Thesis for the degree of Doctor of Philosophy

Maximally Supersymmetric Models in Four and Six Dimensions

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Abstract

We consider two examples of maximally supersymmetric models; the $\mathcal{N} = 4$ Yang-Mills theory in four dimensions and the $(2, 0)$ theory in six dimensions. The first part of the thesis serves as an introduction to the topics covered in the appended research papers, and begins with a self-contained discussion of principal fibre bundles and symplectic geometry. These two topics in differential geometry find applications throughout the thesis in terms of gauge theory and canonical quantization.

Subsequently, we consider the $\mathcal{N} = 4$ supersymmetry algebra and the massless Yang-Mills multiplet representation. In particular, we discuss the vacuum structure of the $\mathcal{N} = 4$ theory in a space-time with the geometry of a torus, and the computation of its weak coupling spectrum. We investigate the case with a gauge group of adjoint form and discuss the implications of non-trivial bundle topology for the moduli space of flat connections. Furthermore, we consider gauging the R-symmetry of the theory by introducing a corresponding background connection. We identify a special class of terms in the partition function, which are BPS and can (in principle) be computed at weak coupling, and discuss the action of S-duality on this sector.

Finally, we consider the $(2, 0)$ theory in six dimensions, provide a general introduction and derive the free tensor multiplet representation of the supersymmetry algebra. We then proceed to study $(2, 0)$ theory defined on a manifold which can be described as a circle fibred over some five-dimensional manifold. We discuss the dimensional reduction of the free $(2, 0)$ tensor multiplet on the circle and derive the (maximally supersymmetric) abelian Yang-Mills theory obtained in five dimensions for the most general metric of such a fibration. We discuss the properties of the Yang-Mills theory corresponding to the superconformal symmetry of the $(2, 0)$ theory and propose a generalization to the interacting (non-abelian) case, where a field theory description in six dimensions is problematic. The case when the circle fibration description becomes singular is also considered and we give a concrete example of such a manifold and discuss the degrees of freedom located at the singularity.

Keywords

Maximal supersymmetry, Yang-Mills theory, Topologically non-trivial connections, Flat connections, $(2, 0)$ theory, Circle fibrations, Taub-NUT space.

This thesis consists of an introductory text and the following five appended research papers, henceforth referred to as PAPER I-V:

I. J. Lindman Hörnlund and F. Ohlsson, *The weak coupling spectrum around isolated vacua in* $\mathcal{N} = 4$ *super Yang-Mills on* T^3 *with any gauge group*, JHEP **07** (2008) 077, `arXiv:0804.0503 [hep-th]`.

II. F. Ohlsson, *Finite energy shifts in* $\mathrm{SU}(n)$ *supersymmetric Yang-Mills theory on* $T^3 \times \mathbb{R}$ *at weak coupling*, Phys. Rev. **D81** (2010) 125018, `arXiv:0912.1556 [hep-th]`.

III. M. Henningson and F. Ohlsson, *BPS partition functions in* $N = 4$ *Yang-Mills theory on* T^4, JHEP **03** (2011) 145, `arXiv:1101.5331 [hep-th]`.

IV. H. Linander and F. Ohlsson, $(2, 0)$ *theory on circle fibrations*, JHEP **01** (2012) 159, `arXiv:1111.6045 [hep-th]`.

V. F. Ohlsson, $(2, 0)$ *theory on Taub-NUT: A note on WZW models on singular fibrations*, `arXiv:1205.0694 [hep-th]`.

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Contents

1
Introduction

The conventional way to introduce a thesis in theoretical high energy physics appears to consist in conscientiously listing a number of great physicists, active in the beginning of the previous century, whose contributions to our understanding of quantum mechanics and general relativity - the cornerstones of fundamental physics - were monumental. Even though this is of course the historically correct starting point (and relativity and quantum mechanics are certainly integral parts of the contents of this thesis) it may be a slightly misguiding one for the purpose of putting the work described below into context. Instead, it is perhaps more appropriate to begin this introduction with the advent of gauge theory in 1954 through the works of C. N. Yang and R. L. Mills [1]. It had previously been discovered that the two constituent particles of the atomic nuclei, the proton and the neutron, had very similar masses and interactions. The observation lead to the proposal that the two are different states associated to a single particle called the nucleon. More specifically, these states correspond to the eigenstates of the projection onto the third component of a conserved quantity known as the isospin[1] of the nucleon. The proton and neutron are related through a rotation in an internal space associated to the isospin quantum numbers. A more technical statement is that the proton and neutron furnish the fundamental two-dimensional representation of the isospin symmetry group SU(2). The theory describing the nucleons is invariant under such SU(2) rotations and the distinction between a proton and a neutron simply amounts to an arbitrary choice of a basis in the module of the fundamental representation.

The realization of Yang and Mills was that in order for the choice of basis to be void of physical significance, as suggested by the local nature of quantum field theory, it must be allowed to be made independently at every point in space-time. (It is of course still possible to consider global continuous symmetries where this is not the case, and the choice of basis at one point determines the distinction between (say)

[1]So named for its resemblance with the ordinary spin of quantum mechanics.

a proton and a neutron throughout space-time. However, such symmetries appear to be absent in nature.) The consequence is a sharpening of the requirement on the theory describing the nucleons and their interactions; it must be invariant under local SU(2) rotations called gauge transformations[2]. In order to accommodate the space-time dependence of the SU(2) rotations one is forced to introduce a bosonic space-time vector field, commonly denoted A_μ (where $\mu = 0, 1, 2, 3$ label the space-time coordinates), transforming in the adjoint representation of SU(2). This so called gauge field is then used to define covariant derivatives with well-defined properties under the local SU(2) symmetry transformations. Consequently, the gauge field A_μ is associated with the force describing the interactions in the theory.

It is straightforward to generalize Yang-Mills theory by replacing SU(2) by an arbitrary Lie group G encoding a local symmetry. The generators in field theory of the gauge symmetries form the Lie algebra $\mathfrak{g}$, encoding the local (or infinitesimal) structure of the symmetry group G. The matter contents of these theories are described by fermionic spinor fields falling into irreducible (but not necessarily fundamental) representations of G while the particles mediating the corresponding forces are described by the gauge field in the adjoint representation of G. In fact, it could be argued that the most successful theory in the history of physics (and possibly in the history of science) is the standard model describing all observed particles of nature[3] and unifying the known forces (excluding gravity) of particle physics: the electromagnetic, the weak nuclear and the strong nuclear forces. All matter particles of the standard model are organized into multiplets of the $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge group and the respective vector fields describe the bosonic particles mediating the corresponding forces. Yang-Mills theories appear frequently in theoretical physics and the concept of gauge symmetry is an immensely powerful and versatile one.

In fact, as we will return to discuss in the introductory part of this thesis, gauge symmetry is not a symmetry of a physical theory in the strict sense of the word, but a redundancy in its description. Investigations of all possible proper symmetries of the S-matrix of a realistic (local and relativistic) quantum field theory in four dimensions resulted in 1967 in a theorem by S. Coleman and J. Mandula [2]. The theorem states that that the most general Lie algebra of generators of continuous such symmetries is given by the Poincaré algebra (encoding symmetry transformations between different inertial reference frames in space-time) together with the generators of a compact internal Lie group[4]. However, in 1975 it was shown by R. Haag, J. Lopuszanski and M. Sohnius [3] that it was possible to introduce a

[2]In fact, as is pointed out in [1], this phenomenon was already established in electromagnetism where the wave function can be transformed by a local phase factor corresponding to a U(1) gauge transformation.

[3]Hopefully, this statement will be rendered incorrect by future discoveries at the LHC.

[4]Here, the internal Lie group encodes global symmetries. In addition, local (gauge) symmetries are compatible with the Coleman-Mandula theorem.

new kind of symmetry, with fermionic generators, which relates particles of different spin. The symmetry generators are then no longer required to form a Lie algebra, thus circumventing the theorem by Coleman and Mandula. The novel symmetry was named supersymmetry and it provides a connection between the seemingly unrelated fermionic matter fields and bosonic force carrier fields of the theory. Just as is the case with other symmetries the particles related by supersymmetry, called superpartners, fall into multiplets. (We will provide a more technical discussion of supersymmetry in later chapters.) Since its inception supersymmetry has been intensely investigated and is (in spite of the fact that it has yet to be experimentally established as a symmetry of nature) expected to solve a number of outstanding problems in theoretical particle physics.

Theories of particle physics in four dimensions are usually restricted to include particles of spin less than or equal to one in order to avoid problems regarding the quantization of gravity, which is associated to particles of spin two. (In fact, gravity is normally several orders of magnitude weaker than the gauge interactions of the theory so to a very high accuracy this is not really a restriction.) As a consequence, the amount of supersymmetry allowed is limited accordingly and there is a notion of a maximally supersymmetric theory. Such theories are highly constrained and have turned out to possess remarkable properties. Furthermore, they frequently appear in the context of string theory and M-theory - seemingly the only viable candidates for a quantum theory of everything - as effective theories in the limit where gravity decouples.

As the title suggests, the main theme of this thesis is the study of theories with maximal supersymmetry. The content of such theories depend on the dimension of space-time in which the corresponding particles propagate and the work presented in this thesis is concerned with two specific models; the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions and the $(2, 0)$ theory in six dimensions. The names, which at this point appear rather cryptical, refer to the algebra of generators of the supersymmetry transformations and will be further elucidated during the course of this introductory part of the thesis. The supersymmetric Yang-Mills theory is quite simply the supersymmetric extension of ordinary Yang-Mills theory obtained by adding the superpartners of the vector gauge field. The $(2, 0)$ theory on the other hand, is perhaps best described as the theory living on certain extended five-dimensional objects, called M5-branes, appearing in M-theory; a perspective which we will not consider further here but return to discuss in more detail in due time.

It turns out that the two models are closely related through a process called dimensional reduction: Consider two out of the six dimensions of the space-time in which $(2, 0)$ theory is defined to be curled up, or compactified, into two tiny circles forming a torus. On energy scales which are small compared to the inverse radii of the circles an observer in the four uncompactified dimensions cannot detect the two extra directions. From the point of view of such an observer the $(2, 0)$ theory therefore appears as an effective four-dimensional theory: the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This connection may seem somewhat contrived but can in fact provide significant insight into both theories involved. For example, the $\mathcal{N} = 4$ theory possesses a remarkable symmetry called S-duality which is geometrically manifested in the compactification as symmetries of the torus.

At present, the title of this thesis would appear to be very appropriate. However, we would like to point out that it can in fact be slightly misleading. The models we consider are indeed maximally supersymmetric but the investigations conducted in the introductory part, as well as in the appended papers, are rarely explicitly incorporating the feature that the amount of supersymmetry is maximal. Furthermore, not even the dimensions appearing in the title are strictly correct: The $(2,0)$ theory in six dimensions in general has no field theory description. It is therefore convenient to consider the dimensional reduction to five dimensions which in analogy with the situation described above yields a (maximally) supersymmetric Yang-Mills theory. Finally, the $(2,0)$ and $\mathcal{N} = 4$ theories share another property called conformal symmetry, that can be thought of as a generalization of scaling invariance of the models, which we will only consider briefly in six dimensions[5]. Nevertheless, the array of interesting properties displayed by the $(2,0)$ and $\mathcal{N} = 4$ theories offers sound motivation for the furthering our understanding of these models and their connections. Contributing to this end is the purpose of the study carried out in this thesis.

Outline

When starting work on the introductory part of this thesis I decided to keep in mind a piece of advice offered by my thesis advisor: To write a text I would have appreciated reading myself when I began my graduate studies. Accordingly, I have taken the liberty to begin by giving an introduction to selected topics in differential geometry in **Chapter 2**. The aim of this section is to provide a rigorous mathematical introduction to certain constructions appearing in gauge theory and canonical quantization. To reflect the more introductory nature and the ambition to make the presentation self-contained, the style of chapter 2 differs from the remaining chapters[6], which are concerned with the original work presented in the appended papers. The selection of included topics is far from complete and is furthermore biased by a personal inclination towards a geometrical description of these matters, which historically was obtained significantly later than the inception in physics of the concepts themselves. A thorough command of the formal aspects of the physical concepts is however often helpful for understanding them. Furthermore, modern research in theoretical particle physics is often concerned with questions bordering

[5]The title of this thesis is in fact inspired by E. Witten's lecture [4] on the supersymmetric and conformally invariant theories considered here.

[6]This fact should not be mistaken for any illusions on the part of the author of being able to offer novel insights into the topics treated. Rather, it is a reflection of a fastidious nature and a fascination with the formalities of mathematical physics.

on pure mathematics which also warrants the establishment of a firm mathematical foundation.

In **Chapter 3** we initiate the study of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We provide a more detailed introduction to supersymmetry and the Yang-Mills theory before specializing to the situation consider in Paper I-II: The $\mathcal{N} = 4$ theory defined on a space-time of the form $M = T^3 \times \mathbb{R}$, where $\mathbb{R}$ denotes time. We consider the vacuum structure of this theory for a gauge group on adjoint form, which implies that topological aspects must be considered, and study the moduli space of flat connections or gauge fields. In particular, we identify certain vacua which admit a perturbative expansion of the theory at weak coupling and study the energy spectrum of the theories located there.

Along similar lines we perform in **Chapter 4** an investigation of $\mathcal{N} = 4$ Yang-Mills theory on a space-time T^4 for special unitary gauge groups. By coupling the theory to a background gauge field for the so called R-symmetry, corresponding to rotations among the supersymmetries, we can identify certain terms in the partition function which are independent of the coupling strength. They can then be computed using a weak coupling approximation and the S-duality mentioned above, which is the topic of Paper III.

Finally, we then consider in **Chapter 5** the $(2, 0)$ theory on a six-dimensional manifold which can be described as a circle fibration over some five-dimensional base manifold, a geometry which is the generalization of the compactification on circle(s) mentioned above. We summarize the results of Paper IV where the dimensional reduction to five dimensions of the simplest $(2, 0)$ theory is derived and the generalization to all $(2, 0)$ theories is discussed. We also consider what happens when the circle fibration becomes singular. A particular example of such a situation is the concern of Paper V.

2

Mathematical preliminaries

The work presented in this thesis, both in the introductory part and in the appended papers, utilizes results from a variety of areas in mathematics, one of the most important of which is differential geometry. During the course of the past century a geometric perspective on several areas of modern physics evolved, which provides a description that is often both compact and powerful. Furthermore, by exhibiting the underlying mathematical structure of physical systems it is possible to gain further insight into their properties.

In this chapter we provide a brief, but self-contained, overview of a selection of concepts in differential geometry and their application to physics. The purpose is two-fold: First, we aim to give working definitions of a number of fundamental concepts that appear throughout this thesis and place them into their geometrical context. Second, we attempt a more detailed introduction to selected topics that appear in the work presented in the following chapters. More specifically we will discuss the theory of fibre bundles, providing the mathematical foundation for gauge theory, and symplectic geometry which finds application in the description of classical dynamical systems and their quantization. We will return to both these topics in later chapters and discuss their application in the works described in Paper I-V. The reader who is well versed in differential geometry and gauge theory may want to omit this chapter and proceed to the next one.

2.1 Differential geometry

This section aims to provide a very brief overview of some basic concepts of differential geometry that are needed for the discussion in the present chapter. For a more detailed account of the material covered here we refer the reader to e.g. [5–7].

2.1.1 Manifolds

The purpose of the discipline of differential geometry is to extend the notion of a curve to curved surfaces of arbitrary dimension and apply the theory of differential calculus to study their geometry. The desired generalization of the curve is provided by the concept of a **manifold**, which can be thought of as a space that locally looks like $\mathbb{R}^n$, where n is called the dimension of the manifold. However, this is not quite sufficient: In order to ensure that the machinery of differential calculus is accessible we need to endow the space with further structure.

Definition 1. *A* **differentiable structure** *or* C^∞**-structure** *on a topological space* X *is an* **atlas** $\{(U_i, \phi_i)\}$, *where* $U_i \subset X$ *are open subsets and* $\phi_i : U_i \to \mathbb{R}^n$ *are homeomorphisms, satisfying*

(i) $X = \bigcup_i U_i$

(ii) $\forall\, U_i, U_j : U_i \cap U_j \neq \emptyset\,,\ \psi_{ij} = \phi_i \circ \phi_j^{-1} \in C^\infty$

The homeomorphisms ϕ_i are called the coordinate maps and the U_i are called the coordinate neighbourhood; together they constitute a **chart** (U_i, ϕ_i). The homeomorphisms ϕ_i is given by a **coordinate presentation** $\phi_i(p) = \{x^\mu\}$, where $\mu = 1, \ldots, n$ and $x^\mu(p)$ are the coordinate functions. In terms of differentiable structures we can now give the rigorous definition of a manifold.

Definition 2. *A* **differentiable manifold** *or* **smooth manifold** M *is a topological space equipped with a* C^∞*-structure.*

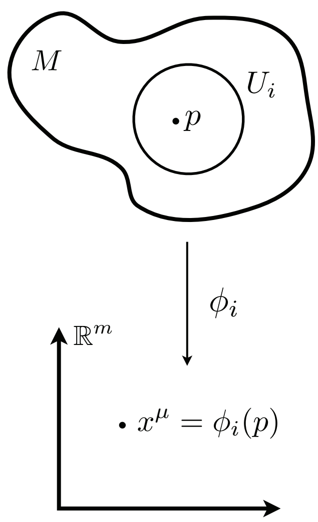


Figure 2.1: Illustration of the basic concept of a manifold.

By definition a manifold is thus homeomorphic to $\mathbb{R}^n$ locally, but globally this is only true if the **transition functions** ψ_{ij} are trivial. The smoothness allows us to apply the machinery of ordinary calculus on M in a coordinate independent way, i.e. independent on which coordinate presentation we choose on $U_i \cap U_j \neq \emptyset$, since the ψ_{ij} defining the change of coordinates are differentiable.

