumptions. In Table 14 we see that, for the specific motor insurance case, a trend prediction error of 2% per quarter will affect the SCR twice as much as the error of not taking correlations between insurance types into account. To make the SCR value meaningful, one somehow must quantify how well an experienced actuary can anticipate future trends in average claim amounts. If the prediction uncertainty is high, then this uncertainty may affect the SCR level more than the data distribution assumptions. A suggestion to insurance companies developing internal models, and supervisory authorities evaluating them, is to consider the effects of the uncertainty in trend prediction in detail in their continuing work.

CEIOPS (2009) suggests that insurance obligations should be segmented into *homogenous risk groups* when calculating technical provisions, where the minimum level of segmentation is lines of business. For the solvency capital requirements, however, an insurance company is not necessarily required to use this segmentation.

In the aggregate method all insurance obligations are aggregated into one risk group. This risk group will contain claims from two different lines of business. The rationale for this aggregation is that, since uncertainty is always present when working with a limited amount of data, reducing the number of parameters will reduce the overall uncertainty of the model. The fact that the aggregate method gives results similar to the individual method in our simulations indicate that this is the way to go. More work is needed to figure out under what circumstances, and to what risk types, the aggregate method can be applied.

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References

CEIOPS. Advice to the European Commission in the framework of the Solvency II project on pillar I issues - further advice: CEIOPS-DOC-08/07, 2007.

CEIOPS. Advice for level 2 implementing measures on Solvency II: Technical provisions - lines of business, on the basis of which (re)insurance obligations are to be segmented: CEIOPS-DOC-22/09, 2009.

S. Coles. *An introduction to statistical modeling of extreme values.* Springer, London, England, 2001. ISBN 1-85233-459-2.

European Commission. QIS5 technical specifications - annex to call for advice from CEIOPS on QIS5: working document, 2010.

D. Filipovic. Multi-level risk aggregation. *ASTIN Bulletin*, 39:565–575, 2009.

Folksam General Insurance. Årsredovisning 2010 (Annual report 2010), 2011.

P. Keller. A primer for calculating the Swiss Solvency Test "cost of capital" for a market value margin: Swiss Federal Office of Private Insurance, 2006.

S. Kotz, N. Balakrishnan, and N. L. Johnson. *Continuous multivariate distributions*, volume 1. John Wiley & Sons, Inc., New York, NY, USA, 2nd edition, 2000. ISBN 978-0-471-18387-7.

T. Mack. Distribution-free calculation of the standard error of chain ladder reserve estimates. *ASTIN Bulletin*, 23(2):214–225, 1993.

A. J. McNeil, R. Frey, and P. Embrechts. *Quantitative risk management: concepts, techniques, and tools*. Princeton University Press, Princeton, NJ, USA, 2005. ISBN 978-0-691-12255-7.

E. Ohlsson and J. Lauzeningks. The one-year non-life insurance risk. *Insurance: Mathematics and Economics*, 45:203–208, 2009.

V. Ronkainen and L. Koskinen. Topical modelling issues in Solvency II. *Scandinavian Actuarial Journal*, 2:135–146, 2007.

R. Salzmann and M. V. Wüthrich. Cost-of-capital margin for a general insurance liability runoff. *ASTIN Bulletin*, 40(2):415–451, 2010.

A. Sandström. Solvency II: Calibration for skewness. *Scandinavian Actuarial Journal*, 2:126–134, 2007.

N. Savelli and G. P. Clemente. Hierarchical structures in the aggregation of premium risk for insurance underwriting. *Scandinavian Actuarial Journal*, 3:193–213, 2011.

M. V. Wüthrich and M. Merz. *Stochastic claims reserving methods in insurance*. Wiley, Chichester, England, 2008. ISBN 978-0-470-72346-3.

M. V. Wüthrich, P. Embrechts, and A. Tsanakas. Risk margin for a non-life insurance run-off. *Statistics & Risk Modeling*, 28:299–317, 2011.

A Actuarial prediction methods

In this paper we assume that payment patterns of different insurance types are independent. Moreover, we assume that, for a fixed insurance type, the payment patterns of different accident periods are i.i.d. These assumptions make it reasonable to let our fictive actuary apply the one-dimensional chain ladder method to each individual insurance type when predicting future claim amounts.

Individual method

In the following we consider a specific insurance type, say n. We drop the superscript (n) to simplify the notation.

The basic assumptions in the (distribution-free and one-dimensional) chain ladder method are:

- Cumulative claim amounts C_{ij} of different accident periods i are independent.
- There are *development factors* $f_0, \ldots, f_{J-1}$ such that

$$\begin{aligned} &\mathrm{E}\left[C_{ij} | C_{i0}, \ldots, C_{i,j-1}\right] \\ &\quad = \mathrm{E}\left[C_{ij} | C_{i,j-1}\right] = f_{j-1} C_{i,j-1}. \end{aligned}$$

These assumptions were proposed by Mack (1993) and are sufficient for motivating the mechanical chain ladder calculations used in this paper. It is easy to show from the assumptions above that if $\mathcal{F}_I$ is the information available at time I, then

$$\begin{aligned} &\mathrm{E}\left[C_{ij} | \mathcal{F}_I\right] = C_{i,I-i} f_{I-i} \cdots f_{j-1}, \\ &i \leq I, \quad j \geq I - i + 1. \end{aligned}$$

The development factors are estimated using data from the last d periods ($d = 12$). Let $\widehat{f}_j^I$ denote the estimated development factor at time I, then

$$\widehat{f}_j^I = \left(\sum_{i=I-j-d}^{I-j-1} C_{i,j+1} \right) \Bigg/ \left(\sum_{i=I-j-d}^{I-j-1} C_{ij} \right). \tag{20}$$

Let $\widehat{\mathrm{E}}\left[C_{ij} | \mathcal{F}_I\right]$ and $\widehat{\mathrm{E}}\left[X_{ij} | \mathcal{F}_I\right]$ denote the predictions of C_{ij} and X_{ij}, respectively, at time I, then

$$\begin{aligned} &\widehat{\mathrm{E}}\left[C_{ij} | \mathcal{F}_I\right] = C_{i,I-i} \widehat{f}_{I-i}^I \cdots \widehat{f}_{j-1}^I, \quad \text{and} \\ &\widehat{\mathrm{E}}\left[X_{ij} | \mathcal{F}_I\right] = \widehat{\mathrm{E}}\left[C_{ij} | \mathcal{F}_I\right] - \widehat{\mathrm{E}}\left[C_{i,j-1} | \mathcal{F}_I\right]. \end{aligned} \tag{21}$$

To use the chain ladder method we need at least one payment, so we cannot use it directly for future accident periods. Instead we look at the normalized ultimate claim amounts $Y_1, \ldots, Y_I$. The first $I - J$ of these amounts $(Y_1, \ldots, Y_{I-J})$ are known at time I, while the others $(Y_{I-J+1}, \ldots, Y_I)$ are predicted by

$$\widehat{Y}_i = \frac{\widehat{\mathrm{E}}\left[C_{iJ} | \mathcal{F}_I\right]}{E_i}, \quad i = I - J + 1, \ldots, I,$$

where E_i is the exposure of accident period i.

To predict the future normalized ultimate claim amounts, a linear function is fitted to the last h of the known or predicted Y_i's ($h = 12$). Let $\widehat{\alpha}$ and $\widehat{\beta}$ denote the intercept and slope, respectively, of the fitted function. Then, $Y_{I+1}, \ldots, Y_{I+K}$ are predicted by

$$\widehat{\mathrm{E}}\left[Y_i | \mathcal{F}_I\right] = \widehat{\alpha} + \widehat{\beta} I + \widehat{b}(i - I),$$

where $\widehat{b}$ is the actuary's guess about the future trend of the normalized ultimate claim amounts. In this paper the actuary always predicts that the trend continues, i.e., $\widehat{b} = \widehat{\beta}$.

The incremental claim amounts are now predicted by

$$\widehat{\mathrm{E}}\left[X_{ij} | \mathcal{F}_I\right] = \widehat{p}_j^I E_i \widehat{\mathrm{E}}\left[Y_i | \mathcal{F}_I\right], \qquad i = I+1, \ldots, I+K,$$

where $\widehat{p}_j^I$ is the chain ladder estimate of the proportion paid in development period j given the information $\mathcal{F}_I$,

$$\widehat{p}_0^I = \prod_{k=0}^{J-1} \frac{1}{\widehat{f}_k^I}, \quad \text{and}$$
$$\widehat{p}_j^I = \widehat{p}_{j-1}^I \widehat{f}_{j-1}^I, \quad j \geq 1.$$

At time $I + K$, we have at least one payment for each accident period, so the actuary just updates the development factors and predicts the unknown claim amounts by replacing I with $I + K$ in (20) and (21), respectively.

Aggregate method

In the aggregate method we treat all insurance policies as if they were of the same type, i.e., we let

$$X_{ij} = \sum_{n=1}^{N} X_{ij}^{(n)}, \quad C_{ij} = \sum_{n=1}^{N} C_{ij}^{(n)},$$
$$\text{and} \quad E_i = \sum_{n=1}^{N} E_i^{(n)}.$$

The calculations in the aggregate method are exactly the same as in the individual method with only one insurance type.

B Data plots

In all figures below, winter quarters are visualized by blue solid circles and summer quarters by red solid triangles.

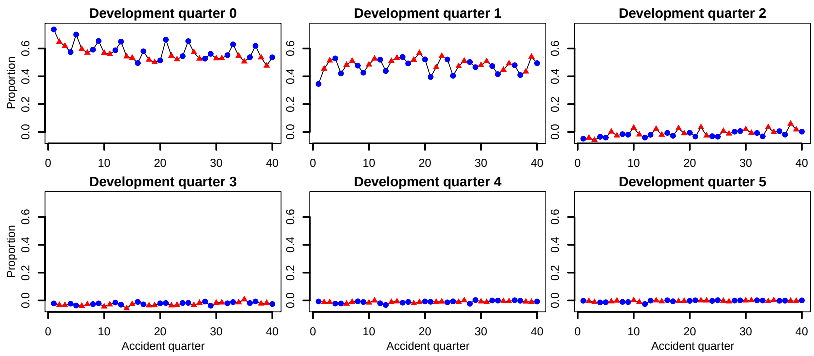

Figure 1: Payment proportions ($p_{ij}^{(n)}$'s) for development quarters 0–5 for collision insurance ($n = 1$).

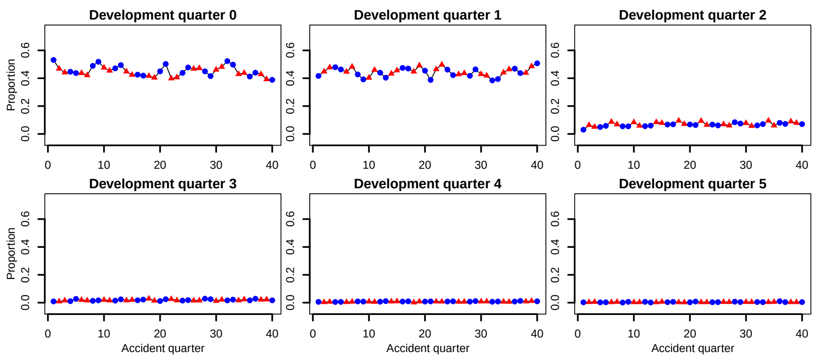

Figure 2: Payment proportions ($p_{ij}^{(n)}$'s) for development quarters 0–5 for major first party insurance ($n = 2$).

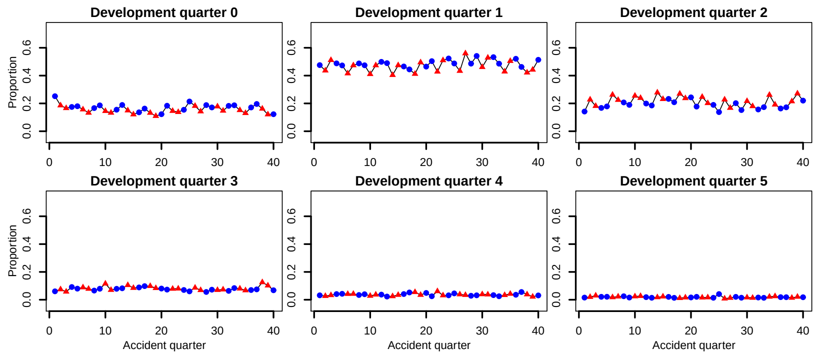

Figure 3: Payment proportions ($p_{ij}^{(n)}$'s) for development quarters 0–5 for third party property insurance ($n = 3$).

