Thesis for the Degree of Doctor of Philosophy

An analytic approach to Briançon-Skoda type theorems

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ABSTRACT

The Briançon-Skoda theorem can be seen as an effective version of the Hilbert Nullstellensatz and gives a connection between size conditions on holomorphic functions and ideal membership. The size conditions are captured algebraically by the notion of integral closure of ideals. Many techniques have been applied to prove the Briançon-Skoda theorem and variations of it. The first proof by Briançon and Skoda used L^2-theory. Later, Lipman and Tessier observed that residue calculus could be used to obtain an alternative proof, and inspired by this approach they generalized the theorem to an algebraic setting. Berenstein-Yger et al. developed further this residue method by introducing a division formula by Berndtsson into the picture. The theory of tight closure, introduced by Hochster and Huneke, was motivated by, and has been used to prove, the Briançon-Skoda theorem. This thesis explores how one can use analytic methods, including residue theory, to obtain Briançon-Skoda type theorems on singular varieties.

Keywords: Briançon-Skoda theorem, Artin-Rees lemma, Singular varieties, Residue calculus, Milnor number

This thesis contains an introduction and the following papers.

Paper 1: An elementary proof of the Briançon-Skoda theorem. Ann. fac. sci. Toulouse, 19 no. 3-4 (2010), p. 675-685
Paper 2: On the Briançon-Skoda theorem on a singular variety. Ann. inst. Fourier, 60 no. 2 (2010), p. 417-432, joint work with Mats Andersson and Håkan Samuelsson.
Paper 3: A Briançon-Skoda type result for a non-reduced analytic space.
Paper 4: A residue calculus approach to the uniform Artin-Rees lemma. To appear in Israel J. Math.
Paper 5: The Briancon-Skoda number of analytic irreducible planar curves.

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Jacob Sznajdman
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INTRODUCTION

1. The ideal membership problem

The problem of deciding whether a polynomial belongs to a given ideal dates back to days of Dedekind and Kronecker. Their approaches to this problem were however quite different. In fact, the precise definition of an ideal is due to Dedekind who generalized Kummer's ideal numbers in number theory. An ideal I in a ring R is defined as a subgroup of R which is closed under multiplication by any element of R. This definition is rather abstract, and if one is given generators $a_1, \dots, a_m$ for an ideal I, meaning that $I = \{u_1 a_1 + \dots + u_m a_m : u_i \in R\}$, it is non-trivial to determine whether an element is a member of the ideal or not. Kronecker's view was that such definitions were not desirable, and he devised another concept which he called 'divisor'.

During the 19th and 20th century, non-constructive mathematics blossomed from mathematicians such as Cantor, Bolzano, Cauchy, Weierstrass, Hilbert and others, and constructive mathematics, such as for example elimination theory, was on the downfall. Today however, with the aid of computers, computational and algorithmic mathematics has gained ground. One algorithm that stands out as being particularily useful is Buchberger's algorithm of finding Gröbner bases. Gröbner bases were in fact invented in order to solve the ideal membership problem. The original algorithm works for a polynomial ring with coefficients in a field, but there has been work to generalize the algorithm. Gröbner bases have also numerous other applications such as solving polynomial equations, elimination of variables, computation of primary decomposition and computation of b-functions, to name a few.

The central theme in this thesis will be theorems that give sufficient conditions for ideal membership. Throughout the thesis, we consider ideals in the ring $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of holomorphic functions at $0 \in \mathbb{C}^n$, or in a quotient $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}$. The most famous example of such a theorem is Hilbert's Nullstellensatz; it states that if $\mathfrak{a}$ is an ideal in $\mathcal{O}_{\mathbb{C}^n,0}$, and $V(\mathfrak{a})$ is the corresponding zero locus, and $I(V(\mathfrak{a}))$ is the set of functions vanishing on $V(\mathfrak{a})$, then $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. In other words, if f vanishes on $V(\mathfrak{a})$, then there exists some integer k such that $f^k \in \mathfrak{a}$. Although this does not strictly speaking give a sufficient condition for $f \in \mathfrak{a}$, one can easily derive such a condition (although non-constructively): Since the ring $\mathcal{O}_{\mathbb{C}^n,0}$ is Noetherian, the ideal $\sqrt{\mathfrak{a}}$ has a finite set of generators, say $b_1, \dots b_m$. Let n_i be integers such that $b_i^{n_i} \in \mathfrak{a}$. Then $f \in \sqrt{\mathfrak{a}}^{\Sigma_i n_i - m + 1}$ implies that $f \in \mathfrak{a}$, because f is a sum of terms like $u \cdot b_1^{k_1} \cdot \dots \cdot b_m^{k_m}$ where $\sum_1^m k_i = \sum_1^m n_i - m + 1$, and $u \in \mathcal{O}_{\mathbb{C}^n,0}$, so applying the pigeon hole principle, we see that any such term contains a factor $b_i^{n_i} \in \mathfrak{a}$. The smallest integer k such that $\sqrt{\mathfrak{a}}^k \subset \mathfrak{a}$ is by definition the *degree of nilpotency* of the ring $\mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{a}$.

If one now tries to find a set of sufficient conditions for ideal membership in a given ideal $\mathfrak{a}$, there are some properties that should be desirable. First, the condition should be as easy as possible to check, and second, the set of elements satisfying the condition should be as close to all of $\mathfrak{a}$ as possible.

One type of conditions for ideal membership involves a notion called integral closure of ideals. We consider only its analytic definition: Pick a set of generators $a_1, \ldots, a_m$ for $\mathfrak{a}$ and set

$$|\mathfrak{a}| = \max_{1 \leq j \leq m} |a_j|. \tag{1}$$

Strictly speaking this is not well defined due to dependence on the choice of generators, but making another choice gives a function that is *equivalent* to $|\mathfrak{a}|$ in a sense that we now make precise. We say that $f \lesssim g$ if there is a constant C such that $f \leq Cg$, and then f and g are equivalent if $f \lesssim g \lesssim f$. The integral closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ is defined as the ideal

$$\overline{\mathfrak{a}} = \{\phi \in \mathcal{O}_{\mathbb{C}^n,0} : |\phi| \lesssim |\mathfrak{a}|\}, \tag{2}$$

which contains $\mathfrak{a}$ and is contained in $\sqrt{a}$. It is clear that this definition only depends on the equivalence class of $|\mathfrak{a}|$.

If one considers ideals of one complex variable z, then an arbitrary ideal has the form $\mathfrak{a} = (z^k)$. Thus $\mathfrak{a} = \overline{\mathfrak{a}}$ for any $\mathfrak{a}$. However, in higher dimensions the inclusion $\mathfrak{a} \subset \overline{\mathfrak{a}}$ may be strict, i.e., the size condition $|\phi| \lesssim |\mathfrak{a}|$ does not guarantee that $\phi \in \mathfrak{a}$. For example in dimension two, $\phi = z_1 z_2$ belongs to $\overline{\mathfrak{a}} \setminus \mathfrak{a}$, where $\mathfrak{a} = (z_1^2, z_2^2)$. The same idea can be extended to higher dimensions.

It follows from the Nullstellensatz that $\overline{\mathfrak{a}} \subset \sqrt{\mathfrak{a}}$. Thus if k is the degree of nilpotency of $\mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{a}$, then $\overline{\mathfrak{a}}^k \subset \sqrt{a}^k \subset \mathfrak{a}$. The Briançon-Skoda theorem gives a sharp criterion for ideal membership along these lines.

Theorem 1.1 (Briançon-Skoda)**.** *Let $m, r \geq 1$ be integers and $\varrho = \min(m, n)$. Then for any ideal $\mathfrak{a} \subset \mathcal{O}_{\mathbb{C}^n,0}$ which can be generated by m functions,*

$$\overline{\mathfrak{a}^{\varrho+r-1}} \subset \mathfrak{a}^r.$$

Note that the ideal $\overline{\mathfrak{a}^{\varrho+r-1}}$ consists of all functions $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $|\phi| \lesssim |\mathfrak{a}|^{\varrho+r-1}$. If we increase the exponent $\varrho+r-1$ in Theorem 1.1 to $n+r-1$, it is *uniform* in $\mathfrak{a}$ and linear in r. This theorem is optimal in the sense that one cannot replace ϱ by a smaller value[1].