Having defined manifolds we can consider a map $f : M \to N$ from an m-dimensional manifold M to an n-dimensional manifold N:

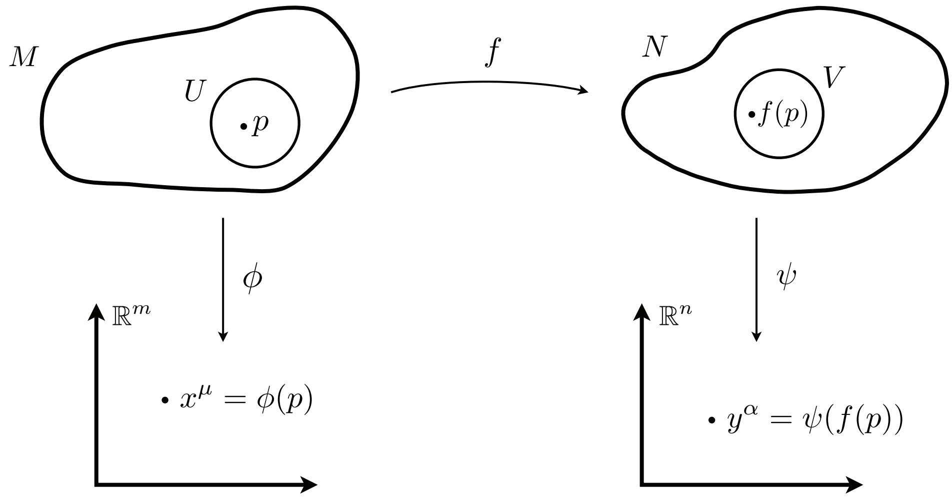


Figure 2.2: A map between two manifolds M and N.

With notation according to figure 2.2, the coordinate presentation of f is given by

$$\begin{array}{rcrcl} \psi \circ f \circ \phi^{-1} & : & \mathbb{R}^m & \to & \mathbb{R}^n \\ & & x^\mu & \mapsto & y^\alpha = \psi \circ f \circ \phi^{-1}(x^\mu)\,. \end{array} \tag{2.1}$$

If the coordinates $y^\alpha(x^\mu)$ are smooth with respect to each x^μ, the map f is smooth. This is of course a coordinate independent property since by assumption the transition functions are smooth. A bijective smooth map with a smooth inverse is called a **diffeomorphism** and such maps define an equivalence relation among manifolds. In fact, we consider all manifolds belonging to the same equivalence class under diffeomorphisms to be identified.

A special case of maps between manifolds, that will be particularly useful for us (and is indeed generally exceptionally useful) is that of smooth maps from an arbitrary manifold to the real line.

Definition 3. *A* **function** *on a manifold M is a smooth map $f : M \to \mathbb{R}$. The set of all functions on M is denoted $\mathcal{F}(M)$.*

2.1.2 Tangent vectors and differential forms

We are now in a position where we can define other geometric objects, in addition to maps, on manifolds. The first such object is the generalization of the familiar notion of a vector in Euclidean space. The usual definition in $\mathbb{R}^n$ must be refined in order to accommodate the more general structure of a manifold.

Definition 4. *Let M be a manifold and let $c : (a,b) \to M$ with $a < 0 < b$ be a curve in M. A* **tangent vector** *X_p at $p = c(0) \in M$ is then defined as a differential operator satisfying*

$$X_p(f) = \left.\frac{df\,(c(t))}{dt}\right|_{t=0} \quad , \quad \forall\, f \in \mathcal{F}(M)\,. \tag{2.2}$$

In local coordinates x^μ the defining relations become

$$X_p(f) = \frac{\partial f}{\partial x^\mu}\left.\frac{dx^\mu\,(c(t))}{dt}\right|_{t=0} \tag{2.3}$$

so that the local expression for the vector X_p is

$$X_p = X_p^\mu \frac{\partial}{\partial x^\mu} \quad , \quad X_p^\mu = \left.\frac{dx^\mu\,(c(t))}{dt}\right|_{t=0} . \tag{2.4}$$

From the definition above it is clear that there may be several curves that give the same differential operator at $p \in M$. We can therefore identify the tangent vector X_p as the equivalence class of such curves. The set of all tangent vectors at $p \in M$ form the **tangent space** T_pM which is a vector space that may thus be identified with the set of all equivalence classes of curves through $p \in M$. A basis of T_pM is provided by $\{\partial/\partial x^\mu\}$. Because T_pM has the structure of a vector space we can define its dual space, called the **cotangent space** T_p^*M at $p \in M$, whose elements are linear maps $\omega_p : T_pM \to \mathbb{R}$. The basis of T_p^*M dual to $\{\partial/\partial x^\mu\}$ is given by the coordinate differentials $\{dx^\mu\}$.

It is possible to extend the definition of vectors and cotangent vectors to the entire manifold M in the following way:

Definition 5. *A* **vector field** *X on a manifold M is a smooth assignment of a vector $X_p \in T_pM\,, \forall\, p \in M$. The set of vector fields is denoted $\chi(M)$.*

Definition 6. *A* **differential 1-form** *ω is a smooth assignment of a cotangent vector $\omega_p \in T_p^*M\,, \forall\, p \in M$, i.e. a map*

$$\begin{array}{rccc} \omega & : & \chi(M) & \to & \mathcal{F}(M) \\ & & X & \mapsto & \omega(X)\,. \end{array} \tag{2.5}$$

The space of 1-forms on M is denoted $\Omega^1(M)$.

The definition of 1-forms generalizes in a straightforward way to **differential r-forms**, which are totally antisymmetric maps

$$\omega : \underbrace{\chi(M) \times \ldots \times \chi(M)}_{r\text{ times}} \to \mathcal{F}(M)\,. \tag{2.6}$$

A basis for the space $\Omega^r(M)$ of r-forms is given by $\{dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}\}$, the elements of which are the totally antisymmetric tensor products of $dx^{\mu_1}, \ldots, dx^{\mu_r}$. In local coordinates we thus have the expression

$$\omega = \frac{1}{r!}\omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r} \tag{2.7}$$

for an r-form. We furthermore identify the space of zero-forms on M with the space of functions $\Omega^0(M) = \mathcal{F}(M)$. Evaluating e.g. a 2-form $\sigma \in \Omega^2(M)$ on two vectors $X = X^\mu \partial/\partial x^\mu$ and $Y = Y^\mu \partial/\partial x^\mu$ gives

$$\sigma(X, Y) = \frac{1}{2}\sigma_{\mu\nu}\left(X^\mu Y^\nu - Y^\mu X^\nu\right) = \sigma_{\mu\nu} X^\mu Y^\nu\,. \tag{2.8}$$

We now consider two of the most basic operations acting on the space $\Omega^r(M)$ of r-forms. The first is the **exterior derivative** $d_r : \Omega^r(M) \to \Omega^{r+1}(M)$ which generalizes the ordinary derivative[1]. In local coordinates this operation acts on $\omega \in \Omega^r(M)$ as

$$d\omega = \frac{1}{r!}\partial_\mu \omega_{\nu_1 \ldots \nu_r} dx^\mu \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_r}\,. \tag{2.9}$$

Forms in the image of d are called **exact** while forms in the kernel of d are called **closed**. The exterior derivative is nilpotent $d^2 = 0$, which implies that every exact form is closed. The converse, however, is only true locally[2]. The topological obstruction to globally extending this result is encoded in the **de Rham cohomology** group defined as

$$H^r(M, \mathbb{R}) = \frac{\ker d_r}{\operatorname{im} d_{r-1}}\,, \tag{2.10}$$

where two closed forms $a, b \in \ker d_r$ are called cohomologous if $a - b \in \operatorname{im} d_{r-1}$.

We can also define the contraction of a r-form with a vector, known as the **interior product** $\iota_X : \Omega^r(M) \to \Omega^{r-1}(M)$ with the vector $X \in \chi(M)$, which is defined by

$$\iota_X \omega(X_1, \ldots, X_{r-1}) \equiv \omega(X, X_1, \ldots, X_{r-1}) \quad , \quad X, X_i \in \chi(M) \tag{2.11}$$

or in local coordinates

$$\iota_X \omega = \frac{1}{(r-1)!} X^\mu \omega_{\mu\nu_2 \ldots \nu_r} dx^{\nu_2} \wedge \ldots \wedge dx^{\nu_r}\,. \tag{2.12}$$

[1]We will often suppress the explicit subscript on d_r for convenience.

[2]In a contractible coordinate neighbourhood any closed form is exact by Poincaré's lemma.

2.1.3 Induced maps and submanifolds

A smooth map between manifolds $f : M \to N$ induces maps of both tangent vectors and r-forms. The induced map of vectors is called the **push-forward**, or differential map, $f_* : T_pM \to T_{f(p)}N$ and is defined by

$$f_* X_p(g) \equiv X_p(g \circ f) \quad , \quad \forall\, g \in \mathcal{F}(N)\,. \tag{2.13}$$

Note that the composition $g \circ f \in \mathcal{F}(M)$ so that both sides of the defining relation make sense. Similarly, we may define the induced map of a r-form using the push-forward of vectors. This map is called the **pullback** $f^* : \Omega^r_{f(p)}(N) \to \Omega^r_p(M)$ and is defined by

$$f^*\omega(X_1, \ldots, X_r) \equiv \omega(f_*X_1, \ldots, f_*X_r) \quad , \quad X_i \in T_pM\,. \tag{2.14}$$

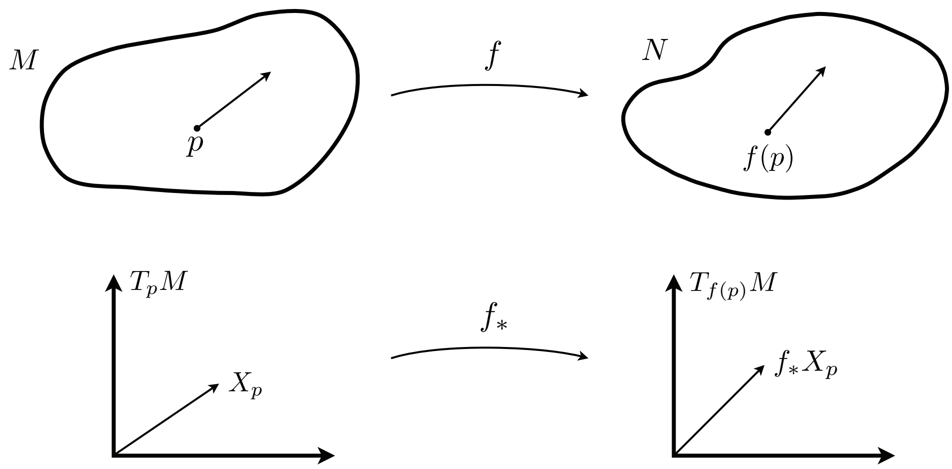


Figure 2.3: The push-forward $f_* : T_pM \to T_{f(p)}N$ induced by the map $f : M \to N$.

Using again the notation of figure 2.2 we can obtain the local expression for the induced maps. The components of a vector V on M and its push-forward $W = f_*V$ are related by

$$W^\alpha = V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \tag{2.15}$$

The Jacobian $(\partial x^\mu / \partial y^\alpha)$ of the map f also relates the components of a r-form ω on N to those of its pullback $\zeta = f^*\omega$ through

$$\zeta_{\mu_1 \ldots \mu_r} = \omega_{\alpha_1 \ldots \alpha_r} \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}}\,. \tag{2.16}$$

We recognize the familiar tensorial transformation properties under diffeomorphisms for both vectors and r-forms.

A concept that will be use repeatedly in the present chapter is that of **submanifolds**, i.e. subsets of manifolds that are themselves manifolds. To make this notion precise we use the induced maps to introduce the following two concepts:

Definition 7. *Let M, N be two manifolds with $\dim(M) \leq \dim(N)$ and let $f : M \to N$ be a smooth map. If f_* is injective, the map f is called an* **immersion**. *If f is an injective immersion it is called an* **embedding**.

The definition of a submanifold is then straightforward.

Definition 8. *The image $f(M)$ of an embedding $f : M \to N$ is a* **submanifold** *of N.*

An important property of the submanifold $f(M)$ is that (at least locally) it may be expressed as the zero locus of $(\dim(N) - \dim(M))$ functions $\{\psi_i\}$.

The conceptual difference between an immersion and an embedding, and the reason why the latter has the structure of a manifold, can be illustrated by the canonical example of a map $f : S^1 \to \mathbb{R}^2$ in figures 2.4 and 2.5.

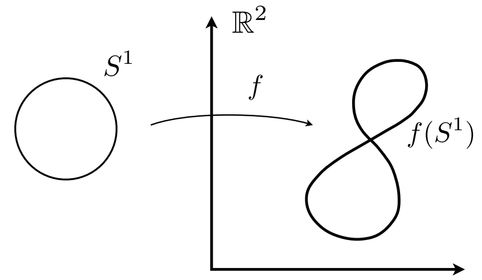


Figure 2.4: Immersion of S^1 into $\mathbb{R}^2$.

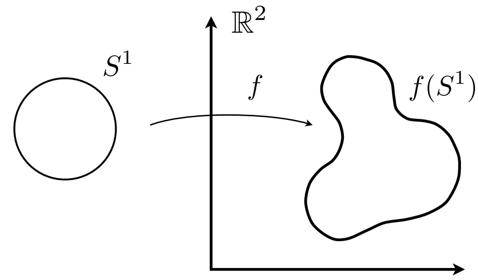


Figure 2.5: Embedding of S^1 into $\mathbb{R}^2$.

It is clear from this simple example that while the immersion maps T_pS^1 to a vector subspace of $T_{f(p)}\mathbb{R}^2$, the fact that f itself is not injective means that $f(S^1)$ is not a manifold. The embedding, on the other hand, has an image $f(S^1)$ that is diffeomorphic to S^1 itself and thus a manifold.

2.1.4 The Lie derivative

Having defined vector fields and differential forms on a manifold M we would now like to extend the definition of the directional derivative of a function in $\mathbb{R}^n$ in order to measure the rate of change of these objects on M. The first thing we need is thus a way to specify the direction on M in which we would like to investigate the change of an object.

Definition 9. *Given a vector field $X \in \chi(M)$ the* **flow generated by** X *is the map $\sigma_t(p) : \mathbb{R} \times M \to M$ solving the initial value problem*

$$\frac{dx^\mu\left(\sigma_t(p)\right)}{dt} = X^\mu(\sigma_t(p)) \quad , \quad \sigma_0(p) = p\,. \tag{2.17}$$

For each $p \in M$ the flow is the **integral curve** σ_t of X which satisfies $\sigma_0 = p$ and whose tangent vector at σ_t is given by $X|_{\sigma_t}$. Furthermore, for fixed t, the flow constitutes a smooth map $\sigma : M \to M$, i.e. a diffeomorphism.

We can now consider the change of a vector Y and a 1-form ω along the flow $\sigma_t(p)$ generated by X. However, evaluating Y and ω at two infinitesimally separated points p and $\sigma_\epsilon(p)$ is not sufficient since this produces elements of the distinct vector spaces $T_pM(T^*_pM)$ and $T_{\sigma_\epsilon(p)}M(T^*_{\sigma_\epsilon(p)}M)$ respectively. Fortunately, for a given $p \in M$ the flow itself induces the maps needed to obtain well defined differences in T_pM and T^*_pM. The result can then be used to define the desired **Lie derivative** $\mathcal{L}_X$ along the flow generated by X

$$\mathcal{L}_X Y = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\sigma_{-\epsilon}(p)_* Y|_{\sigma_\epsilon(p)} - Y_p\right) \tag{2.18}$$

and

$$\mathcal{L}_X \omega = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\sigma_{\epsilon}(p)^* \omega|_{\sigma_\epsilon(p)} - \omega_p\right)\,. \tag{2.19}$$

The definition of the Lie derivative can be extended in a straightforward manner to tensors of arbitrary rank. We also note that $\mathcal{L}_X$ reproduces the familiar directional derivative for a function $f \in \mathcal{F}(M)$ and satisfies the useful relation

$$\mathcal{L}_X = d\iota_X + \iota_X d\,. \tag{2.20}$$

An important property of the Lie derivative is that it endows the set of vector fields $\chi(M)$ with the structure of a Lie algebra. We define the **Lie bracket** $[\cdot,\cdot] : \chi(M) \times \chi(M) \to \chi(M)$ by

$$[X, Y] = \mathcal{L}_X(Y)\,. \tag{2.21}$$

In local coordinates the bracket takes the form

$$[X, Y] = (X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu)\partial_\nu\,, \tag{2.22}$$

in which bilinearity and antisymmetry are manifest. Furthermore, the Lie bracket is easily shown to satisfy the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0\,, \tag{2.23}$$

constituting the last defining property of a Lie algebra.

2.1.5 Lie groups

A concept closely related to Lie algebras is that of Lie groups. In the context of this chapter we will be mostly interested in the geometric description of Lie groups and their connection to Lie algebras. For an introduction to Lie algebras, their representation theory and application in encoding symmetries in physics we refer to the rich literature available, see e.g. [8–10].

A Lie group is quite simply an object that is simultaneously a group and a manifold. The definition ensures that the two structures defining these two classes of objects are compatible.

Definition 10. *A* **Lie group** G *is a (smooth) manifold equipped with a* **differentiable group structure***, i.e. such that the group operations multiplication and inverse are differentiable.*

Using the group structure of G we can define two particular diffeomorphisms by the left and right action[3] of G on itself, also called the **left-translation** $L_g : G \to G$ and the **right-translation** $R_g : G \to G$ by an element $g \in G$

$$L_g(h) = gh \quad , \quad R_g(h) = hg\,. \tag{2.24}$$

The push-forward map $(L_g)_* : T_hG \to T_{gh}G$ between the tangent spaces of G can be used to define the notion of a **left-invariant vector field** $X \in \chi(G)$ satisfying

$$(L_g)_* X_h = X_{gh}\,. \tag{2.25}$$

From the definition of the push-forward it follows that

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] \tag{2.26}$$

which implies that the set of left-invariant vector fields closes to a Lie sub-algebra of $\chi(M)$ under the Lie bracket. The set of left-invariant vector fields are referred to as the Lie algebra $\mathfrak{g}$ of the Lie group G and is isomorphic to T_eG by (2.25), where $e \in G$ denotes the identity[4]. In particular, given a basis of left-invariant vector fields $\{T^a\}$, where $a = 1, \ldots, \dim(\mathfrak{g})$, called the generators of the Lie algebra, a basis of T_gG is given by $\{T^a_g\}$. Conversely, given a basis $\{U^a\}$ of T_eG, $\{(L_g)_*U^a\}$ provides a basis of T_gG and consequently also a basis of the vector space $\mathfrak{g}$. The closure of $\mathfrak{g}$ implies that

$$[T^a, T^b] = f^{ab}{}_c T^c\,, \tag{2.27}$$

where the $f^{ab}{}_c$ are the **structure constants** of $\mathfrak{g}$. We will always use antihermitian generators T^a so that the structure constants are real. Note that because of (2.25) they are indeed constants on G.

[3] We will always use the standard notation where juxtaposition denotes group multiplication.

[4] We will identify these two isomorphic spaces when convenient.