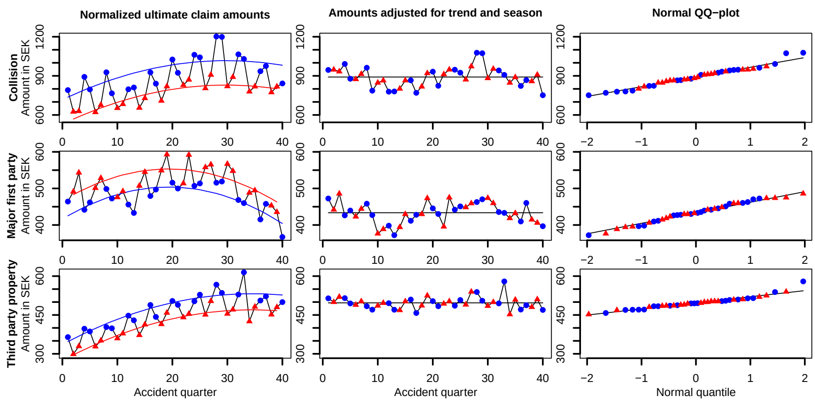

Figure 4: Empirical (normalized) ultimate claim amounts. Original values (left column), trend and seasonally adjusted values (middle column) and normal QQ plot of trend and seasonally adjusted values (right column).

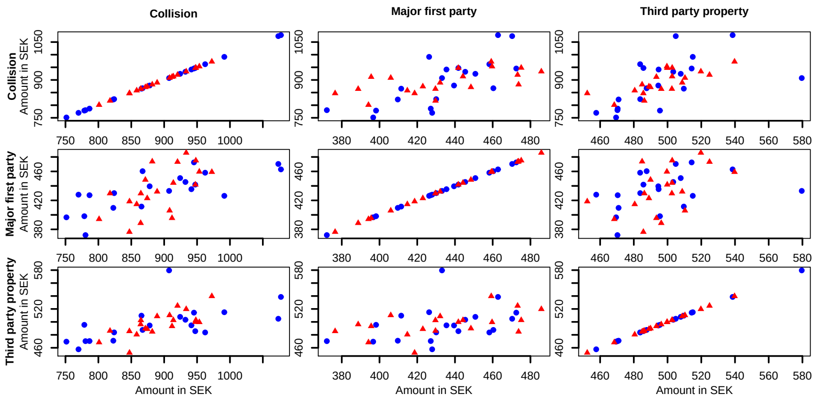

Figure 5: Scatter plots of trend and seasonally adjusted amounts.

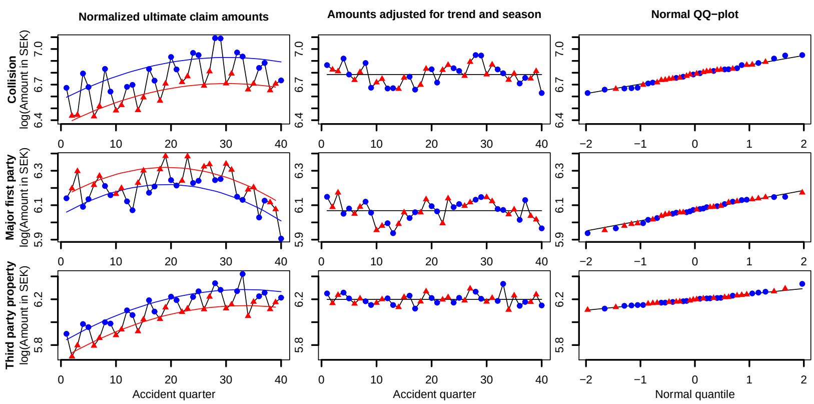

Figure 6: Logarithmized empirical (normalized) ultimate claim amounts. Original values (left column), trend and seasonally adjusted values (middle column) and normal QQ plot of trend and seasonally adjusted values (right column).

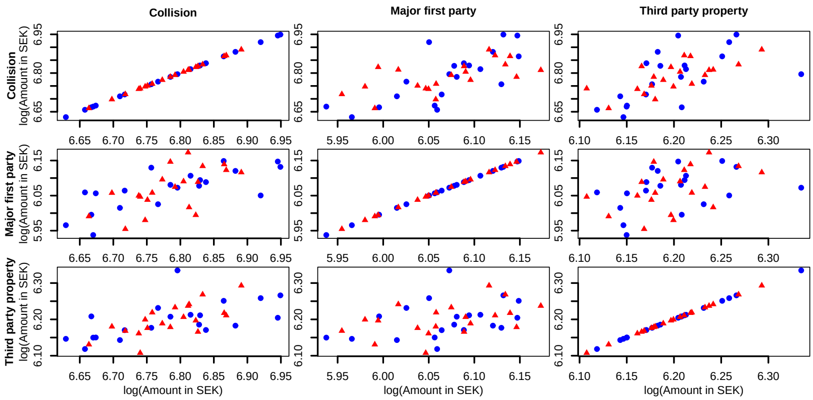

Figure 7: Scatter plots of trend and seasonally adjusted logarithmized amounts.

Paper II

Foreign-currency interest-rate swaps in asset-liability management for insurers

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January 8, 2013

Abstract

We consider an insurer with purely domestic business whose liabilities towards its policy holders have long durations. The relative shortage of domestic government bonds with long maturities makes the insurer's net asset value sensitive to fluctuations in the zero rates used for liability valuation. Therefore, in order to increase the duration of the insurer's assets, it is common practice for insurers to take a position as the fixed-rate receiver in an interest-rate swap. We assume that this is not possible in the domestic currency but in a foreign currency supporting a larger market of interest-rate swaps. Monthly data over 16 years are used as the basis for investigating the risks to the future net asset value of the insurer from using foreign-currency interest-rate swaps as a proxy for domestic ones in asset-liability management. We find that although a suitable position in swaps may reduce the standard deviation of the future net asset value it may significantly increase the exposure to tail risk that has a substantial effect on the estimation of the solvency capital requirements.

1 Introduction

Typically, a life insurer has a liability portfolio with longer duration than its bond portfolio. This makes the insurer's balance sheet vulnerable to a sudden fall in interest rates. To increase the duration of the asset portfolio, and hence reduce the interest-rate risk, the insurer may engage in an interest-rate swap as the fixed-rate receiver. For an insurer operating in a small country with its own currency (for example Sweden) it may not be possible to find a counterparty willing to enter a swap agreement in the local currency. The insurer may then

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choose to enter a swap agreement in a major currency in a neighboring market (for example the euro). This introduces some new risks, most notably spread risk (i.e., the risk that the spread between the domestic zero-rate curve and the foreign swap zero-rate curve changes) and exchange-rate risk. The aim of this paper is to study these risks in detail, and identify situations where the balance sheet may be affected for the worse.

We consider an insurer with a purely domestic insurance business. In order to obtain a good return on capital the insurer is exposed to the domestic stock market (e.g., OMX Stockholm 30 for a Swedish insurer) and in order to manage interest-rate risk the insurer takes positions in liquid interest-rate swaps in a foreign currency (e.g., the euro) closely linked to the domestic currency.

This is the outline of the paper: In Section 2 we introduce the notation, set up an expression for the insurer's net asset value, and discuss how the size of the swap position may be chosen. Section 3 consists of a peaks-over-threshold analysis of empirical future net asset values for an insurer with a stylized portfolio under four different settings. In Section 4 we replace the domestic zero-rate curve and the foreign-domestic zero-rate spread curve with a domestic yield and a yield spread, respectively, and then analyze the empirical distribution of of the 4-dimensional vector consisting of monthly changes in the risk factors: yield, yield spread, exchange rate and stock index. We also investigate how the insurer's net asset value is affected by these risk-factor changes. Conclusions are presented in Section 5, and all figures are found in Section 6.

2 The insurer's assets and liabilities

In this section we introduce the notation used throughout this paper and set up expressions for the values of an insurer's liabilities, bond portfolio and swap position, respectively. We arrive in an expression for the change in the insurer's net asset value given changes in zero rates, swap spreads, stock prices and exchange rate.

Consider an insurer with a random liability cash flow $(C^L_{1/2}, C^L_{2/2}, \dots, C^L_{n/2})$, where C^L_t refers to the random amount to be paid to policy holders in t years from today. In reality, the insurer's payments to policy holders will occur more often than every six months. However, in order to simplify the presentation we consider only cash flows at the frequency of six months. The cash-flow structure is assumed to be stationary since the age distribution of the policy holders does not change markedly in a short time frame.

The insurer's assets are assumed to consist of domestic non-defaultable government bonds, domestic stocks, and interest-rate swaps in a foreign currency. The bonds give rise to a deterministic cash flow $(c^B_{1/2}, c^B_{2/2}, \dots, c^B_{n/2})$. Also this cash-flow structure is assumed to be stationary over time. The liability cash flow we consider here stretches further into the future than the bond cash flow, and therefore $c^B_{k/2} = 0$ for $k > n_0$ for some $n_0 < n$. We focus on the modeling of the values of assets and liabilities at a time $\Delta t \in (0, 1/2)$, e.g., one month from today.

Taking taxes into account[1], the values of the insurer's liabilities at times 0 and Δt are given by

$$L_0 = \sum_{k=1}^{n} \mathrm{E}[C^L_{k/2}] e^{-(1-T) r_{k/2} k/2}, \tag{1}$$

$$L_{\Delta t} = \sum_{k=1}^{n} \mathrm{E}_{\Delta t}[C^L_{k/2}] e^{-(1-T) r_{\Delta t, k/2+\Delta t} k/2}, \tag{2}$$

respectively, where T is the tax level in terms of the zero rates obtained from domestic government bonds, $\mathrm{E}_{\Delta t}$ denotes conditional expectation given the information available at time Δt, and r_t and $r_{\Delta t, t}$ refers to the domestic zero rates at time 0 and Δt, respectively, for maturity t. We assume that no information is available at time Δt that gives the insurer reasons to re-estimate the expected liability cash flow, and set $c^L_t = \mathrm{E}[C^L_t] = \mathrm{E}_{\Delta t}[C^L_t]$.

The value of the insurer's assets is the sum of the values of the positions in bonds, stocks, cash, and swaps. Here, cash may be interpreted as non-defaultable zero-coupon no-discount bonds maturing at time Δt. The values of the bond portfolio at times 0 and Δt are

$$A^B_0 = \sum_{k=1}^{n} c^B_{k/2} e^{-r_{k/2} k/2}, \quad A^B_{\Delta t} = \sum_{k=1}^{n} c^B_{k/2} e^{-r_{\Delta t, k/2+\Delta t} k/2}$$

respectively, and the value of the stocks at time Δt is $A^{St}_{\Delta t} = A^{St}_0 e^{y^{St}_{\Delta t}}$, where A^{St}_0 is the stock value at time 0, and $y^{St}_{\Delta t}$ is the stock log return from time 0 to time Δt. The amount held in cash is denoted by K.

At time 0, the insurer takes a position as the fixed-rate receiver in an m-year foreign-currency interest-rate swap with nominal amount N in the foreign currency. The swap zero rates are expressed as $r_t + s_t$, where the spread s_t is simply the difference between the swap's zero rate and the domestic government zero rate. The insurer relies on that the spread does not vary much over time. The insurer receives the yearly fixed amount cN at times $1, \dots, m$ and makes semi-annual floating-rate payments on the nominal amount N at times

[1] Each year a Swedish life insurer must pay a tax amount equal to 15% of the *government borrowing rate* times the estimated value of the liabilities towards its policy holders. The government borrowing rate is a weighted average of market rates during the previous year of government bonds with a maturity of 5 years or more. Setting $T = 0.15$ in (1) and (2) yields good approximations of the total liabilities (i.e., liabilities towards the policy holders plus liabilities towards the tax enforcement administration) at times 0 and Δt, respectively.

