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A Concave-Convex Procedure for TDOA Based Positioning

Mohammad Reza Gholami, Sinan Gezici, and Erik G. Ström

Abstract—This letter investigates the time-difference-of-arrival based positioning problem in wireless sensor networks. We consider the least-mean absolute, i.e., the ℓ_1 norm, minimization of the residual errors and formulate the positioning problem as a *difference of convex functions (DC) programming*. We then employ a concave-convex procedure to solve the corresponding DC programming. Simulation results illustrate the improved performance of the proposed approach compared to existing methods.

Index Terms– Wireless sensor network, time-difference-of-arrival, DC programming, concave-convex procedure.

I. INTRODUCTION

Time-difference-of-arrival (TDOA) based positioning has been proposed in the literature as an effective technique in removing the clock offset imperfection [1]. A number of researchers have investigated the positioning problem based on TDOA measurements. The maximum likelihood estimator (MLE) for this positioning problem poses a difficult global optimization problem [2]. To avoid the difficulty in obtaining the MLE, a few suboptimal approaches have been proposed in the literature. For instance, the authors in [3] formulate the TDOA based positioning as a semidefinite programming relaxation (SDR) problem. To formulate an SDR approach with low complexity, the authors in [4] consider a minimax approach and propose two suboptimal algorithms. Another approach based on a linear least squares (LLS) technique is introduced in [5], which achieves good performance for low noise variances. In addition, [6] presents a method based on the squared-range least squares, which has similar performance to the LLS. To provide a good coarse estimate as a starting point for the MLE, an efficient technique is proposed in [2] based on projection onto convex sets. Recently a method based on geometric circle fitting is studied in [7], which shows good performance for sufficiently small noise

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variances. Although the proposed suboptimal algorithms are efficient in terms of complexity, there is still some room to improve their performance.

In this letter, we study the single node positioning problem based on TDOA measurements. With the aim to derive an efficient and robust approach with superior performance compared to the existing approaches, especially for low numbers of reference nodes, we consider the ℓ_1 norm minimization of the residuals and then formulate the TDOA based positioning problem as a *difference of convex functions (DC)* programming. We then employ a *concave-convex procedure* (CCCP) [8] to solve the problem. In particular, we need to solve a sequence of second order cone programs to find an estimate of the target position. We also simplify the problem to a linear program and solve the corresponding CCCP in a sequential manner. Simulation results show the promising performance of the proposed approach compared to the optimal and existing suboptimal estimators. Numerical results also illustrate that only a few sequential updatings are required for the proposed technique to converge. Thus, the proposed approaches have similar complexities compared to existing suboptimal estimators.

II. SYSTEM MODEL

Consider an m-dimensional network ($m = 2$ or 3) with N reference (anchor) nodes located at known positions $\mathbf{a}_i \in \mathbb{R}^m$, $i = 1, ..., N$ and with one target node placed at the unknown position $\mathbf{x} \in \mathbb{R}^m$. Suppose that the target node transmits a signal at time instant T_0, which is unknown to the reference nodes. Then, the TOA measurement at reference node i can be modeled as [9]

$$t_i = T_0 + \frac{d(\mathbf{a}_i, \mathbf{x})}{c} + \tilde{n}_i, \qquad i = 1, \ldots, N \quad (1)$$

where $d(\mathbf{a}_i, \mathbf{x}) \triangleq \|\mathbf{x} - \mathbf{a}_i\|_2$ is the Euclidian distance between reference node i and the point $\mathbf{x}$, c is the speed of propagation, and $\tilde{n}_i$ is the TOA estimation error at reference node i for the signal transmitted from the target node. The estimation error is often modeled by a zero-mean Gaussian random variable with variance σ_i^2/c^2; i.e., $\tilde{n}_i \sim \mathcal{N}(0, \sigma_i^2/c^2)$ [10].