A left-invariant vector field X defines an integral curve $\sigma_t(e)$ through $e \in G$ which allows us to construct a map from the Lie algebra to G called the exponential map $\exp : T_eG \to G$ by

$$\exp(tX_e) = \sigma_t(e)\,, \tag{2.28}$$

The exponential map provides an important connection between the Lie group and its Lie algebra since it allows the local structure of G to be described in terms of $\mathfrak{g}$. The integral curve of X through an arbitrary $g \in G$ is recovered as $\sigma_t(g) = g\exp(tX_e)$, so that the flow of X (c.f. (2.17)) is given by $\sigma_t = R_{\exp(tX_e)}$.

On every Lie group manifold there exists a particular Lie algebra-valued one-form called the **Maurer-Cartan form** $\omega_{\mathrm{MC}} \in \Omega^1(G, \mathfrak{g})$ such that $\omega_{\mathrm{MC}}|_g : T^*_gG \to T^*_eG\,, \forall g \in G$. It is uniquely defined by left-translation of the identity map $\mathbb{1}_e : T_eG \to T_eG$ at the identity

$$\omega_{\mathrm{MC}}|_g = (L_{g^{-1}})^*\mathbb{1}_e\,. \tag{2.29}$$

From the definition it follows that ω_{MC} is left-invariant and acts on $X_g \in T_gG$ by $\omega_{\mathrm{MC}}(X_g) = (L_{g^{-1}})_*X_g = X_e$. The Maurer-Cartan form is commonly denoted $g^{-1}dg$ and can be thought of as containing the same local information about G as the Lie algebra $\mathfrak{g}$. We will find applications for ω_{MC} in the discussion of fibre bundles below.

2.2 Fibre bundles

Having established a number of important concepts in differential geometry we are now ready to proceed towards the first goal of this chapter: The formulation of gauge theory in terms of the geometric construction called a fibre bundle. In this section we will give an overview of the basic concepts of fibre bundles and some selected properties that will be useful throughout this thesis. There are several excellent references, including [5, 6, 11, 12], detailing various aspects of fibre bundles.

2.2.1 Definitions

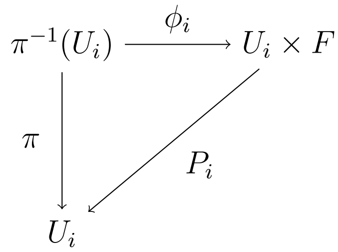
A **fibre bundle** is a manifold which locally looks like a direct product of two manifolds called the **base space** B and the **fibre** F. We can think of the fibre bundle as a copy of the manifold F attached to each point $b \in B$. However, the global topology of E can differ from $M \times F$. This notion can be formalized in the following definition[5]:

Definition 11. *A* **fibre bundle** *with* **fibre** F *over the* **base manifold** B *consists of a* **total space** E*, a surjective* **projection** $\pi : E \to B$ *and a Lie group* G *called the* **structure group** *with the following properties:*

[5]In line with the previous section we will always assume that manifolds and maps involved in the bundle constructions are smooth.

(i) *The structure group G acts on F from the left.*

(ii) *Given a covering $\{U_i\}$ of B there exists a* **local trivialization** $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ *such that (with P_i projecting to the U_i factor) the following diagrams commute:*

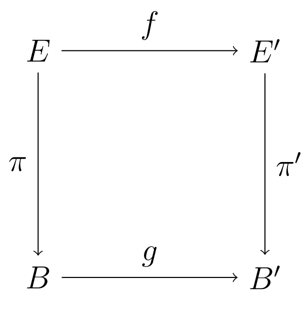


Denoting the components of the local trivialization by $\phi_i = (\pi, f_i)$ we obtain a diffeomorphism $\pi^{-1}(b) \to F$ for each $b \in B$ by restricting to f_i. The inverse image is therefore called the fibre at b and is denoted $F_b = \pi^{-1}(b) \cong F$. The topology of the fibre bundle is encoded in the obstruction to globally extending the local trivialization. A convenient way to describe this obstruction is by means of the **transition functions** $g_{ij} : \mathrm{U}_i \cap U_j \to G$ defined by

$$f_i = g_{ij} f_j \, , \tag{2.30}$$

relating different trivializations $\phi_i = (\pi, f_i)$ and $\phi_j = (\pi, f_j)$ on an overlap $\mathrm{U}_i \cap U_j \neq \emptyset$. The transition functions can therefore be viewed as a prescription for glueing together the (topologically trivial) local pieces to form the total space E of the bundle. The transition functions satisfy the cocycle condition $g_{ij} g_{jk} = g_{ik}$ on triple overlaps $U_i \cap U_j \cap U_k \neq 0$, and completely determine the topology of the fibre bundle[6]. In particular, if (and only if) there exists a local trivialization such that all $g_{ij} = e$, where as before $e \in G$ is the identity, the bundle is **topologically trivial**: $E = B \times F$. In the application to gauge theory described in later chapters, an important construction is a (local) **section** of a bundle (E, π, B), which we define as a map $\sigma : U_i \to E$ such that $\pi \circ \sigma = \mathbb{1}_{U_i}$.

A **bundle map** between two bundles (E, π, B) and (E', π', B') is a pair of maps $f : E \to E'$ and $g : B \to B'$ such that the following diagram commutes:



[6] In fact, an equivalent description of a fibre bundle is given by specifying the covering $\{U_i\}$ of B and the corresponding transition functions g_{ij}.

(In fact, a map $f : E \to E'$ which maps the fibre F_b onto a fibre $F_{b'}$ induces a map $g : B \to B'$.) If $B = B'$ and $\pi' \circ f = \pi$ the map is called a **bundle isomorphism** and if furthermore $E = E'$ we refer to the isomorphism as a **bundle automorphism**. We will consider isomorphic bundles as equivalent (c.f. diffeomorphic manifolds) and classify bundles according to their isomorphism class.

2.2.2 Principal bundles

We will now introduce an important type of fibre bundles, that will in particular play a central role in the remainder of this thesis.

Definition 12. *A* **principal bundle** $\pi : P \to B$ *is a fibre bundle which admits a free right action of* G*, such that* $\pi(pg) = \pi(p)$ *for* $p \in P$ *and* $g \in G$*, whose restriction to* $\pi^{-1}(b)$, $b \in B$ *is transitive.*

An equivalent way of defining a principal bundle (or a G-bundle) is thus as a bundle where the fibre F is isomorphic to the structure group. Furthermore, the base space can be identified with the quotient P/G and the fibre F_b is (isomorphic to) the G-orbit of any point $p \in \pi^{-1}(b)$.

The existence of a right action[7] is a feature of principal bundle which will be essential when we consider connections below. It also provides a convenient description of bundle automorphisms $f : P \to P$. An element of the group $\mathrm{Aut}(P)$ of such automorphisms is called a **gauge transformation** and defines a map $g : P \to G$ such that $f(p) = pg(p)$. In particular, a gauge transformation induces an action on sections of P according to $\sigma_i \to \sigma_i g$, which we will return to below.

Given a principal bundles we can construct other bundles with the same structure group and base space, but with a fibre given by a space which admits an action G from the left:

Definition 13. *Let* $\pi : P \to B$ *be a principal* G*-bundle and let* F *be a manifold which admits a left action of* G*. The* **associated bundle** *is* $\pi_{E_P} : E_P \to B$ *with total space* $P \times F/\sim$*, where the equivalence relation is defined by the simultaneous action of* G *in both factors as* $(p, f) \sim (pg^{-1}, gf)$*, and projection* π_{E_P} *given by* $\pi(p)$*.*

In later chapters we will be concerned with the particular case when $F = V$ is a G-module carrying a representation $\rho : G \to End(V)$. Then the **associated vector bundle**, denoted

$$P \times_\rho V\,, \tag{2.31}$$

is obtained by the identification $(p, v) \sim (pg^{-1}, \rho(g)v)$. We will later encounter the bundle

$$\mathrm{ad}(P) = P \times_{\mathrm{ad}} \mathfrak{g} \tag{2.32}$$

[7] In fact, the right action is globally defined in contrast to the left action of the transition function which depends on a choice of local trivializations ϕ_i.

associated to P through the adjoint action of G on its Lie algebra $\mathfrak{g}$. The transition functions relating local trivializations of the associated vector bundle on overlapping coordinate neighbourhoods $U_i \cap U_j \neq \emptyset$ are determined in terms of the corresponding quantities of the principal bundle as $\rho(g_{ij})$. Since the topological class (or isomorphism class) is completely determined by the transition functions this implies that the associated bundles inherit the topology of the principal bundle P from which they are constructed.

2.2.3 Topology of principal bundles

We will now briefly consider the problem of characterizing the topological class, or isomorphism class, of principal bundles. In doing so we restrict attention to the case when the structure group G is a compact simple Lie group (which is the case we will consider almost exclusively in this thesis[8]). As mentioned above the transition functions completely determine the global structure of the bundle, but there is a more convenient description of the topology in terms of **characteristic classes**.

For the purpose of this subsection it is sufficient to let $\dim(B) \leq 4$ and consider bundles where the structure group is on the adjoint form $G = \tilde{G}/C$, where $\tilde{G}$ is the simply connected cover of G and C is the center subgroup of $\tilde{G}$. The non-trivial homotopy groups of G relevant for bundles over B are then $\pi_1(G) \cong C$ and $\pi_3(G) \cong \mathbb{Z}$. The restriction of the bundle P to a two-sphere S^2 in B can be described by trivial bundles over the two hemispheres U_N and U_S with a transition function $g_{NS} : U_N \cap U_S = S^1 \to G$ describing the topology of the (restricted) bundle by defining an element in $\pi_1(G)$. Using the isomorphism $H^2(S^2, C) \cong C$ we can thus describe the restriction using an element $v \in H^2(S^2, C)$. Similarly, the restriction of P to an S^4 in B can be described by glueing trivial bundles over the two hemispheres using a transition function $g_{NS} : U_N \cap U_S = S^3 \to G$ defining an element of $\pi_3(G)$. Here, $H^2(S^4, C)$ is trivial but $H^4(S^4, \mathbb{Z}) \cong \mathbb{Z}$ which means that the bundle topology is characterized by an element $k \in H^4(S^4, \mathbb{Z})$.

In general, the topological class of the bundle can be completely characterized by two characteristic classes: The 't Hooft flux $v(P) \in H^2(B, C)$ and the fractional instanton number (or second Chern class) $k(P) \in H^4(B, \mathbb{Q})$. However, they are not independent, which is the reason for k generally taking its values in $\mathbb{Q}$ rather than $\mathbb{Z}$, but related through [13]

$$k - \frac{1}{2} v \cdot v \in H^4(B, \mathbb{Z})\,. \tag{2.33}$$

The product $v \cdot v$ is defined by composition of the cup product and the non-degenerate symmetric pairing $C \times C \to \mathbb{R}/\mathbb{Z}$. (We assume that the base manifold B admits a spin structure, since we will always consider supersymmetric gauge theories, in which

[8] In the final chapter 5 we will consider U(1) fibrations in some detail. However, there we will not be concerned with global properties of the fibrations.

case the product $v \cdot v$ is divisible by two in a canonical way.) We will encounter both these classes in the following chapters of the thesis and also see some examples from gauge theory below. In particular, chapter 3 and PAPER I-II are concerned with bundles over a three-dimensional base which implies that the topological class is determined uniquely by the magnetic 't Hooft flux. The instanton number and the pairing $v \cdot v$ is discussed in the context of PAPER III in chapter 4.

2.3 Connections and curvature

Having introduced the notion of a principal fibre bundle P over some base space B in the previous subsection, we will now proceed to consider its geometry in more detail. In particular, we will consider parallel transport of a point $p \in P$ along a curve γ in the base B. In order to accomplish this we need to introduce a notion of what parallel means, which is accomplished by the definition of a **connection** on P. The present exposition will be somewhat technical. However, the patient reader will be rewarded by physical applications to gauge theory in the final subsection.

2.3.1 Connections

In order to define parallel transport we first define the notion of a vector in the fibre direction.

Definition 14. *The* **vertical subspace** $V_pP \subset T_pP$ *at* $p \in P$ *is* $V_pP = \ker(\pi_p)_*$.

Thus, V_pP is the component of T_pP along the fibre F_b at $b = \pi(p)$. The lack of a canonical definition of a corresponding horizontal subspace H_pP, complementary to V_pP, is what requires the introduction of further structure on P.

Definition 15. *A* **connection** *on P is a smooth decomposition of T_pP into a vertical and a horizontal subspace which is equivariant with respect to the right action of G on P. In other words, it is a smooth assignment of H_pP such that*

(i) $T_pP = V_pP \oplus H_pP$

(ii) $H_{pg}P = (R_g)_* H_pP \;\; \forall g \in G\,.$

In particular, this means that any vector $X \in T_pP$ is decomposed as $X = X^V + X^H$. Given a connection we can now define the unique parallel transport of $p \in P$ along a curve γ in B in the following way:

Definition 16. *Let* $\gamma : [0,1] \to B$ *be a curve in* B*. A* **horizontal lift** *of* γ *is a curve* $\tilde{\gamma} : [0,1] \to P$ *such that* $\pi(\tilde{\gamma}(t)) = \gamma(t)$ *and the tangent of* $\tilde{\gamma}(t)$ *is an element of* $H_{\gamma(t)}P$.

In particular, there exists a unique horizontal lift $\tilde{\gamma}(t)$ through $p \in \pi^{-1}(\gamma(0))$ and consequently a unique parallel transport of p given by $\tilde{\gamma}(1)$.

We will now begin to make contact with the gauge field that we usually associated to a connection in gauge theory. This is done by giving an alternative (but equivalent) definition of a connection. First we consider the vertical subspace V_pP in more detail. In particular, we can construct V_pP by considering the curve in P

$$\sigma_t(p) = p \exp(tA) \tag{2.34}$$

obtained from the exponentiation of an element $A \in \mathfrak{g}$ (not to be confused with the local connection one-form introduced below) through the right action of G on P. The vector tangent to this curve is $X_A|_p \in T_pP$ given (c.f. the definition (2.2)) as

$$X_A|_p(f) = \left.\frac{df(\sigma_t(p))}{dt}\right|_{t=0} . \tag{2.35}$$

In fact, $X_A|_p$ is an element of V_pP and $\sigma_t(p)$ a curve in the fibre F_p, since π is invariant under the action of G on the right. The definition can be smoothly extended over all of P to define the left-invariant **fundamental vector field** X_A associated to A and generating the flows $\sigma_t(p)$.

Definition 17. *A* **connection one-form** $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ *is a Lie-algebra valued one-form satisfying* $\omega|_{F_b} = \omega_{MC}$ *and* $R_g^*\omega = Ad_{g^{-1}}\omega\,, \forall g \in G$.

(Here, the restriction to $F_b = \pi^{-1}(b)$ is obtained by pulling back ω with the embedding of G into P using its right action.) The fact that ω acts like the Maurer-Cartan form in the fibre implies that $\omega(X_A) = A$ (recall that ω_{MC} is left-invariant). We now define the horizontal subspace H_pP as the kernel of the connection

$$H_pP = \ker \omega|_p \,. \tag{2.36}$$

The properties of ω ensures the equivalence with the definition as a separation of the tangent space of P.

An appealing feature of the connection one-form is that it is defined globally on P. When considering gauge theory we will, however, be concerned with quantities in the base manifold B. It is therefore useful to define the equivalent of the connection one-form on B, a construction which is accomplished using sections of P. However, for topologically non-trivial bundles we can only define local sections and the one-forms on B are therefore inherently local:

Definition 18. *Let* σ_i *be a local section with respect to some covering* $\{U_i\}$ *of* B. *The local connection or* **gauge potential** $A_i \in \Omega^1(U_i) \otimes \mathfrak{g}$ *is defined as*

$$A_i = \sigma_i^* \omega \,. \tag{2.37}$$

In order to ensure that this definition is invertible, i.e. that a set of local connections A_i uniquely defines a connection one-form on P, we define

$$\omega_i = g_i^{-1}\pi^* A_i g_i + g_i^{-1} d_P g_i \,, \tag{2.38}$$

where g_i is the canonical local trivialization relative to the section σ_i (defined by $\phi_i(p) = (\pi, g_i)$ with $p = \sigma_i(p)g_i$) and d_P is the exterior derivative on P. It can be shown that (2.38) satisfies (2.37) and the requirements on a connection one-form on P. In order to obtain a globally defined ω we require that on $U_i \cap U_j \neq \emptyset$ we have $\omega_i = \omega_j$ which implies that the gauge potentials are related by

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} d g_{ij} \tag{2.39}$$

where d is now the exterior derivative on $U_i \cap U_j$ and g_{ij} are the transition functions of the bundle. This transformation property of A_i thus contains the global information encoded in the connection ω.

A similar transformation is obtained by considering a different choice σ_i' of the (arbitrarily chosen) section σ_i appearing in (2.37) for a coordinate chart U_i. The two sections are related through a bundle automorphism (or gauge transformation), given by some element of the structure group G, as $\sigma_i'(p) = \sigma_i(p)g(p)$. The corresponding local one-form gauge potentials are then related through

$$A_i' = g^{-1} A_i g + g^{-1} dg \tag{2.40}$$

which we recognize as a local **gauge transformation** of the potential A_i. This implies that we can identify gauge invariance as the redundancy in the description of a connection of the principal bundle in terms of the local gauge potential, corresponding to a choice of local coordinates on P.

Finally, we can now return to the parallel transport, motivating the introduction of the connection, and express it in terms of the local connection A_i. Let γ be a curve in U_i and $\tilde{\gamma}$ its horizontal lift through $p \in \pi^{-1}(\gamma(0))$. Given a section σ_i such that $\sigma_i(\gamma(0)) = \tilde{\gamma}(0)$ the lift can be expressed as $\tilde{\gamma}(t) = \sigma_i(\gamma(t))g_i(t)$ and the fact that the tangent of $\tilde{\gamma}$ is horizontal translated to a differential equation for $g_i(t)$ defining parallel transport:

$$\frac{dg_i}{dt} + A_i(X) g_i(t) = 0 \,, \tag{2.41}$$

where $A_i = \sigma_i^* \omega$ and X is tangent to γ at $\gamma(0)$. Integrating this we obtain

$$g_i(t) = P \exp\left(- \int_{\gamma(0)}^{\gamma(t)} A_i \right) \,, \tag{2.42}$$

where P denotes path-ordering. In particular, if we let γ be a loop in B we obtain the **holonomy** of the connection around γ

$$g[\gamma] = P \exp\left(- \int_\gamma A \right) \,, \tag{2.43}$$

which describes the change in the fibre as $p = \tilde{\gamma}(0)$ is parallel transported around γ. We will return to this concept in the next chapter when we describe vacua in Yang-Mills theory.