Let now k be the smallest integer such that $\overline{\mathfrak{a}^k} \subset \mathfrak{a}$ for a fixed ideal $\mathfrak{a}$. By the Briançon-Skoda theorem we know that $k \leq \min(m, n)$. One

[1]This can be seen for $n = 2$ by considering again the example $\phi = z_1 z_2$, $\mathfrak{a} = (z_1^2, z_2^2)$, and similarly for higher dimensions.

defines the Łojasiewicz exponent $\nu = \nu(\mathfrak{a})$ as

$$\nu = \inf\{\theta : \forall \phi \in \sqrt{\mathfrak{a}} \quad |\phi|^{\theta} \lesssim |\mathfrak{a}|\}. \tag{3}$$

If $\phi \in \sqrt{\mathfrak{a}}$, it follows that $|\phi| \lesssim |\mathfrak{a}|^{1/(\nu+\epsilon)}$ for any small $\epsilon > 0$. A consequence of this is that $\sqrt{\mathfrak{a}}^{\lceil k(\nu+\epsilon)\rceil} \subset \mathfrak{a}$, where $\lceil\cdot\rceil$ denotes rounding upwards to the nearest integer. In particular, $\sqrt{\mathfrak{a}}^{\nu \min(m,n)+1} \subset \mathfrak{a}$.

In [23] Section 10.5, Lazarsfeld also uses (a slight variation of) the Briançon-Skoda theorem to obtain an effective Nullstellensatz, which in particular gives an upper bound for the degree of nilpotency.

A different, but related, result is the Artin-Rees lemma, which is an important result in commutative algebra from the 1950s. It states the following:

Theorem 1.2 (Artin-Rees)**.** *Let A be a Noetherian ring and M a finitely generated A-module. Given an ideal $I \subset A$ and a submodule $N \subset M$, there exists a number μ such that*

$$I^{\mu+r}M \cap N = I^r(I^{\mu}M \cap N), \tag{4}$$

for all integers $r \geq 0$.

This result was used to prove the exactness of the I-adic completion functor, see [8]. As Atiyah and MacDonald writes, one can use transcendental methods in algebraic geometry over the complex numbers by regarding a rational function as a power series about a point. In more abstract algebraic geometry, this is not possible, but through completion some problems can be expressed in formal power series, which is also very useful.

For most applications, including the exactness of completion, the slightly weaker statment

$$I^{\mu+r}M \cap N \subset I^r N \tag{5}$$

suffices.

In the uniform Artin-Rees lemma, one imposes the extra requirement that the number μ should not depend on the ideal I. This variant of the theorem has its origin in the paper [17] by Eisenbud and Hochster on a generalization of Zariski's main lemma on holomorphic functions, [37]. The uniform Artin-Rees lemma[2] was stated as an open question which was needed for an alternative version of their proof. The uniform Artin-Rees lemma has been proven in various versions for example in [30] by O'Carroll, in [11] by Bierstone-Milman and in [21] by Huneke.

2. The history of the Briancon-Skoda theorem

We return now to the Briançon-Skoda theorem. The first proof of the theorem was given in 1974 by Joël Briançon and Henri Skoda, [13]. Their proof was based on an L^2-division theorem by Skoda, [35]. After

[2]In their formulation, it was assumed that I should be a maximal ideal.

that, Lipman and Sathaye, [26], proved algebraically that the theorem holds for any regular Noetherian ring. A little earlier, Lipman and Tessier proved in [27] that

$$\overline{\mathfrak{a}^{n+r-1}} \subset \mathfrak{a}^r \tag{6}$$

for a "reasonable" pseudo-rational n-dimensional ring R (reasonable means that the localization at each prime is also pseudo-rational). However, the improvement for few generators works only[3] for special ideals $\mathfrak{a}$, for example if $\mathfrak{a}$ has a reduction (i.e., a subideal with the same integral closure) generated by a regular sequence. They also proved that the class of pseudo-rational rings includes all regular rings. In [27], Lipman and Tessier wrote

> The proof given by Briançon and Skoda of this completely algebraic statement is based on a quite trancendental deep result by Skoda in [..]. The absence of an algebraic proof has been for algebraists something of a scandal—perhaps even an insult—and certainly a challenge.

This challenge was actually made explicit by Hochster at a CBMS conference held at Georege Mason University in 1979, where he was the principal speaker and concentrated on the Briançon-Skoda theorem. It is interesting to note that the algebraic approach by Lipman and Tessier is in fact inspired by an analytic method by which they obtain the result for the original ring $\mathcal{O}_{\mathbb{C}^n,0}$. We explain briefly the latter method which is based on local duality. Let $f_1, \ldots, f_n$ be a regular sequence of elements in the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{\mathbb{C}^n,0}$ and set $I = (f)$. The *residue bilinear pairing* $\mathrm{Res}_f : \mathcal{O}_{\mathbb{C}^n,0}/I \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/I \to \mathbb{C}$ associated with $f = (f_1, \ldots, f_n)$ is induced by the pairing $\mathcal{O}_{\mathbb{C}^n,0} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} \to \mathbb{C}$ given by

$$\mathrm{Res}_f(\phi, h) = \int_{\{|f_j|=\epsilon\}} \phi h \frac{dz_1 \wedge \ldots \wedge dz_n}{f_1 \ldots f_n}, \tag{7}$$

which is independent of ϵ if ϵ is sufficiently small. Local duality states that this pairing is non-degenerate, which means that if $\mathrm{Res}_f(\phi, h) = 0$ for all $h \in \mathcal{O}_{\mathbb{C}^n,0}$, then $\phi \in I$. If we assume that $|\phi| \lesssim |I|^n$, it follows that $\phi h/(f_1 \ldots f_n)$ is bounded, so

$$|\mathrm{Res}_f(\phi, h)| \lesssim \int_{\{|f_j|=\epsilon\}} |dz_1 \wedge \ldots \wedge dz_n|,$$

which must be zero since the sets $\{|f_j| = \epsilon\}$ shrink to zero as ϵ approaches zero. This proves (6) under the assumption that $f_1, \ldots, f_n$ is a regular sequence of elements in $\mathfrak{m}$ and that $r = 1$. It is however possible to reduce the general case to this situation.

[3] The generalization was fully proven two decades later in [1], by Aberbach and Huneke.

In the late eighties, Craig Huneke and Melvin Hochster introduced the notion of tight closure, which is a closure operation on ideals, such that the tight closure is always contained in the integral closure. Tight closure works naturally in rings of characteristic p, but it can be used to prove statements in characteristic 0 by reduction to characteristic p. This method has been quite successful, and proofs of various statements are often remarkably short in positive characteristic. In Chapter 13 of [22] by Huneke and Swanson, it says

> The Briançon-Skoda theorem has played an important role in the development of many techniques in commutative algebra. These developments range from the theorem of Lipman and Sathaye, Theorem 13.3.3, to contributing to the development of tight closure, as well as Lipman's development of adjoint ideals.

The first tight closure proof of the Briançon-Skoda theorem for a regular ring appeared in [18]. Schoutens, [34], gave an elementary proof, based on the tight closure approach, by using ultrafilters to simplify the reduction to characteristic p.

In [21], Huneke showed in a general setting that the Artin-Rees lemma holds in a uniform sense, meaning that the constant μ in Theorem 1.2 can be chosen independently of I and r. This is a much more delicate matter than merely showing the existence of μ for each fixed I. In the same paper, Huneke showed for a quite general Noetherian reduced local ring R a uniform version of the Briançon-Skoda theorem. It states that there exists an integer N such that $\overline{\mathfrak{a}^{N+r-1}} \subset \mathfrak{a}^r$, for all ideals $\mathfrak{a} \subset R$ and $r \geq 1$. In particular this applies to the local ring $\mathcal{O}_{X,0}$ of a singular variety X. The author is unaware of any proofs using L^2-theory on singular varieties.

It is easy to see that for the cusp $z^2 = w^{2k+1}$, a lower bound for the number N is $k+1$. Thus for singular varieties, the dimension does not give an upper bound for N.

Lazarsfeld, [23], gives a proof of the original Briançon-Skoda theorem based on multiplier ideal sheaves, vanishing theorems and log-resolutions. In Remark 9.6.29, he writes "experience shows that algebraic statements established by L^2-methods or multiplier ideals can also be understood via tight closure."