The preceding measurement model indicates that in order to obtain an estimate of the distance between the target node and a reference node, the parameter T_0

should be estimated as well, which makes the problem quite challenging. One way to get rid of this unknown parameter is to subtract the TOA measurements at reference nodes i and j, and form a TDOA measurement assuming synchronized reference nodes. In this study, we assume that the TDOA measurements are computed by subtracting all the TOA measurements, except the first one, from the first TOA. Consequently, we obtain the range-difference-of-arrival (RDOA) measurements as

$$z_{i,1} = c(t_i - t_1) = d_{i,1} + n_i - n_1, \quad i = 2, \ldots, N \quad (2)$$

where $n_i = c\tilde{n}_i$ and $d_{i,1} = d(\mathbf{a}_i, \mathbf{x}) - d(\mathbf{a}_1, \mathbf{x})$. We collect the measurements $z_{i,1}$ in (2) into a vector $\mathbf{z}$ as

$$\mathbf{z} = [z_{2,1} \; \ldots \; z_{N,1}]^T \in \mathbb{R}^{(N-1)} \quad (3)$$

The MLE for the location based on the TDOA measurements in (3) poses a difficult optimization problem [1]. In the next section, we propose an efficient suboptimal estimator to solve the positioning problem.

III. PROPOSED TECHNIQUE

In this section, we take the ℓ_1 norm minimization of the residuals into account and propose a technique to solve the positioning problem. We consider the least-mean absolute errors of the residuals as follows:

$$\underset{\mathbf{x} \in \mathbb{R}^m}{\text{minimize}} \; \|\mathbf{r}\|_1 \quad (4)$$

where $\mathbf{r} = [r_2 \ldots r_N]^T$ with $r_i = z_{i,1} - d(\mathbf{a}_i, \mathbf{x}) + d(\mathbf{a}_1, \mathbf{x})$. Note that for high signal-to-noise ratios (low standard deviations of noise), the ℓ_2 and ℓ_1 minimization approaches have similar performance. In addition, the ℓ_1-based minimization in (4) is a suitable approach for dealing with the positioning problem in the presence of outliers [11].

Using a dummy vector $\mathbf{q} = [q_2 \; \ldots \; q_N]^T$, the optimization problem in (4) can be written (in the epigraph form) as [11]

$$\underset{\mathbf{x}, \mathbf{q}}{\text{minimize}} \; \sum_{i=2}^{N} q_i$$
$$\text{subject to} \; z_{i,1} - d(\mathbf{a}_i, \mathbf{x}) + d(\mathbf{a}_1, \mathbf{x}) \leq q_i$$
$$-z_{i,1} - d(\mathbf{a}_1, \mathbf{x}) + d(\mathbf{a}_i, \mathbf{x}) \leq q_i. \quad (5)$$

The problem in (5) is a nonconvex problem and difficult to solve. Here we employ a technique from the optimization literature to solve the problem in a sequential manner. The technique is called the concave-convex procedure (CCCP) and aims at solving a nonconvex problem including the difference of convex functions (DC) [8]. The general form of the DC programming is as follows:

$$\underset{\mathbf{x}}{\text{minimize}} \; f_0(\mathbf{x}) - g_0(\mathbf{x})$$
$$\text{subject to} \; f_i(\mathbf{x}) - g_i(\mathbf{x}) \leq 0, \; i = 1, \ldots, M \quad (6)$$

where $f_i(\mathbf{x})$ and $g_i(\mathbf{x})$ are smooth convex functions for $i = 1, \ldots, M$. A method to solve (6) is to approximate the concave term with a convex one. We consider an affine approximation of the concave function $(-g_i(\mathbf{x}))$. Let us consider a point $\mathbf{x}^k$ in the domain of the problem in (6) and linearize the concave function around $\mathbf{x}^k$ and write the optimization problem (6) as

$$\underset{\mathbf{x}}{\text{minimize}} \; f_0(\mathbf{x}) - g_0(\mathbf{x}^k) - \bigtriangledown g_0(\mathbf{x}^k)^T(\mathbf{x} - \mathbf{x}^k)$$
$$\text{subject to} \; f_i(\mathbf{x}) - g_i(\mathbf{x}^k) - \bigtriangledown g_i(\mathbf{x}^k)^T(\mathbf{x} - \mathbf{x}^k) \leq 0,$$
$$i = 1, \ldots, M. \quad (7)$$