$1/2, 1, \ldots, m$. The fixed swap rate c and the swap zero rates $r_k + s_k$ are related via the expression

$$c \sum_{k=1}^{m} e^{-(r_k+s_k)k} + e^{-(r_m+s_m)m} - 1 = 0$$

reflecting that the initial market value of the swap is zero. However, the investment bank setting up the swap agreement charges the insurer a fee by increasing the floating rate by a fixed amount. The market values of the swap for the insurer at times 0 and Δt are therefore

$$\begin{aligned}
A_0^{Sw} =& N_0 \left(c \sum_{k=1}^{m} e^{-(r_k+s_k)k} + e^{-(r_m+s_m)m} - 1 - d \sum_{k=1}^{2m} e^{-(r_{k/2}+s_{k/2})k/2} \right), \\
A_{\Delta t}^{Sw} =& N_0 e^{y_{\Delta t}^E} \left(c \sum_{k=1}^{m} e^{-(r_{\Delta t,k}+s_{\Delta t,k})(k-\Delta t)} + e^{-(r_{\Delta t,m}+s_{\Delta t,m})(m-\Delta t)} - 1 \right. \\
& \left. -d \sum_{k=1}^{2m} e^{-(r_{\Delta t,k/2}+s_{\Delta t,k/2})(k/2-\Delta t)} \right),
\end{aligned}$$

respectively, where N_0 is the initial nominal swap amount in the domestic currency (i.e., N times the initial exchange rate), $y_{\Delta t}^E$ is the exchange-rate log return from time 0 to time Δt. Note that the swap's initial market value is negative for the insurer, and that the additional interest rate payments, determined by the value d, makes the future value of the swap more exposed to exchange rate changes. Notice also that a simultaneous increase in swap zero rates and the exchange rate is the worst scenario for the value of the insurer's swap position.

The insurer's net asset value, i.e., the difference in value between assets and liabilities, at time Δt is

$$\begin{aligned}
A_{\Delta t} - L_{\Delta t} =& \sum_{k=1}^{n} \left(c_{k/2}^B e^{-r_{\Delta t,k/2+\Delta t} k/2} - c_{k/2}^L e^{-(1-T) r_{\Delta t,k/2+\Delta t} k/2} \right) \\
& + A_0^{St} e^{y_{\Delta t}^{St}} + K + N_0 e^{y_{\Delta t}^E} \left(c \sum_{k=1}^{m} e^{-(r_{\Delta t,k}+s_{\Delta t,k})(k-\Delta t)} \right. \\
& + e^{-(r_{\Delta t,m}+s_{\Delta t,m})(m-\Delta t)} - 1 \\
& \left. -d \sum_{k=1}^{2m} e^{-(r_{\Delta t,k/2}+s_{\Delta t,k/2})(k/2-\Delta t)} \right).
\end{aligned} \tag{3}$$

This net asset value must exceed the insurer's solvency capital requirements (SCR) if the insurer is to be considered solvent. In Solvency I the SCR is equal to 4% of the present liability value (SCR $= 0.04L_{\Delta t}$), but in Solvency II the SCR depends on the distributions of both assets and liabilities. As a rule of thumb, the insurer will have some trouble with the supervising authorities, and may be forced to sell off stocks, if the net asset value is less than 25% of the present liability value. More details about solvency regulation proposals and practical implementation issues can be found in [2] and [5].

2.1 Portfolio settings

In this subsection we set up stylized cash-flow structures for an insurer's liabilities and bonds, respectively. Given such structures, the change in net asset value over some time period depends on the size of the swap position, and the amount of money invested in stocks and held in cash, respectively. We introduce the concept of a *portfolio setting* as a choice of swap-stock-cash position, and discuss how the swap position may be chosen to be, in some sense, optimal.

We consider an insurer with liability-cash-flow structure

$$c^L_{k/2} = \begin{cases} c^L, & k = 1, \dots, 50, \\ c^L\left(1 - \frac{1}{30}\left(\frac{k}{2} - 25\right)\right), & k = 51, \dots, 109, \\ 0, & k \geq 110. \end{cases} \tag{4}$$

Given an initial domestic zero-rate curve and a tax rate, we normalize c^L such that $L_0 = 100$, and get

$$c^L = \frac{100}{\sum_{k=1}^{109} e^{-(1-T)r_{k/2}k/2} - \frac{1}{30}\sum_{i=51}^{109}\left(\frac{k}{2} - 25\right)e^{-(1-T)r_{k/2}k/2}}. \tag{5}$$

We suppose that the insurer has 140 to invest at time 0, and that the insurer invests 100 in a bond portfolio with cash-flow structure

$$c^B_{k/2} = \begin{cases} c^B, & k = 1, \dots, 30, \\ 0, & k \geq 31, \end{cases} \tag{6}$$

to balance its liabilities. This choice corresponds to choosing c^B such that $A^B_0 = L_0 = 100$:

$$c^B = \frac{100}{\sum_{k=1}^{30} e^{-r_{k/2}k/2}}. \tag{7}$$

The remainder is invested in either stocks or cash, i.e., $(A^{St}_0, K) = (40, 0)$ or $(A^{St}_0, K) = (0, 40)$. The insurer also enters a position as the fixed receiver in a 5-year foreign-currency interest-rate swap ($m = 5$) with initial nominal amount N_0 in the domestic currency. There is no cost for taking the swap position, but since the insurer will pay a floating rate that is higher than the market rate, the initial market value of the swap position A_0^{Sw} is negative. We call the triple (A_0^{St}, K, N_0) a *portfolio setting* for the insurer.

We set the tax rate $T = 0.15$ and the additional swap-rate payment corresponds to $d = 0.001$. The initial domestic zero rates, and the initial spreads, for maturities 1/2, 2, 5, 7 and 10 years are set to the market rates at December 30, 2011:

$$(r_{1/2}, r_2, r_5, r_7, r_{10}) = (0.01333, 0.00918, 0.01153, 0.01353, 0.01621),$$
$$(s_{1/2}, s_2, s_5, s_7, s_{10}) = (0.00294, 0.00402, 0.00580, 0.00731, 0.00779).$$

For $t < 1/2$ we set $r_t = r_{1/2}$ and $s_t = s_{1/2}$, and for $t > 10$ we set $r_t = r_{10}$ and $s_t = s_{10}$. To get zero rates and spreads for other maturities, we use linear interpolation. These values imply that $c^L \approx 1.649$ and $c^B \approx 3.732$. The resulting liability and bond cash flows, and their corresponding market values, are shown in Figure 1. The initial value of the insurer's swap position decreases linearly in $N_0 d$. For $N_0 = 100$ and $d = 0.001k$ the initial value, for the insurer, of the swap position is $-0.96k$ (see Table 1 for the case $d = 0.001$).

N_0	L_0	A_0^B	$A_0^{St} + K$	A_0^{Sw}	A_0	$A_0 - L_0$
0	100.00	100.00	40.00	0.00	140.00	40.00
100	100.00	100.00	40.00	-0.96	139.04	39.04
200	100.00	100.00	40.00	-1.92	138.08	38.08

Table 1: Initial asset and liability values for the insurer for different sizes N_0 of the swap position.

There are various approaches to choosing the size N_0 of the swap position. A common text-book approach is to consider the asset and liability values as functions of the domestic zero-rate curve and choose N_0 to be the value that makes the net asset value immune to an instantaneous parallel shift in the zero-rate curve. Assuming that the value of the stocks and the exchange rate are independent of interest-rate movements in the short term, the condition can be written

$$\nabla\left(A_0^B\right)^T \mathbf{1} + \nabla\left(A_0^{Sw}\right)^T \mathbf{1} = \nabla(L_0)^T \mathbf{1}, \tag{8}$$

where $\nabla = \left(\frac{\partial}{\partial r_{1/2}}, \frac{\partial}{\partial r_{2/2}}, \dots, \frac{\partial}{\partial r_{n/2}}\right)$, which yields

$$N_0 = \frac{\frac{1}{2}\sum_{k=1}^{n}\left((1-T)kc_{k/2}^L e^{-(1-T)r_{k/2}k/2} - kc_{k/2}^B e^{-r_{k/2}k/2}\right)}{c\sum_{k=1}^{m} ke^{-(r_k+s_k)k} + me^{-(r_m+s_m)m} - \frac{d}{2}\sum_{k=1}^{2m} ke^{-(r_{k/2}+s_{k/2})k/2}}.$$

For the insurer considered here, under the market conditions defined in the previous section, this approach produces an optimal N_0-value of 178.

Notice that although an appropriate swap position may reduce the sensitivity of the net asset value to interest-rate changes, a large swap position leads to an exposure to potentially dangerous fluctuations in the spread between the swap zero rates and the domestic zero rates. In particular, a reduction in variance of the net asset value do not necessarily imply a reduction in the estimates of the overall solvency capital requirements for the insurer. The solvency capital requirements are essentially a high quantile of the loss distribution over some time period, where the loss is the negative change in net asset value. We return to this important issue in Section 3.

3 Data and extreme-value analysis

In this section we construct a data set of risk-factor changes, and conduct a peaks-over-threshold analysis of empirical future net asset values for an insurer with the stylized liability and bond cash-flow structures defined in Section 2.1 under four different portfolio settings. We calculate point estimates and confidence intervals of the parameters in the generalized Pareto distribution, and of the 2400-month return level which is related to the solvency capital requirements (SCR) in Solvency II.

Market values of Swedish government bonds and Eurozone interbank swap rates from 196 consecutive months (from September 1995 to December 2011) are transformed, using a standard bootstrap procedure, into a data set consisting of monthly changes in domestic zero rates and interest-rate-swap spreads for maturities 1/2, 2, 5, 7 and 10 years. Market values of the exchange rate *SEK per euro* and the stock index *OMX Stockholm 30* from the same time period are transformed into monthly exchange-rate log returns and stock log returns, respectively, and added to the data set of rate and spread changes.

The data form a multidimensional sample of risk-factor changes. Given a portfolio setting, i.e., a triple (A_0^{St}, K, N_0), the net asset value in one month may be computed for each sample point using (3). Thus, each portfolio setting yields a sample of size 196 of future net asset values. Histograms of these net asset values are shown in Figure 2 for one portfolio setting with stocks and no cash: $(40, 0, 100)$, and three portfolio settings with cash and no stocks: $(0, 40, 0)$, $(0, 40, 100)$ and $(0, 40, 200)$, respectively. The histograms indicate, as expected, that increasing the size of the swap position reduces the variance of the future net asset value. Figure 3 shows the empirical estimate of the standard deviation of the future net asset value as a function of N_0 given that (A_0^{St}, K) is either $(40, 0)$ or $(0, 40)$. It is clear that swaps can be useful as a variance-reduction tool. Moreover, we find that swap positions corresponding to nominal amounts 210 (when the amount 40 is invested fully in stocks) and 140 (when the amount 40 is held in cash) give the optimal variance reduction. However, what really could affect the solvency of the insurer for the worse is the behavior of the left tail of the net asset value distribution. A thorough analysis, presented below, is required to get an idea of this behavior.

Given a portfolio setting, each sample point generates a net asset value in one month. We are interested in analyzing small net asset values leading to insolvency or near insolvency for the insurer. For practical reasons we consider net liability values, i.e., net asset values with a minus sign. The ith sample point of the historical multidimensional sample of risk-factor changes gives rise to the net liability value z_i. In order to analyze the right tail of the distribution of net liability values we conduct a peaks-over-threshold analysis on the sample $\{z_1, \ldots, z_{196}\}$. We choose a threshold u such that 20% of the z_is exceed this level. Whether this threshold choice is good or not will be examined later in this section. Since we have 196 observations, the 39 z_i-values exceeding u are $z_{(158)}, \ldots, z_{(196)}$, where $z_{(i)}$ is the ith smallest order statistic. We set $u = (z_{(157)} + z_{(158)})/2$, and define excess i by $y_i = z_{(157+i)} - u$ for $i = 1, \ldots, 39$. Under the assumption that the net liability value is in the maximum domain of attraction of an extreme-value distribution, the appropriately scaled excesses $y_1, \ldots, y_{39}$ fit well to a generalized Pareto distribution with distribution function given by

$$G(y; \sigma, \xi) = \begin{cases} 1 - (1 + \xi y/\sigma)^{-1/\xi}, & \xi \neq 0, \\ 1 - \exp(-y/\sigma), & \xi = 0, \end{cases}$$

for $y > 0$ and $\sigma > 0$. Maximum-likelihood estimation is used to estimate the parameters of the generalized Pareto distribution, and the parameter estimates are denoted by $\hat{\sigma}$ and $\hat{\xi}$, respectively.

