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Stochastic systems with locally defined dynamics: convergence and limiting properties

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Abstract

This thesis considers two large classes of models related to the dynamical point processes. The first is the locally interactive sequential adsorption, or LISA, models. We provide the general LISA framework, show that a lot of well-understood models can be described within the framework, such as Polya urn schemes, fragmentation processes, cooperative sequential adsorption. We study several particular new examples of LISA processes which possess the feature of scalability. Our results describe the limiting behaviour of empirical measures of such processes.

The second class is Bit Flipping models, where we study a behaviour of a sequence of independent bits, each flipping between several states at given intensity p_k. We investigate conditions on p_k at which the model switches from transient to recurrent behaviour, prove the central limit theorem for the transient case, and provide a bound for moments of the recurrence time in the recurrent case.

Keywords: point process, sequential adsorption, stopping set, random measure, Polya urn, convergence of empirical measures, bit flipping, recurrence, mixing times

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I thank my friends and family for their support. I thank Lena for the pie.

Anton Muratov
Göteborg, April 24, 2012

List of Papers

The licentiate thesis includes the following papers.

I. **Muratov, A.**, Zuyev, S. (2012). LISA: Locally Interacting Sequential Adsorption *Submitted*.

II. **Muratov, A.**, Zuyev, S. (2012). Bit flipping models and time to recover *Work in progress*

Contents

1 Introduction

This thesis consists of the two connected papers about dynamically constructed processes. The main points of interest are the limiting behaviour of the models when time goes to infinity, the existence of the stationary regime and its properties. In the first model that turns into the questions of convergence for the empirical measures of the process. In the second we address the questions of recurrence.

The thesis is organized in a following way. Section 2 is a review of the paper on LISA processes, submitted for publication in Stochastic Models. It starts with an original R. Darling construction, which gave us an idea to consider the LISA models. In the term "Locally Interacting Sequential Adsorption" first two words emphasize the local character of the dependence in the structure, "sequential" means that we mainly consider the processes in the discrete time, and "adsorption" stands for the variant of a branching process in which the particles don't die but rather are generated one-by-one in a stepwise manner. After the motivation comes the formulation of a general LISA setting we are going to work with, and the particular examples we consider, along with the brief overview of some other models falling into the general setting. The topics include Polya's urn models and their connection to Dirichlet processes, branching random walks, unity-splitting process and some other fragmentation processes, as well as the cooperative sequential adsorption models. In the end of the first section we outline our main results on the proposed examples.

The following Section 3 is devoted to the fascinating Bit Flipping models, work in progress. We begin with our original motivation, explaining how flipping bits helps to handle a certain LISA-related problem. We indicate a link between bit flipping and the areas of dynamic percolation, random walks on Cayley graphs, and analysis of the algorithms. We then proceed with formulating the two models, Binary Flipping (BF) and Damaged Bits (DB), and our results. As a conclusion, we pose several open questions and indicate the possibilities for further research.

2 LISA: Locally Interacting Sequential Adsorption

The main reason to consider LISA models was a particular problem introduced by Richard Darling to my supervisor Sergey Zuyev. The original model is defined as follows.

Let $X_0 = \{x_1, x_2, \ldots, x_{n_0}\} \subset \mathbb{R}^2$ be the starting configuration, where all the points are disjoint. At each time step $n = n_0 + 1, n_0 + 2, \ldots$ we are adding a new point to a configuration according to the following algorithm:

- Sample χ_n – independent of the past, uniformly distributed over the indices $1, 2, \ldots, n$. That is the number of point to generate a new one during this time step.
- Find a smallest ball $B(x_{\chi_n}, R)$, containing $m, m \geq 3$ points from $X_n \setminus x_{\chi_n}$. Such ball is an example of a *stopping set*, which we define somewhat later.
- Assume that m closest neighbours of x_{χ_n} lying in $B(x_{\chi_n}, R)$ come from a normal distribution with mean x_{χ_n}. Estimate the covariance matrix C of that distribution from those m points.
- Sample a new point x_{n+1} from a normal distribution $\mathrm{N}(x_{\chi_n}, C)$ with mean x_{χ_n} and the estimated covariance matrix C.
- Add a new point to the configuration: $X_{n+1} = X_n \cup \{x_{n+1}\}$