2.3.2 Curvature

In order to provide a geometric interpretation for the gauge field strength similar to the one provided for the vector potential A_i in the previous subsection, we define a covariant derivative of a Lie-algebra valued form using the connection on P:

Definition 19. *Let $\zeta \in \Omega^r(P) \otimes \mathfrak{g}$ and $X_1, \ldots, X_{r+1} \in \chi(P)$. The* **covariant derivative** $D\zeta \in \Omega^{r+1}(P) \otimes \mathfrak{g}$ *of ζ is defined by*

$$D\zeta(X_1, \ldots, X_{r+1}) = d_P\zeta(X_1^H, \ldots, X_{r+1}^H)\,. \tag{2.44}$$

(Here, d_P acts in the differential form factor of ζ.) The differentiation is indeed covariant with respect to the connection on P since, by definition, the connection determines the decomposition $T_pP = V_pP \oplus H_pP$.

We can then define the Lie-algebra valued curvature on the principal bundle P as the covariant derivative of the connection:

Definition 20. *The* **curvature two-form** $\Omega \in \Omega^2(P) \times \mathfrak{g}$ *is defined as*

$$\Omega = D\omega\,. \tag{2.45}$$

Just as the connection one-form, the curvature transforms in the adjoint under right translation

$$R_g^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega \ , \ \ \forall g \in G \tag{2.46}$$

and satisfies **Cartan's structure equation**

$$\Omega = d_P\omega + \frac{1}{2}[\omega, \omega] = d_P\omega + \omega \wedge \omega\,, \tag{2.47}$$

where we define the Lie bracket between two Lie-algebra valued forms $\zeta \in \Omega^r(P) \otimes \mathfrak{g}$ and $\eta \in \Omega^s \otimes \mathfrak{g}$ as

$$[\zeta, \eta] = \zeta \wedge \eta - (-1)^{rs}\eta \wedge \zeta = \zeta^a \wedge \eta^b \otimes f_{ab}{}^cT_c\,. \tag{2.48}$$

The local curvature is then defined in analogy with the local connection:

Definition 21. *Let σ_i be a local section with respect to some covering $\{U_i\}$ of B. The local curvature or* **gauge field strength** $F_i \in \Omega^2(P) \otimes \mathfrak{g}$ *is defined as*

$$F_i = \sigma_i^*\Omega\,. \tag{2.49}$$

The gauge transformation properties of F_i under a change of the section σ_i to $\sigma_i' = \sigma_i g$ is given by the relation

$$F_i' = g^{-1} F_i g \,, \tag{2.50}$$

and the local descriptions on overlapping charts $U_i \cap U_j$ are related analogously. Using the fact that the pull-back and exterior derivatives commute we immediately obtain the local relation between the curvature and the connection as

$$F_i = dA_i + A_i \wedge A_i \,, \tag{2.51}$$

which in local coordinates on B reproduces the familiar definition

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc} A^b_\mu A^c_\nu \,. \tag{2.52}$$

From (2.47) we derive the local form of the **Bianchi identity** $D\Omega = 0$ as

$$dF + [A, F] = 0 \,. \tag{2.53}$$

In the previous subsection we encountered the (second) Chern class, which was introduced abstractly as a certain element in cohomology characterizing the isomorphism class of a bundle. We can now give a more tangible definition in terms of the curvature of the bundle[9]:

Definition 22. *Let $\pi : P \to B$ be a principal bundle with curvature F. The* **total Chern class** *is defined by*

$$c(P) = \det\left(I + \frac{i}{2\pi} F\right) \,. \tag{2.54}$$

Expanding $c(P)$ we obtain

$$c(P) = 1 + c_1(P) + c_2(P) + \ldots \tag{2.55}$$

where $c_n(F)$ is called the n**-th Chern class**.

It can be shown that $c_n(P) \in H^{2n}(B)$ and that the cohomology class is in fact independent on the curvature of the bundle, motivating the notation $c(P)$. Furthermore, when integrated over a $2n$-cycle Σ in B yields the **Chern numbers**

$$C_n(P) = \int_\Sigma c_n(F) \tag{2.56}$$

which are topological invariants of P. Since we will mainly focus on bundles where $\dim(B) \leq 4$ the only non-vanishing Chern classes are $c_1(F)$ and $c_2(F)$ (the latter of

[9]The 't Hooft flux on the other hand cannot be expressed in terms of F.

which we encountered above as the fractional instanton number k). The expressions for these classes are given by

$$c_1(F) = \frac{i}{2\pi}\mathrm{Tr}F \quad , \quad c_2(F) = \frac{1}{8\pi^2}\left[\mathrm{Tr}(F \wedge F) - \mathrm{Tr}F \wedge \mathrm{Tr}F\right] \tag{2.57}$$

and we will return to compute them in explicit examples in gauge theory below.

We now return to the vector bundles $E_P = P \times_\rho V$ associated to P and use the connection on P to define a covariant derivative of sections of E_P. Such a section σ is described by (the equivalence class of) a pair $[(p, v)]$, with $p : B \to P$ and $v : B \to V$, and we define the covariant derivative $D : \Omega^0(B, E_P) \to \Omega^1(B, E_P)$ by

$$D\sigma(X) = [(p, dv(X) + \rho(\omega(p_* X))v)] \quad , \quad \forall X \in \chi(B)\,. \tag{2.58}$$

In local coordinates on E_P and B a section $\phi \in \Omega(B, E_P)$ is interpreted as a field ϕ^i on B in the representation ρ of the structure (gauge) group G. The covariant derivative then takes the familiar form

$$D_\mu \phi^i = \partial_\mu \phi^i + \rho(T_a)^i{}_j A^a_\mu \phi^j\,, \tag{2.59}$$

where we let x^μ be coordinates on B. In particular, for the associated bundle $\mathrm{ad}(P) = P \times_{\mathrm{ad}} \mathfrak{g}$ we will encounter in subsequent chapters ρ is the adjoint representation in which the representation matrices are given by the structure constants and we recover the local expression

$$D_\mu \phi^a = \partial_\mu \phi^a + f^a{}_{bc} A^b_\mu \phi^c\,. \tag{2.60}$$

Finally, we now complement the definition of curvature given above with a geometric interpretation. We found that the horizontal lift $\tilde{\gamma}$ of a closed loop γ in B is generically not closed. The curvature measures this failure of closure of $\tilde{\gamma}$ as indicated by the local relation

$$[D_\mu, D_\nu]\phi^a = \frac{1}{2} f^a{}_{bc} F^b_{\mu\nu} \phi^c\,. \tag{2.61}$$

Consequently, the operation of parallel transport using the covariant derivative commutes only when the curvature vanishes. A connection for which $\Omega = 0$, and the gauge field strength consequently vanishes, is referred to as a **flat connection**. Parallel transport using such a connection is independent of the path between the two points $\gamma(0)$ and $\gamma(1)$ in the base manifold B. Flat connections are of special interest in this thesis through their connection to vacuum states in gauge theory.

2.3.3 Gauge theory

Having endured a substantial amount of abstract geometry it is now time for the zealous reader to reap the reward: An inherently geometric definition of gauge theory. As we have hinted at several times before, gauge theory is essentially the study of connections (or gauge vector potentials) A_μ on principal bundles over some base manifold B, which we will always assume is equipped with a metric. We can then define the Hodge duality operator $* : \Omega^r(B) \to \Omega^{d-r}(B)$, where d is the dimension of B, by its action on an r-form $\omega \in \Omega^r(B)$ which for a flat metric takes the form

$$* \, \omega = \frac{-1}{r!(d-r)!} \, \omega^{\mu_1 \ldots \mu_r} \epsilon_{\mu_1 \ldots \mu_r \mu_{r+1} \ldots \mu_d} dx^{\mu_{r+1}} \wedge \ldots \wedge dx^{\mu_d} \,, \tag{2.62}$$

with $\epsilon^{\mu_1 \ldots \mu_d}$ being the totally antisymmetric tensor density. (The generalization to curved manifolds is treated in the context of Riemannian geometry in section 5.1.) The Hodge dual appears frequently throughout the remainder of this thesis.

The structure group G encodes gauge symmetries of the theory, which correspond to bundle automorphisms and are therefore more accurately thought of as a redundancy of the description, rather than an ordinary symmetry of a physical system corresponding to a conserved quantity. To be more specific, states of the physical theory are required to be invariant under the connected subgroup $\mathrm{Aut}_0(P)$ of gauge transformations homotopically equivalent to the trivial transformation.

In order to formulate a physical theory, we must then supplement the geometric Bianchi identity for A_μ (or rather the field strength $F_{\mu\nu}$) with some dynamical way of determining the evolution of the gauge potential. This is done in the Lagrangian formulation (see section 2.5.1 for further details) by specifying an action functional on the space of gauge inequivalent connections. The action is then required to be stationary under arbitrary variations of the field A_μ. For ordinary Yang-Mills theory the action is given by

$$S = \frac{1}{2} \int_B \mathrm{Tr}(F \wedge *F) \,, \tag{2.63}$$

where $*F$ is the dual field strength so that the integrand is proportional to $F^a_{\mu\nu} F^{\mu\nu}_a$. The equation of motion derived from the variation of (2.63) is

$$D * F = 0 \tag{2.64}$$

or equivalently

$$D_\mu F^{\mu\nu} = 0 \,. \tag{2.65}$$

We will now proceed to consider two classical[10] examples of topologically non-trivial gauge bundles (i.e. connections obtained by solving the equations of motion). In particular, we will compute the Chern class of the two examples using the curvature.

The Dirac monopole bundle

We first consider the bundle describing a magnetic Dirac monopole [14] located at the origin of $\mathbb{R}^3$. This is accomplished by a principal U(1) bundle over S^2 (or equivalently

[10] In the sense that they are well-known and often recurring examples; not in the sense of classical as opposed to quantum.

$\mathbb{R}^3 \setminus \{0\}$). Since S^2 cannot be covered by a single coordinate neighbourhood we consider the atlas consisting of the two hemispheres U_N and U_S, with intersection $U_N \cap U_S = S^1$ along the equator. The bundle is then defined by the transition function $g_{NS} = e^{-in\phi}$, where we require $n \in \mathbb{Z}$ for consistency.

One particular solution to Maxwell's equations is given by the gauge potentials

$$A_N = i\frac{n}{2}(1 - \cos\theta)d\phi \quad , \quad A_S = i\frac{n}{2}(-1 - \cos\theta)d\phi \, , \tag{2.66}$$

where (θ, ϕ) are the standard coordinates on S^2 and the integer n is called the charge of the monopole. The potentials are indeed related through the transformation

$$A_S = A_N + g_{NS}^{-1} dg_{NS} \tag{2.67}$$

along the equator $U_N \cap U_S$, and consequently define a connection on the bundle described above. This is Dirac's solution describing a magnetic monopole in the center of the ball bounded by the S^2. We note that the potentials are not well-defined over all of S^2. In particular, when considered in $\mathbb{R}^3 \setminus \{0\}$ the connection A_N has a stringlike singularity along the negative z-axis (and similarly for A_S) which is known as the Dirac string. The resolution to this singular behaviour is to consider A_N and A_S as defined only on U_N and U_S respectively, in agreement with the definition of local connection one-forms given above.

We can now consider the first Chern number of the monopole bundle by defining $F_N = dA_N$ and $F_S = dA_S$ and computing using (2.57)

$$C_1 = \frac{i}{2\pi}\left(\int_{U_N} F_N + \int_{U_S} F_S\right) = \frac{i}{2\pi}\int_{S^1}(A_N - A_S) = \frac{i}{2\pi}\int_{S^1} g_{NS}^{-1} dg_{NS} = -n \, . \tag{2.68}$$

Thus, we find that the monopole charge must be integral and is given by (minus) the first Chern number.

The instanton bundle

The next example is provided by SU(2) Yang-Mills theory in $\mathbb{R}^4$, which implies that adding a point at infinity we consider principal bundles over $B = S^4$. We will consider the (anti-)self-dual instanton solutions [15] to the Euclidean Yang-Mills equations which satisfy $*F = \pm F$. In order for the corresponding Yang-Mills action to be finite we must require that the connection satisfies

$$A \to g^{-1} dg \quad , \quad |x| \to \infty \, . \tag{2.69}$$

In the compactification of $\mathbb{R}^4$ this defines the two hemispheres U_N and U_S of S^4 as being respectively the complement of and a neighbourhood of the south pole (at $|x| \to \infty$). On the overlap $U_N \cap U_S = S^3$ we can then take

$$A_N = A = g^{-1} dg \quad , \quad A_S = 0 \, , \tag{2.70}$$

where the transition function $g : S^3 \to \mathrm{SU}(2)$ defining the topology of the bundle and appearing in the gauge transformation of A_S is given by

$$g(x) = \left[\frac{1}{r}(x^0 + ix^i\sigma_i)\right]^k . \qquad (2.71)$$

Here, σ_i are the Pauli matrices, r is the S^3 radius and $k \in \mathbb{Z}$. In order to compute the second Chern class[11] we note that locally $\mathrm{Tr}F \wedge F = dCS(A)$, where the Chern-Simons form is defined by

$$CS(A) = \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) . \qquad (2.72)$$

Consequently, we have for the second Chern number

$$C_2 = \frac{1}{8\pi^2}\int_{S^4} \mathrm{Tr}F \wedge F = \frac{1}{8\pi^2}\int_{S^3} CS(A) = -\frac{1}{24\pi^2}\int_{S^3} \mathrm{Tr}A \wedge A \wedge A\,. \qquad (2.73)$$

where we have used $F = 0$ on S^3. For $k = 1$ we have $\mathrm{Tr}A \wedge A \wedge A = 12\mathrm{Vol}_{S_3}$ and thus $C_2 = -1$; for arbitrary k one similarly finds $C_2 = -k$. Thus, the integer k characterizing the winding of the transition function $g : S^3 \to \mathrm{SU}(2)$ in the gauge group is again given by the (negative) Chern number in analogy with the monopole bundle above. (In the previous section the fractional instanton number was denoted $k \in H^4(B, \mathbb{Q})$. For $G = \mathrm{SU}(n)$ which is simply connected the 't Hooft flux is trivial and the instanton number is really an integral class which integrated over B gives $k \in \mathbb{Z}$.)

2.4 Symplectic geometry

The geometric formulation of classical mechanics and the process of canonical quantization is the second objective of this chapter. Here, we develop some further geometrical concepts that will be needed for this application, which turns out to involve a certain class of manifolds, called symplectic manifolds, equipped with an additional structure. Further reading on the theory of symplectic manifolds and canonical quantization can be found in e.g. [6, 16, 17].

2.4.1 Symplectic manifolds

We begin by giving the definition of a symplectic manifold:

Definition 23. *A* **symplectic manifold** *is a pair (M, σ), where M is a manifold of dimension $2n$ and $\sigma \in \Omega^2(M)$ is* **closed** *and* **non-degenerate**[12].

[11] Since $G = \mathrm{SU}(n)$ the generators are traceless and $c_1(P) = \frac{i}{2\pi}\mathrm{Tr}F = 0$.
[12] The form σ is non-degenerate if $\sigma(X, Y) = 0\,, \forall Y \in \chi(M)$ implies $X = 0$.

The 2-form σ is called a **symplectic form**, and is said to equip M with a symplectic structure. An important consequence of the existence of the symplectic form is that it associates a vector field to each element of $\mathcal{F}(M)$.

Definition 24. *Given a function $f \in \mathcal{F}(M)$ the associated* **Hamiltonian vector field** X_f *is defined by*

$$\iota_{X_f}\sigma + df = 0 \,. \tag{2.74}$$

In the next chapter we will consider the application of symplectic geometry to Hamiltonian mechanics, motivating the name Hamiltonian vector field. The non-degeneracy of σ implies that the defining equation for X_f, which can also be expressed as

$$\sigma(X_f, Y) + df(Y) = 0 \quad , \quad \forall Y \in \chi(M) \,, \tag{2.75}$$

has a unique solution. From the properties $d\sigma = 0$ and (2.20) it follows that

$$\mathcal{L}_{X_f}\sigma = d\iota_{X_f}\sigma + \iota_{X_f}d\sigma = 0 \,, \tag{2.76}$$

which means that the flow generated by X_f preserves the symplectic form σ. Consequently, the pair (M, σ) is invariant under the flow which is therefore called a symplectomorphism. Another important feature of the associated vector field is that it endows $\mathcal{F}(M)$ with an algebraic structure.

Definition 25. *The* **Poisson bracket** *of two functions $f, g \in \mathcal{F}(M)$ is given by*

$$\{f, g\} = \sigma(X_f, X_g) \,. \tag{2.77}$$

For arbitrary elements $f, g, h \in \mathcal{F}(M)$ the Poisson bracket, which is bilinear and antisymmetric by construction due to the properties of differential forms, satisfies the Jacobi identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ and consequently endows $\mathcal{F}(M)$ with the structure of a Lie algebra. In fact, (2.74) constitutes a homomorphism of Lie algebras $\mathcal{F}(M) \to \chi(X)$.

There is a particular interpretation of the Poisson bracket that makes it central in the application to classical mechanics below: From the definition of the Lie derivative it follows that $\{f, g\} = \mathcal{L}_{X_f}(g)$, that is $\{f, g\}$ gives the change of the function g along the flow generated by the Hamiltonian vector field associated to f (and vice versa). A special case is $\{f, g\} = 0$ which implies that g is preserved by the flow generated by X_f.

Before proceeding to explore symplectic geometry and the algebraic structure of $\mathcal{F}(M)$ further, it is convenient at this point to make a slight detour to discuss submanifolds of symplectic manifolds. To this end we consider vector spaces that carry a symplectic structure.

Definition 26. *A* **symplectic vector space** (V, σ) *is a vector space V equipped with a skew-symmetric and non-degenerate bilinear form $\sigma : V \times V \to \mathbb{R}$.*

In particular, the tangent space T_pM at each point $p \in M$ of a symplectic manifold is given the structure of a symplectic vector space by restriction of the symplectic form σ on M. Due to its non-degeneracy, σ provides a notion of orthogonality in the context of symplectic vector spaces.

Definition 27. *Let (V, σ) be a symplectic vector space and let $W \subseteq V$ be a subspace of V. The* **symplectic complement** *of W is*

$$W^{\perp} = \{X \in V \,|\, \sigma(X, Y) = 0\,, \forall Y \in W\}\,. \tag{2.78}$$

The elements $X \in W^{\perp}$ are said to be **symplectically orthogonal** to W. Using the symplectic complement we can now define a special class of submanifolds that will appear in the discussion of constrained dynamical systems below.