The convex problem in (7) can now be efficiently solved. Denoting the solution of (7) as $\mathbf{x}^{k+1}$, next we go for further improving the solution by convexifing (6) for new point $\mathbf{x}^{k+1}$ similar to the procedure performed for $\mathbf{x}^k$. This sequential programming procedure continues for a number of iterations. The convergence behavior of the CCCP approach has been thoroughly studied in the literature, e.g., [8], [12]. Note that if $g_i(\mathbf{x})$ is not differentiable at $\mathbf{x}^k$, we can replace the $\bigtriangledown g_i(\mathbf{x}^k)$ term by a subgradient[1] of $g_i(\mathbf{x})$ at $\mathbf{x}^k$.

Now applying the CCCP technique to the problem in (5), we solve the following optimization problem to obtain $\mathbf{x}^{k+1}$ from $\mathbf{x}^k$:

$$\underset{\mathbf{x}, \mathbf{q}}{\text{minimize}} \; \sum_{i=2}^{N} q_i$$
$$\text{subject to} \; \|\mathbf{x} - \mathbf{a}_1\|_2 - \mathbf{h}_{i,k}^T \mathbf{x} + b_{i,k} - q_i \leq 0$$
$$\|\mathbf{x} - \mathbf{a}_i\|_2 - \mathbf{h}_{1,k}^T \mathbf{x} + c_{i,k} - q_i \leq 0 \quad (8)$$

where $\mathbf{h}_{i,k} = (\mathbf{x}^k - \mathbf{a}_i)/d(\mathbf{a}_i, \mathbf{x}^k)$, $b_{i,k} = \mathbf{h}_{i,k}^T \mathbf{x}^k + z_{i,1} - d(\mathbf{a}_i, \mathbf{x}^k)$, and $c_{i,k} = \mathbf{h}_{1,k}^T \mathbf{x}^k - z_{i,1} - d(\mathbf{a}_1, \mathbf{x}^k)$. The optimization problem in (8), which is called the second order cone programming (SOCP), can be efficiently solved. We call the corresponding CCCP as CCCP-SOCP. Note that the approximations used in this study are different from other approaches considered in the positioning literature, e.g., [3], in which convex relaxations, which may not be sufficiently tight in some scenarios, are used to convert the MLE into a convex problem. On the other hand, the DC programming often finds the global solution and has been considered as an efficient and robust technique applied to a class of nonconvex problems [14].

In the sequel, we propose another simplification to the problem in (8). Namely, we replace the feasible set by an outer linear approximation. The main reason for dealing with such an approximation is to decrease the complexity in solving the problem in (8). In particular, we linearize the nonlinear convex function in (8) and

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[1]Let $\mathcal{D}$ be a nonempty set in $\mathbb{R}^n$. A vector $\mathbf{g} \in \mathbb{R}^n$ is a subgradient of a function $f : \mathcal{D} \to \mathbb{R}$ at $\mathbf{x} \in \mathcal{D}$ if $f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{g}^T(\mathbf{y} - \mathbf{x})$ for all $\mathbf{y} \in \mathcal{D}$ [13].

express the problem as a linear program (LP)

$$\underset{\mathbf{x},\mathbf{q}}{\text{minimize}} \quad \sum_{i=2}^{N} q_i$$
$$\text{subject to} \quad \mathbf{g}_{i,k}^T \mathbf{x} + z_{i,1} + m_{i,k} - q_i \leq 0$$
$$-\mathbf{g}_{i,k}^T \mathbf{x} - z_{i,1} - m_{i,k} - q_i \leq 0 \quad (9)$$

where $\mathbf{g}_{i,k} = \mathbf{h}_{1,k} - \mathbf{h}_{i,k}$ and $m_{i,k} = \mathbf{g}_{i,k}^T \mathbf{x}^k + d(\mathbf{a}_i, \mathbf{x}^k) - d(\mathbf{a}_1, \mathbf{x}^k)$. We call the resulting CCCP as CCCP-LP.