So, at each step we choose the point to give a birth to a new one uniformly over already existing points, then sample a new point from a normal distribution with mean x_{χ_n} and covariance matrix estimated from the closest m points to x_{χ_n} from $X_n \setminus x_{\chi_n}$, for $m \geq 3$, and add the new point to the configuration.

Simulations of this iterative process (fig. 1 at pp. 3) show, that the resulting model has several main properties:

(i) The rules of adding the new particle depend only on the local configuration around that particle.

(ii) The geometry of a configuration tends to positively reinforce itself.

In other words (i) means, the dynamics of a system is defined locally, hence we can for example embed the whole process in a continuous time in a natural way, letting every particle generate new ones with fixed intensity, and distributing them around itself according to local rules. That reflects somehow the possible real-world applications of a model: cities or bacteria growth etc.

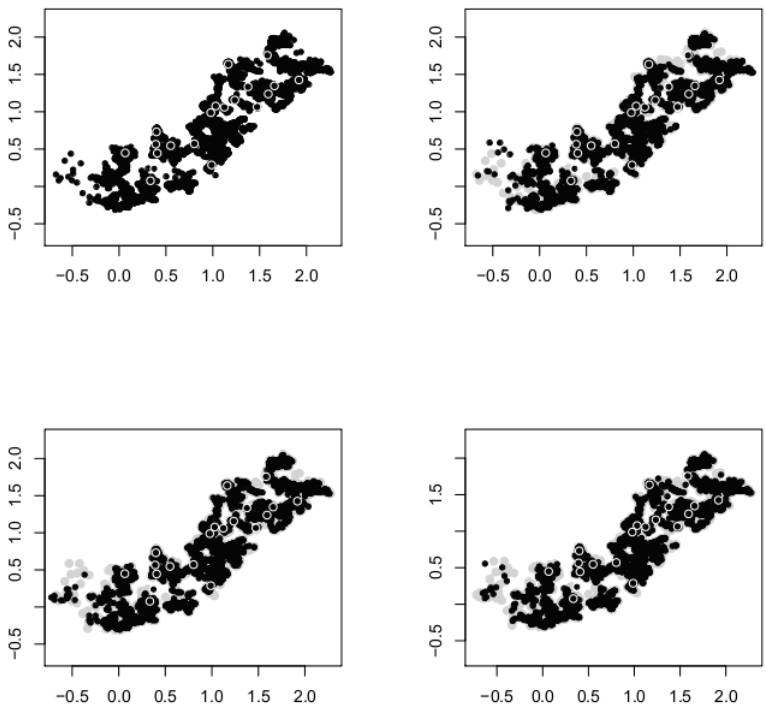

Figure 1: The sequence shows the simulated process after $n = 10^4, 2{\cdot}10^4, 3{\cdot}10^4, 4{\cdot}10^4$ steps. Newly added particles are shown in dark, previously existing – in light grey, initial particles are contoured void circles.

On the other hand, (ii) means that on a global scale the particles try to behave similar to how their ancestors did: if we start from a round-shaped configuration, then the covariance matrices will be isotropic and hence the whole configuration will likely remain round-shaped. But every now and then the configuration tends to shoot out "beams" of particles, in such "beam" areas the correspondent covariance matrices estimated are highly anisotropic, i.e. with high concentration along some vector, and that behaviour is also self-reinforced.

One more significant feature of our model is scaling invariance, which might imply stochastic self-similarity and various fractal properties for the limiting empirical measure.

All in all, that seems to be a very interesting behaviour, however hard to analyze rigorously on that level of complexity.