Lipman's notion of adjoint ideals, introduced in [25], is a generalization of multiplier ideals. One may therefore speculate that the theorem of Briançon-Skoda has also contributed to the development of multiplier ideals, and their algebraic formulation. An argument that supports this view is that Skoda's theorem is actually a statement about multiplier ideals, although this notion had not yet been introduced at that time.

A different approach to proving the Briançon-Skoda theorem and its generalizations is to use division formulas and residue calculus. This approach was first taken by Berenstein, Gay, Vidras, and Yger in [9] to prove the original version of the theorem. The authors used a division formula by Berndtsson, [10], which was developed further by Passare in [32]. The residues used by these authors and the present author differ from the ones used by Lipman et al.; the latter are viewed as cohomological objects, whereas the former are currents which are analytical objects.

In summary, quite many theories, some of them vastly differing from the others, can be used to prove the Briançon-Skoda theorem. This thesis explores how the residue method can be used to prove Briançon-Skoda and Artin-Rees type theorems. We give an elementary proof of the classical Briançon-Skoda theorem, obtain analytic proofs of both of Huneke's theorems mentioned above, and in the case of a plane curve, we determine the optimal value for the constant N in the Huneke-Briançon-Skoda theorem.

3. The residue calculus approach to the ideal membership problem

In this section we explain how one can use residue calculus to solve the ideal membership problem for an ideal $J \in \mathcal{O}_{\mathbb{C}^n,0}$ which is generated by functions $a_1, \dots, a_m$. The question is how we can determine if $\phi \in J$ for a function $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$. The method which we describe can be traced back to Hörmander's solution to various ideal membership problems, including the corona problem, in [19], where it was assumed that $a_1, \dots, a_m$ have no common zeros, that is, $J = (1)$.

Later, Passare [32] and Dickenstein and Sessa [15] proved the Duality theorem for the case when J is a complete intersection; it states that $\phi \in J$ if and only if ϕ *annihilates* a residue current (the Coleff-Herrera product) associated to the ideal J. This means that the product of ϕ and the residue current is zero (in the sense of currents).

In [2] and [3], Andersson gave a method to show ideal membership in the non-complete intersection case. In this method, one associates a residue current to a complex of vector bundles, and annihilating the residue current, for a suitable choice of complex, is sufficient for ideal membership. We will use the formalism from the paper [3].

We now begin to explain the method. Assume that we want to show that $\phi \in J$. We take a generically exact complex of trivial vector bundles

$$\cdots \to E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \to 0, \tag{8}$$

where $\operatorname{rank} E_0 = 1$ and $\operatorname{rank} E_1 = m$ and such that the map f_1 is given by the row matrix $[a_1, \dots, a_m]$. We then have a corresponding complex of $\mathcal{O}_{\mathbb{C}^n,0}$-modules

$$\cdots \to \mathcal{O}(E_2) \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0) \to \mathcal{O}_{\mathbb{C}^n,0}/J \to 0. \tag{9}$$

Our goal is to find a holomorphic section v_1 of E_1 such that $f_1 v_1 = \phi$. We begin with the case that $J = (1)$, or equivalently that there is no point at which $a_1, \ldots, a_m$ all vanish. In this case one can make sure that the complex (8) is point-wise exact; for example the Koszul complex will do. As an intermediate step towards the solution, we find a not necessarily holomorphic solution v_1 to $f_1 v_1 = \phi$. A smooth choice is $v_1 = (\overline{a_1}, \ldots, \overline{a_m})\phi/|a|^2$, where $a = (a_1, \ldots, a_m)$. If it happens to be the case that $\overline{\partial} v_1 = 0$, we are done, but of course in general this fails. We therefore try to find a modified solution of the form $\tilde{v}_1 = v_1 - A$, where $f_1 A = 0$ and $\overline{\partial} v_1 = \overline{\partial} A$. An easy way to satisfy the first condition is to let $A = f_2 g$ for some section g of E_2. Thus we need to solve the equation $f_2 \overline{\partial} g = \overline{\partial} v_1$. That is, we are looking for a $\overline{\partial}$-exact solution v_2 to the equation $f_2 v_2 = \overline{\partial} v_1$. By the Dolbeault lemma it is sufficient to find a $\overline{\partial}$-closed solution. As before, we begin by finding an arbitrary solution v_2 which is not necessarily $\overline{\partial}$-closed. Note that $\overline{\partial} v_1$ is in the kernel of f_1, since $f_1 \overline{\partial} v_1 = \overline{\partial} f_1 v_1 = \overline{\partial} 1 = 0$. Thus the exactness of the complex implies that a solution v_2 exists. Again we have to modify the solution to obtain a $\overline{\partial}$-closed one. This gives a new equation: $f_3 v_3 = \overline{\partial} v_2$, for which we need a $\overline{\partial}$-exact solution v_3, and so on. We have that $f_j \overline{\partial} v_j = \overline{\partial}(f_j v_j) = \overline{\partial}^2 v_{j-1} = 0$, so since (8) is exact, $\overline{\partial} v_j$ is in the image of f_{j+1}. This means that we can solve each of the equations

$$
\begin{aligned}
& f_1 v_1 = \phi \\
& f_{j+1} v_{j+1} = \overline{\partial} v_j.
\end{aligned} \tag{10}
$$

Since v_j is a $(0, j)$-form, after at most n steps we will get $\overline{\partial} v_j = 0$, so the procedure will terminate.

We will now show how to find explicit solutions to the equations (10) in order to later generalize the procedure when J is arbitrary. We assume that $\phi = 1$, because if we solve $f_1 u_1 = 1$, then $v_1 = \phi u_1$ solves $f_1 v_1 = \phi$. Similarily, let u_j be solutions to (10) when $\phi = 1$.

We bestow each of the bundles E_j with a hermitian metric, for example the trivial metric with respect to some fixed frame.

Definition 3.1. Define σ_i pointwise as the minimal-norm inverse of f_i on the image of f_i and extend σ_i by zero on the orthogonal complement of the image.

Example 3.2. For the Koszul complex, each σ_i will be the mapping $\xi \mapsto \sigma \wedge \xi$ where $\sigma = \sum \overline{a_i} e_i / |a|^2$ and $\{e_i\}$ is a frame for E_1.

Using that σ_i is (on the image of f_{i+1}) a section of the map f_{i+1}, we can take $u_1 = \sigma_1$ and $u_{j+1} = \sigma_j \overline{\partial} u_j$ as solutions to (10). This gives that

$$
u_j = \sigma_j \overline{\partial} \sigma_{j-1} \wedge \ldots \wedge \overline{\partial} \sigma_1. \tag{11}
$$

To simplify notation, we set

$$u = \sum u_j, \qquad f = \sum f_j,$$
$$E = \sum E_j, \qquad \sigma = \sum \sigma_j.$$

Consider the operator $\nabla_f = f - \overline{\partial}$. The equations (10) can then be written more concisely as

$$\nabla_f u = 1. \tag{12}$$

Let k be an integer so that $\overline{\partial} u_k = 0$. This implies that locally $u_k = \overline{\partial} v_k$ for some form v_k. It follows by (10) that $\overline{\partial}(u_{k-1} - f_k v_k) = 0$, so we can solve $u_{k-1} - f_k v_k = \overline{\partial} v_{k-1}$. Continuing like this for each $1 \leq j < k$, we get that $\overline{\partial}(u_j - f_{j+1} v_{j+1}) = 0$. Finally, we obtain the modified holomorphic solution at the first step: $f_1(u_1 - f_2 v_2) = 1$.

We now move on to the case of a general ideal J and mimic the case $J = (1)$. The construction above is local, so outside of the analytic set Z on which (8) is not exact, u is defined and satisfies (12). Since the form u could be used to solve the ideal membership problem when $J = (1)$, we want to give a meaning to u across Z. To do this, one can regularize u and pass to the limit. One then obtains a current U that extends u. In fact, the extension is canonical and it is the only 'reasonable' extension of u. There are however several possible ways to regularize it, but we will use the approach of smooth cut-off functions. This way of regularizing principal value currents, and residue currents, was introduced by Passare in [33].