In the CCCP approach, a solution (not exact) in every step can be obtained and used for linearizing the nonlinear terms. We here consider a simple updating approach based on the subgradient technique for the problem in (9). To that aim, we express (9) as

$$\underset{\mathbf{x}}{\text{minimize}} \quad \|\mathbf{G}_k^T \mathbf{x} + \mathbf{b}_k\|_1 \quad (10)$$

where $\mathbf{G}_k = [\mathbf{g}_{2,k}^T \cdots \mathbf{g}_{N,k}^T]^T$ and $\mathbf{b}_k = [z_{2,1} + m_{2,k} \cdots z_{N,1} + m_{N,k}]^T$. The objective function in (10) is a nondifferentiable function and we use the following updating rule for solving the problem:

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \mathbf{g}^k, \quad (11)$$

where α_k is a step size (fixed or time variant) and $\mathbf{g}^k$ is a subgradient of $\|\mathbf{G}_k^T \mathbf{x} + \mathbf{b}_k\|_1$ in (10). A subgradient of $\|\mathbf{G}_k^T \mathbf{x} + \mathbf{b}_k\|_1$ at $\mathbf{x}^\ell$ can be computed as

$$\mathbf{g}^k = \mathbf{G}_k^T \text{sgn}(\mathbf{G}_k \mathbf{x}^\ell + \mathbf{b}_k), \quad (12)$$

with $\text{sgn}(\mathbf{x}) = [\text{sgn}(x_1), \ldots, \text{sgn}(x_N)]^T$, where $\text{sgn}(\cdot)$ denotes the signum function. For a discussion on different rules for selecting the step size in the subgradient method, see, e.g., [13]. Note that although the convergence of the modified CCCP problem in (9) is observed through simulations, the convergence proof needs future analysis, which is considered as a future work.

IV. NUMERICAL RESULTS

We consider a 80 by 80 square meters area with a number of reference nodes that are located at fixed positions $\mathbf{a}_1 = [40\ 40]$, $\mathbf{a}_2 = [40\ -40]$, $\mathbf{a}_3 = [-40\ 40]$, $\mathbf{a}_4 = [-40\ -40]$, $\mathbf{a}_5 = [40\ 0]$, $\mathbf{a}_6 = [0\ 40]$, and $\mathbf{a}_7 = [-40\ 0]$ (all in meters). In the simulations, we pick n reference nodes as $\mathbf{a}_1, \ldots, \mathbf{a}_n$. One target node is randomly distributed inside the area. To assess the proposed technique, we implement the MLE [1] using Matlab function *lsqnonlin* [15] initialized with the true target location (as a benchmark), the SDR [3], the MLE initialized with the SDR estimate, the linear least squares (LLS) [5], the linear least squares followed by a correction technique, and the Cramér-Rao lower bound (CRLB) [1]. To simulate the RDOA, we first add Gaussian noise to the true distance between the target and
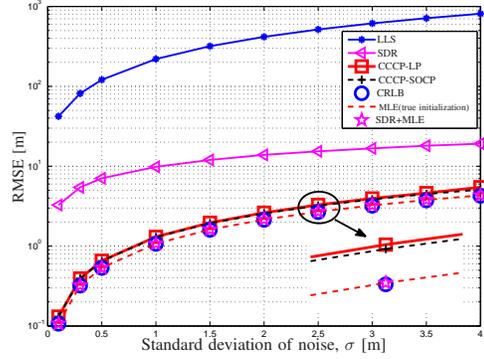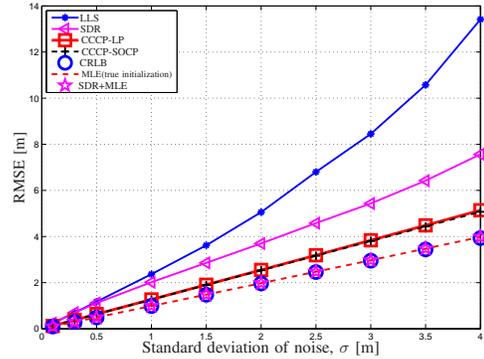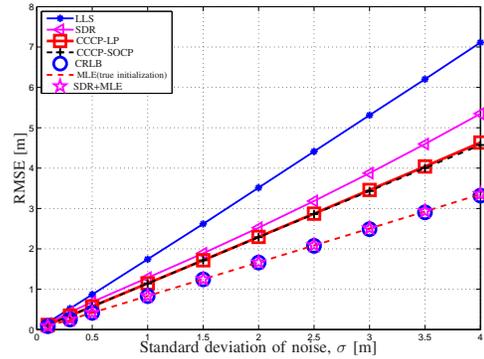