2.1 General LISA setting

Above we mention the so called *stopping set*. We believe that the stopping set terminology is the right way to assess the spatial processes with local dependence structure. It extends the notion of a stopping time for Markov processes in one time-dimension. The definitions are borrowed from [19].

Let E be a locally compact separable topological space, with $\mathbb{K}$ and $\mathbb{F}$ being its system of compact and closed sets, respectively. Let $(\Omega, \{\mathcal{F}_K\}, \mathbf{P})$ be a filtered probability space. Filtration $\{\mathcal{F}_K\}$ here is a collection of σ-algebras $\mathcal{F}_K$ indexed by compact sets $K \in \mathbb{K}$ that has the following properties:

- monotonicity: $\mathcal{F}_{K_1} \subseteq \mathcal{F}_{K_2}$ for any two compact $K_1 \subseteq K_2$;
- continuity from above: $\mathcal{F}_K = \cap_{n=1}^{\infty} \mathcal{F}_{K_n}$ if $K_n \downarrow K$.

A random closed set Ξ is a measurable mapping $\Xi{:}(\Omega, \{\mathcal{F}_K\}, \mathbf{P}) \mapsto [\mathbb{F}, \sigma_f]$, where σ_f is the σ-algebra generated by the system $\{F{\in}\mathbb{F} : F \cap K \neq \varnothing\}$, $K \in \mathbb{K}$. We say that Ξ is $\{\mathcal{F}_K\}$-adapted, if the random set $\Xi \cap K$ is $\mathcal{F}_K$-measurable for all $K \in \mathbb{K}$. We will consider only adapted random sets.

Definition 1. A random compact set δ is called a stopping set, if the event $\{\delta \subseteq K\}$ is $\mathcal{F}_K$-measurable for all $K \in \mathbb{K}$.

Having the stopping sets at hand, we can now define the general setting for LISA processes. First, we need a sample space W. We consider LISA processes in $W = \mathbb{R}^d$, $d \geq 1$. We have an initial configuration of points in space, $X_{n_0} = \{x_1, x_2, \ldots, x_{n_0}\}$. The core of the process is the procedure of generation of new points. We follow the general scheme:

- Pick a parent point x_{χ_n} uniformly among the existing at time step n, $n = n_0, n_0 + 1, n_0 + 2, \ldots$.

- Find a stopping set $S_n = S(x_{\chi_n}, X_n)$ defined by the configuration at time step n and previously picked x_{χ_n}.

- Sample a random variable ζ_{S_n}, whose distribution is defined by the geometry of a stopping set and which is otherwise not dependent on X_n:
$$\mathbf{P}(\zeta_{S_n} \in B | X_n) = \mathbf{P}(\zeta_{S_n} \in B | S_n)$$

- Add the new point $x_{n+1} = \zeta_{S_n}$ to the configuration:
$$X_{n+1} = X_n \cup \{x_{n+1}\}$$

The natural idea for the continuous-time alternative, which tracks back to [1], is to embed the procedure of generation of the new points into a continuous-time branching process, so that every point $x \in X_t$ generates children with the same fixed intensity. The location at which the children are placed is then controlled by the geometry of a stopping set $S(x, X_t)$ at the moment of birth.

As we see further, the above setting is quite flexible, in particular, we can get any kind of a stepwise growing Markov point process by just setting $S_n = X_n, n = n_0, n_0 + 1, \ldots$. Tweaking S_n and ζ_{S_n}, one could also obtain variants of branching random walks, segmentation processes, Polya's urn models, etc.

In our paper we analyze several particular examples of LISA models.

Example 1 Put $X = [0, 1] \subset \mathbb{R}$, $n_0 = 1$, $X_{n_0} = \{0\}$, $r(x) := \min\{y \in X_n \backslash \{x\} \cup 1 : y > x\}$ and $S(x, X_n) = [x, r(x)]$ That is, the stopping set of the point is the interval to the right from it. Put $\zeta(S) \sim \mathrm{Unif}(S)$, so that the new point is distributed uniformly on an interval to the right from the parent point.