Definition 28. *Let $M_0 \subset M$ be a submanifold of a symplectic manifold (M, σ). M_0 is a* **symplectic submanifold** *if*

$$T_pM_0 \cap (T_pM_0)^{\perp} = \emptyset\,, \forall p \in M_0\,. \tag{2.79}$$

An appealing feature of symplectic submanifolds (and the reason for their name) is that the symplectic structure σ on M induces a symplectic structure on M_0. Let $i : M_0 \to M$ be the inclusion map. Then (2.79) implies that the pullback $\sigma_0 = i^*\sigma \in \Omega(M_0)$ is non-degenerate and consequently (since d commutes with the pullback) that (M_0, σ_0) is a symplectic manifold called the **symplectic restriction** of (M, σ)[13]. Finally, a symplectic submanifold induces a decomposition of the tangent space

$$T_pM = T_pM_0 \oplus (T_pM_0)^{\perp}\,, \forall p \in M_0\,. \tag{2.80}$$

The components of a vector field $X \in \chi(M)$ relative to this decomposition are denoted

$$X = X^T + X^{\perp}\,. \tag{2.81}$$

2.4.2 Canonical coordinates

Before proceeding to the prime example of symplectic manifolds in physics we state, without proof, a theorem by Darboux which allows for a convenient local formulation of symplectic geometry: On any symplectic manifold (M, σ) it is possible to find local coordinates (q^i, p_i), $i = 1, \ldots, n$, such that the symplectic form is given by

$$\sigma = \sum_i dp_i \wedge dq^i\,. \tag{2.82}$$

[13]The symplectic restriction can be generalized to the case when $(T_pM_0)^{\perp} \subseteq T_pM_0\ \forall p \in M_0$. Such submanifolds are called coisotropic and the corresponding construction is called symplectic reduction.

In other words; on each coordinate patch U_i in the covering of M it is possible to choose the homeomorphism ϕ_i so that σ takes the standard form of Darboux. The coordinates (q^i, p_i) are referred to as **canonical coordinates** for reasons that will become apparent below.

We can now derive the coordinate expression for the geometric quantities introduced above as a first step towards making contact with the familiar Hamiltonian formulation of classical mechanics considered in the next section. From the defining relation (2.74) we find the coordinate expression

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} \tag{2.83}$$

for the Hamiltonian vector field associated to the function $f \in \mathcal{F}(M)$. For the Poisson bracket on the other hand we obtain, in local coordinates, the familiar form

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}\,. \tag{2.84}$$

2.4.3 The cotangent bundle

We are now ready to consider the (perhaps) most important examples of symplectic manifolds in physics, namely that of cotangent bundles. In fact, as we will see shortly, cotangent bundles are also important in their own right through the connection with the local description of general symplectic manifolds provided by Darboux's theorem.

Definition 29. *Let Q be an n-dimensional manifold. The* **cotangent bundle** *$\pi : T^*Q \to Q$ is the vector bundle over Q defined as the disjoint union of the cotangent spaces T_q^*Q*

$$T^*Q = \bigcup_{q\in Q} \{q\} \times T_q^*Q\,. \tag{2.85}$$

From the definition[14] it is clear that the fibre over a point $q \in Q$ is $F_q = T_q^*Q$ and that a point $m \in M = T^*Q$ is a pair (q, p), where $q \in Q$ and $p \in T_q^*Q$. The projection operator is simply given by the projection to the first element $\pi(q, p) = q$.

There is a **canonical symplectic structure** on the cotangent space $M = T^*Q$ for any manifold Q. This structure is provided by the **tautological one-form** $\tau \in \Omega^1(M)$ which is most transparently defined by its value when evaluated on an arbitrary vector $X_m \in T_mM$ at a point $m = (q, p)$:

$$\tau(X_m) \equiv p(\pi_*(X_m))\,. \tag{2.86}$$

[14] To make contact with the previous section we note that the definition does not imply that T^*Q is globally a direct product (trivial bundle) for a generic manifold Q. The non-triviality of the cotangent bundle is encoded in transition functions valued in $\mathrm{GL}(n, \mathbb{R})$ corresponding to coordinate transformations on Q.

The vector X_m and its push-forward are illustrated in figure 2.6. Taking the exterior derivative of the tautological one-form we obtain

$$\sigma = d\tau \,, \tag{2.87}$$

which is a closed and non-degenerate two-form making (M, σ) a symplectic manifold.

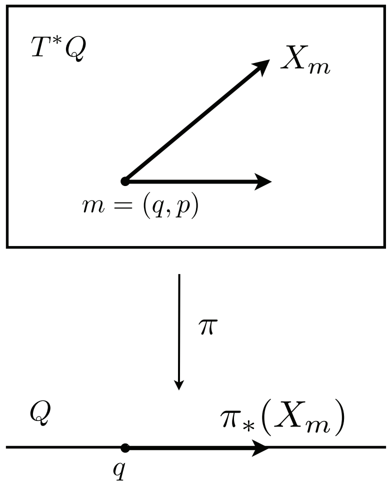


Figure 2.6: Schematic illustration of the push-forward of a vector $X_m \in T_m M$, where $M = T^*Q$, by the projection map $\pi : M \to Q$.

To prove the claim that σ is a symplectic form on $M = T^*Q$ it is convenient to work in local coordinates. Given coordinates $\{q^i\}$ on a patch $U_i \subset Q$ we recall that for $q \in U_i$ a basis for the cotangent space $T^*_q Q$ is provided by $\{dq^i\}$. A cotangent vector $p \in T^*_q Q$ is then written in terms of components as $p = p_i dq^i$. Thus, on on $U_i \times T^* U_i$ we have local coordinates

$$x^\mu = (\underbrace{q^1, \ldots, q^n}_{x^i}, \underbrace{p_1, \ldots, p_n}_{x^\alpha}) \,. \tag{2.88}$$

In these coordinates an arbitrary vector $X_m \in T_m M$ is expanded as

$$X_m = X^i_m \frac{\partial}{\partial x^i} + X^\alpha_m \frac{\partial}{\partial x^\alpha} \,, \tag{2.89}$$

and evaluating a one-form $\tau \in \Omega^1(M)$ on the vector X_m gives

$$\tau(X_m) = \tau_i X^i_m + \tau_\alpha X^\alpha_m \,. \tag{2.90}$$

The coordinate expression (2.15) for the push-forward becomes

$$\pi_*(X_m) = X^i_m \frac{\partial q^k}{\partial x^i} \frac{\partial}{\partial q^k} + X^\alpha_m \frac{\partial q^k}{\partial x^\alpha} \frac{\partial}{\partial q^k} = X^i_m \frac{\partial}{\partial q^i} \tag{2.91}$$

and evaluating $p \in T_q^*Q$ on this vector yields

$$p(\pi_*(X_m)) = X_m^i p_i \,. \tag{2.92}$$

Recalling the definition $\tau(X_m) \equiv p(\pi_*(X_m))$ of the tautological one-form we obtain

$$\tau = p_i dq^i \,, \tag{2.93}$$

and applying the exterior derivative to τ we get the local expression for $\sigma \in \Omega^2(Q)$ as

$$\sigma = d\tau = \sum_i dp_i \wedge dq^i \,, \tag{2.94}$$

which is indeed a symplectic two-form on the standard form of Darboux.

We now see the significance of cotangent bundles in symplectic geometry: By Darboux's theorem every symplectic manifold locally looks like a cotangent space. The canonical coordinates on an arbitrary symplectic manifold are then those in which the symplectic form σ takes the form of the canonical symplectic form on the corresponding cotangent bundle.

Finally, we note that although much of the discussion in this section has been concerned with the local coordinate formulation, the tautological one-form (and consequently also the canonical symplectic structure of T^*Q) is defined in a coordinate independent way.

2.5 Canonical quantization

We are now ready to consider the application to classical mechanics of the theory of symplectic manifolds developed in the previous section. We will assume familiarity with both Lagrangian and Hamiltonian formulations of classical mechanics[15] and show how they can be conveniently formulated in terms of geometric quantities. In particular, the Hamiltonian formulation, its connection to symplectic geometry and the process of canonical quantization are explored.

2.5.1 Hamiltonian mechanics

Any description of a classical dynamical system, i.e. any formulation of classical mechanics, can be seen as originating with the **configuration space** Q of a mechanical system. The points in Q describe the (generalized) positions or configurations accessible to the system. In other words at any given time the system will be described by some $q \in Q$. We will assume that Q is a smooth manifold which is generically the case for ordinary mechanical systems. With time the system may change its position

[15]Detailed accounts of the material reviewed in the first part of this section can be found most textbooks on classical mechanics e.g. [16, 18].

in configuration space and the corresponding trajectory is a curve $\gamma : \mathbb{R} \to Q$ that completely determines the evolution of the system.

In order to describe the state of the system at some instant, Lagrangian mechanics uses the tangent vector of the trajectory $\gamma(t)$ parametrized by the time t. That is, instantaneously the system is completely characterized by the position $q = \gamma(t)$ and generalized velocity $\dot{q} = \gamma_*(d/dt) \in T_{\gamma(t)}Q$ which together define a point in the tangent bundle TQ (defined in analogy with the cotangent bundle as the disjoint union of all tangent spaces of Q so that a point $m \in TQ$ is described by the pair $m = (q, \dot{q})$).

The **Lagrangian function** $L(q, \dot{q})$ on the tangent bundle determines the dynamics of the system as a initial value problem by requiring that the trajectory $\gamma(t)$ extremizes the action functional

$$S[q, \dot{q}] = \int dt\, L\left(q, \dot{q}\right) \,. \tag{2.95}$$

This requirement, referred to as the variational principle, implies the familiar **Euler-Lagrange equations of motion**

$$\frac{d}{dt}\left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}}\right) - \frac{\partial L(q, \dot{q})}{\partial q} = 0 \,. \tag{2.96}$$

Given an initial position and velocity $(q, \dot{q})$ these equations determine uniquely the trajectory $\gamma(t)$ of the system.

To transition to the Hamiltonian formulation, which is the formulation used in canonical quantization below, we begin by defining the (generalized) **conjugate momentum** $p(q, \dot{q})$ as the derivative of $L(q, \dot{q})$ in the fibre direction of the tangent bundle[16]

$$p(q, \dot{q}) = \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \,. \tag{2.97}$$

Using the momentum $p(q, \dot{q}) \in T_q^*Q$ we can now characterize the state of the dynamical system at some time as an element (q, p) of the cotangent bundle rather than the tangent bundle. Thus, the **phase space** $\mathcal{P}$ of the dynamical system is precisely the cotangent bundle T^*Q of the configuration space Q.

Under the technical assumption that (2.97) is invertible, so that we can solve for the velocity as a function of (q, p), we can define the **Hamiltonian** function of the system on the cotangent bundle

$$H : T^*Q \to \mathbb{R} \tag{2.98}$$

[16]To make this expression more precise we note that since the Lagrangian is a function on TQ the desired quantity p must be in the dual space of the variation $\delta\dot{q} \in T_qQ$, which implies $p \in T_q^*Q$ and that the defining relation is $p(\delta\dot{q}) = \delta L$.

through the Legendre transformation[17]

$$H(q,p) = p(\dot{q}) - L(q,\dot{q})\,. \tag{2.99}$$

The Hamiltonian $H(q,p)$ gives the total energy of the system at some point (q,p) of phase space. The equations of motion in the Hamiltonian formulation can be derived from the Euler-Lagrange equations. The equations are called **Hamilton's equations** and in canonical coordinates they are given by

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \quad , \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}\,. \tag{2.100}$$

Previously, we showed that the phase space $\mathcal{P} = T^*Q$ together with the canonical symplectic form σ is a symplectic manifold $(\mathcal{P},\sigma)$. We can therefore consider the Hamiltonian vector field X_H associated to the Hamiltonian through the symplectic form σ, i.e.

$$\iota_{X_H}\sigma + dH = 0\,. \tag{2.101}$$

Recalling that in canonical coordinates this implies that the Hamiltonian vector field is given by

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{2.102}$$

and inserting Hamilton's equations we obtain[18]

$$X_H = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i} = \frac{d}{dt}\,. \tag{2.103}$$

Thus X_H is the vector field that generates time translations on the phase space $\mathcal{P} = T^*Q$ of the dynamical system, or equivalently; the system evolves along the flow on $\mathcal{P}$ generated by X_H.

Physical observables of the system in classical mechanics then correspond to functions $f \in \mathcal{F}(\mathcal{P})$. Thus, there is a canonical algebraic structure on the set of physical observables induced by the symplectic structure of the phase space, namely the Poisson bracket discussed in the previous chapter. In particular, the time evolution of an observable $f \in \mathcal{F}(M)$ is given by

$$\frac{df}{dt} = \{H, f\} \tag{2.104}$$

and an observable satisfying $\{H,f\} = 0$ is a constant of motion for the system. A special case is $\{H,H\} = 0$ which entails conservation of energy. Conversely, the antisymmetry of the Poisson bracket implies that given a flow that preserves the

[17] Here, $p(\dot{q})$ denotes the cotangent vector p evaluated on the vector $\dot{q}$ at $q \in Q$.

[18] The final equality is somewhat schematic; d/dt is of course a vector on the domain of the curve $\gamma(t)$ and we are referring to the push-forward of d/dt to a vector tangent to T^*Q as defined above.

Hamiltonian H (and thus constitutes a continuous symmetry of the system) there is a corresponding conserved charge, associated through (2.74) to the vector that generates the flow. This correspondence is simply Noether's theorem for Hamiltonian mechanics [17, 18].

2.5.2 Canonical quantization

Equipped with the geometric formulation of classical mechanics considered in some detail in the previous section we can now take the first step towards the corresponding quantum theory. As we saw above the classical theory is described by the **phase space** $\mathcal{P}$ where physical observables are functions $f \in \mathcal{F}(\mathcal{P})$. The symplectic structure of the phase space induces a canonical algebraic structure of $\mathcal{F}(\mathcal{P})$, making it a Lie algebra with respect to the Poisson bracket$\{\,,\}: \mathcal{F} \times \mathcal{F} \to \mathcal{F}$.

The quantum theory on the other hand is described by a **Hilbert space** $\mathcal{H}$, which is a linear space over $\mathbb{C}$ equipped with a sesquilinear inner product. Physical states correspond to rays in Hilbert space, i.e. one-dimensional subspaces of $\mathcal{H}$. A set of observables of the quantum theory is given by a subset of endomorphisms $\hat{f} \in \hat{\mathcal{F}} = \mathrm{End}(\mathcal{H})$, called quantum operators, acting on the states of the Hilbert space. Multiplication of operators $\hat{f}, \hat{g}$ can be defined by successive action on states in $\mathcal{H}$ and the space of endomorphisms closes to an algebra under the ordinary commutator $[,]: \hat{\mathcal{F}} \times \hat{\mathcal{F}} \to \hat{\mathcal{F}}$.

Canonical quantization of a classical theory then amounts to an algebra homomorphism $\{f\} \to \{\hat{f}\}$, implying that observables of the classical theory are mapped to quantum operators that satisfy commutation relation given by the Poisson bracket

$$[\hat{f}, \hat{g}] = i\widehat{\{f, g\}}\,. \tag{2.105}$$

Thus, the quantization preserves (to the extent possible) the canonical algebraic structure of the classical theory, induced by the canonical symplectic structure of phase space $\mathcal{P}$.

There are several remarks regarding canonical quantization to be made at this point. The fact that the algebraic structure of the classical theory is carried over to the quantum theory implies that the quantization respects the continuous symmetries of the classical theory (barring quantum anomalies which we will not discuss here). Conserved quantities, i.e. constants of motion, of the classical theory are mapped to quantum operators that act as symmetry generators of the quantum theory through commutation. This can be viewed as the purpose of requiring the quantization to preserve the canonical algebraic structure.

However, it is important to recognize that canonical quantization is not a well-defined procedure. First of all, it is in general not possible to associate quantum operators to all functions $f \in \mathcal{F}(\mathcal{P})$ that are potential physical observables. Thus, canonical quantization involves a choice of a subset (or rather a subalgebra) of observables of the quantum theory. Perhaps more severely, there is an ordering ambiguity in the quantum theory: We define the product on $\mathcal{F}(\mathcal{P})$ by point wise multiplication of functions, which is commutative. However, because of (2.105) the multiplication of operators in the quantum theory (which is itself well-defined as we saw above) is not commutative whenever the Poisson bracket of the two corresponding function is non-vanishing. Thus, given an observable of the classical theory that is expressed as a function of the canonical coordinates (q^i, p_i), the corresponding operator is not well-defined as a product of the operators $\hat{q}^i, \hat{p}_i$.

Finally, one of the distinctive features of the Hamiltonian formulation (which is the setting of canonical quantization) is that is singles out time by describing it as the parameter of the trajectory of a system in phase space, while the position parametrizes the configuration space Q. Thus, manifest Lorentz invariance is lost when applying canonical quantization to a quantum field theory. I order to keep Lorentz invariance manifest we must consider path integral quantization in the Lagrangian formulation.

2.5.3 Constraints

So far we have always implicitly assumed in our treatment of the Hamiltonian formulation and canonical quantization of classical dynamical systems that the system is not subject to any constraints. This means that the available phase space is the entire cotangent bundle T^*Q over the configuration space Q. The evolution according to Hamilton's equations, or equivalently the flow generated by the corresponding Hamiltonian vector field, can then in principle reach any point of $\mathcal{P}$ given appropriate initial conditions. In this context it is important to recognize that conservation of the constants of motion, i.e. the conservation of total energy and any other physical observable that has vanishing Poisson bracket with H, should not be considered as constraints. From the geometric perspective they are simply invariants under the flow generated by X_H.

A set of **constraints** on a dynamical system corresponds to a set of functions $\{\psi_\alpha\}$ on the symplectic manifold (M, σ), where $M = T^*Q$ is the cotangent bundle of the configuration space Q of the system as before, for which we require that $\psi_a = 0$. The phases space of the constrained system is therefore not the cotangent bundle M but rather the submanifold

$$M_0 = \{p \in M | \psi_a(p) = 0 \, , \forall \, a\} \ , \tag{2.106}$$

i.e. the **zero locus** of the set of constraints $\{\psi_a\}$[19].

Below we will review the classification of constraints due to Dirac [19, 20] and discuss the properties of the corresponding submanifolds. The purpose is to arrive at

[19]The representation of the constraints as at set $\{\psi_a\}$ of functions is not unique. Physically, any set of functions defining the same submanifold, and consequently the same phase space for the dynamical system, must be considered equivalent.

a geometric understanding of Dirac's procedure for quantizing constrained classical systems, which involves using the so called Dirac bracket instead of the Poisson bracket. In addition to Dirac's classic lecture notes, an excellent reference for the geometrical part of this discussion is [21], to which we try to conform in terms of notation.