Let χ be any smooth function $[0, \infty) \to \mathbb{R}$ such that $\chi \equiv 0$ on a neighbourhood of 0 and $\chi \equiv 1$ on a neighbourhood of ∞. An extension U is then given as

$$U = \lim_{\epsilon \to 0} \chi(|F|^2/\epsilon^2) u, \tag{13}$$

where F is any non-trivial tuple of analytic functions whose zero locus contains Z. The fact that this limit converges in the sense of currents, that $U = u$ outside of Z and that U does not depend on χ and F is proved by Andersson in [2].

Example 3.3. Consider the complex of line bundles

$$0 \to \mathbb{C} \xrightarrow{f} \mathbb{C} \to 0,$$

where f is a single holomorphic function. This gives that $u = \sigma = 1/f$. The current U is then the principal value $[1/f]$. Note that outside of $\{f = 0\}$, f is an isomorphism on each fiber, so $u = 1/f$ is the only possible section of the map $f : \mathbb{C} \to \mathbb{C}$, regardless of which metric we use for the complex.

Clearly, $\nabla_f U = 1$ cannot hold in general as the method above then would give that any ϕ belongs to J. However, it is natural to study the difference $1 - \nabla_f U$.

Definition 3.4. The difference $R := 1 - \nabla_f U$ is called the residue current associated to the Hermitian complex (8).

Since $\nabla_f U = \nabla_f u = 1$ outside of Z, we see that R has support on Z. It is readily checked that

$$R = \lim_{\epsilon \to 0}(1 - \chi(|F|^2/\epsilon^2)) + \bar{\partial}\chi(|F|^2/\epsilon^2) \wedge u.$$

The first term has of course limit zero, so it may seem peculiar to include it, but it will matter later when we define products of residue currents.

Remark 3.5. In Example (3.3), $R = \bar{\partial}[1/f]$.

Now if ϕ is a function so that $R\phi = 0$ as a current, then

$$\nabla_f U\phi = \phi - R\phi = \phi, \tag{14}$$

but this means that the equations (10) are solvable. By the argument that we used for the case $J = (1)$, we know that there is a $\bar{\partial}$-closed current solution u_1 to $f_1 u_1 = 0$, but by Weyl's lemma, u_1 is holomorphic. We conclude that $\phi \in J$.

Define the annihilator of R, $\operatorname{ann} R$, as the set of all functions ϕ such that $R\phi = 0$. The result above is then that $\operatorname{ann} R \subset J$.

A main result of [7] is that if the complex (8) is chosen so that (9) is exact, then $\operatorname{ann} R = J$.

4. Membership in product ideals and ideals on singular varieties

In this thesis we will need to solve membership problems on singular varieties on one hand, and on the other hand we have membership problems with respect to products of ideals.

We look first at the setting of a singular variety Z whose ring of local sections is $\mathcal{O}_{Z,0} = \mathcal{O}_{\mathbb{C}^n,0}/J$. Let $\mathfrak{a} \subset \mathcal{O}_{Z,0}$ be the ideal with respect to which we consider the membership problem. Let $a = (a_1, \ldots, a_m)$ be a tuple of functions in $\mathcal{O}_{\mathbb{C}^n,0}$ whose images in $\mathcal{O}_{Z,0}$ generate $\mathfrak{a}$. We choose a suitable complex E^a such that (a) is the image of the first map of the corresponding sheaf complex. If $\mathfrak{a}$ is an arbitrary ideal, it is natural to let E^a be the Koszul complex, but if $\mathfrak{a}$ itself is a product of ideals, we use a generalization of the Koszul complex which is given in Section 5. For the associated residue current R^a, we have that $\operatorname{ann} R^a \subset a$. We also need a residue current which is associated to Z. By choosing a free resolution of $\mathcal{O}_{\mathbb{C}^n,0}/J$ as our complex, the corresponding current R^Z satisfies $\operatorname{ann} R^Z = J$ according to [7].

If we identify ϕ with one of its representatives in $\mathcal{O}_{\mathbb{C}^n,0}$, the membership problem, phrased in the ambient space, is to show that $\phi \in (a)+J$, given some conditions on ϕ.

We want to have a residue current whose annihilator is contained in $(a) + J$. Once this is achieved, the membership problem is reduced to checking that the sufficient conditions on ϕ imply that ϕ annihilates this current.

The residue current is obtained as the product of the currents R^a and R^Z, which we will now define. The associated product complex (for the sum ideal $(a) + J$) is defined as

$$E_k = \bigoplus_{i+j=k} E_i^a \otimes E_j^Z. \tag{15}$$

We also define the total map $f = f^a + f^Z$ for the complex E, where f^a and f^Z are 'extended' to E in a way compatible with a so called super structure on E, meaning that

$$f(\xi \wedge \eta) = f^a(\xi) \wedge \eta + (-1)^{\deg \xi} \xi \wedge f^Z(\eta), \tag{16}$$

where ξ and η are differential forms with values in E^a and E^Z respectively.

Let $\chi_\epsilon^a = \chi(|a|^2/\epsilon^2)$. For any $\epsilon > 0$, the product $\chi_\epsilon^a u^a \wedge R^Z$ is a well-defined current with values in E since any current can be multiplied by smooth functions. We define

$$U^a \wedge R^Z = \lim_{\epsilon \to 0} \chi_\epsilon^a u^a \wedge R^Z. \tag{17}$$

Again, resolution of singularities can be used to see that this is well-defined and independent of χ. Similarily, we define the desired product residue as

$$R^a \wedge R^Z = \lim_{\epsilon \to 0} ((1 - \chi_\epsilon^a) + \overline{\partial} \chi_\epsilon^a \wedge u^a) \wedge R^Z. \tag{18}$$

It remains to see that the annihilator of this current is contained in $(a) + J$. Note first that by construction, the image of the first map in the sheaf complex $\mathcal{O}(E)$ is indeed $(a) + J$. As described in the previous section, it suffices to see that if ϕ annihilates $R^a \wedge R^Z$, then ϕ is ∇_E-exact, where $\nabla_E = f - \overline{\partial} = f^a + f^Z - \overline{\partial}$. In analogy with the construction of residue currents from complexes of vector bundles, we wish to find a current U such that

$$\nabla_E U = 1 - R^a \wedge R^Z. \tag{19}$$

We verify that $U = U^Z + U^a \wedge R^Z$ is an admissible choice, where U^Z is 'extended' to E in the sense of (16). At least formally, we have

$$\nabla_E(U^Z + U^a \wedge R^Z) = (1 - R^Z) + \nabla_E(U^a) \wedge R^Z - U^a \wedge (\nabla_E R^Z) =$$
$$= (1 - R^Z) + (1 - R^a) \wedge R^Z = 1 - R^a \wedge R^Z.$$

This calculation is justified by replacing U^a by $U_\epsilon^a = \chi_\epsilon^a u^a$, and using that $\nabla_E U_\epsilon^a = \nabla_{E^a} U_\epsilon^a = 1 - R_\epsilon^a$, where $R_\epsilon^a = (1 - \chi_\epsilon^a) + \overline{\partial} \chi_\epsilon^a \wedge u^a$, and then letting $\epsilon \to 0$.

In Paper 4 we shall need a complex associated with the product of two ideals I and J. The techinque is to construct residues R^I and R^J so that $\operatorname{ann} R^I \subset I$ and $\operatorname{ann} R^J \subset J$. Again, we form a product of the two underlying complexes E^I and E^J, but in a slightly different way. Namely, let the product complex $E = E^I \diamondsuit E^J$ be defined as

$$(20) \qquad \ldots \stackrel{f_3^I \oplus f_2^I \oplus f_3^J \oplus f_2^J}{\longrightarrow} E_1^I \otimes E_2^J \oplus E_2^I \otimes E_1^J \stackrel{f_2^I \oplus f_2^J}{\to}$$
$$\to E_1^I \otimes E_1^J \stackrel{f_1^I \otimes f_1^J}{\to} E_0^I \otimes E_0^J \to 0,$$

that is, at step k we take the direct sum of all $E_i^I \otimes E_j^J$ such that $i + j = k + 1$, and the map from E_{k+1} to E_k is given by $f_2^I \oplus \ldots \oplus f_{k+1}^I \oplus f_2^J \oplus \ldots \oplus f_{k+1}^J$, except for the first map $f_1^I \otimes f_1^J$. If one then defines $u = u^I \wedge u^J$ and extends u in the usual way to a current U, then one can show that the corresponding residue $R = 1 - \nabla_E U$ has the following form

$$(21) \qquad R = R^I \wedge U^J - U^I \wedge R^J,$$

where the products are defined in analogy with (18), in particular,

$$U^I \wedge R^J = \lim_{\epsilon \to 0} \chi(|I|^2/\epsilon^2) u^I \wedge R^J.$$

This formula enables us to prove membership in product ideals by annihilation of the current in (21). In fact, the same method also yields membership in tensor products of submodules of E_0^I and E_0^J, also when the latter have rank more than one.