This is one of the many versions of a stick-breaking process, it is also directly related to a construction of Dubbins-Freedman [7], particularly, if we try now to draw the distribution function for the resulting limiting measure, it will be distributed exactly as the random distribution function defined in [7]. This example in a bit more generality is thoroughly analyzed in [17], the singular limiting measure is proven to exist and the Hausdorff dimension of its support is computed. We use this example as an illustration for the interesting properties of certain LISA processes.

Example 2 Put $W = \mathbb{R}$, $n_0 \geq 2$, fix $X_{n_0} \subset \mathbb{R}$ consisting of n_0 distinct particles. Define $d(x, X) = \min\limits_{y \in X \backslash \{x\}} |x - y|$, $S(x, X) = [x - d(x, X), x + d(x, X)]$, and $\zeta(S) \sim \mathrm{Unif}(S)$. So, the new point is added at the uniform distance from the parent point, scaled with the distance from the parent point to its closest neighbour from already existing points.

Example 3 Put $W = \mathbb{R}^d$, $n_0 \geq 2$, fix $X_{n_0} \subset \mathbb{R}^d$. As before, $d(x, X) = \min_{y \in X \setminus \{x\}} ||x - y||$ is the Euclidean distance to the closest neighbour. The stopping set $S(x, X)$ is defined as a closed ball with center in x with minimal radius, containing at least one point of $X \setminus \{x\}$. $d(x, X)$ is then the radius of the ball. Define $\zeta(S(x, X))$ to be distributed as $x + \psi d(x, X)$, where ψ is some $\mathbb{R}^d$-valued random variable.

The examples 2 and 3 are the simplifications of the original R. Darling model, where we restrict to depending on only one closest neighbour. We provide our results for these two examples further in the text, after a brief review of the related topics.

2.2 Polya's urn schemes

The simplest version of a Polya's urn model is defined as follows. Let the urn initially contain k black balls and l white balls. Let $n_0 = k + l$. At each step, draw a ball from the urn, look at its color and return it back to the urn, along with one more ball of the same color. Then the proportion of black balls in the urn tends to a random limit, which is distributed as Beta(k, l). In particular, if $k = l = 1$, the limit is uniform on $(0, 1)$ interval.

That version of Polya urn model is the "degenerate" case of LISA setting, obtained by putting the sample space W to be the set of two colors, the stopping set $S(x, X) = x$ and $\zeta_{S(x,X)}$ to be degenerate in x. It is also a border case of R. Darling model: let us say that instead of throwing its' child away on normally distributed distance scaled with the distance to the closest neighbour(s) the mother point lets a child stay at her place. Then the points of initial configuration can be regarded as a set of colors, and the starting configuration as an urn, containing one ball of each of the n_0 colors. The rest of dynamics is the same: pick a point to reproduce, add one child to its location.

Polya's urn model has lots of variations, with different amount of colors and various rules for reinforcement. One generalization is having the replacement controlled by a reinforcement matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

That means, when we pull out a black ball from the urn, we return it back along with a additional black and b white balls, whereas if the pulled ball's color is white, we return it together with c black and d white balls. Then, depending on the relations between a, b, c, d there are different cases. For the details one should refer to [12, 6, 9, 14]. See [1] for the embedding of the Polya's urns into a continuous-time branching processes, and the related results on limit behaviour. For more limit theorems see [10, 9, 3]. For the general survey on the processes with reinforcement, see [15]

One of the nicest implications of a Polya's urn model is the probabilistic insight on how to construct random measures in general, [5], along with a simple way to simulate the Dirichlet processes, a rich class of priors, widely used in statistics due to their rigorous tractability. The construction in [5] involves an infinite amount of colors with the limiting measure being a sample from Dirichlet process with a corresponding parameter measure on the space of colors.