Let ℓ_k denote the k-observation return level, i.e, the level that is exceeded on average once every kth observation. Under the assumption that the excess Y above the threshold u has distribution function G, the k-observation return level is given by

$$\ell_k = \begin{cases} u + \frac{\sigma}{\xi}\left[(k\zeta_u)^{\xi} - 1\right], & \xi \neq 0, \\ u + \sigma \log(k\zeta_u), & \xi = 0, \end{cases} \tag{9}$$

where $\zeta_u = \mathrm{P}(Z > u)$. We estimate ζ_u by $\hat{\zeta}_u = 39/196 \approx 0.1990$, and get an estimate $\hat{\ell}_k$ of ℓ_k by replacing σ, ξ and ζ_u by $\hat{\sigma}$, $\hat{\xi}$ and $\hat{\zeta}_u$, respectively, in the above expression.

In Solvency II, the solvency capital requirements (SCR) are related to the 200-year return level of the distribution of the net liability value: essentially the probability that the net liability value in one year takes a positive value must not exceed 0.5%. Here we model the net liability value in one *month* and not in one year. There is no obvious relation between these two future net liability values unless we make further assumptions about the evolution of the net liability value over time. The 2400-month return level ℓ_{2400} is a net liability value with the property that it will be exceeded on average once every 200 years (if monthly updated and monthly exceedances of the return levels are independent). We choose ℓ_{2400} as a measure of risk for the insurer. A negative value means a solvent company. In Figure 4 the estimates of σ, ξ and ℓ_{2400} are plotted against the swap position N_0 given that (A_0^{St}, K) is either $(40, 0)$ (stocks but no cash) or $(0, 40)$ (no stocks but cash). The lower right plot in Figure 4 suggests that if there is no investment in stocks, then engaging in swap contracts makes the insurer less solvent, so the optimal choice in terms of extreme risk is $N_0 = 0$. This is rather counterintuitive since there is covariation between the domestic government bond rate and the foreign swap rate. What may be even more counterintuitive is that when adding stocks to the portfolio (lower left plot) the optimal choice becomes $N_0 = 100$. However, the $\hat{\ell}_{2400}$-curve is rather flat between $N_0 = 0$ and $N_0 = 200$, so we should not focus too much on this value.

Now, consider four different settings for the insurer's portfolio; (i) the amount 40 invested in stocks, no cash and no swaps; (ii) the amount 40 invested in stocks, no cash and and a swap position $N_0 = 178$; (iii) no stocks, the amount 40 held in cash and no swaps; and (iv) no stocks, the amount 40 held in cash and a swap position $N_0 = 178$; respectively. The four portfolio settings correspond to the following values for the triple (A_0^{St}, K, N_0):

$$\begin{array}{llll} \text{(i)}: & (40,0,0), & \text{(ii)}: & (40,0,178), \\ \text{(iii)}: & (0,40,0), & \text{(iv)}: & (0,40,178). \end{array} \tag{10}$$

We choose $N_0 = 178$ since this is the value we get in the naïve text-book approach for choosing swap position in Section 2.1. Maximum-likelihood point estimates of the parameters in these four cases are shown in Table 2. The generalized Pareto distribution with the ML estimates as parameters fits the data relatively well as seen in the QQ plots in Figures 5 and 6.

(A_0^{St}, K, N_0)	$\hat{\xi}$	$\hat{\sigma}$	$\hat{\ell}_{2400}$
$(40, 0, 0)$	-0.55	3.47	-31.0
$(40, 0, 178)$	-0.29	2.37	-29.9
$(0, 40, 0)$	-0.06	1.52	-30.2
$(0, 40, 178)$	0.21	0.81	-27.2

Table 2: Maximum-likelihood estimates of the parameters of the generalized Pareto distribution for the four settings (i)–(iv) in (10).

Point estimates of the generalized Pareto parameters define a model for the threshold excesses. To get information about the model uncertainty, we calculate confidence intervals of the parameters ξ and σ, and the return level ℓ_{2400}, respectively. We use profile log-likelihood intervals since they take the asymmetry of the return-level uncertainty into account. By the asymmetry of return-level uncertainty we mean that if we have a small sample of heavy-tailed data, we are more certain about the lower bound of the return level than of the upper bound. This is true since we have at least a few observations around the lower bound, but very often no observations at all around the upper bound. Figures 5 and 6 contain plots of the profile log-likelihood functions of ξ, σ and ℓ_{2400}, respectively, for the settings (i)–(iv) in (10). In each plot, the cross shows the point estimate, and the intersections between the profile log-likelihood function and the horizontal line show the bounds of an approximate 95% confidence interval.

In this paragraph, and the one following, we explain the idea behind profile log-likelihood confidence intervals, and comment on some of the issues when the sample size is small or moderate. Let $l(\sigma, \xi)$ denote the log-likelihood function for the estimation of the parameters of the generalized Pareto distribution. Then $l_p(\xi) = \max_\sigma l(\sigma, \xi)$ is the profile log-likelihood function for ξ. By instead maximizing with respect to ξ, we get the profile log-likelihood function for σ. From (9) it follows that σ may be expressed as a function of ξ and ℓ_{2400} (assuming that ζ_u is deterministic with $\zeta_u = \hat{\zeta}_u$). Therefore, the generalized Pareto distribution may be re-parameterized as having parameters ξ and ℓ_{2400} which gives rise to a profile log-likelihood function for ℓ_{2400}. The calculated confidence intervals for the values of ξ, σ and ℓ_{2400} are based on Theorem 2.6 in [3]. An approximate confidence interval for the value of ξ (and similarly for σ and ℓ_{2400}) stems from the fact that the deviance function $D_p(\xi) = 2\{l(\hat{\sigma}, \hat{\xi}) - l_p(\xi)\}$ is approximately χ_1^2-distributed if the sample of excesses is sufficiently large. This is proven by Wilks in [13]. Therefore, $\{\xi : D_p(\xi) \le c_\alpha\}$, where c_α is the $(1-\alpha)$ quantile of the χ_1^2-distribution, is an approximate $(1-\alpha)$ confidence interval for the unknown parameter value of ξ.

For small to moderate sample sizes these confidence intervals may be improved upon by introducing correction factors. In [6] Lawley introduces a general method to calculate such factors given a sample size n. In view of constructing profile log-likelihood confidence intervals for a one-dimensional parameter θ, the method boils down to finding $\varepsilon_p(\theta)$ such that $(1-\varepsilon_p(\theta))D_p(\theta)$ has the same moments as a χ_1^2-distributed random variable if quantities of order $O(1/n^2)$ are neglected. In [12] Tajvidi uses Lawley's results to find explicit formulas for caluculating $\varepsilon_p(\xi)$ and $\varepsilon_p(\sigma)$ given ξ and n, for the parameters ξ and σ in the generalized Pareto distribution (it turns out that both $\varepsilon_p(\xi)$ and $\varepsilon_p(\sigma)$ are independent of σ). We do not use these correction factors when constructing the profile log-likelihood confidence intervals. However, to get a better understanding of how large the correction factors are it is worth mentioning that for $\xi = 0.2$ and $n = 40$ we get $\varepsilon_p(\xi) \approx 0.191$ and $\varepsilon_p(\sigma) \approx 0.275$ using Tajvidi's formula. In this same paper Tajvidi also conducts a simulation study which suggests that (for $\xi = 0.2$) a two-sided confidence interval for ξ

which should be on the 95% level for a large sample is approximately on the 92% level without correction factor, and on the 95% level with correction factor, if $n = 40$. The corresponding confidence interval for σ is on the 95% level without correction factor, and on the 97% level with correction factor.

The plots in Figures 5 and 6 suggest that the net liability value is in the maximum domain of attraction of a Weibull distribution in case (i) and a Fréchet distribution in case (iv), while case (ii) and (iii) could be Weibull, Gumbel or Fréchet. One of the most important observations from these figures is that the right endpoint of the confidence interval for ℓ_{2400} increases rapidly with the heaviness of the right tail (i.e., increases as ξ decreases). The value of the right endpoint provides information about the degree of uncertainty in the calculation of solvency capital requirements given that the model is correct.

The analysis presented above and the point estimates in Table 2 are valid for the threshold value corresponding to 20% of the monthly net liability values above the threshold. An obvious question is whether the conclusions are robust or not to changes in the threshold value. The answer to that question is provided by Figures 7 and 8 which show estimates of ξ, σ, and ℓ_{2400} as functions of the threshold value. From these figures we draw the conclusion that the exposure to foreign-currency interest-rate swaps is the main driver of extreme risk for the net asset value of the insurer. The initial choice of threshold value is in fact a choice that may underestimate the risk of near insolvency: increasing the threshold value (moving further into the tail) leads to higher estimates of ℓ_{2400} in the setting (iv) with a large exposure to swap-spread risk.

Although the approximate confidence intervals for ξ, σ and ℓ_{2400} in Figures 5 and 6 are based on well-established statistical methodology, a sound scepticism to the accuracy of the approximation of the deviance function by a χ_1^2-distributed random variable may call for alternative approaches to assessing the values of ξ, σ and ℓ_{2400}. Figure 9 shows histograms of point estimates of ξ, σ and ℓ_{2400} based on parametric bootstrap for settings (i)–(iv). The sample in each of the plots is formed by drawing with replacement 196 times from the original sample of multivariate historical risk-factor changes to form a new sample, on which the peaks-over-threshold method is applied by fitting a generalized Pareto distribution to the excesses corresponding to the worst 20% of the sample. The conclusions drawn from Figure 9 turns out to be completely compatible with those drawn from Figures 5 and 6.

The findings so far can be summarized as follows. By taking a position in foreign-currency interest-rate swaps as the fixed-rate receiver, the insurer can reduce the sensitivity of the net asset value to fluctuations in the domestic zero-rate curve. This is common practice for insurers. However, the fact that the foreign-currency interest-rate swap zero rates do not move in tandem with domestic zero rates introduces a swap-spread risk, and to a lesser extent an exchange-rate risk. Although the use of such swaps stabilize fluctuations in the insurer's solvency ratio over time it can increase the risk of extreme drops in the net asset value that threatens the solvency of the insurer. The result is very much in line with the conclusions drawn from Figures 5 and 6.

4 Key risk factors and extreme scenarios

In Section 3 the historical data set of risk-factor changes was transformed into a parametric family of empirical distributions of future net asset values with the portfolio setting (A_0^{St}, K, N_0) as parameter vector, where A_0^{St} is the initial market value of stocks, K the initial amount of cash, and N_0 the size of the swap position. Given that (A_0^{St}, K) was chosen to be either $(40, 0)$ or $(0, 40)$, we treated N_0 as the single parameter. Extreme-value analysis of these empirical distributions led to a better understanding of which factors drive extreme risk, and quantified extreme risk in terms of return-level estimates. The domestic government zero-rate curve and the foreign-swap zero-rate curve were formed from vectors of zero rates using interpolation. Here we replace the domestic zero-rate curve by a domestic *yield* - a flat term structure. Moreover, we replace the 'spread curve' by a single *yield spread* that represents the spread between the domestic yield and the foreign yield. The reason for this simplification is that we want to clarify the key risk drivers and illustrate the dependence among them.