5. A generalized Koszul complex

We will now construct a complex E^{a^r} associated to the ideal $a^r \subset \mathcal{O}_{\mathbb{C}^n,0}$, where $r \geq 1$ is an integer. This is in fact a special case of the diamond product of complexes. Let E^i, $1 \leq i \leq r$, be trivial vector bundles over an open set $X \subset \mathbb{C}^n$ with frames $\{e_j^i\}_{j=1}^m$. The Koszul complex $E_\bullet^i$ is then

$$(22) \qquad \cdots \longrightarrow \bigwedge^2 E^i \stackrel{\delta_i}{\longrightarrow} \bigwedge^1 E^i \stackrel{\delta_i}{\longrightarrow} \mathbb{C} \times X \to 0,$$

where δ_i is interior multiplication with the section $a_1 {e_1^i}^* + \ldots + a_m {e_m^i}^*$ of the dual bundle E^{i*}. We define δ_i' to be equal to δ_i on E_1^i and zero on all other components. We also define δ_i'' so that

$$\delta_i = \delta_i' + \delta_i'',$$

that is, δ_i'' is zero on E_1^i and equal to δ_i on E_k^i for $k > 1$. As in [4], we get generically defined forms

$$(23) \qquad u_i^a = \sum_{j=1}^{\min(n+1,m)} \sigma_i \wedge (\overline{\partial} \sigma_i)^{j-1},$$

where

$$(24) \qquad \sigma_i = \sum_{j=1}^m \frac{\overline{a_j} e_j^i}{|a|^2},$$

and $a_1, a_2, \ldots, a_m$ is a set of generators for $\mathfrak{a}$. Let $\nabla_\delta = \delta - \overline{\partial}$. We have a current extension U_i^a of u_i^a given by

$$U_i^a = \lim_{\epsilon \to 0} \chi_\epsilon^a u_i^a \tag{25}$$

and the accompanying residue current

$$R_i^a = [\lim_{\epsilon \to 0} R_{i,0,\epsilon}^a + \overline{\partial}\chi_\epsilon^a \wedge u_a^i], \tag{26}$$

where $R_{i,0,\epsilon}^a = 1 - \chi_\epsilon^a$.

The product complex E^{a^r} is

$$\cdots \to \bigoplus_{\substack{\sum k_i = k-1 \\ k_i \geq 0}} \bigwedge_{i=1}^r \bigwedge^{1+k_i} E^i \to \cdots \to \bigoplus_{\substack{\sum k_i = 1 \\ k_i \geq 0}} \bigwedge_{i=1}^r \bigwedge^{1+k_i} E^i \to \tag{27}$$
$$\to E^1 \wedge E^2 \wedge \cdots \wedge E^r \to \mathbb{C} \times X \to 0.$$

Thus for $k \geq 1$,

$$E_k^{a^r} = \bigoplus_{\substack{\sum k_i = k-1 \\ k_i \geq 0}} \bigwedge_{i=1}^r \bigwedge^{1+k_i} E^i.$$

We also have to specify the maps for the complex E^{a^r}. The total map is decomposed as $\delta = \delta' + \delta''$, where δ' is defined on $E_1^{a^r}$ and δ'' is defined on $E_k^{a^r}$ for $k \geq 2$. These maps are given by

$$\delta' = \delta_1' \wedge \delta_2' \wedge \ldots \wedge \delta_r' \tag{28}$$

and

$$\delta'' = \sum_{i=1}^r \delta_i''. \tag{29}$$

We define

$$u^{a^r} = u_1^a \wedge u_2^a \wedge \cdots \wedge u_r^a. \tag{30}$$

We then have $\nabla_\delta u^{a^r} = 1$ outside of $Z(\mathfrak{a})$, which the following calculation shows:

$$\nabla_\delta u^{a^r} = \delta' u^{a^r} + \nabla_{\delta''} u^{a^r}$$
$$\delta' u^{a^r} = (\delta_1' u_1^a) \wedge \ldots \wedge (\delta_r' u_r^a) = 1$$
$$\nabla_{\delta''} u^{a^r} = \sum_{i=1}^r (-1)^i u_1^a \wedge \ldots \wedge u_{i-1}^a \wedge \nabla_{\delta_i''} u_i^a \wedge \ldots \wedge u_r^a =$$
$$= \sum_{i=1}^r (-1)^i u_1^a \wedge \ldots \wedge u_{i-1}^a \wedge (\nabla_{\delta_i} - \delta_i') u_i^a \wedge \ldots \wedge u_r^a = 0,$$

because $(\nabla_{\delta_i} - \delta_i') u_i^a = 1 - 1$.

As usual, there is a current extension

$$U^{a^r} = \lim_{\epsilon \to 0} \chi_\epsilon^a u^{a^r} \tag{31}$$

and a corresponding residue current

$$R^{a^r} = \lim_{\epsilon\to 0} R^{a^r}_{\epsilon} = \lim_{\epsilon\to 0} R^{a^r}_{0,\epsilon} + R^{a^r}_{1,\epsilon} + \ldots = \lim_{\epsilon\to 0}(1-\chi^a_\epsilon) + \overline{\partial}\chi^a_\epsilon u^{a^r}. \tag{32}$$

It follows that

$$\begin{aligned} R^{a^r} &= \lim_{\epsilon\to 0} R^{a^r}_{0,\epsilon} + \overline{\partial}\chi^a_\epsilon \wedge u^a_1 \wedge u^a_2 \wedge \cdots \wedge u^a_r \\ &= \lim_{\epsilon\to 0} R^{a^r}_{0,\epsilon} + \sum_{\sum j_i \le C} \overline{\partial}\chi^a_\epsilon \wedge \sigma^1 \wedge \left(\overline{\partial}\sigma^1\right)^{\wedge j_1} \wedge \cdots \wedge \sigma^r \wedge \left(\overline{\partial}\sigma^r\right)^{\wedge j_r}. \end{aligned} \tag{33}$$

The constant is $C = \min(m,n)$. Since

$$\operatorname{Im}(\delta_a : E^{a^r}_1 \to E^{a^r}_0) = \mathfrak{a}^r,$$

the residue current R^{a^r} has by construction the property that if $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$ annihilates R^{a^r}, then $\phi \in a^r$. The converse does not hold in general.

From (33) one can see that the modulus of $R^{a^r}_\epsilon$ is controlled by $|a|^{-(r+2C+1)}$, which is better than what we would get from the ordinary Koszul complex for a^r. However, our proofs of the Briançon-Skoda and Artin-Rees theorems rely on using resolution of singularities to show that the singularity is in fact no worse than

$$|a|^{-(r+C+1)} = |a|^{-(r+\min(m,n))}.$$

6. Noetherian operators and Coleff-Herrera currents

A germ of a holomorphic differential operator L is called *Noetherian* with respect to in ideal $J \subset \mathcal{O}_{\mathbb{C}^n,0}$ if $L\phi \in \sqrt{J}$ for all $\phi \in J$. We say that $L_1,\ldots,L_M$ is a *defining* set of Noetherian operators for J, if $\phi \in J$ if and only if $L_1\phi,\ldots,L_M\phi \in \sqrt{J}$. These notions are needed in Paper 3 to formulate the main result. The existence of a defining set for any ideal J is due to Ehrenpreis [16] and Palamodov [31], see also [12], [20] and [29].