2.3 Random Splitting

Example 1 above is a variant of a stick-breaking process. This is a sequential random splitting of $[0,1]$ interval where at each step the uniformly picked sub-interval is split in two uniformly. As mentioned above, this process in a slightly more general setting is studied in detail in [17].

The splitting of the unity in [17] is a variation of Kakutani's splitting procedure [13], and formally defined as follows. Let $\{\chi_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables such that χ_n is uniform over $\{0,1,2,\ldots,n\}$. Let $\{W_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with values in $(0,1)$. Define X_n in a following way: $X_1=\{0,1\}$ and if $\xi_0\leq\ldots\leq\xi_n$ are the elements of X_n, enumerated in the increasing order, then

$$X_{n+1}=X_n\cup\{\xi_{\chi_n}+W_n(\xi_{\chi_n+1}-\xi_{\chi_n})\}$$

As the limit for the empiric measures, we get a random singular measure, with probability one sitting on a (random) set of Hausdorff dimension D which is found explicitly in [17]. Note that in the Kakutani's procedure we do not observe this kind of behaviour, but rather get a deterministic uniform measure in the limit, because of different selection methods. In [13] the longest interval is always selected for splitting, when as in [17] the next interval to split is selected uniformly over those existing at the moment. In continuous time that corresponds to giving weights to the intensities of splitting proportional to the intervals' lengths, and giving all of the intervals splitting intensities 1, respectively. See [4] for the entropies behaviour in the splitting procedure.

2.4 Cooperative Sequential Adsorption

The other class of the models included in our setting is that consisting of various versions of adsorption models. The cooperative sequential adsorption (CSA), for example, can be obtained by setting $S(x,X)=B(x,R)$, a ball with a fixed radius R, which is called the interaction radius. Then the random variable $\zeta_{S(x,X)}$ is set to have the density proportional to $\beta_{K(S(x,X))}$, where $K(S(x,X))$ is the number of particles in the ball and $\{\beta_k\}_{k\geq 1}$ is the fixed parameter set. CSA model is quite popular in the applied physics, and the results known are quite vast, see

[8, 16] and references therein. Especially interesting result in our context is obtained in [18], where the limiting measures are studied.

2.5 Main results

The main points of interest in our paper are those related to the characteristics of the limiting measure. At which conditions does the configuration stay inside a compact region? At which conditions does the limiting measure exist, what are its properties?

In the first paper we formulate and prove the following results for the Examples 2, 3.

Theorem 2. *Denote*

$$m_n = \min\{x : x \in X_n\},$$
$$M_n = \max\{x : x \in X_n\}$$

for the LISA model in Example 2. Then almost surely

$$-\infty < \lim m_n \le \lim M_n < +\infty$$

Thus the configuration of the points in Example 2 stays almost surely bounded. We also prove the existence of the limiting measure. Let $\nu_n = \sum_{1\le k\le n} \delta_{x_k}$ denote the empiric measure on the n-th step.

Theorem 3. *Almost surely exists μ^* – a random probability measure such that $\nu_n \xrightarrow{w} \mu^*$*

Moving on to Example 3, for which we come up with a sufficient condition for the boundedness of the process. In the formulation of Example 3, let $\eta = |\psi|$, and introduce $C = \mathbf{E}\,\eta$, $\hat{C} = \mathbf{E}\,\hat{\eta}$.

Theorem 4. *If $C + \hat{C} < 1$, then $\sup\limits_n |\xi_n| < \infty$ a.s. Moreover,*

$$\mathbf{E} \sup_n |\xi_n| \le A_0 + \frac{n_0 D_0 C}{1 - \hat{C} - C}$$

where $A_0 = \max\limits_{k\le n_0} |x_k|$, and D_0 is the maximal spacing of X_{n_0}

We also provide a couple of estimates for the behaviour of the maximal spacing of the model in the Example 3.