Given a domestic zero-rate curve, the market values A^B and L of the insurers bond portfolio and liability portfolio, respectively, can be calculated. Since it is assumed that the present values of the bond portfolio and that of the liabilities are equal, the domestic yield r and the yield spread s are obtained as the solution to

$$0 = \sum_{k=1}^{n} \left(c_{k/2}^{B} e^{-rk/2} - c_{k/2}^{L} e^{-(1-T)rk/2} \right),$$

$$0 = c \sum_{k=1}^{m} e^{-(r+s)k} + e^{-(r+s)m} - 1.$$

The values of c_t^B and c_t^L are given in (4)–(7). Therefore, $n = 109$, $m = 5$, and $c = 0.01740$ give $r = 0.01695$, $s = 0.00030$. Let $\Delta t = 1/12$ (in one month) and denote by X_1 the change in the domestic yield from time 0 to time Δt, by X_2 the change in the yield spread, by X_3 the exchange-rate log return, and by X_4 the stock-index log return. Then the net asset value at time Δt is given by

$f(\mathbf{X})$, where

$$
\begin{aligned}
f(\mathbf{x}) = &\sum_{k=1}^{n} c^{B}_{k/2} e^{-(r+x_1)k/2} - \sum_{k=1}^{n} c^{L}_{k/2} e^{-(1-T)(r+x_1)k/2} + A^{St}_0 e^{x_4} + K \\
&+ N_0 e^{x_3} \Big(c \sum_{k=1}^{m} e^{-(r+x_1+s+x_2)(k-\Delta t)} + e^{-(r+x_1+s+x_2)(m-\Delta t)} - 1 \\
&- d \sum_{k=1}^{2m} e^{-(r+x_1+s+x_2)(k/2-\Delta t)} \Big)
\end{aligned}
$$

From the samples of future net values of the bond portfolio and the liabilities, a sample of size 196 of X_1-values is produced. Given this sample and a sample of future values of the swap position, a sample of X_2-values is produced. By also including log returns of the exchange rate and stock index, a sample of size 196 of values of vectors (X_1, X_2, X_3, X_4) is produced. The corresponding samples of net asset values $f(\mathbf{X})$ are the same as those analyzed in Section 3, i.e., the settings (i)–(iv) in (10).

The sample mean and sample covariance of the vector of risk factor changes (the X_ks) are given by $\widehat{\boldsymbol{\mu}}$ and $\widehat{\Sigma} = DRD$, respectively, where D is a diagonal matrix of sample standard deviations and R is the sample correlation matrix. The estimates are $\widehat{\boldsymbol{\mu}} \approx 10^{-4}(-4.21, 1.41, -2.10, 54.4)^{\mathrm{T}}$, $D \approx 10^{-3}\mathrm{diag}(2.46, 1.92, 17.4, 63.4)$, and

$$
R \approx \begin{pmatrix} 1 & -0.52 & -0.09 & 0.10 \\ -0.52 & 1 & 0.09 & 0.23 \\ -0.09 & 0.09 & 1 & -0.29 \\ 0.10 & 0.23 & -0.29 & 1 \end{pmatrix}.
$$

By looking at the the sample means and comparing them to the sample standard deviations we can say that $\boldsymbol{\mu} \approx \mathbf{0}$ appears reasonable. The sample standard deviations for the yield and foreign-domestic yield-spread data are far smaller then those for the exchange rate and stock index. However, the bond portfolio and liability cash flow have long durations and therefore the effect of the variability in the yield and yield spread has a similarly large effect on the future net asset value as that of the stock index, see (11) below and also Figure 2 for a graphical illustration. The effect of the variability in the exchange rate is rather small and we see from the expression for f above that it is necessary to have a simultaneous increase in both the exchange rate and the foreign-domestic yield spread in order to achieve an outcome that is negative for the insurer. From the sample correlations and the scatter plots in Figure 10 we can really only say that there are strong indications that the yield changes and yield-spread changes are negatively correlated. Such a negative correlation is bad news for the insurer since it indicates that a simultaneous drop in the domestic yield (reduces the net value of the bond portfolio and liabilities) and a rise in the swap yield spread (reduces the value of the swap position for the fixed-rate receiver) is rather likely.

The sensitivity of the net asset value to changes in the risk factors can be investigated by linearizing $f(\mathbf{x})$ in a neighborhood of $\boldsymbol{\mu} = \mathbf{0}$,

$$f(\mathbf{X}) \approx f(\mathbf{0}) + \nabla f^{\mathrm{T}}(\mathbf{0})\mathbf{X},$$

and computing $\nabla f^{\mathrm{T}}(\mathbf{0})$ for settings (i)–(iv), i.e., the triple (A_0^{St}, K, N_0) equals $(40, 0, 0)$, $(40, 0, 178)$, $(0, 40, 0)$ and $(0, 40, 178)$, respectively. We first need to compute the partial derivatives of f with respect to the x_ks.

$$\begin{aligned}
\frac{\partial f}{\partial x_1}(\mathbf{x}) = &-\sum_{k=1}^{n} \frac{k}{2}\Big(c_{k/2}^{B} e^{-(r+x_1)k/2} - c_{k/2}^{L}(1-T)e^{-(1-T)(r+x_1)k/2}\Big) \\
&- N_0 e^{x_3}\Big(c\sum_{k=1}^{m}(k-\Delta t)e^{-(r+x_1+s+x_2)(k-\Delta t)} \\
&+ (m-\Delta t)e^{-(r+x_1+s+x_2)(m-\Delta t)} \\
&- d\sum_{k=1}^{2m}(k/2-\Delta t)e^{-(r+x_1+s+x_2)(k/2-\Delta t)}\Big), \\
\frac{\partial f}{\partial x_2}(\mathbf{x}) = &-N_0 e^{x_3}\Big(c\sum_{k=1}^{m}(k-\Delta t)e^{-(r+x_1+s+x_2)(k-\Delta t)} \\
&+ (m-\Delta t)e^{-(r+x_1+s+x_2)(m-\Delta t)} \\
&- d\sum_{k=1}^{2m}(k/2-\Delta t)e^{-(r+x_1+s+x_2)(k/2-\Delta t)}\Big),
\end{aligned}$$

corresponding to changes in the yield and swap-yield spread, respectively, and

$$\begin{aligned}
\frac{\partial f}{\partial x_3}(\mathbf{x}) = &N_0 e^{x_3}\Big(c\sum_{k=1}^{m} e^{-(r+x_1+s+x_2)(k-\Delta t)} + e^{-(r+x_1+s+x_2)(m-\Delta t)} - 1 \\
&- d\sum_{k=1}^{2m} e^{-(r+x_1+s+x_2)(k/2-\Delta t)}\Big), \\
\frac{\partial f}{\partial x_4}(\mathbf{x}) = &A_0^{St} e^{x_4},
\end{aligned}$$

corresponding to effects due to log returns of the exchange rate and stock index, respectively. Inserting initial values for the risk factors and those for the cash-flow structure in the four different settings yields

$$\nabla f^{\mathrm{T}}(\mathbf{0}) \approx \begin{cases} (835.4, 0, 0, 40), & (A_0^{St}, K, N_0) = (40, 0, 0), \\ (-6.611, -842.0, -1.444, 40), & (A_0^{St}, K, N_0) = (40, 0, 178), \\ (835.4, 0, 0, 0), & (A_0^{St}, K, N_0) = (0, 40, 0), \\ (-6.611, -842.0, -1.444, 0), & (A_0^{St}, K, N_0) = (0, 40, 178). \end{cases}$$

Relevant measures of the sensitivities of the net asset value to unfavorable deviations of the X_ks away from 0 are obtained by computing $\partial f(\mathbf{0})/\partial x_k F_{X_k}^{-1}(p)$ for, say, $p = 0.05$ if $\partial f(\mathbf{0})/\partial x_k > 0$ and $p = 0.95$ if $\partial f(\mathbf{0})/\partial x_k < 0$. For settings (i)–(iv), the vectors $\mathbf{y} = \mathbf{y}(A_0^{St}, K, N_0)$ of such sensitivities are

$$\mathbf{y} \approx -\begin{cases} (23.63, 0, 0, 5.00), & (A_0^{St}, K, N_0) = (40, 0, 0), \\ (0.02, 2.32, 0.03, 5.00), & (A_0^{St}, K, N_0) = (40, 0, 178), \\ (23.63, 0, 0, 0), & (A_0^{St}, K, N_0) = (0, 40, 0), \\ (0.02, 2.32, 0.03, 0), & (A_0^{St}, K, N_0) = (0, 40, 178). \end{cases} \tag{11}$$

We see above that adding an appropriate swap position effectively removes much of the portfolio's sensitivity to yield changes in the way described by many text books. However, instead we get a large sensitivity to spread changes.

A peaks-over-threshold analysis of the four X_k samples, similar to that presented in Section 3, reveals that only the X_2 sample has excesses over high thresholds that fits well to a generalized Pareto distribution with a positive parameter ξ. This statement remains valid when the threshold is varied over a range of reasonable threshold values. In particular, for more extreme deviations away from 0 of the X_ks than those corresponding to 5% or 95% quantile values, extreme drops in the net asset value are rather likely to be due to large unfavorable outcomes of the swap spread. This finding is in line with the findings in Section 3.

4.1 Extreme scenarios and outcomes

There is an increasing attention of regulators to complement internal model building for estimation of solvency capital with scenario analysis. Most relevant is to identify scenarios that lead to insolvency or near insolvency and to identify the most likely scenarios in such a set of extreme scenarios. The difficulty lies in developing a structured and credible approach to identifying sets of equally likely extreme scenarios.

We want to determine the minimizer $\mathbf{x}^*$ of $f(\mathbf{x})$ over a set of scenarios that in some appropriate sense are equally likely. Under the assumption that $\mathbf{X}$ is elliptically distributed with stochastic representation $\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + AR\mathbf{U}$, where $R \geq 0$ and $\mathbf{U}$ are independent, and $\mathbf{U}$ is uniformly distributed on the unit sphere, an appropriate set of equally likely scenarios is given by a set of scenarios of the form $\mathbf{x} = \mathbf{x}(\mathbf{u}) = \boldsymbol{\mu} + A\lambda\mathbf{u}$ with $\mathbf{u}$ an element on the unit sphere. The matrix A satisfies $AA^{\mathrm{T}} = \Sigma$, where Σ is the covariance matrix of $\mathbf{X}$. This scenario set is the ellipsoid $\{\mathbf{x} : (\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) = \lambda^2\}$, where the severity parameter λ represents a suitably extreme quantile value of R. Scenario generation via ellipsoidal and more general scenario sets are well studied, see, e.g., [1], [8], and [11]. If the stock-cash position (A_0^{St}, K) is fixed, then for a given nominal amount N_0 of the swap position and a severity value λ we can determine the minimizer $\mathbf{u}^*(N_0, \lambda) = \operatorname{argmin}_{\mathbf{u}} f(\mathbf{x}(\mathbf{u}))$ that corresponds to the worst case scenario $\mathbf{x}(\mathbf{u}^*)$, and investigate how $\mathbf{x}(\mathbf{u}^*)$ varies with the swap position size N_0 and the severity value λ. The essential question at this point is whether ellipsoidal scenario sets are relevant for describing extreme scenarios for the insurer's risk factors $X_1, \dots, X_4$. A peaks-over-threshold analysis shows that the right (and left) tail of the distribution of X_2 (changes in the yield spread) appears to be heavier than those of the other variables. This is not consistent with assuming that $\mathbf{X}$ has an elliptical distribution. That $\mathbf{X}$ is elliptically distributed, $\mathrm{P}(\mathbf{X} = \boldsymbol{\mu}) = 0$, and has an invertible dispersion matrix is equivalent to $\mathbf{Y}/|\mathbf{Y}|$, where $\mathbf{Y} = A^{-1}(\mathbf{X}-\boldsymbol{\mu})$, being uniformly distributed on the unit sphere. In particular, we may test for ellipticality by transforming the 4-dimensional sample points accordingly and checking how well the transformed sample fits to the uniform distribution on the sphere. Here, the fit is good enough to accept the assumption of an elliptical distribution as a reasonably accurate approximation.