To illustrate how a defining set might look like, we consider the example $J = (w^k)$ where $z_1,\ldots z_{n-1}, w$ are coordinates for $\mathbb{C}^n$, $n \ge 1$. To determine if $\phi \in J$ for a function $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$, we can write down the Taylor expansion of ϕ in the w-direction. Thus $\phi \in J$ if and only if $\partial^j_w\phi(z,0) \equiv 0$ for $0 \le j \le k-1$. Note that $Z(w^k) = \{w=0\}$, so by Hilbert's Nullstellensatz $\partial^j_w\phi(z,0) \equiv 0$ is equivalent to $\partial^j_w\phi \in \sqrt{J}$. Therefore $1, \partial_w, \ldots, \partial^{k-1}_w$ is a defining set for $J = (w^k)$.

We will now discuss *Coleff-Herrera* currents and their relation to Noetherian operators. Assume that X is a germ of an analytic set of pure codimension p at $0 \in \mathbb{C}^n$. Let μ be a current of bidegree $(0,p)$ with support on X. We let χ be a smooth function such that $\chi \equiv 0$ on $[0,x_1]$ and $\chi \equiv 1$ on $[x_2,\infty)$ for some $0 < x_1 < x_2 < \infty$. One says that μ has the standard extension property (SEP) if $\mu = \lim_{\varepsilon\to 0}\chi(|h|^2/\varepsilon^2)\mu$ for any $h \in \mathcal{O}_{\mathbb{C}^n,0}$ that does not vanish identically on any component of X.

Definition 6.1. A current of bidegree $(0,p)$ with support on X is a Coleff-Herrera current on X if it is $\bar{\partial}$-closed, has SEP and is annihilated by $\overline{\phi}$ if $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishes on X.

The set of all Coleff-Herrera currents on X is an $\mathcal{O}_{\mathbb{C}^n,0}$-module which we denote by $\mathcal{CH}_X$. One also considers the submodule of all Coleff-Herrera currents annihilated by a given ideal J such that $Z(J) = X$. We will denote this module by $\mathcal{CH}_X(J)$. The model type for a Coleff-Herrera current is the Coleff-Herrera product

$$\mu^f = \left[\bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p}\right], \tag{34}$$

where $f = (f_1, f_2, \ldots, f_p)$ is a complete intersection such that $Z(f) = X$. We illustrate by the case $f_j = w_j^{1+M_1}$, where $w_1, \ldots, w_p$ are the last p coordinates for $\mathbb{C}^n$ and M_j are given integers. The action on a $(n, n-p)$ test form ξ is given by

(35)
$$\left[\bar{\partial}\frac{1}{w_1^{1+M_1}} \wedge \cdots \wedge \bar{\partial}\frac{1}{w_p^{1+M_p}}\right].\xi = M!(2\pi i)^p \int_{w=0} \partial_{w_1}^{M_1} \ldots \partial_{w_p}^{M_p} \left(\frac{\partial}{\partial w} \lrcorner\, \xi\right),$$

where $\partial/\partial w = \partial/\partial w_1 \wedge \ldots \wedge \partial/\partial w_p$, the symbol $\lrcorner$ denotes interior multiplication, and the derivative symbols refer to Lie derivatives.

The current μ^f was introduced by Coleff and Herrera in [14]. The machinery for defining products of residue currents described in Section 4, cf. (18), can be applied here to give a definition of the product (34). In fact, each $\bar{\partial}[1/f_j]$ is the residue current obtained from the Koszul complex for the principal ideal (f_j), cf. Example 3.3, and the product of these is equal to μ^f. A remarkable fact is that μ^f is a generator for $\mathcal{CH}_X((f))$, see Theorem 4.2 in [5].

If Z is a pure dimensional variety of codimension p, then one can show that there exists a reduced complete intersection $X = Z(f)$ such that Z is a union of some of the irreducible components of X. It follows from basic distribution theory that if $\mu \in \mathcal{CH}_Z$, then $\mu \in \mathcal{CH}_X((f^N))$ for some big integer N, where $f^N = (f_1^N, \ldots, f_p^N)$. Thus $\mu = A\mu^{f^N}$, so for pure dimensional varieties, all Coleff-Herrera currents are up to a holomorphic factor Coleff-Herrera products. However, μ^{f^N} has in general larger support than μ. A result by Björk, which is cited as Theorem 2.2 in Paper 4, gives that μ can be represented as a principal value integral on Z as

$$\mu.\xi = \lim_{\epsilon \to 0} \int_Z \chi(|h|^2/\epsilon^2) \frac{Q(\mathcal{S} \lrcorner\, \xi)}{h}, \tag{36}$$

where Q is a holomorphic differential operator, $\mathcal{S}$ is a smooth section of $\bigwedge^p T(\mathbb{C}^n)$, and h is a generically non-vanishing holomorphic function on Z. Note that (35) is a special case of this formula. Conversely, if Q is a holomorphic differential operator and $\mathcal{S}$ is a smooth section of $\bigwedge^p T(\mathbb{C}^n)$, then the current defined by (36) belongs to $\mathcal{CH}_Z$ if it is $\overline{\partial}$-closed.

Noetherian operators and Coleff-Herrera currents have a close connection. If $\mu \in \mathcal{CH}_Z(J)$ for some ideal $J \subset \mathcal{O}_{\mathbb{C}^n,0}$ such that $Z(J) = Z$, then the operator Q that arises from the proof of Björk's theorem is Noetherian with respect to J. By [7], given a pure dimensional ideal J, the construction in Section 3 gives a residue current R such that $\operatorname{ann} R = J$ when the complex (9) is a free resolution of $\mathcal{O}_{\mathbb{C}^n,0}/J$. Let $\mu_1, \ldots, \mu_l$ be the components of R so that $J = \cap_1^l \operatorname{ann} \mu_j$. Given that $\mathcal{O}_{\mathbb{C}^n,0}/J$ is Cohen-Macualay, it is shown in [7] that $\mu_j \in \mathcal{CH}_Z$, so for each μ_j we have a representation as in (36). Given $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$, the Leibniz rule yields that

$$\phi\mu_j.\xi = \lim_{\epsilon \to 0} \sum_k \int_Z \chi(|h|^2/\epsilon^2) \frac{L_{jk}(\phi) K_{jk}(\mathcal{S} \lrcorner \xi)}{h},$$

where L_{jk} and K_{jk} are holomorphic differential operators. Björk's theorem also states that $\{L_{jk}\}_k$ is a defining set for $\operatorname{ann} \mu_j$, and thus the union of these sets is a defining set for J.

7. Puiseux's theorem and the Milnor number of a plane curve

This section aims to explain some of the theory used in Paper 5. Let $C \in \mathbb{C}^2$ be (a germ of) an irreducible analytic curve through the origin. For curves, the notions of normalization and resolution of singularities are identical. Puiseux's theorem states that there is a locally defined 1-1 map $\pi : \mathbb{C} \to C$ which parametrizes C. Moreover, one can choose coordinates (z, w) for $\mathbb{C}^2$ and a coordinate t for $\mathbb{C}$ such that $\pi(t) = (t^m, g(t))$, where m is the multiplicity of C at the origin and $g(t)$ is an analytic function with a zero at the origin of multiplicity at least m.

The function g can be understood from the geometry of the curve C. By Weierstrass' preparation theorem, we can assume that C is the zero locus of an irreducible polynomial

$$P(z, w) = w^m + a_1(z)w^{m-1} + \ldots + a_m(z),$$

where each a_j vanishes at the origin. Using that P is irreducible, one can show that (unless C is smooth) the discriminant of P has an isolated zero at the origin. Thus the projection from $\mathbb{C}^2$ to the z-plane exhibits C as a branched covering of the z-plane. The covering has m sheets that intersect at the origin. Locally outside of the origin, each sheet is the graph of an analytic function f_j. By choosing an ordering of the m sheets near some point $z_0 \neq 0$, one can define $g(t) = f_1(t^m)$, where $z = t^m$. We can describe an analytic continuation of g to a neighbourhood of 0 and the extension is determined uniquely by how we defined it near z_0. When t traverses a circular arc of $2\pi/m$ radians around the origin, $g(t)$ will follow the graph of f_1 a whole circle around the origin. Of course, f_1 may not be defined along the whole circle, so we may have to stop earlier at a point $z_1 \neq 0$. Close to z_1 we can find new functions $\tilde{f}_j$ that parametrize the sheets locally. Continuing this way, and letting t traverse a whole circle, we will go around the z-plane m times. One can show that we will end up at the point $(z_0, f_1(z_0))$ that we started at. Thus we have extended $g(t)$ to a punctured disc. Further analysis shows that g is bounded (in fact it vanishes to order at least m) on $\mathbb{C} \setminus \{0\}$, so by the Riemann extension theorem, g is analytic across 0.