3 Bit Flipping Models

The second paper included in the thesis is about Bit Flipping models. The original model considered was born in the discussion with Sergey Foss when we were looking for an example of LISA model having the limiting measure with unbounded support.

In LISA setting, let $W = \{0, 1, 2, \dots\}$, $X_{n_0} = X_1 = \{0\}$, let η be the non-negative integer-valued random variable with distribution $\mathcal{P}$, let $\psi_{S(x,X)}$ be distributed as η, if x is a single entry in X and be degenerate in 0 if x is a multiple entry. Here we allow multiple entries, so the same point can present in the configuration in many instances. Now take a look at all of the particles generated by the first one. The question we asked was, what is the probability for all the points generated by the first one to be multiple at some moment of time? It is clear that the above probability is always greater than zero, but is it ever one? And in which cases is it less than one? The answer is given by the analysis of the Damaged Bits model which follows below.

The term "Bit Flipping" term is adopted from [2] where it is used in the context of analyzing the behaviour of a random-edge simplex algorithm on a Klee-Minty cube. There, in a sequence of bits indexed with $\{1, 2, 3, \dots\}$ each one is flipping its state with intensity one, and when the bit changes its state, all of the bits to the right from it do so as well. Note that although there is infinite amount of events happening during any finite time interval, any finite consecutive subset of bits starting with the first is still a Markov process in a continuous time with a finite state space. Our models differ from the one in [2]: we do not have dependency structure as of now, however, we let the bits have different intensities of flipping, which yields some interesting behaviour too.

Bit Flipping models are somewhat related to the dynamical percolation processes, see [11]. In that context, a particularly interesting question might be, what if the intensities $\{p_k\}$ of flipping grow really fast? Question is, can the probability of existence of a non-empty set of times τ such that all the bits are idle at time τ be positive? What can we tweak in the model to make it positive? We do not address these questions in the current paper, but rather leave it for the possible future work.

3.1 Model description, main results

First, we are going to define the Binary Flipping (BF) and Damaged Bits (DB) models in the discrete time setting. Then we show how to embed them into continuous time naturally. In fact, that embedding, hinted by R. Pemantle, turns out to be the main tool in the analysis. Both models contain an infinite sequence of elements, or "bits", that change their states one by one, according to certain dynamics. The number of a next bit to change the state is sampled each time from a fixed distribution

on positive integers. That distribution is the parameter in the model.

Binary Flipping (BF) In that version of a model, we let the bits switch between the idle and active states. Let $\zeta_k^n, k = 1, 2, \dots, n = 0, 1, 2, \dots$ denote the state of k-th bit at time n. Put $\zeta_k^n = 0$, that is, all the bits start in the idle state. Let $\mathcal{P} = \{p_1, p_2, \dots\}$ be the probability distribution on positive integers. We assume all of the p_k to be positive and ordered, $p_1 > p_2 > p_3 > \dots > 0$ That is not a restriction, since we always can throw out the null entries and reorder the rest (unless only the finite number of p_k's are non-zero, which seems to be the trivial case). At each time step n sample the index ξ_n from $\mathcal{P}$ independently of everything, and change the state of ξ_n-th bit according to the dynamics

$$\textit{idle(0)} \leftrightarrow \textit{active(1)}$$

That is, if a chosen bit was idle, it becomes active, and vice versa.

Damaged Bits (DB) In that version there is three possible states: idle, active and damaged. Again, let $\zeta_k^n, k = 1, 2, \dots, n = 0, 1, 2, \dots$ denote the state of k-th bit at time n and $\zeta_k^n = 0$. At each time step n sample the index ξ_n from $\mathcal{P}$ independently of everything, and change the state of ξ_n-th bit according to the dynamics

$$\textit{idle(0)} \to \textit{active(1)} \to \textit{damaged(2)}$$

So, if we pick a damaged bit, nothing happens, the bit remains damaged.