Under the assumption that $(A_0^{St}, K) = (0, 40)$ and either $N_0 = 0$ or $N_0 = 178$, i.e., settings (iii) or (iv) in (10), $f(\mathbf{x})$ represents the net asset value in one month as a result of a parallel shift x_1 in the (flat) zero curve and x_2 in the (flat) spread curve, and a log return x_3 of the exchange rate. Based on 196 monthly yield changes, yield-spread changes, and exchange-rate log returns, the covariance matrix Σ_{-4} of $(X_1, X_2, X_3)^{\mathrm{T}}$ is estimated and the Cholesky decomposition of Σ_{-4} yields an estimate of A_{-4} satisfying $A_{-4}A_{-4}^{\mathrm{T}} = \Sigma_{-4}$:

$$\Sigma_{-4} \approx 10^{-6} \begin{pmatrix} 6.03 & -2.46 & -3.64 \\ -2.46 & 3.67 & 3.13 \\ -3.64 & 3.13 & 302.48 \end{pmatrix}$$

Any element $\mathbf{u} \in \mathbf{R}^3$ with $\mathbf{u}^{\mathrm{T}}\mathbf{u} = 1$ corresponds to a unique pair $(\theta_1, \theta_2) \in [0, \pi] \times [0, 2\pi)$ such that $(u_1, u_2, u_3) = (\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2, \cos\theta_1)$. In particular, to any $\mathbf{x} = \mathbf{x}(\mathbf{u})$ there is a pair (θ_1, θ_2) such that

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \approx \lambda 10^{-3} \begin{pmatrix} 2.46 & 0 & 0 \\ -1.00 & 1.63 & 0 \\ -1.48 & 1.01 & 17.30 \end{pmatrix} \begin{pmatrix} \sin\theta_1 \cos\theta_2 \\ \sin\theta_1 \sin\theta_2 \\ \cos\theta_1 \end{pmatrix}$$

Given a swap position N_0 and a severity λ, we define the worst area at level $\alpha \in (0, 1)$ as

$$Q_\alpha = \{(\theta_1, \theta_2) \in [0, \pi] \times [0, 2\pi) : f(\mathbf{x}(\theta_1, \theta_2)) \le q_\alpha\},$$

where q_α is chosen so that $\int_{Q_\alpha} \sin\theta_1 d\theta_1 d\theta_2 = 4\pi\alpha$. The worst area Q_α is to be interpreted as the set of equally extreme directional scenarios, that correspond to risk-factor scenarios $\mathbf{x}(\theta_1,\theta_2) = \lambda A\mathbf{u}(\theta_1,\theta_2)$ for $(\theta_1,\theta_2) \in Q_\alpha$, that are most likely to lead to bad outcomes $f(\mathbf{x}(\theta_1,\theta_2))$ for $(\theta_1,\theta_2) \in Q_\alpha$. This interpretation only makes sense if Q_α is rather stable when varying λ. For small α we may determine the location of the set Q_α on the unit sphere by considering the limit set $\lim_{\alpha\to 0} Q_\alpha$. For most functions f one is likely to encounter in practice, this limit set exists and consists of the point $\operatorname{argmin}_{(\theta_1,\theta_2)} f(\mathbf{x}(\theta_1,\theta_2))$. Notice that if $f(\mathbf{x})$ is linear in $\mathbf{x}$, then the limit $\lim_{\alpha\to 0} Q_\alpha$ does not depend on λ. Here, the approximation $f(\mathbf{x}) \approx f(\mathbf{0}) + \nabla f^{\mathrm{T}}(\mathbf{0})\mathbf{x}$ is reasonably accurate for the λ-values that correspond to the sample points that are most unfavorable to the insurer's future net asset value.

Figure 11 shows the worst area at level 0.01 as a gray region, and contour lines of the type $f(\mathbf{x}(\theta_1,\theta_2)) = \text{const.}$, when $\lambda = 2, 5$. The two lower rows of plots correspond to settings (iii) and (iv) treated above. The values of $q_{0.01} = q_{0.01}(\lambda)$ are shown in Table 3. The angles (θ_1 and θ_2) corresponding to the four worst observations are marked with crosses in Figure 11. The severity and future net asset value of each of these observations are shown in Table 4.

Now suppose that $X_3 = 0$, corresponding to no change in the exchange rate, and the settings (i) and (ii), i.e., (A^{St}, K, N_0) is either $(40, 0, 0)$ or $(40, 0, 178)$. The covariance matrix Σ_{-3} of (X_1, X_2, X_4) is estimated by

$$\Sigma_{-3} \approx 10^{-6} \begin{pmatrix} 6.03 & -2.46 & 14.91 \\ -2.46 & 3.67 & 28.05 \\ 14.91 & 28.05 & 4019.78 \end{pmatrix},$$

and the worst area, contour lines and angles of the four worst observations are seen in the two upper rows of Figure 11. The shape of the worst area is rather stable when varying λ, but the location changes slightly towards smaller values of θ_1 as λ increases. The values of $q_{0.01}$ are shown in Table 3, and the four worst observations are shown in Table 4.

An interesting remark is that $q_{0.01}$ increases, and hence the risk decreases, for $\lambda = 5$ (see Table 3) when increasing the swap position. This is the opposite of what we saw in Section 3 where increasing the swap position lead to a heavier tail of the net-asset-value distribution. The reason for this is that the assumption of elliptical distribution forces the tails of the X_ks to be equivalent up to affine transformations which here leads to an underestimate of the heaviness of the tail of the spread distribution.

The gray area $Q_{0.01}$ in Figure 11 tells us where we are most likely to find bad outcomes (in terms of a low future net asset value) given a severity λ. Essentially this area consists of the worst combinations of risk-factor movements. To get a grasp of the worst combinations in this elliptical model we map one pair of angles (θ_1, θ_2) in the gray region into a point $\mathbf{u} = (u_1, u_2, u_3)^{\mathrm{T}}$

(A_0^{St}, K, N_0)	$q_{0.01}(2)$	$q_{0.01}(5)$
$(40, 0, 0)$	33.32	22.92
$(40, 0, 178)$	33.45	25.91
$(0, 40, 0)$	35.63	27.56
$(0, 40, 178)$	35.30	29.74

Table 3: Values of $q_{0.01} = q_{0.01}(\lambda)$ for settings (i)–(iv) and $\lambda = 2, 5$.

on the unit sphere for each setting (i)–(iv). In setting (i), $(3\pi/4, \pi)$ is mapped to $\mathbf{u} \approx (-0.71, 0, -0.71)^{\mathrm{T}}$ corresponding to $\mathbf{x} \approx \lambda 10^{-2}(-0.17, 0.07, -4.64)^{\mathrm{T}}$. Here the worst combination is falling yield and falling stock index. In setting (ii), $(5\pi/6, 5\pi/6)$ is mapped to $\mathbf{u} \approx (-0.43, 0.25, -0.87)^{\mathrm{T}}$ corresponding to $\mathbf{x} \approx \lambda 10^{-2}(-0.11, 0.08, -4.90)$. Here the worst combination is falling yield, rising spread and falling stock index. In setting (iii), $(\pi/2, \pi)$ is mapped to $\mathbf{u} \approx (-1, 0, 0)^{\mathrm{T}}$ and $\mathbf{x} \approx \lambda 10^{-2}(-0.25, 0.10, 0.15)^{\mathrm{T}}$. Here only falling yield matters. And in setting (iv), $(\pi/2, 3\pi/4)$ is mapped to $\mathbf{u} \approx (-0.71, 0.71, 0)^{\mathrm{T}}$ corresponding to $\mathbf{x} \approx \lambda 10^{-2}(-0.17, 0.19, 0.18)^{\mathrm{T}}$. Here falling yield and rising spread is the worst combination and changes in the exchange rate are essentially irrelevant.

In reality a bad outcome may be due to a large λ paired with a bad combination of risk-factor movements. This is the case for most of the observations in Table 4. However, a bad outcome may also be due to a very large λ paired with a not-as-bad combination of risk-factor movements. This is the case especially for the second observation in setting (ii), where the very large yield and spread movements dwarf the 7% stock index rise. In settings (iii) and (iv), this becomes the worst observation since we remove the damping stock-index movement. In these settings we have both a very large λ and a bad risk-factor combination. Classical large-deviation heuristics says that very small future net asset values are due to the most likely extreme (unfavorable) scenarios. Under the assumption of an elliptical distribution for the vector $\mathbf{X}$ of risk-factor changes, the heuristics says that if the radial variable R in the stochastic representation of $\mathbf{X}$ is light-tailed, then a bad risk-factor combination (corresponding to an extreme directional scenario in the set Q_α) is required to obtain a bad outcome for the future net asset value. However, if R is heavy-tailed, then a bad risk-factor combination is unlikely to be the cause of a very small future net asset value.

As mentioned above, the approach we have considered to the analysis of extreme scenarios is based on an assumption of an elliptical distribution, an assumption which is only partially supported by the data. Other approaches exist, but none of them offer simple alternatives that are more appropriate here. If the tails of the distributions of the X_ks were found to fit well to regularly varying distributions with similar tail indices, than it would be reasonable to

			$(A^{St}, K, N_0) = (40, 0, 0)$				
λ	θ_1	θ_2	x_1	x_2	(x_3)	x_4	$f(\mathbf{x})$
3.928	2.055	1.836	−0.224	−0.456	(1.131)	−18.466	31.310
2.650	2.491	2.480	−0.311	−0.034	(1.336)	−15.387	31.553
2.655	2.158	2.318	−0.369	−0.114	(−0.397)	−13.060	31.816
2.299	2.331	2.966	−0.403	0.117	(0.840)	−11.032	32.211
			$(A^{St}, K, N_0) = (40, 0, 178)$				
λ	θ_1	θ_2	x_1	x_2	(x_3)	x_4	$f(\mathbf{x})$
3.953	2.606	1.320	0.123	0.269	(1.229)	−15.845	30.430
5.337	1.548	2.287	−0.860	1.008	(−2.829)	7.029	31.875
2.567	2.678	2.374	−0.203	0.213	(3.348)	−12.505	32.013
3.294	2.005	2.091	−0.365	0.572	(2.679)	−3.739	32.104
			$(A^{St}, K, N_0) = (0, 40, 0)$				
λ	θ_1	θ_2	x_1	x_2	x_3	(x_4)	$f(\mathbf{x})$
5.760	1.957	2.287	−0.860	1.008	−2.829	(7.029)	31.649
3.217	1.932	3.022	−0.734	0.241	−1.558	(4.138)	33.030
3.494	0.829	2.764	−0.588	0.085	4.345	(0.442)	34.558
2.390	1.760	2.861	−0.554	0.120	−0.509	(−3.455)	34.903
			$(A^{St}, K, N_0) = (0, 40, 178)$				
λ	θ_1	θ_2	x_1	x_2	x_3	(x_4)	$f(\mathbf{x})$
5.760	1.957	2.287	−0.860	1.008	−2.829	(7.029)	29.037
3.247	1.169	2.091	−0.365	0.572	2.679	(−3.739)	33.486
2.460	2.023	2.682	−0.487	0.359	−1.466	(−3.509)	35.214
2.503	2.062	2.704	−0.491	0.353	−1.651	(7.584)	35.258

Table 4: The four worst observations for settings (i)–(iv). The x_k-values are in % and the values appearing as $(\cdot)$ have no influence on the net asset value in the rightmost column for the particular portfolio setting.

assume that the vector $\mathbf{X}$ is multivariate regularly varying, see e.g. [9], with a non-degenerate limit measure, which is a good starting point for multivariate extreme-value statistics. Here, such an assumption is not supported by the data. The right tail of the distribution of X_2 appear heavier than those of the other X_ks. However, large values of the vector $\mathbf{X}$ are not likely to be due to large values of X_2 and average-size values of the other X_ks. A multivariate method of threshold exceedances, similar in spirit to the peaks-over-threshold method and based on a multivariate generalized Pareto distribution, could be useful for analyzing extreme scenarios. Although multivariate generalized Pareto distributions have been studied, see e.g. [10], the application to the 3- or 4-dimensional setting considered here is not straightforward. Other approaches are found in Chapter 8 in [4] and in [7], but the statistical analysis needed to successfully apply these approaches here is beyond the scope of this paper.