Once the function g is found, one can define the Puiseux characteristics. Assume that $g(t) = \sum_{k=m}^{\infty} c_k t^k$. Set $e_0 = m$ and define inductively

$$\beta_j = \min\{k \in \mathbb{N} : c_k \neq 0, e_{j-1} \nmid k\} \tag{37}$$

and

$$e_j = \gcd(e_{j-1}, \beta_j) = \gcd(m, \beta_1, \beta_2, \dots, \beta_j).$$

It is not hard to see that $\beta_\bullet$ is strictly increasing and $e_\bullet$ is strictly decreasing. The construction stops when, for some integer M, one has $e_M = 1$. Note that if we would have $\lim e_\bullet = e_\infty > 1$, then all of the exponents of $g(t)$ would be divisible by e_∞, so $\pi(t)$ would be a function of t^{e_∞}, and then it would not be 1-1.

Let us now introduce the Milnor number μ_P of the polynomial P defining the curve. There are various interpretations of this number. The Milnor fibration is given by the map

$$F: \; S_\epsilon \setminus C \to S^1$$
$$F = \frac{P}{|P|},$$

where S_ϵ is some sufficiently small sphere centered at the origin. Milnor, [28], has shown that F has no critical points on $S_\epsilon \setminus C$, which implies that each fiber $F_\theta = F^{-1}(e^{i\theta})$ is a smooth manifold of real dimension 2. Furthermore, each of the fibers has the homotopy type of a bouquet of 1-spheres, and the number of such spheres is the Milnor number. We can therefore view this number more or less as the number of 'handles' that each fiber has. Thus the Milnor number is defined purely topologically in terms of the Milnor fibration.

There is however an algebraic way to compute μ_P:

$$\mu_P = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/(P'_z, P'_w),$$

which generally leads to an easier calculation than the topological definition does.

In Milnor's book [28], a formula for μ_P in terms of the Puiseux characteristics is given in the case P is irreducible, however without proof. It states that

$$\mu_P = \sum_{j=1}^{M} (\beta_j - 1)(e_{j-1} - e_j). \tag{38}$$

In Paper 5, the right hand side of (38) appeared when analyzing the optimal bound in the Briançon-Skoda theorem for a planar curve. Using this formula, it was possible to connect the Milnor number to the Briançon-Skoda theorem.

8. Overview of Papers

8.1. Paper 1 – An elementary proof of the Briançon-Skoda theorem.

As commented in Section 2, the first proof of the Briançon-Skoda theorem relied on Skoda's L^2-division theorem. Let $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ be an ideal which can be generated by m elements, and let $\hat{I}^{(k)}$ be the ideal of analytic functions which consists of all $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$ such that

$$\int \frac{|\phi|^2}{|I|^{2(k+\epsilon)}} < \infty.$$

Skoda's theorem (in fact a special case) says that

$$\hat{I}^{(\min(m,n+1)+r-1)} \subset I^r \hat{I}^{(\min(m,n+1)-1)}.$$

Using the fact that one can always find an ideal $J \subset I$ generated by $\min(m,n)$ elements, such that $|J| \sim |I|$ (one says then that J is a reduction of I), the Briançon-Skoda theorem follows:

$$\overline{I^{\min(m,n)+r-1}} = \overline{J^{\min(m,n)+r-1}} \subset \hat{J}^{(\min(m,n)+r-1)} \subset J^r \subset I^r,$$

since J can be generated by $\min(m,n)$ elements.

Residue calculus provides an alternative method for proving the Briançon-Skoda problem which is in some ways simpler than Skoda's method, but one needs to apply Hironaka's resolution of singularities to even show the existence of residue currents.

The purpose of this paper is to prove that $\hat{I}^{(\min(m,n)+r-1)} \subset I^r$ with residue calculus (although the use of residues is only implicit), but without using resolution of singularities. This claim is slightly stronger than the Briançon-Skoda theorem, but we do not give L^2-estimates for the coefficient functions in the representation of the membership, which Skoda's theorem does.

The key point in the proof is to show, for $1 \leq r \leq m$, that

$$\frac{|\phi| \cdot |\partial f_1 \wedge \cdots \wedge \partial f_r|}{|f|^{k+r}}$$

is locally integrable if $\phi \in \hat{I}^{(k)}$, where $I = (f_1, \ldots, f_m)$. This can be shown by using the Chern-Levine-Nirenberg inequalities.

In a way the proof follows the setup discussed in Section 3, but the residue current appears only implicitly as we consider a sequence of forms that converge to $R\phi$. The reason that it is possible to avoid Hironaka's theorem is that we approach $R\phi$ and not R by it self. Given the hypothesis of the Briançon-Skoda theorem, $R\phi$ is the zero current, which explains why it is easier to show that its regularization converges.

8.2. **Paper 2 – On the Briançon-Skoda theorem on a singular variety.** In this paper, we give an analytic proof of the Briançon-Skoda-Huneke theorem introduced in Section 2. As explained in Section 3, the membership problem on a singular variety is reduced to showing that $\phi R^{a^r} \wedge R^Z = 0$, given suitable size conditions on ϕ. To state the actual theorem, we need to introduce some notions.

Let $Z \in \mathbb{C}^n$ be a germ of an analytic space of pure codimension p. We choose (9) to be a free resolution of $\mathcal{O}_{\mathbb{C}^n,0}/I(Z)$. Let Z^k be the set where the corresponding mapping f_{p+k} of (8) does not have optimal rank. One can show that the sets Z^k are independent of the embedding of Z. We also set $Z^0 = Z_{sing}$. These sets form a monotonic sequence $\dots Z^{k+1} \subset Z^k \subset \dots \subset Z^1 \subset Z^0$.

Theorem 8.1 (Theorem 1.1 in Paper 2)**.**
(i) There is a natural number μ, only depending on Z, such that for any ideal $\mathfrak{a} = (a_1, \dots, a_m)$ in $\mathcal{O}_{Z,0}$ and $\phi \in \mathcal{O}_{Z,0}$,

$$|\phi| \leq C|\mathfrak{a}|^{\mu+r-1} \tag{39}$$

implies that $\phi \in \mathfrak{a}^r$.
(ii) If for a given ideal $\mathfrak{a} = (a_1, \dots, a_m)$

$$\operatorname{codim}(Z^k \cap Z(\mathfrak{a})) \geq m + 1 + k, \quad k \geq 0, \tag{40}$$

then for any $\phi \in \mathcal{O}_{Z,0}$,

$$|\phi| \leq C|\mathfrak{a}|^{m+r-1} \tag{41}$$

implies that $\phi \in \mathfrak{a}^r$.

Part (ii) of the theorem means that we have sharper results when the intersections $Z^k \cap Z(\mathfrak{a})$ are small and the number of generators m is not too big. Assume that $k = k_0$ is the highest integer for which $Z^k \cap Z(\mathfrak{a}) \neq \emptyset$. Note that if $Z_{sing} \cap Z(\mathfrak{a}) = \emptyset$, we can see immediately that (41) implies $\phi \in \mathfrak{a}^r$, because then the germ of Z is either smooth, or $\mathfrak{a} = (1)$. We get from (40) that $m \leq \dim Z - k_0 - 1$, so part (ii) says nothing for ideals with more generators than this.

In order to analyze the current $\phi R^{a^r} \wedge R^Z$, we express R^Z as a principal value integral on the variety Z, cf. Proposition 3.1 in Paper 2. After a suitable resolution of singularities $X \to Z$, the current R^Z is locally the principal value of a monomial in the coordinates of X. By integration by parts, the order of the monomial can be reduced to 1 in each coordinate, at the expense of derivatives falling on

$$\phi R_\epsilon^{a^r} = \phi \bar\partial \chi_\epsilon^a \wedge u^{a^r}.$$

By explicitly calculating the derivatives of $\phi R^{a^r}_\epsilon$, one can show that the result is bounded for some sufficiently large constant μ. Dominated convergence can then be used to see that $\phi R^{a^r} \wedge R^Z = 0$.