As mentioned above, both BF and DB models have natural continuous time implementations. That is, we can consider a sequence $\{\zeta_k(t)\}_{k\geq 1}$ of independent continuous time Markov processes, each with the state space {idle, active} (BF) or {idle, active, damaged} (DB). Then we let $\zeta_k(t)$ start in the idle state and have the intensity p_k of transitioning to the next state, $k = 1, 2, \dots$ Then if we denote by $\{T_n\}_{n\geq 0}$, the sequence of transition times of a correspondent Markov process $\{(\zeta_1(t), \zeta_2(t), \dots)\}_{t\geq 0}$ with $T_0 = 0$, we have the following distributional equality:

$$\{(\zeta_1^n, \zeta_2^n, \zeta_3^n, \dots)\}_{n\geq 0} \stackrel{\mathcal{D}}{=} \{(\zeta_1(T_n{+}0), \zeta_2(T_n{+}0), \zeta_3(T_n{+}0), \dots)\}_{n\geq 0}$$

Here on the left stands the discrete-time version of a corresponding model and on the right stands the continuous-time version taken at the times of transitions. Further we use both discrete- and continuous-time setting to formulate our results, depending on which of the two is more convenient at the moment.

One of the main questions for the bit flipping models is, if we start from the ground state where no bits are active, what's the probability to return to that state? It is quite transparent that the latter probability is greater than zero, but can it be one? In other words, is the so called

ground state recurrent? If it is, is the expectation of the recurrence time finite? Can we give some estimate for the moments? Then, if the ground state is not recurrent, how does the number of active bits grow as time goes to infinity?

The model is clear and simple, and yet the answers to most of the above questions turn out to be quite peculiar. First, we address the question of recurrence. By recurrence in this context we mean the recurrence of the ground state. In BF it means the recurrence of a corresponding Markov chain in the usual sense. In DB it rather means the Markov chain jumping out of some subset of states with probability one eventually. More precisely, we say that the BF or DB model is recurrent if $\tau = \min\{n \geq 1 : \zeta_n \text{ is not active}\}$ is almost surely finite, otherwise we say that the model is transient.

It turns out, that both BF and DB models exhibit recurrent or transient behaviour, depending on the speed of the decay of p_k. We start with proving the sufficient conditions for recurrence and transience of the BF model.

Theorem 5. *If the distribution $\mathcal{P} = \{p_1, p_2, \dots\}$ is such that:*

(i) $\limsup_{k\to\infty} 2^k p_k = C < \infty$, *then BF model is recurrent, i.e.* $\mathbf{P}\{\tau < \infty\} = 1$,

(ii) $\liminf_{k\to\infty} (2+\varepsilon)^k p_k = C > 0$ *for some* $\varepsilon > 0$, *then BF model is transient, i.e.* $\mathbf{P}\{\tau = \infty\} > 0$.

Theorem 5 means, that the "critical" decay is that of a geometric distribution with parameter $p = 0.5$. It is not a threshold in the usual sense, since the strict criteria would have been connected with the convergence of some series. However, one could still make sense to the criticality of geometric decay in BF model, with the help of the deterministic version of the process.

Imagine we have an infinite sequence of bits in state 0, numbered with $\{1, 2, \dots\}$. Then, let k-th bit alternate its state every 2^{k-1} seconds deterministically. That means that the frequency (or intensity) of switching is 2^{-k+1} for the k-th bit, which corresponds to the "critical" decay in BF model. Let us see what we have in first few steps of the process:

$$\begin{array}{c} \\ t=0 \\ t=1 \\ t=2 \\ t=3 \\ t=4 \\ t=5 \\ \dots \end{array} \begin{array}{c} \begin{array}{ccccc} k=1 & k=2 & k=3 & k=4 & \dots \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \end{array}$$

One can observe that the sequence of bits is the binary representation of the amount of whole seconds passed! Therefore, at all times at least one bit will be active, because that's how the clock works. However, if we now increase the decay speed, make it $(2+\varepsilon)^{-k+1}$ for some $\varepsilon > 0$, then the first k bits will have time to go through all the possible combinations (including all of the bits being idle) before the $k+1$-st bit becomes active.