5 Conclusions and discussion

We have considered an insurer with purely domestic business whose liabilities to policy holders have long durations. The relative shortage of domestic government bonds with long maturities makes the net asset value sensitive to fluctuations in the zero rates that are used for valuation of the insurer's liabilities. Therefore, the insurer wants to take a suitable position as the fixed-rate receiver in an interest-rate swap. We have assumed that this is not possible in the domestic currency but in a foreign currency supporting a larger market of interest-rate swaps. Monthly data over 16 years have been used as the basis for investigating the risks to the net asset value of the insurer from using foreign-currency interest-rate swaps as a proxy for domestic ones in asset-liability management.

Fluctuations in domestic zero rates and returns on stocks, a part of the insurer's asset portfolio, are obviously responsible for much of the uncertainty in the future net asset value. However, positions in foreign-currency interest-rate swaps introduces new risk factors: fluctuations in the spread between the domestic zero-rate curve and the foreign-currency-swap zero-rate curve, and also fluctuations in the exchange rate. Whereas the latter is seen to be of very little importance, the former is not. Although a suitable swap position reduces significantly the standard deviation of the future net asset value, it also has the unpleasant side effect of making the left tail of the distribution of the future net asset value substantially heavier.

In view of the Solvency II framework, a point estimate of the solvency capital requirements (SCR) may be expressed as $\widehat{\mathrm{SCR}} = A_0 - L_0 + \hat{\ell}_{2400}$, where A_0 and L_0 are present values of assets and liabilities, respectively, and ℓ_{2400} denotes a value that the net liability value in one month exceeds on average once every 200×12 months. Since A_0 and L_0 are known, we get a confidence interval of SCR directly from the profile log-likelihood confidence interval of ℓ_{2400} (see Figure 5).

It is reasonable to use a point estimate as a basis for solvency capital requirement if ℓ_{2400} could be estimated with acceptable accuracy. As seen in Figures 5 and 6, and Table 5, this may be the case in setting (i), but certainly not in setting (iv). The heavy right tail of the spread-change distribution makes the confidence interval for ℓ_{2400}, and hence SCR, wider when increasing the swap position. Although different settings of the insurer's asset portfolio leads to somewhat similar point estimates of ℓ_{2400}, the confidence intervals for ℓ_{2400} are very different.

A risk measure that takes the uncertainty of the return-level estimate into account can be constructed by using the upper bound of a confidence interval instead of a point estimate. Such a measure would discourage investment strategies heavily exposed to tail risk. More research is needed to find out the details about how this measure should be defined in order to have sound theoretical properties and be useful in practice. Due to the limited amount of data, one probably have to choose m lower than 2400 for the return level ℓ_m, and a confidence level lower than the 95% used in this paper.

(A^{St}, K, N_0)	$\hat{\mathrm{SCR}}$	$(\hat{\mathrm{SCR}}^{lower}, \hat{\mathrm{SCR}}^{upper})$
$(40, 0, 0)$	9.00	$(8.60, 12.90)$
$(40, 0, 178)$	8.39	$(7.29, 17.69)$
$(0, 40, 0)$	9.80	$(7.40, 29.30)$
$(0, 40, 178)$	11.09	$(6.19, 66.39)$

Table 5: Point estimates and confidence intervals of SCR for settings (i)–(iv).

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References

[1] T. Breuer, M. Jandacka, K. Rheinberger, and M. Summer: How to find plausible, severe, and useful stress scenarios. *Working Paper 150, Österreichische Nationalbank* (2009)

[2] CEIOPS: Consultation paper no. 20 (2007) (Available at: `https://eiopa.europa.eu/fileadmin/tx_dam/files/publications/submissionstotheec/CEIOPS-DOC-08-07AdviceonPillarI-Issues-FurtherAdvice.pdf`, 2012-12-14)

[3] S. Coles: *An Introduction to Statistical Modeling of Extreme Values.* Springer Series in Statistics (2001)

[4] L. de Haan and A. Ferreira: *Extreme Value Theory: an Introduction.* Springer Series in Operations Research and Financial Engineering (2006).

[5] European Commission: QIS5 technical specifications (2010) (Available at: `https://eiopa.europa.eu/fileadmin/tx_dam/files/`

consultations/QIS/QIS5/QIS5-technical_specifications_20100706.pdf, 2012-12-14)

[6] D. N. Lawley: A general method for approximating to the distribution of likelihood ratio criteria. *Biometrika* **43**, pp. 295–303 (1956)

[7] A. Ledford and J. A. Tawn: Statistics for near independence in multivariate extreme values. *Biometrika* **83**, pp. 169–187 (1996).

[8] A. J. McNeil and A. D. Smith: Multivariate stress scenarios and solvency. *Insurance: Matematics and Economics* **50**, pp. 299–308 (2012)

[9] S. I. Resnick: *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling.* Springer Series in Operations Research and Financial Engineering (2007)

[10] H. Rootzen and N. Tajvidi: The multivariate generalized Pareto distribution. *Bernoulli* **12**, pp. 917–930 (2006)

[11] G. Studer: Market risk computation for nonlinear portfolios. *Journal of Risk* **1**, pp. 33–53 (1999)

[12] N. Tajvidi: Confidence intervals and accuracy estimation for heavy-tailed generalized Pareto distributions. *Extremes* **6**, pp. 111–123 (2003).

[13] S. S. Wilks: The large-sample distribution of the likelihood ratio for testing composite hypotheses. *Annals of Mathematical Statistics* **9/1**, pp. 60–62 (1938)

6 Figures

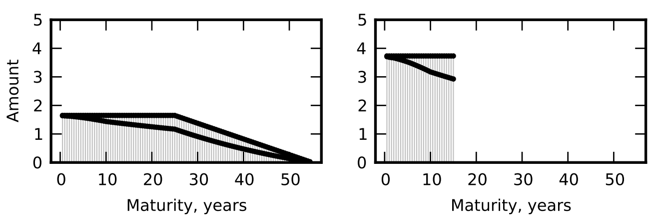

Figure 1: Left: Liability cash flows (upper dots) and their corresponding tax-adjusted 'market values' (lower dots). Right: Bond cash flows and their corresponding market values.

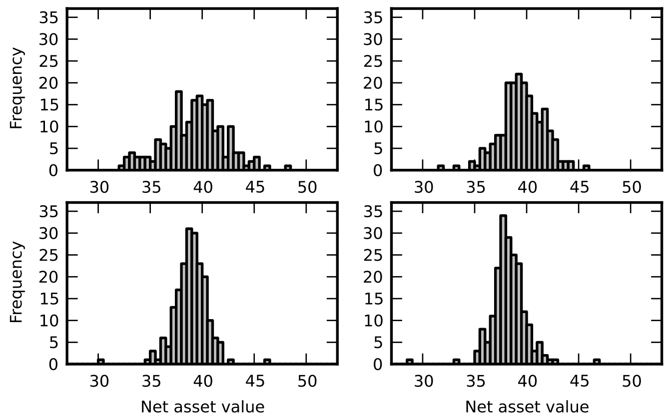

Figure 2: Empirical distribution of future net asset value $A_{\Delta t} - L_{\Delta t}$ for the benchmark portfolio for $N_0 = 100$ (upper left), and the benchmark portfolio with no investment in stocks and the amount 40 held in cash for $N_0 = 0, 100, 200$ (upper right, lower left and lower right, respectively).

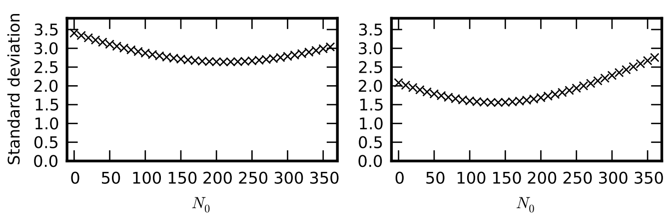

Figure 3: Estimates of standard deviations for the future net value with swap position size N_0, with and without investments in the stock index. The minima are obtained for $N_0 = 210$ and $N_0 = 140$, respectively.

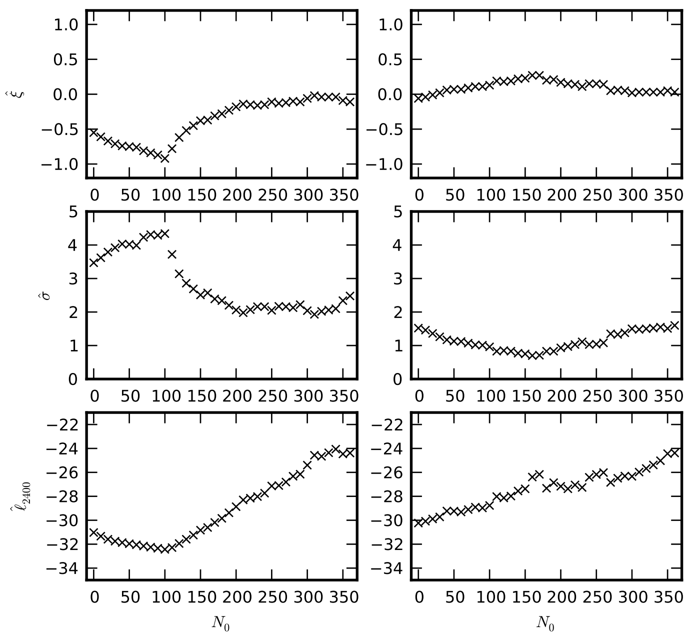

Figure 4: Left: Maximum likelihood estimates of ξ, σ and ℓ_{2400}, respectively, for an insurer with 40 in stocks, no cash and a swap position of size N_0. Right: Corresponding plots for an insurer with no stocks and the amount 40 held in cash.

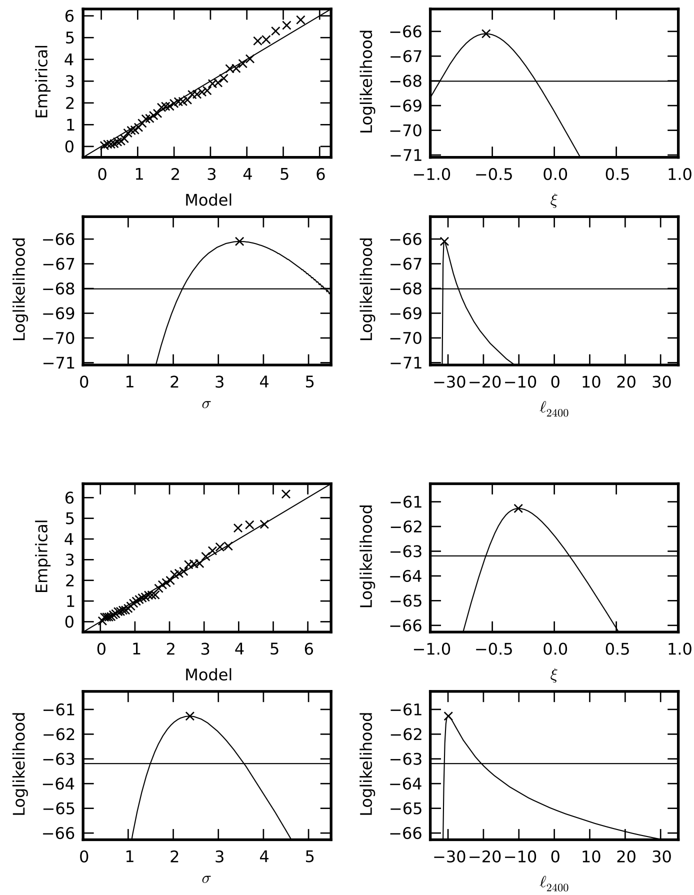

Figure 5: Portfolio settings (i) and (ii). Upper left: QQ plot for generalized Pareto distribution with ML parameters. Upper right: Profile log-likelihood for ξ. Lower left: Profile log-likelihood for σ. Lower right: Profile log-likelihood for ℓ_{2400}. The horizontal lines represent the boundary for 95% confidence intervals, the crosses are ML point estimates.