Let $\Psi \subset \mathcal{F}(M)$ be the linear span of the constraint functions $\{\psi_a\}$ and $I \subset \mathcal{F}(M)$ be the ideal in $\mathcal{F}(M)$ generated by $\{\psi_a\}$. In other words, I is the set of functions that vanishes when restricted to the subspace $\psi_a = 0$. According to Dirac we then let $F \subset \Psi$ denote a maximal subspace of functions whose Poisson bracket with the constraints belong to this ideal

$$\{F, \Psi\} \subset I\,. \tag{2.107}$$

We refer to a basis $\{\phi_i\}$ of F, by definition satisfying $\{\phi_i, \psi_a\}|_{M_0} = 0\,, \forall a$, as **first class constraints** and basis $\{\chi_\alpha\}$ of the complement of F in Ψ as **second class constraints**. First class constraints are always associated to a group of symmetries of the dynamical system, generated by the Hamiltonian vector fields of the constraint functions. Correspondingly, there is a redundancy in the canonical description of the system which can be used to eliminate the first class constraints through a **choice of gauge**, resulting in a system of only second class constraints. (Below we will always assume this gauge fixing has been carried out.) Detailed accounts of these results, which we will not discuss further here, can can be found in e.g. [20, 22, 23].

A system of second class constraints $\{\chi_\alpha\}$ are by definition such that no linear combination of their Poisson bracket vanishes even after restriction to M_0. Let $\{Z_\alpha\}$ be the Hamiltonian vector fields corresponding to the constraints

$$\iota_{Z_\alpha}\sigma + d\chi_\alpha = 0\,. \tag{2.108}$$

By definition, a vector field X is tangent to M_0 iff $d\chi_\alpha(X) = 0\,, \forall\alpha$, or using (2.108)

$$\sigma(X, Z_\alpha) = 0\,, \forall\alpha\,. \tag{2.109}$$

The tangent bundle TM_0 of the submanifold (2.106) is therefore given by the symplectic complement of the span of $\{Z_\alpha\}$. Taking the symplectic complement of this relation we find

$$TM_0^\perp = \langle Z_\alpha\rangle\,, \tag{2.110}$$

i.e. $\{Z_\alpha\}$ is a (local) basis of $TM_0^\perp$. From the second class property it follows [19] that $\{\chi_\alpha, \chi_\beta\}$ is non-singular, or correspondingly by (2.77) that $\sigma(Z_\alpha, Z_\beta)$ is non-degenerate. We can now express Dirac's result in terms of symplectic geometry: A system of second class constraints defines a submanifold M_0 that satisfies $TM_0 \cap TM_0^\perp = \emptyset$; a symplectic submanifold.

Since the phase space of the mechanical system is M_0 rather than the full cotangent bundle, its dynamics is described in the Hamiltonian formulation by the flow on (M_0, σ_0) generated by the Hamiltonian. We are therefore interested in the explicit form of σ_0 in terms of σ, or in other words a description in which the constraints of the dynamical system are manifestly satisfied. In particular, we can use the construction in 2.4.1 to compute the Poisson bracket on M_0 defined in terms of σ_0.

Let $f, g \in \mathcal{F}(M_0)$ be functions which extend to functions $f, g \in \mathcal{F}(M)$ and denote the corresponding Hamiltonian vector fields w.r.t. (M, σ) by X_f, X_g. By (2.81) they decompose as

$$X_f = X_f^T + X_f^\perp \quad , \quad X_g = X_g^T + X_g^\perp \, , \tag{2.111}$$

From the definition of σ_0 and $TM_0^\perp$ we can express the Poisson bracket on (M_0, σ_0) as

$$\{f, g\}_0 = \sigma(X_f^T, X_g^T) = \sigma(X_f, X_g) - \sigma(X_f^\perp, X_g^\perp) \, . \tag{2.112}$$

The orthogonal component of a vector $X \in \chi(M)$ is expanded in terms of the basis $\{Z_\alpha\}$ of $TM_0^\perp$ as

$$X^\perp = \sum_\alpha \zeta^\alpha Z_\alpha \tag{2.113}$$

yielding

$$\sigma(X, Z_\alpha) = \sigma(X^\perp, Z_\alpha) = \sum_\beta \zeta^\beta \sigma(Z_\beta, Z_\alpha) \, . \tag{2.114}$$

Now, since M_0 is a symplectic submanifold the matrix

$$C_{\alpha\beta} = \sigma(Z_\alpha, Z_\beta) = \{\chi_\alpha, \chi_\beta\} \tag{2.115}$$

is invertible and, denoting its inverse by $C^{\alpha\beta}$, we can solve for the coefficients in (2.113)

$$\zeta^\beta = \sum_\alpha \sigma(X, Z_\alpha) C^{\alpha\beta} \, . \tag{2.116}$$

Finally, we can then compute

$$\begin{aligned} \sigma(X_f^\perp, X_g^\perp) &= \sum_{\alpha\beta} \sigma(\zeta_f^\alpha Z_\alpha, \zeta_g^\beta Z_\beta) \\ &= \sum_{\alpha\beta} \sigma(X_f, Z_\alpha) C^{\alpha\beta} \sigma(Z_\beta, X_g) \\ &= \sum_{\alpha\beta} \{f, \chi_\alpha\} C^{\alpha\beta} \{\chi_\beta, g\} \end{aligned} \tag{2.117}$$

and obtain the Poisson bracket on the symplectic submanifold (M_0, σ_0) as

$$\{f, g\}_0 = \{f, g\} - \sum_{\alpha\beta} \{f, \chi_\alpha\} C^{\alpha\beta} \{\chi_\beta, g\} \, . \tag{2.118}$$

This is the Dirac bracket constructed in [19]. The derivation above in terms of symplectic submanifolds provides a geometrical interpretation of its role in the Hamiltonian description of constrained dynamical systems[20].

[20]The geometric derivation is also significantly simpler than the original one due to Dirac. In

2.5.4 Quantization of constrained systems

Applying the canonical quantization procedure to a constrained classical system is straightforward given the developments in the preceding section: The phase space $\mathcal{P}$ and its canonical symplectic structure comprise the symplectic manifold (M_0, σ_0) defined by the constraints in (2.106) and the commutation relation in (2.105) is consequently given by the Poisson bracket on M_0

$$[\hat{f}, \hat{g}] = i\widehat{\{f, g\}}_0 \,. \tag{2.119}$$

The discussion in 2.5.2, which is solely based on the symplectic structure of $\mathcal{P}$, remains valid even when the phase space is submanifold of the cotangent bundle. In this sense it is trivial to extend canonical quantization to constrained systems, which is not surprising; a constraint can indeed be viewed as a redundancy in the description in terms of the configuration space Q and the Dirac bracket can be viewed as the Poisson bracket defined for the physical degrees of freedom parametrizing the actual phase space.

From a more pragmatic point of view, however, it can be advantageous to retain the redundant description of the classical system in terms of Q and an accompanying set of constraints. The "problem" of canonical quantization is then to ensure that the constraint equations $\chi_\alpha = 0$ are consistent with the commutation relations $[\hat{f}, \hat{g}]$ assigned. Consider for example the simple system

$$q_1 = 0 \quad , \quad p^1 = 0 \tag{2.120}$$

of second class constraints. This is obviously inconsistent with the ordinary commutation relation $[\hat{q}_i, \hat{p}^j] = -i\delta_i^j$ obtained from the ordinary Poisson bracket. In this case the inconsistency can be remedied by eliminating the degree of freedom corresponding to the index value $i = 1$ and redefining the Poisson bracket in terms of the remaining canonical coordinates. (Indeed, this provides the simplest example of symplectic restriction of the cotangent bundle.) Generalizing the example, it is possible to derive the property

$$[\hat{f}, \chi_\alpha] = 0 \,, \forall \alpha \tag{2.121}$$

using the expression (2.118), which guarantees the consistency of the Dirac bracket with the constraints $\chi_\alpha = 0$. Thus, canonical quantization of a system subject to second class constraints is consistent when the commutation relations are provided by the Dirac bracket. In chapter 3 we will use this construction quantize the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory located at isolated vacua at weak coupling, which is the concern of PAPER I-II.

fact, Dirac himself pointed out [20] that there ought to be a simpler way of establishing the Jacobi identity for the new bracket than a brute force computation. In the geometric formulation this follows immediately from the definition $\sigma_0 = i^*\sigma$.

3

Isolated vacua in Yang-Mills theory on T^3

After the mathematical introduction of the previous chapter we begin the discussion of the first of the two main topics of this thesis: The maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory in four dimensions. As is indicated by the title of the present chapter we will be particularly interested in the case when the spatial part of $M_4 = B \times \mathbb{R}$ is a flat torus $B = T^3 = \mathbb{R}^3/\mathbb{Z}^3$, which is the case considered in PAPER I-II. We will first consider the supersymmetry algebra in four dimensions and construct the massless $\mathcal{N} = 4$ vector multiplet, whose dynamics is described by the Yang-Mills theory. Subsequently, we will consider the vacuum structure of the Yang-Mills theory on T^3, and in particular the moduli space $\mathcal{M}$ of flat connections of principal bundles over T^3. The fact that T^3 is not contractible implies the existence of topologically non-trivial bundles and $\mathcal{M}$ is decomposed into disjoint subspaces according to the topological class of the bundle.

In particular, there exist for some gauge groups G isolated points in $\mathcal{M}$ which correspond to normalizable vacua of the classical Yang-Mills theory and are well suited for performing a perturbative analysis in the weak coupling regime. PAPER I is concerned with computing the energy spectrum of the free theory in the weak coupling limit at all such isolated points of $\mathcal{M}$ for arbitrary simple gauge group. PAPER II is devoted to an attempt to give a perturbative description of the interacting Hilbert space of the theories located at isolated points for $\mathrm{SU}(n)$ gauge groups.

3.1 Supersymmetric Yang-Mills theory

The first order of business is to properly introduce the concept of supersymmetry mentioned in chapter 1. It generalizes the ordinary notion of a symmetry transformation of a theory by the introduction of fermionic supersymmetry generators that exchange bosonic and fermionic fields. This can be contrasted to e.g. a translation of a field which is related to the same field at a different point in space-time. In this chapter and the next we consider supersymmetry in four dimensions and denote vector indices of the Lorentz group SO(3, 1) by $\mu = 0, \ldots, 3$. We will denote by γ^μ the matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ where $\eta^{\mu\nu}$ is the (inverse) Minkowski metric. (We consider exclusively flat space-times in this chapter since our present investigations concern the flat T^3.) There are a large number of references complementing the material covered in this section, including e.g. [24–26].

3.1.1 The supersymmetry algebra

In the introduction we mentioned that the theorem of Coleman and Mandula [2] states that the most general algebra of generators of continuous symmetries in quantum field theory is given by the Poincaré algebra together with the algebra of a (finite) internal Lie group[1]. In the case of a theory describing massless particles the Poincaré algebra can in fact be enlarged to accommodate conformal symmetry generators. We will however not consider conformal symmetry at the algebraic level in this thesis.

We also mentioned how Haag, Lopuszanski and Sohnius found a generalization [3] by removing the assumption that the symmetry generators form an ordinary Lie algebra. More specifically, the supersymmetry algebra is obtained by considering a $\mathbb{Z}_2$ graded Lie algebra with an bosonic (even) part X consisting of the Poincaré generators P_μ and $M_{\mu\nu}$, and the internal symmetry generators T_a according to the Coleman-Mandula theorem, while the fermionic (odd) part Q consist of the supersymmetry generators. The Lie bracket is given by commutation and anticommutation for the bosonic and fermionic parts respectively. In particular, in order to respect the grading we must have

$$[X, Q] \subseteq Q \tag{3.1}$$

which implies that the supersymmetry generators furnish a representation of the bosonic subalgebra. From the super-Jacobi identity (generalizing the ordinary Jacobi identity to the $\mathbb{Z}_2$-graded case) it follows that the supersymmetry generators transform as spinors under the Lorentz group SO(3, 1) while they commute with P_μ and are consequently space-time translation invariant. In four dimensions, consistency of the algebra requires the T_a to generate the internal symmetry group SU($\mathcal{N}$), in the supersymmetric context commonly called the R-symmetry group[2], where the cases $\mathcal{N} = 1$ and $\mathcal{N} > 1$ are referred to as minimal and extended supersymmetry. The supersymmetry generators Q transform in the $\mathcal{N}$-dimensional representation

[1]Here, internal refers to the generators being invariant under the Poincaré subalgebra.

[2]In fact, the R-symmetry group allowed by the algebra is SU(N) × U(1). The U(1) factor is anomalous and broken in the quantum theory and we will not consider it further here.

of the R-symmetry group. Supersymmetry in various space-time dimensions is described in e.g. [24, 27, 28] and in particular all possible supersymmetry algebras relevant for particle physics are classified in [29] together with their representations (see also [30] for a comprehensive treatment).

We will be exclusively interested in the $\mathcal{N} = 4$ algebra in the present chapter, which is in fact the maximally extended[3] algebra in four dimensions [29], with R-symmetry group $\mathrm{SU}(4) \cong \mathrm{SO}(6)$. The supersymmetry generators are Majorana spinors under the Lorentz group and their chiral and anti-chiral parts are conventionally distinguished by the vertical position of the R-symmetry index as Q_i and Q^i, with $i = 1, \ldots, 4$. We then have $Q = Q_i + Q^i$ where the two terms are related by complex conjugation. In the present case Q_i transforms in the $(\mathbf{2}_+; \mathbf{4})$ representation of the bosonic subalgebra $\mathfrak{so}(3,1) \oplus \mathfrak{su}(4)$, where $\mathbf{2}_+$ is a chiral Lorentz spinor and $\mathbf{4}$ is the fundamental spinor representation of SU(4).

In order to complete the description of the supersymmetry algebra we must now specify the anticommutation relations for the Q's. From the properties of the bosonic subalgebra it follows that

$$\{Q_i, Q^j\} = \frac{1}{2}(1+\gamma)\gamma^\mu P_\mu \delta_i^j \,, \tag{3.2}$$

where γ is the chirality operator in four dimensions. It is also possible to have non-zero anticommutators

$$\{Q_i, Q_j\} = \frac{1}{2}(1+\gamma)Z_{ij} \quad , \quad \{Q^i, Q^j\} = \frac{1}{2}(1-\gamma)Z_{ij} \tag{3.3}$$

with Z_{ij} an antisymmetric matrix of central charges of the algebra. We will, however, almost exclusively consider massless representations in this thesis for which Z_{ij} must vanish.

3.1.2 The $\mathcal{N} = 4$ vector multiplet

Having introduced the $\mathcal{N} = 4$ supersymmetry algebra in the previous subsection we now proceed to construct the massless representation of this algebra which will be considered below. To find this representation we use Wigner's method of induced representations, by studying the little group which leaves invariant the light-like momentum of a massless particle and can be put on the standard form $P_\mu = (E, 0, 0, E)$. Since the internal generators T_a commute with P_μ the little group in the present case is $\mathrm{SO}(2) \times \mathrm{SU}(4)$. We are therefore interested in the decomposition of the supersymmetry generator Q under the subgroup $\mathrm{SO}(1,1) \times \mathrm{SO}(2) \times \mathrm{SU}(4)$ of the bosonic group $\mathrm{SO}(3,1) \times \mathrm{SU}(4)$.

[3] As mentioned in the introduction maximal supersymmetry in a theory of particle physics is due to the restriction to particles of spin less than or equal to one.

In general, the little group for massless representations induces a decomposition of the supersymmetry generators according to

$$Q = Q_{1/2} + Q_{-1/2} \tag{3.4}$$

where the subscript denotes the SO(1, 1) weight. From the supersymmetry algebra it can be shown that for light-like momenta the $Q_{-1/2}$ generators anticommute with all other generators, implying that they must be represented by zero when acting on the physical Hilbert space of states. Consequently, only the so called active (see e.g. [30]) supersymmetry generators $Q_{1/2}$ are associated to physical states of the representation.

In four dimensions we find that $Q_{1/2}$ transform under the little group as

$$(\mathbf{1}_+, \mathbf{4}) \oplus (\mathbf{1}_-, \bar{\mathbf{4}}) \tag{3.5}$$

where the first factor contains SO(2) chiral spinors and the SO(1, 1) weight is suppressed. From the anticommutation relation (3.2) it follows that the two parts constitute the creation and annihilation operators of a Clifford algebra. The representation is thus obtained by successively acting with the creation operators (taken to be the ones which increase the helicity) as

$$(\mathbf{1}_{-1} \oplus \mathbf{1}_1) \oplus \mathbf{6}_0 \oplus (\mathbf{4}_{1/2} \oplus \bar{\mathbf{4}}_{-1/2}) , \tag{3.6}$$

where the subscript denotes the helicity eigenvalue. The multiplet with the above representation content can be represented in terms of fields as a vector gauge field A_μ, six scalar fields ϕ_A in the $\mathbf{6}$ vector representation of SO(6) and a Majorana spinor $\psi = \psi_i + \psi^i$ transforming in the $(\mathbf{2}_+, \mathbf{4}) \oplus (\mathbf{2}_-, \bar{\mathbf{4}})$ as the supersymmetry generators. The multiplet obtained is called the Yang-Mills multiplet due to the highest spin particle being a massless gauge vector potential. We can no return to the statement made in the previous subsection regarding $\mathcal{N} = 4$ being the maximally extended algebra: By the above procedure for constructing representations, any $\mathcal{N} > 4$ multiplet will necessarily include particles of helicity greater than one.

3.1.3 A ten-dimensional formulation

Having determined the representation content of the $\mathcal{N} = 4$ vector multiplet, we will now consider its field theory realization. If the gauge field A_μ is the connection of a non-abelian principal bundle the dynamics of the theory is given by a Lagrangian in four dimensions which, in addition to the ordinary gauge invariant kinetic terms for the fields, include a ϕ^4 self-interaction term and a Yukawa coupling term (schematically given by) $\phi\bar{\psi}\psi$.