Fig. 1. The RMSE of different approaches for (a) 4 reference nodes, (b) 5 reference nodes, and (c) 7 reference nodes.

every reference node and then we subtract the noisy distance measurements from the first range measurement. The proposed approaches are implemented by using the CVX toolbox [16]. For every realization of the network, we run CVX six times to find an estimate of the target location. In the simulations, we assume that $\sigma_i = \sigma$ for $i = 1, \ldots, N$. We initialize the CCCP approaches with the mean of the locations of the reference nodes. Fig. 1 illustrates the root-mean-square-error (RMSE) of different approaches versus the standard deviation of noise for different numbers of reference nodes. The figure shows that the proposed approach achieves high
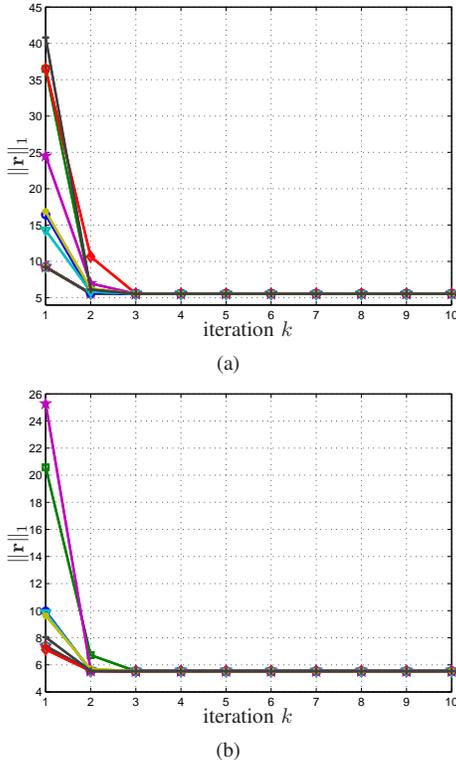
Fig. 2. The convergence speed of the proposed approaches for random initializations for $N = 4$ and $\sigma = 3$ m for (a) CCCP-LP and (b) CCCP-SOCP.

performance compared to the SDR and LLS, especially for small numbers of reference nodes. As suggested in [3], we can improve the SDR estimate by employing a refining approach. From the figure, we observe that the MLE initialized with the SDR estimate attains the CRLB. However, such an approach has significantly higher complexity than the proposed approaches in this study. The LLS algorithm has the worst performance, especially for low numbers of reference nodes. As the number of reference nodes increases, the SDR gets closer to the proposed estimators. Surprisingly, it is observed that the performance of the CCCP-LP is very close to that of the CCCP-SOCP. Fig. 2 shows the convergence speed of the proposed approaches for a realization of the network with 4 reference nodes and $\sigma = 3$ [m]. For every estimate generated by the CCCP-SOCP or CCCP-LP algorithm, we compute the residual $\|\mathbf{r}\|_1$, where $\mathbf{r}$ is given in (4). We randomly choose the initial point $\mathbf{x}^0$ and run the CCCP approach for 10 sequential updatings. For every updating, we need to solve an optimization problem as described in Section III. In the simulations, we consider 4 reference nodes. As it can be seen, the CCCP approach converges very fast, approximately in three sequential updatings. It is also observed that both CCCP-SOCP and CCCP-LP have similar convergence

behaviors.

V. CONCLUSION

In this letter, we have proposed computationally efficient suboptimal positioning algorithms based on TDOA measurements. We have first applied an ℓ_1 norm minimization of the residuals and have formulated the problem as a DC programming. We have then employed a concave-convex procedure to solve the corresponding DC problem. Simulation results show that the proposed approaches outperform the existing suboptimal estimators, in some scenarios, e.g., for low numbers of reference nodes and when the target is in the convex hull of the reference nodes.

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