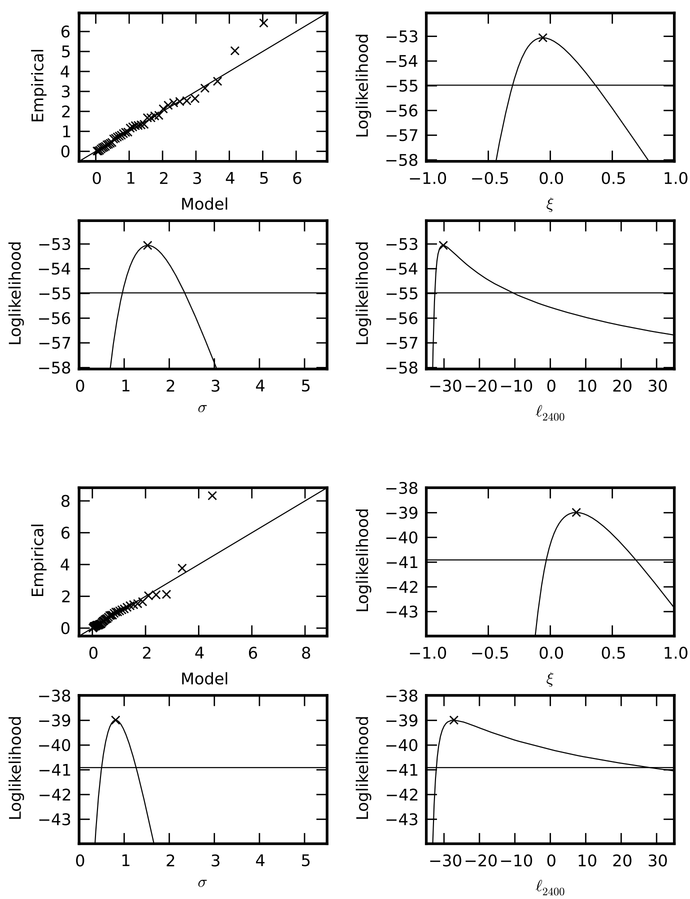

Figure 6: Portfolio settings (iii) and (iv). Upper left: QQ plot for generalized Pareto distribution with ML parameters. Upper right: Profile log-likelihood for ξ. Lower left: Profile log-likelihood for σ. Lower right: Profile log-likelihood for ℓ_{2400}. The horizontal lines represent the boundary for 95% confidence intervals, the crosses are ML point estimates.

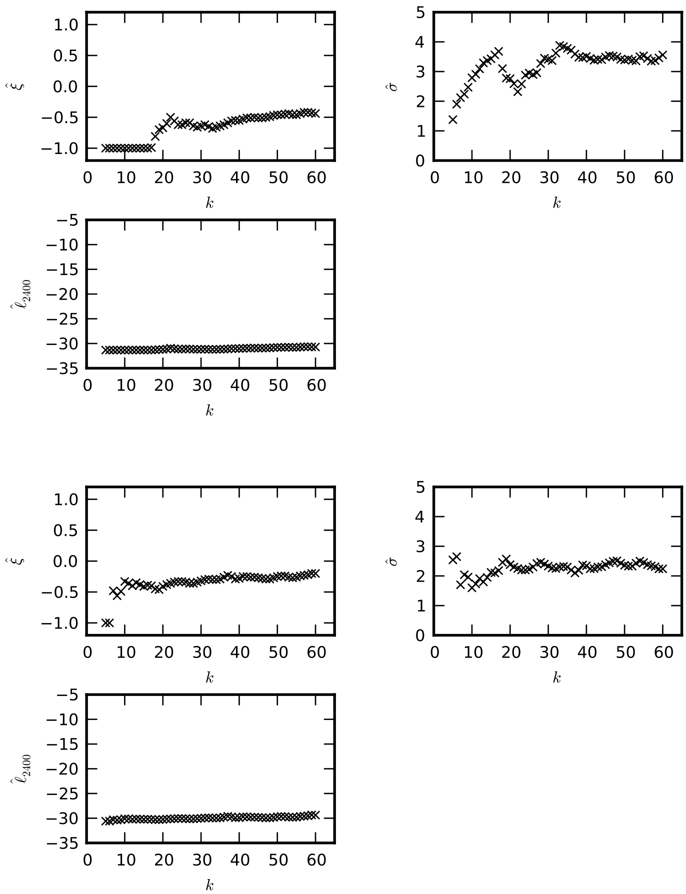

Figure 7: ML estimates of ξ, σ, and ℓ_{2400} as functions of the number of exceedances k. The threshold is set to the mean of the kth and $(k+1)$th largest net liability value. The three upper plots correspond to setting (i), and the three lower plots correspond to setting (ii) in (10).

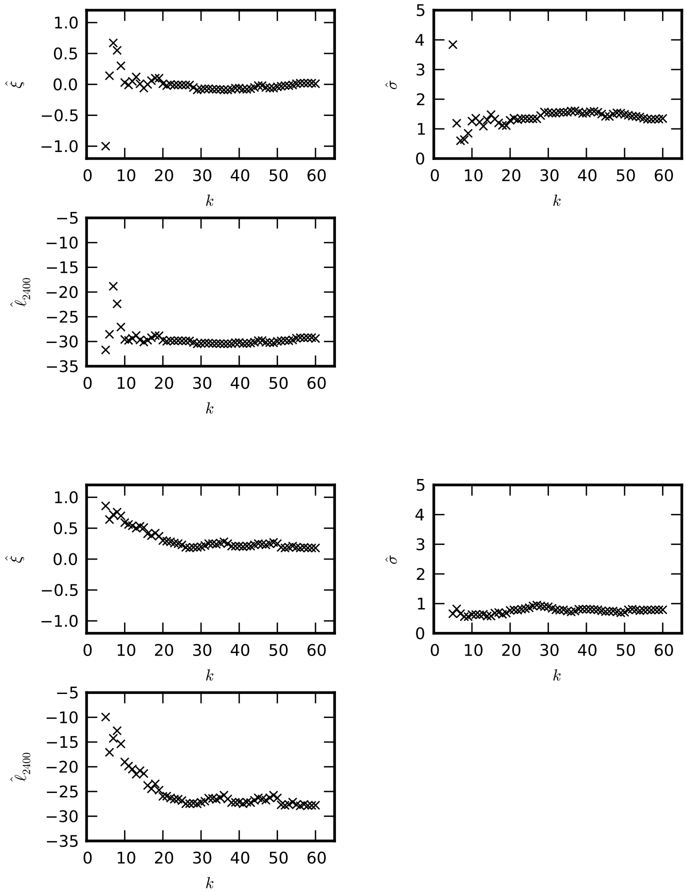

Figure 8: ML estimates of ξ, σ, and ℓ_{2400} as functions of the number of exceedances k. The threshold is set to the mean of the kth and $(k+1)$th largest net liability value. The three upper plots correspond to setting (iii), and the three lower plots correspond to setting (iv) in (10).

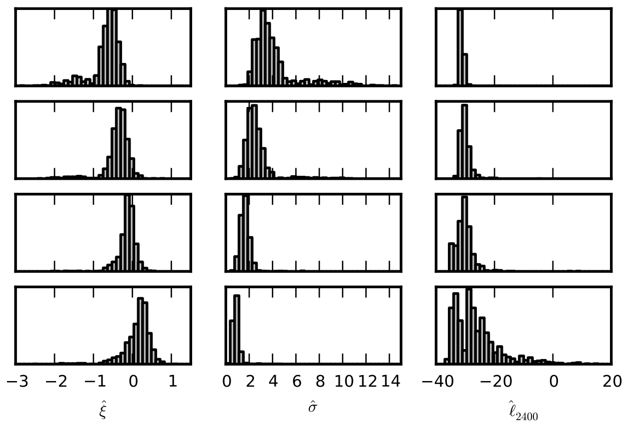

Figure 9: Histograms of ML estimates of ξ, σ, and ℓ_{2400} from parametric bootstrap. The plots in the first row (counting from the top) correspond to setting (i), the plots in the second row correspond to setting (ii), the plots in the third row correspond to setting (iii), and the plots in the fourth row correspond to setting (iv) in (10).

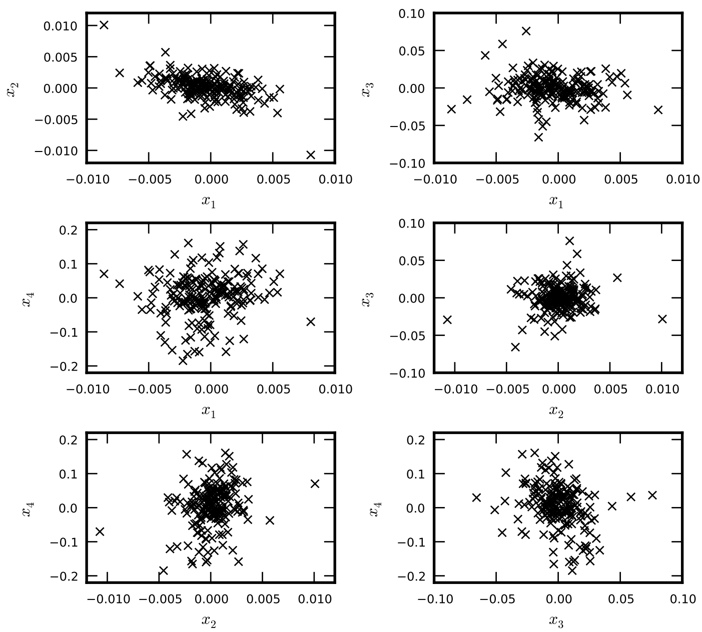

Figure 10: Scatter plots corresponding to (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4), (x_3, x_4).

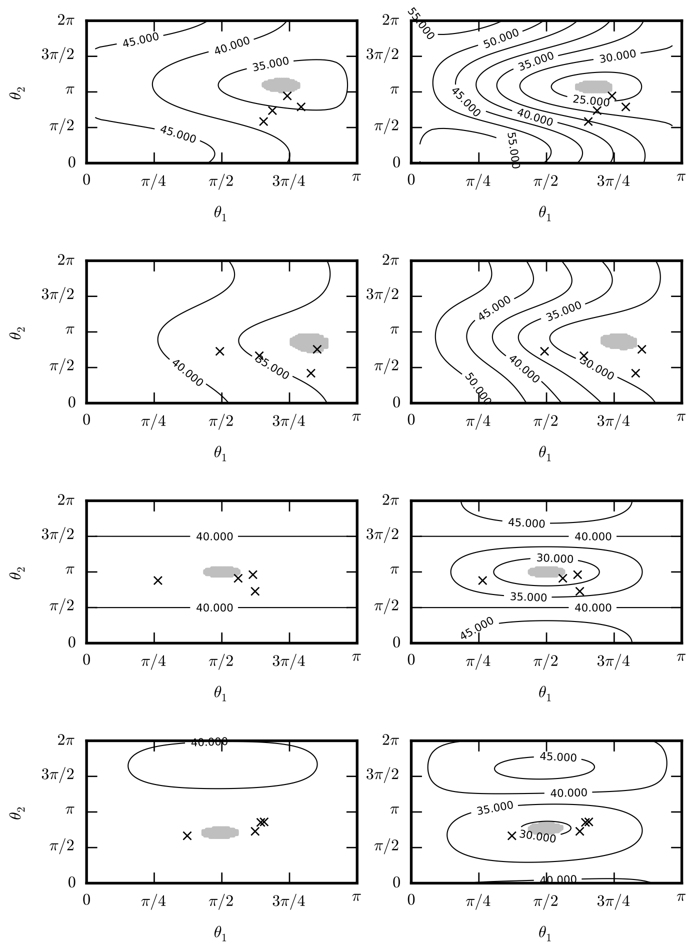

Figure 11: Worst areas for settings (i)–(iv). The plots in the uppermost row correspond to (i), the second uppermost row correspond to (ii), and so on. The plots show contour lines and worst area at level 0.01 (gray region); plots to the left correspond to $\lambda = 2$ and plots to the right correspond to $\lambda = 5$.