A convenient way to describe this theory, which we will find useful below, is as the dimensional reduction of the $\mathcal{N} = 1$ vector multiplet in ten dimensions [31]. The field content of this theory is a gauge vector field A_M and a Majorana-Weyl spinor

ψ transforming respectively in the representations **10** and $\mathbf{16}_+$ of the Lorentz group SO(9, 1), and both transforming in the adjoint representation of the Lie algebra $\mathfrak{g}$ of the gauge group G. Here, the vector index is $M = 0, \ldots, 9$. The Lagrangian density of the supersymmetric Yang-Mills theory is

$$S = \frac{1}{g^2} \int d^{10}x \, \mathrm{Tr} \left(-\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} \overline{\psi} \Gamma^M D_M \psi \right) \tag{3.7}$$

where Γ^M are the generators of the ten-dimensional Clifford algebra and g is the gauge coupling constant. The gauge field strength and covariant derivative D_M are defined as usual by (c.f. section 2.3)

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + f^a{}_{bc} A^b_M A^c_N \tag{3.8}$$

and

$$D_M \psi^a = \partial_M \psi^a + f^a{}_{bc} A^b_M \psi^c \,, \tag{3.9}$$

where a takes it values in the adjoint representation of $\mathfrak{g}$. (We will frequently suppress the Lie algebra index when the transformation properties are clear from the context.) The action is invariant under the supersymmetry transformations

$$\delta A_M = \frac{1}{2} \overline{\eta} \Gamma_M \psi \quad , \quad \delta \psi = \frac{1}{4} F_{MN} \Gamma^{MN} \eta \,, \tag{3.10}$$

where η is the constant fermionic parameter of the transformation in the same spinor representation as ψ of the Lorentz group. The variations indeed represent the ten-dimensional $\mathcal{N} = 1$ supersymmetry algebra since on-shell they satisfy (up to numerical constants)

$$[\delta_1, \delta_2] \psi = \overline{\eta}_2 \Gamma^P \eta_1 D_P \psi \quad , \quad [\delta_1, \delta_2] F_{MN} = \overline{\eta}_2 \Gamma^P \eta_1 D_P F_{MN} \,. \tag{3.11}$$

The dimensional reduction to four dimensions implies the splitting of the vector index in ten dimensions according to $M = (\mu, A)$. Here, $\mu = 0, \ldots, 3$ correspond to the coordinates x^μ of the four-manifold $M_4 = T^3 \times \mathbb{R}$ while $A = 4, \ldots, 9$ correspond to the six transverse coordinates. In order to have a dimensional reduction (rather than a compactification) we demand that $\partial_A = 0$ so that there is no dependence on the transverse coordinates remaining in the four-dimensional theory. In particular, this implies a restriction of the (spatial) momentum eigenvalues of the ten-dimensional theory to the form $p_I = (p_1, p_2, p_3, 0, \ldots, 0)$, where $I = 1, \ldots, 9$ denotes all spatial directions in the higher-dimensional theory in contrast to $i = 1, 2, 3$ which we will now reserve[4] for the spatial directions on T^3.

The decomposition of the vector index corresponds to a decomposition of the Lorentz group according to $\mathrm{SO}(9,1) \to \mathrm{SO}(3,1) \times \mathrm{SO}(6)$, where the second factor

[4]The R-symmetry index i used above will be suppressed for the rest of the present chapter. It should always be clear from the context which of the notationally coinciding indices we refer to.

corresponds to rotations in the transverse directions constituting a global symmetry of the theory on M_4. The fields of the ten-dimensional theory decompose under the reduction according to their respective representations as

$$\mathbf{10} \to (\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{6}) \tag{3.12}$$

and

$$\mathbf{16}_+ \to (\mathbf{2}_+, \mathbf{4}) \oplus (\mathbf{2}_-, \bar{\mathbf{4}}) \,, \tag{3.13}$$

which we recognize as the representation content of the $\mathcal{N} = 4$ multiplet with the SO(6) transverse rotational symmetry identified with the R-symmetry. For the vector field the decomposition corresponds to interpreting the A_μ components as a gauge vector field on M_4 and the transverse components A_A as six scalars ϕ_A. The spinor is interpreted as the chiral components of a Majorana spinor ψ in four dimensions. Performing the dimensional reduction one obtains from the action (3.7), in addition to the gauge invariant kinetic terms of the vector multiplet (A_μ, ϕ_A, ψ), the Yukawa and ϕ^4 terms from the transverse part of the gauge field.

3.2 Vacuum structure

Having introduced the $\mathcal{N} = 4$ Yang-Mills theory in the previous section we will now consider its vacuum structure, that is field configurations of vanishing energy. Throughout the remainder of the chapter we will restrict our attention to compact simple gauge groups on adjoint form, that is $G = \tilde{G}/C$ where $\tilde{G}$ is the simply connected cover of G and C is its center subgroup as in section 2.2.3. Furthermore, we will consider a Hamiltonian formulation (as hinted at by the product structure of $M_4 = B \times \mathbb{R}$) and work in temporal gauge $A_0 = 0$. In particular, this allows us to consider the spatial part A_i of the gauge field as the Lie algebra valued local connection one-form $A = A_i dx^i$ of a principal G-bundle P over B (rather than over the full space-time M_4). The scalar and spinor fields are then related to the space $\Gamma(E)$ of sections of the bundle

$$E = \mathrm{ad}(P) = P \times_{\mathrm{ad}} \mathfrak{g} \,, \tag{3.14}$$

associated to P through the adjoint action of G on its Lie algebra $\mathfrak{g}$, since the supersymmetry require them to transform in the adjoint representation just as the gauge field A_μ^a[5]. Consequently, vacuum states of the Yang-Mills theory are characterized by the topological class of the bundle P (which we recall is inherited by the associated bundles).

[5]More precisely, the scalars are sections of E while the spinors are sections of $E \otimes S$ where S is a spinor bundle over space-time.

3.2.1 The moduli space of flat connections

A zero energy configuration in supersymmetric Yang-Mills theory requires all contributions to the (manifestly non-negative) Hamiltonian to vanish independently. In particular, the contribution to the energy from the gauge field is given by the sum of the electric and magnetic energies, proportional respectively to $\mathrm{Tr}(F^{ij}F^{ij})$ and $\mathrm{Tr}(F^{0i}F^{0i})$. Since both contributions are manifestly positive vacuum states must have $F^{ij} = 0$ and $F^{0i} = 0$. The first of these conditions implies that vacua are supported on the moduli space $\mathcal{M}$ of gauge inequivalent flat connections on P. The second condition, together with the fact that the conjugate momentum of A_i is F^{0i}, implies that the wave functions of the vacua are constant on (each component of) $\mathcal{M}$. In addition to the gauge field the $\mathcal{N} = 4$ multiplet also contains scalars and spinors; only the covariantly constant modes of these fields give vanishing contributions to the energy and feature in the description of a vacuum state.

Since B is three-dimensional we recall from section 2.2.3 that the topological class of the bundle P is determined by the 't Hooft flux

$$m \in H^2(T^3, C)\,, \tag{3.15}$$

which measures the obstruction to lifting P to a principal $\tilde{G}$-bundle. In the Hamiltonian formalism we are currently employing m is really to be thought of as part of the 't Hooft flux v of the full space time bundle. Following 't Hooft [32, 33], m is therefore referred to as the magnetic 't Hooft flux. The other components of v are similarly referred to as the electric 't Hooft flux $e \in H^1(T^3, C)$ and describe the transformation properties of states in the Hilbert space under large transformation properties [33, 34]. We will not be concerned with these transformation properties in the context of the present chapter or that of Paper I-II.

Through the restriction to the T^2 in the ij directions the magnetic 't Hooft flux m defines a triple of elements $m_{ij} \in H^2(T^2, C) \cong C$ which provides an isomorphism $H^2(T^3, C) \cong C^3$

$$m = (m_{23}, m_{31}, m_{12}) \in C^3\,. \tag{3.16}$$

The triple transforms as a vector under the $\mathrm{SL}(3, \mathbb{Z})$ mapping class group of the torus [34]. If the center subgroup C is cyclic it is possible to put the magnetic flux on the form $m = (\mathbb{1}, \mathbb{1}, m)$. In this chapter we will only consider such cases and identify the $\mathrm{SL}(3, \mathbb{Z})$ equivalence class of $m \in H^2(T^3, C)$ with the $m = m_{12}$ component, which uniquely determines the topological class of the bundle P. The moduli space $\mathcal{M}$ is correspondingly decomposed into disjoint subspaces as

$$\mathcal{M} = \bigcup_m \mathcal{M}(m)\,, \tag{3.17}$$

where $\mathcal{M}(m)$ are the moduli space components corresponding to different topological sectors. Below we will consider the structure of the components $\mathcal{M}(m)$ in more detail.

3.2.2 Almost commuting triples

In order to proceed with the study of vacuum states and the structure of the moduli space $\mathcal{M}$ we use the fact that the space of flat connections is parametrized by the holonomies (see section 2.3.1)

$$U_i = P \exp\left(\int_{\gamma_i} A\right) \tag{3.18}$$

around the three homotopically inequivalent generators γ_i of the fundamental group $\pi_1(T^3)$. They constitute well-defined coordinates since the connection is flat and the holonomy U_i is therefore independent on the choice of representative in the homotopy class of γ_i. However, in order to consider only gauge inequivalent flat connections we must account for the fact that the holonomies transform as

$$U_i \to g^{-1} U_i g \tag{3.19}$$

under a gauge transformation in $\mathrm{Aut}_0(P)$ with parameter g (evaluated at the common base point of the curves γ_i). Thus, the moduli space $\mathcal{M}$ is parametrized by the holonomies (U_1, U_2, U_3) modulo simultaneous conjugation.

In terms of these coordinates we can describe the topology of the bundle by considering the curve $\gamma = \gamma_i \gamma_j \gamma_i^{-1} \gamma_j^{-1}$. Because it is trivial in $\pi_1(T^3)$ the holonomy around γ is also trivial and the holonomies U_i are consequently mutually commuting elements in G. However, their lift $(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3)$ to the covering group will satisfy

$$m_{ij} = \tilde{U}_i \tilde{U}_j \tilde{U}_i^{-1} \tilde{U}_j^{-1} \,, \tag{3.20}$$

where $m_{ij} \in C$ are the components in (3.16) of the magnetic 't Hooft flux [34]. The elements $(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3)$ are therefore referred to as an almost commuting triple of elements of $\tilde{G}$. (In fact, $(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3)$ with $m = 0$ is referred to as a commuting triple when it is clear from the context that we are considering the lift.) Different lifts of the holonomies are related through multiplication of the $\tilde{U}_i$ by elements of the center C, so the relation (3.20) is well-defined. Furthermore, the action of the mapping class group $\mathrm{SL}(3, \mathbb{Z})$ on $\pi_1(T^3)$ induces an action on the triple of holonomies (U_1, U_2, U_3) which in turn induces an action on the lift $(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3)$ and the magnetic 't Hooft flux $m \in H^2(T^3, C)$ [34, 35]. We also see that the m_{ij} describes the obstruction to lifting P to a flat $\tilde{G}$-bundle over T^3 and determines the topological class of the G-bundle P.

As we saw above, the moduli space $\mathcal{M}$ is decomposed into disjoint subspaces $\mathcal{M}(m) \subset \mathcal{M}$ of flat connections on bundles belonging to distinct isomorphism classes. We can obtain further structural information by considering that the gauge group G is broken by a flat connection down to its centralizer $H \subset G$, defined as the subgroup of G that simultaneously commute with all three holonomies (U_1, U_2, U_3).

The rank of H can be shown to be locally constant on the components $\mathcal{M}(m)$, which can thus be further decomposed as

$$\mathcal{M}(m) = \bigcup_a \mathcal{M}_{r_a}\,, \tag{3.21}$$

where r_a denote the rank of H on $\mathcal{M}_{r_a}$. The ranks r_a satisfy the relation

$$\sum_a (r_a + 1) = g^\vee\,, \tag{3.22}$$

where $g^\vee$ is the dual Coxeter number of the Lie algebra $\mathfrak{g}$. The relation, which was first found by Witten for the topologically trivial component in [36] and later extended to non-trivial m [37], can be understood in terms of the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in the following way [34]: For simplicity we consider only the component $\mathcal{M}(0)$ of topologically trivial flat connections. By arguments involving the breaking of chiral symmetry it can be shown that the Witten index (or supersymmetric index) $\mathrm{Tr}(-1)^F$ [38] is equal to $g^\vee$ for arbitrary simple gauge group G. On the other hand, it can be shown that on a component with unbroken gauge group H of rank r_a there are $r_a + 1$ unpaired vacua contributing $r_a + 1$ to the Witten index. Summing the contribution from all components of $\mathcal{M}(0)$ one obtains the relation (3.22). An adaptation of the Witten index argument to the $\mathcal{N} = 4$ theory, where the analysis is complicated by the absence of a mass gap, was considered in [39, 40] by counting normalizable bound states at threshold.

Historically the conjugacy classes of almost commuting triples were first studied in mathematics [41, 42] as a property of simple gauge groups. In [36] Witten then used the insight that the moduli space $\mathcal{M}(0)$ was disconnected to resolve the mismatch in the computation of supersymmetric index $\mathrm{Tr}(-1)^F$, discussed in the preceding paragraph, for the $\mathrm{Spin}(n)$ groups. A complete classification of commuting triples in all simple gauge groups was soon obtained [43–46]. Borel et. al. [37] then extended the analysis to the topologically nontrivial components of $\mathcal{M}$ and provided a complete classification of almost commuting triples in simple gauge groups, subsequently used by Witten [34] for completing the analysis of the supersymmetric index in four-dimensional gauge theories.

3.2.3 The structure of moduli space

We will now review the construction of the moduli space $\mathcal{M}(m)$ of flat connections, or equivalently almost commuting triples, for arbitrary topological class m [37]. We will only concern ourselves with finding the disconnected components $\mathcal{M}_a$ and determine the corresponding rank r_a of the unbroken gauge group. Further details of the construction are discussed in Paper I but will not be repeated here.

Let $\tilde{D}$ be the extended Dynkin diagram of $\mathfrak{g}$ and Σ be the group of diagram automorphisms $\sigma : \tilde{D} \to \tilde{D}$. The σ acts by permutation on the nodes of $\tilde{D}$ with identical dual Coxeter labels g_α and it can be shown [37] that $\Sigma \cong C$, allowing us to associate an element $\sigma(m) \in \Sigma$ to each topological class of principal G-bundles. The relevant diagram for the component $\mathcal{M}(m)$ is then the quotient diagram $\tilde{D}/\sigma(m)$ with dual Coxeter labels $g_{\overline{\alpha}} = n_\alpha g_\alpha$, where n_α is the cardinality of the orbit of nodes with label g_α. For each integer $k \geq 1$ dividing at least one of the $g_{\overline{\alpha}}$ the moduli space $\mathcal{M}(m)$ contains $\varphi(k)$ components, where $\varphi(k)$ is the number of integers[6] coprime to k, with $r_a + 1$ given by the number of $g_{\overline{\alpha}}$ such that $k|g_{\overline{\alpha}}$. The integer k is called the order of the components.

As an example let us consider the case of $G = G_2$, which we will return to below. Since G_2 is simply connected the moduli space of flat connections contains only the topologically trivial component. This is also apparent from the extended Dynkin diagram in figure 3.1 which has no non-trivial automorphisms.

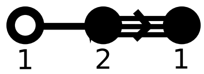


Figure 3.1: The extended Dynkin diagram $\tilde{D}$ of G_2.

We see that $k = 1$ divides all three g_α and since $\varphi(1) = 1$ we obtain the identity component of rank $r_a = 2$. (It is generally true that the moduli space contains one maximal rank component since $k = 1$ divides any integer. For commuting triples this is the identity component with $\mathrm{rank}(H) = \mathrm{rank}(G)$.) Furthermore, $k = 2$ divides only $g_\alpha = 2$ corresponding to an additional component of rank $r_a = 0$ where the gauge group is completely broken down to a finite H. The total moduli space of flat connections for G_2 can thus be written

$$\mathcal{M}_{G_2} = \mathcal{M}_2 \cup \mathcal{M}_0 \tag{3.23}$$

and since $g^\vee_{G_2} = 4$ we see that (3.22) is indeed satisfied.

The dimension of each component of $\mathcal{M}(m)$ is given by $3r_a$ since the holonomies locally vary in a maximal torus of the unbroken gauge group. In particular, this implies that components with $r_a = 0$ are isolated points in $\mathcal{M}$. The corresponding holonomies break all generators of the gauge group (c.f. the $\mathcal{M}_0$ for the case of G_2 discussed above) and are therefore referred to as isolated triples or rank zero triples. Below, we will consider exclusively such points in $\mathcal{M}(m)$. A complete classification of rank zero triples in simple gauge groups was obtained in [37] and is the basis for the considerations of Paper I.

3.3 The weak coupling spectrum

In this section we will consider a weak coupling expansion around a certain class of vacuum states of the $\mathcal{N} = 4$ theory along the lines of Paper I. As we saw above,

[6] The number $\varphi(k)$ is also known as the Euler φ-function of k.

the scalar and spinor fields appearing in the low energy theory are required to be covariantly constant. Consequently, the corresponding zero modes are valued in the maximal torus of the unbroken gauge group. The most general form of the Lie algebra of H is $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{u}(1)^r$, where $\mathfrak{s}$ is semi-simple. However, the presence of zero modes complicates the description of the vacuum states located at the subspaces $\mathcal{M}^H \subset \mathcal{M}(m)$. In particular, the $\mathfrak{u}(1)$ factors corresponds to flat directions in phase space and plane wave solutions for the scalar zero modes which cannot be normalized. To avoid the problems with zero modes we restrict our investigation to the isolated points in $\mathcal{M}(m)$, discussed above, where the gauge group is completely broken by the holonomies. Due to the absence of spinor and scalar zero modes such a rank zero triple of holonomies completely describes the isolated vacuum state.

Let (U_1, U_2, U_3) be the holonomies of an isolated flat connection that we will denote by $\mathcal{A}$. Since the holonomies are mutually commuting and the center C acts trivially on $\mathfrak{g}$ in the adjoint representation it is possible to introduce a basis $\{T_z\}$ of $\mathfrak{g}$ where the action of the $\tilde{U}_i$ are diagonal

$$\tilde{U}_i^{-1} T_z \tilde{U}_i = z_i T_z \,. \tag{3.24}$$

The action of the holonomies U_i is of course also diagonal, with the same eigenvalues z_i in this basis. Since the fundamental group $\pi_1(T^3)$ of the torus is cyclic, the holonomies are elements of finite order in G, and the eigenvalues z_i are consequently complex roots of unity. A distinctive feature of a rank zero triple is the absence of an eigenvalue vector $\vec{z} = (1,1,1)$ since all generators of the gauge group have been broken[7]. Furthermore, the $\mathrm{SL}(3,\mathbb{Z})$ action on the holonomies induces a multiplicative action in each component of $\vec{z}$. In the case of trivial topology of the bundle P the spectrum of z_i is $\mathrm{SL}(3,\mathbb{Z})$-invariant, while the twisting by m in the non-trivial case breaks this symmetry to an $\mathrm{SL}(2,\mathbb{Z})$ in the first two components of $\vec{z}$.