8.3. **Paper 3 – A Briançon-Skoda type result for a non-reduced analytic space.** We consider here an analytic space Z which is embedded into $\mathbb{C}^n$ so that $\mathcal{O}_{Z,0} = \mathcal{O}_{\mathbb{C}^n,0}/J$ for some ideal $J \subset \mathbb{C}^n$. The difference from Paper 2 is that the ring $\mathcal{O}_{Z,0}$ may be non-reduced. That is, there may exist an element $0 \neq \phi \in \mathcal{O}_{Z,0}$ such that a power of ϕ vanishes.
It is now not possible that an implication like $|\phi| \lesssim |\mathfrak{a}|^N \implies \phi \in \mathfrak{a}$ holds for all ideals $\mathfrak{a} \in \mathcal{O}_{Z,0}$. In fact, if $\mathfrak{a}$ is an ideal that does not contain the nilradical $\sqrt{0} \subset \mathcal{O}_{Z,0}$ and $\phi \in \sqrt{0} \setminus \mathfrak{a}$, then ϕ is identically zero on Z and, a fortiori, $|\phi| \lesssim |\mathfrak{a}|^N$. An example of an ideal that does not contain the nilradical is the ideal (w^2) in $\mathcal{O}_{\mathbb{C}^n,0}/(w^3)$, where the nilradical is (w).

A size condition on an element $\phi \in \mathcal{O}_{Z,0}$ merely gives information about ϕ as a function evaluated on Z. In the reduced setting this says everything about ϕ, but to even know if ϕ is zero or not in $\mathcal{O}_{Z,0}$, we need to know all of the functions $L_1(\phi), \dots, L_M(\phi)$ for some defining set $L_1, \dots, L_M$ for J.

The method of this Paper borrows from Paper 2, but the extra ingredient are the Notherian operators and their connections to the residue currents that we use. For techincal reasons, we need the assumption that $\mathcal{O}_{Z,0}$ is Cohen-Macaulay. The main result is the following:

Theorem 8.2 (Theorem 1.2 in Paper 3)**.** *Let Z be a germ of an analytic space at $0 \in Z_{sp}$ such that $\mathcal{O}_{Z,0}$ is Cohen-Macaulay. Then there exists an integer N and Noetherian operators $L_1, \dots, L_M$ with respect to J such that for all ideals $\mathfrak{a} \subset \mathcal{O}_{Z,0}$ and all $r \geq 1$,*

$$|L_j\phi| \leq C|\mathfrak{a}|^{N+r-1}, \quad 1 \leq j \leq M, \tag{42}$$

implies that $\phi \in \mathfrak{a}^r$.

It follows from the case $\mathfrak{a} = 0$ that $L_1, \dots, L_M$ is a defining set for J. In the final section of Paper 3, an improvment is given for the case when the underlying space of Z is smooth. Then the constant N above is the value needed in the smooth reduced case, that is $\min(\dim Z, m)$, plus an extra term which is the maximal distribution order of any Coleff-Herrera current in $\mathcal{CH}_Z(J)$. This extra term is thus a measurement of how much the non-reducedness of Z contributes to the Briançon-Skoda number.

8.4. **Paper 4 – A residue calculus approach to the uniform Artin-Rees lemma.** The main result of the paper is the following:

Theorem 8.3. *Assume that X is a germ of an analytic variety at a point x, that M is a finitely generated module over the local ring $\mathcal{O}_{X,x}$,* *and that $N \subset M$ is a submodule. Then there exists a number μ such that for any ideal I of $\mathcal{O}_{X,x}$, the inclusion*

$$I^{\mu+r} M \cap N \subset I^r N$$

holds for all integers $r \geq 0$.

By a very simple reduction, one can assume that the variety is smooth and that M is a free module. The main contribution of this paper is to introduce the diamond product for complexes of vector bundles. This product can be used to study membership problems in products of ideals, or more generally, in tensor products of modules.

As we saw in (21), the residue current associated to a diamond product $E^1 \diamondsuit \dots \diamondsuit E^r$ can be expressed in terms of residue currents corresponding to the factors E^i. We apply this construction to the product $E^p \diamondsuit E^N$, where E^N is the complex obtained from a free resolution of M/N and E^p is the generalized Koszul complex from Section 5. In fact, E^p is the diamond product of r isomorphic copies of the Koszul complex with respect to I. The technique of analyzing the residue and showing that it is annihilated by $I^{\mu+r} M \cap N$ is similar to Paper 3.

8.5. **Paper 5 – The Briançon-Skoda number of analytic irreducible planar curves.** The main result of Paper 2 gives the existence of a constant N such that $|\phi| \lesssim |\mathfrak{a}|^{N+r-1}$ implies that $\phi \in \mathfrak{a}^r$, where $\mathfrak{a}$ is an ideal on a variety Z. The Briançon-Skoda number $\operatorname{bs}(Z)$ of Z is the smallest such integer N. It is an open problem to describe the Briançon-Skoda number in terms of invariants of Z, or to give an effective algorithm to compute this number.

In Paper 5, the case when $Z = C$ is an irreducible plane curve is treated. We obtain the Briançon-Skoda number in this case and show that $\operatorname{bs}(C) = 1 + \lceil \mu/m \rceil$, where μ is the Milnor number of a function that defines C, m is the multiplicity of C and $\lceil \cdot \rceil$ denotes the operation of rounding upwards to the nearest integer.

As an example consider the curve defined by $z^p = w^q$, where $p > q$ are relatively prime. The Milnor number is then $\mu = (p-1)(q-1)$ and the multiplicity is q. To see that $1 + \lceil \mu/m \rceil$ is a lower bound for the Briançon-Skoda number, we let $\mathfrak{a}$ be the ideal (z). Then $\phi = w^{q-1}$ does not belong to the ideal, but $|w^{q-1}| = |z|^{\frac{p(q-1)}{q}} = |z|^{1+(p-1)(q-1)/q-1/q}$ holds on the curve, so the Briançon-Skoda number is strictly greater than $1 + \lfloor (p-1)(q-1)/q - 1/q \rfloor$, where $\lfloor \cdot \rfloor$ denotes rounding downwards to the nearest integer. By using that $1 + (p-1)(q-1)/q$ belongs to the lattice $\frac{1}{q}\mathbb{Z}$, it follows that indeed $\operatorname{bs}(C) \geq 1 + \lceil \mu/m \rceil$.

This paper also relies on residue theory, but the set up of the proof is slightly different than in the previous papers. It is readily seen from the existence of a normalization of the curve that membership only has to be shown with respect to a principal ideal, i.e., an ideal generated by one function, say f. Given the size condition $|\phi| \lesssim |f|^N$, it follows that ϕ/f is a weakly holomorphic function, that is, holomophic on the regular part of the curve and bounded near the singular point. Now ϕ belongs to (f) if and only if ϕ/f is strongly holomorphic, that is, extends to a holomorphic function in a neighbourhood in $\mathbb{C}^2$. Hence, the ideal membership problem is reformulated as a question about deciding when a weakly holomorphic function is strongly holomorphic. There exists a very useful criterion for strong holomorphicity by Tsikh, [36]. We reformulate this criterion slightly via Leray's residue formula to obtain the criterion below. Let $P \in \mathcal{O}_{\mathbb{C}^2,0}$ be an irreducible function such that $C = Z(P)$ and define ω be the pull-back of dz/P'_w under the inclusion map $C_{reg} \hookrightarrow \mathbb{C}^2$. We choose coordinates so that $P(0, w)$ is not identically equal to zero. The form ω is a meromorphic form on C, but it can also be viewed as a current acting on test forms on C; the latter are smooth forms on C_{reg} that are pull-backs of test forms in the ambient space.

Theorem 8.4 (Theorem 2.2 in Paper 5)**.** *If ψ is any meromorphic function on C, then ψ is strongly holomorphic if and only if $\psi\omega$ is $\overline{\partial}$-closed.*

To use this criterion, we calculate the pull-back of the form ω along the normalization π. The singularity of this form is computed in terms of the Puiseux characteristics of the curve. The result is then reinterpreted using Milnor's formula relating these numbers and the Milnor number. It turns out that $\pi^*\omega = u(t)t^{-\mu}dt$, where $u(t)$ is a holomorphic unit.

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