We also prove the similar theorem for the DB model.

Theorem 6. *If the distribution $\mathcal{P}$ is such that:*

(i) $\limsup_{k\to\infty} \exp(\alpha k) p_k < \infty$, *for some positive α, then the DB model is recurrent,*

(ii) $\liminf_{k\to\infty} \exp(k^\gamma) p_k > 0$ *for some $\gamma \in (0,1)$, then the DB model is transient.*

After we are done with the recurrence/transience question, we move on to the different properties. We prove a central limit theorem for $\eta(t)$ – the number of active bits in the continuous-time version of BF, DB models.

Theorem 7. *Assume the BS or DB model is transient. Then*

$$\mathbf{E}\,\eta(t), \mathbf{var}\,\eta(t) \to \infty, t \to \infty, \quad \frac{\eta(t) - \mathbf{E}\,\eta(t)}{\sqrt{\mathbf{var}\,\eta(t)}} \xrightarrow{\mathcal{D}} N(0,1), t \to \infty$$

Here both $\mathbf{E}\,\eta(t)$ and $\mathbf{var}\,\eta(t)$ are given explicitly in a form of series.

Finally, we prove a bound for certain moments of recurrence time τ_{BF} in a recurrent case of the BF model.

Theorem 8. *Let $p_k = (1-p)p^{k-1}$ for some positive $p < 1$. Then $\mathbf{E}\,\tau_{\mathrm{BF}}^r$ exists for any r: $0 < r < r_0(p)$ whenever $p < p_0$. Here*

$$p_0 = \frac{3-\sqrt{5}}{2}, \quad r_0 = 1 - \frac{\log \frac{2-p}{1-p}}{\log \frac{1}{p}}$$

Moreover, for such p, r and any positive δ there exists a constant $C = C(p, r, \delta)$, such that

$$\mathbf{E}(\tau_{\mathrm{BF}}^{r} | \max A(0) = a) \leq Cap^{-(r+\delta)a}$$

The proofs can be found in paper II.

3.2 Conclusion, future work

The Bit Flipping models turn out to be posing interesting questions, some of which are analytically tractable. The most interesting phenomenon in our opinion is the existence of the threshold for the decay, bordering the transient and recurrent cases in both of the models. The ironic part is that "recurrence" in this context corresponds to the lack of mixing, due to the mixing times' distributions being very inhomogeneous. One can picture it as the whole process being unable to reach the stability due to highly inhomogeneous intensities of mixing of its independent components. That said, we come up with several possible extensions for our results.

One variant of an extension is to give the bits in BF different intensities $\{p_k\}$ and $\{q_k\}$ for turning active and idle, respectively. Then several interesting behaviours arise, depending on the relations between those parameters, and their (non-)summability. This is a promising modification we are working on currently.

The other option is to assume we are dealing with series of independent identical copies of some Markov chain, in which the time is scaled differently, according to parameters $\{p_k\}$. The question one can ask is, do those chains ever "meet" in the same state. Mixing times and distributional characteristics of various sorts can be of interest as well.

The modification that might be interesting from random geometry point of view is adding some kind of spatial dependence structure into the model. For instance, one could assign different intensities for turning active, according to a number of neighbour bits that are active at the time — in a fashion of contact processes.

One more way to extend is to interpret the bit turning on as a call coming to the k-th service station with intensity p_k. Then a call can be transferred to a connected station with intensity r_k or leave the system with intensity q_k. This can lead to a model similar to those considered in a loss-network setting, but with non-homogeneous servers. Usual queueing theory questions about performance of the system can be asked then.

Overall, the class of connected problems is quite rich and seems very promising!

4 References

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