The self-adjoint covariant derivative with respect to the connection $\mathcal{A}$ is denoted $i\mathcal{D}_i$ and, as was mentioned in section 2.3.2, it acts on the space of sections $\Gamma(E)$ of the associated adjoint bundle. The eigensections $u_p(x)$ of $i\mathcal{D}_i$, defined by

$$i\mathcal{D}_i u_p(x) = 2\pi p_i u_p(x) \,, \tag{3.25}$$

form a basis of $\Gamma(E)$ with the properties

$$\bar{u}_p = u_{-p} \quad , \quad (u_p, u_{p'}) = \delta_{p,p'} \,. \tag{3.26}$$

Here we have introduced the sesquilinear inner product on $\Gamma(E)$ defined by

$$(\alpha, \beta) = \int_{T^3} d^3x \mathrm{Tr}(\bar{\alpha}\beta) \,. \tag{3.27}$$

[7]Indeed, the diagonalization of the adjoint action of the holonomies can be performed for an arbitrary almost commuting triple in $\mathcal{M}$. The unbroken generators will then correspond to the generators T_z with $\vec{z} = (1,1,1)$.

In PAPER I the eigensections u_p are constructed and the eigenvalues p_i are related to the z_i of the previous paragraph through

$$p_i = \frac{1}{2\pi}\mathrm{Arg} z_i + k_i\,, \tag{3.28}$$

where Arg denotes the principal argument and $k_i \in \mathbb{Z}$ correspond to dual lattice vectors on T^3. It is interesting to note that this relation implies that in the theory located at isolated vacua the fact that $\vec{z} \neq (1,1,1)$ introduces an IR cutoff by shifting all reciprocal lattice vectors by a non-zero rational number. In the last subsection we will discuss the eigenvalues z_i in more detail. First, however, we will consider the weak coupling quantum theory located at the isolated point of moduli space corresponding to the flat connection $\mathcal{A}$.

3.3.1 The quantum theory at isolated vacua

We now utilize the formulation of the $\mathcal{N} = 4$ theory in terms of the minimally supersymmetric Yang-Mills theory in $9+1$ dimensions, in order to study the weak coupling expansion around an isolated vacuum state. The background connection is then identified with the flat connection $\mathcal{A} = \mathcal{A}_i dx^i$ on T^3, while the components corresponding to scalars in $\mathcal{N} = 4$ theory vanish. Expanding around this background (which also has vanishing fermionic fields) at weak coupling we obtain

$$A = \mathcal{A} + ga \quad , \quad \psi = g\lambda\,. \tag{3.29}$$

We denote the covariant derivative with respect to the full connection A by D to distinguish it from the $\mathcal{D}$ introduced above. In order to stay in temporal gauge we impose $a_0 = 0$ so that $a = a_I dx^I$, and impose the (background) Coulomb gauge condition $\mathcal{D}_I a^I = 0$ to fix the remaining gauge invariance. To lowest order in the coupling constant g the Yang-Mills action is then

$$S = \int_{T^3} d^3x \left(-\frac{1}{2}\mathcal{D}_M a_N \mathcal{D}^M a^N + \frac{1}{2}\overline{\lambda}\Gamma^M \mathcal{D}_M \lambda \right)\,. \tag{3.30}$$

which describes the free effective theory in the weak coupling regime, i.e. when $g \to 0$. In order to perform canonical quantization of this theory we must then consider the transition to a Hamiltonian formulation. The momenta conjugate to the dynamical fields a_I and λ are $\pi^I = \dot{a}^I$ and $\pi^{(\lambda)} = \frac{1}{2}\overline{\lambda}\Gamma^0$ so the Hamiltonian of the system is given by

$$H = \int_{T^3} d^3x \left(\frac{1}{2}\pi_I \pi^I + \frac{1}{2}\mathcal{D}_I a_J \mathcal{D}^I a^J - \frac{1}{2}\overline{\lambda}\Gamma^I \mathcal{D}_I \lambda \right)\,. \tag{3.31}$$

We can then proceed to expand the fields and their conjugate momenta in the basis of complete eigenfunctions $u_p(x)$ of the covariant derivative $i\mathcal{D}_i$ as

$$a_I(x) = \sum_p a_I(p) u_p(x) \ , \ \pi^I(x) = \sum_p \pi^I(p) u_p(x) \ , \ \lambda(x) = \sum_p \lambda(p) u_p(x)\,. \tag{3.32}$$

Inserting the expansions and using the properties of the eigenfunctions $u_p(x)$ we obtain the expression for the Hamiltonian in momentum space

$$H = \frac{1}{2}\sum_p \left(\pi_I(-p)\pi^I(p) + 4\pi^2 a_I(-p)a^I(p) - 2\pi\lambda(p)^\dagger\Gamma^0\Gamma^I p_I\lambda(p)\right) , \tag{3.33}$$

where we have also used that reality[8] of the a and λ imply that the Fourier modes satisfy the reality conditions

$$a_I(p)^* = a_I(-p) \ \ , \ \ \lambda(p)^* = \lambda(-p) . \tag{3.34}$$

We can now proceed with the canonical quantization of the free theory according to the procedure outlined in chapter 2. The fermionic sector is not subject to any constraints that make inconsistent the assignment of the ordinary anticommutation relation

$$\{\bar{\epsilon}\lambda(x), \overline{\lambda}(x')\epsilon'\} = c_F\bar{\epsilon}\Gamma^0\epsilon'\delta^{(3)}(x-x') , \tag{3.35}$$

where we have introduced bosonic spinors ϵ, ϵ' to contract the spinor indices and an overall normalization constant c_F.

For the bosonic fields we must use the Dirac bracket $\{f,g\}_0$ of section 2.5.4 in order to obtain consistent commutation relations. Using the temporal gauge $a_0 = 0$ and the antisymmetry of the gauge field strength, which implies $\pi^0 = 0$, we can begin by eliminating the corresponding degrees of freedom. The bosonic sector is then described by the components a_I and π^I which parametrize the cotangent bundle of configuration space and are subject to the constraints[9]

$$\chi^a_{1x} = (\mathcal{D}_I a^I(x))^a \quad , \quad \chi^a_{2x} = (\mathcal{D}_I \pi^I(x))^a . \tag{3.36}$$

These constraints are second class since the matrix defined in (2.115), whose independent component is given by

$$C^{ab}_{1x,2y} = \int_{T^3} d^3x' \left[\frac{\delta\chi^a_{1x}}{\delta a^c_I(x')}\frac{\delta\chi^b_{2y}}{\delta\pi^I_c(x')} - \frac{\delta\chi^b_{2y}}{\delta a^c_I(x')}\frac{\delta\chi^a_{1x}}{\delta\pi^I_c(x')}\right] = -\mathcal{D}_I\mathcal{D}^I\delta^{(3)}(x-y)\delta^{ab} , \tag{3.37}$$

is invertible. Using the definition (2.118) we find the commutation relations

$$[a^a_I(x), \pi^b_J(x')] = c_B\left(\delta_{IJ}\delta^{ab}\delta^{(3)}(x-x') - \sum_p \frac{p_I p_J}{|p|^2}\bar{u}^a_p(x)u^b_p(x')\right) \tag{3.38}$$

[8]Assuming that we have chosen a representation of the Clifford algebra in which the Majorana condition implies reality of the spinor λ.

[9]In momentum space the constraints in the bosonic sector translate into the conditions $p_I a^I = 0$ and $p_I\pi^I = 0$ for the Fourier coefficients.

where c_B is again a normalization constant, and

$$[a_I^a(x), a_J^b(x')] = 0 \quad , \quad [\pi_I^a(x), \pi_J^b(x')] = 0 \,. \tag{3.39}$$

The presence of the second term in (3.38) is the consequence of the second class constraints which cannot be removed by gauge fixing as discussed in section 2.5.2.

We finally wish to obtain the momentum space description of the quantum theory. This is accomplished by introducing creation and annihilation operators acting in the Hilbert space $\mathcal{H}$ of the weak coupling effective theory. We first introduce the bosonic operators

$$\alpha_I(p) = \frac{1}{\sqrt{2}}|p|^{-1/2}\pi_I(p) + \sqrt{2}\pi i|p|^{1/2}a_I(p) \tag{3.40}$$

$$\alpha_I^\dagger(p) = \frac{1}{\sqrt{2}}|p|^{-1/2}\pi_I(p) - \sqrt{2}\pi i|p|^{1/2}a_I(p) \tag{3.41}$$

which (with the appropriate choice of c_B) satisfy the commutation relations

$$[\alpha_I(p), \alpha_J^\dagger(p')] = \left(\delta_{IJ} - \frac{p_I p_J}{|p|^2}\right)\delta_{p+p',0} \,. \tag{3.42}$$

Thus, a bosonic single-excitation state of momentum p is created by $\alpha_I^\dagger(p)$ and annihilated by $\alpha_I(-p)$.

The fermionic operators can be found by considering the Hermitian operator

$$\Gamma_p = |p|^{-1}p_I\Gamma^0\Gamma^I \,, \tag{3.43}$$

which anticommutes with all the Γ^M. Since it squares to unity it acts as a projection operator corresponding to the decomposition of the ten-dimensional Lorentz group $\mathrm{SO}(9,1) \to \mathrm{SO}(1,1) \times \mathrm{SO}(8)_p$ which singles out the spatial direction of p from the directions transverse to it. Under this decomposition the chiral spinor representation of $\mathrm{SO}(9,1)$ decomposes according to

$$\mathbf{16}_+ \to \mathbf{8}_+ \oplus \mathbf{8}_- \,. \tag{3.44}$$

Consequently, the fermionic Fourier coefficients decompose as

$$\lambda(p) = \lambda_+(p) + \lambda_-(p) \tag{3.45}$$

where the terms transform respectively in the $\mathbf{8}_+$ and $\mathbf{8}_-$ representations of $\mathrm{SO}(8)_p$ and satisfy

$$\Gamma_p\lambda_\pm(p) = \pm\lambda_\pm(p) \,. \tag{3.46}$$

Furthermore, since $\Gamma_{-p} = -\Gamma_p$ we have that $\Gamma_p\lambda_\pm(-p) = \mp\lambda_\pm(-p)$ and we see that $\lambda_+(-p)$ and $\lambda_-(-p)$ transform in the $\mathbf{8}_-$ and $\mathbf{8}_+$ representations of $\mathrm{SO}(8)_p$, which is simply a consequence of the fact that reversing p exchanges the positive and negative chiralities. We also have

$$\lambda^*_\pm(p) = \lambda_\mp(-p) \tag{3.47}$$

from the reality properties of the Fourier coefficients $\lambda(p)$. The creation and annihilation operators are required to have the same transformation properties suggesting that they should be defined in terms of $\lambda_+(p)$ and $\lambda_-(p)$. Indeed, from the anticommutation relation (3.35) it follows that

$$\{\lambda^m_+(p), \lambda^n_-(p')\} = \delta^{mn}\delta_{p+p',0} \tag{3.48}$$

where we have introduced the index $m = 1, \ldots, 8$ labeling the operators creating and annihilating a state of momentum p.

The full Hilbert space $\mathcal{H}$ is then obtained by acting with the creation operators $\alpha^\dagger_I(p)$ and $\lambda^m_\mp(p)$ on the vacuum state $|0\rangle$. Inserting the definition of the creation and annihilation operators into (3.33) we obtain

$$H = \sum_p |p| \left(\alpha^\dagger_I(p)\alpha^I(-p) + \delta^{mn}\lambda^m_+(p)\lambda^n_-(-p)\right). \tag{3.49}$$

This is the Hamiltonian of eight fermionic and eight bosonic harmonic oscillators for each p whose energies are given by $|p|$. Consequently, the spectrum of the theory (i.e. the eigenvalues of H acting in the Hilbert space) are completely determined in terms of the momentum eigenvalues p_i of the covariant derivative $i\mathcal{D}_i$ with respect to the flat background connection $\mathcal{A}$ on T^3. These eigenvalues are computed for all isolated vacua in simple gauge groups in Paper I while Paper II is concerned with an investigation of perturbative corrections in the regime of weak, but finite, gauge coupling g.

3.3.2 Momentum eigenvalues

In the final part of this chapter we will give an overview of the results of Paper I where the eigenvalues z_i of (3.24) are found by diagonalizing the adjoint action of the rank zero almost commuting triples $(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3)$ defining the isolated vacua. The momentum eigenvalues (and the energy spectrum of the theory) are then given by (3.28). For the classical matrix groups $\mathrm{SU}(n)$, $\mathrm{Spin}(n)$ and $\mathrm{Sp}(n)$ the almost commuting triples have been explicitly constructed [43–46] (see also [39, 40]) and the eigenvalues are simply found by diagonalizing their adjoint action on the Lie algebra. The exceptional groups are treated by embedding the almost commuting triple in maximal regular subgroups of $\tilde{G}$. We will consider two examples, namely $\tilde{G} = \mathrm{SU}(n)$ and $\tilde{G} = G_2$. The $\mathrm{SU}(n)$ example is particularly important since the result is used in the treatment of the exceptional groups. The G_2 examples was treated in Paper I using a description of G_2 as the subgroup of $\mathrm{Spin}(7)$ stabilizing a fixed spinor.

The SU(n) example

Using the method for constructing the moduli space [37] described above it is found that $\mathcal{M}(m)$ contains rank zero components when m is a generator of the center $C_{\mathrm{SU}(n)}$. The corresponding element $\sigma(m) \in \Sigma$ acts regularly on the extended Dynkin diagram $\tilde{D}$ of A_{n-1} in figure 3.2. The quotient diagram $\tilde{D}/\sigma(m)$ is a single node

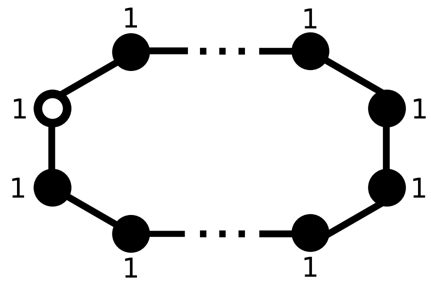


Figure 3.2: The extended Dynkin diagram $\tilde{D}$ of A_{n-1}.

with dual Coxeter label $\mathfrak{g}_{\overline{\alpha}} = n$, implying that the moduli space $\mathcal{M}(m)$ consists entirely of isolated points. Using the property

$$\sum_{k|n} \varphi(k) = n \tag{3.50}$$

of the $\varphi(k)$ function we find that more precisely $\mathcal{M}(m)$ contains n isolated points of orders k such that $k|n$.

The corresponding almost commuting triples can be constructed by taking the pair $(\tilde{U}_1, \tilde{U}_2)$ to be the unique elements [37,44] in SU(n) satisfying $\tilde{U}_1\tilde{U}_2 = m\tilde{U}_2\tilde{U}_1$. The n distinct triples are then obtained by choosing $\tilde{U}_3$ in the commutant of $(\tilde{U}_1, \tilde{U}_2)$ which is the center $C_{\mathrm{SU}(n)} \cong \mathbb{Z}_n$. The adjoint action of these triples on the Lie algebra $\mathfrak{su}(n)$ is diagonalized in Paper I and the resulting eigenvalues are given by

$$(z_1, z_2, z_3) \in \{(\xi_1, \xi_2, 1) | \xi_i^n = 1\}^\dagger \tag{3.51}$$

where we make use of the compact notation introduced in Paper I, where a set of eigenvalue vectors from which $\vec{z} = (1,1,1)$ has been excluded is denoted by $\{(\xi_1, \xi_2, \xi_3)\}^\dagger$. Indeed, the number of eigenvalue vectors is $n^2 - 1$ in agreement with the dimension of SU(n), the z_i are complex roots of unity and the spectrum exhibits SL$(2, \mathbb{Z})$ symmetry in the first two components as expected.

The G_2 example

We will finally use the $\tilde{G} = G_2$ to illustrate a useful method for determining the spectra for the exceptional groups. As mentioned above, it consist in embedding the pair $(\tilde{U}_1, \tilde{U}_2)$, satisfying the almost commutation relation (3.20), in a maximal regular subgroup S of $\tilde{G}$ in such a way that they completely break the group S. The final element of the triple is then obtained by choosing an element $\tilde{U}_3$ that breaks $\tilde{G}$ to S. The moduli space $\mathcal{M}_{G_2}$ was constructed above and found to contain a single isolated point of order $k = 2$. The relevant maximal regular subalgebra is obtained by deleting the node with $g_\alpha = 2$ (for commuting triples the order k is given by the dual Coxeter label of the deleted node [37]) from the extended Dynkin diagram in figure 3.1 giving $\mathfrak{s} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. (The relevant subalgebra is always a sum of $\mathfrak{su}(n_i)$ terms [34] which is consistent with the result of [37] that the only groups containing almost commuting pairs are products of simple $\mathrm{SU}(n_i)$ factors.)

The embedding $\iota : \mathrm{SU}(2) \times \mathrm{SU}(2) \hookrightarrow G_2$ of the exponentiation of $\mathfrak{s}$ into G_2 has a non-trivial kernel $K \subset C^2_{\mathrm{SU}(2)}$ which must be accounted for in the description of the subgroup S [34]. In order to determine K we consider the fundamental representation of G_2 which decomposes under the subalgebra $\mathfrak{s}$ as

$$\mathbf{7} = (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{2}) \,. \tag{3.52}$$

Since the elements $c_1, c_2 \in C_{\mathrm{SU}(2)} \cong \mathbb{Z}_2$ act trivially on the adjoint representation $\mathbf{3}$ and by an overall phase on the fundamental representation $\mathbf{2}$ it follows that the kernel is $K = \{(\mathbb{1}, \mathbb{1}), (-\mathbb{1}, -\mathbb{1})\}$. The subgroup of G_2 in which we wish to embed the commuting triple is thus

$$S = \mathrm{SU}(2) \times \mathrm{SU}(2)/K \subset G_2 \,. \tag{3.53}$$

The first two elements of the triple are constructed in analogy with the $\mathrm{SU}(n)$ case as $\tilde{U}_1 = (A, A)$ and $\tilde{U}_2 = (B, B)$, where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.54}$$

are the $\mathrm{SU}(2)$ matrices satisfying the almost commutation relation $ABA^{-1}B^{-1} = -\mathbb{1}$ discussed above. According to the results for $\mathrm{SU}(n)$ we know that these elements completely break $\mathrm{SU}(2)$ so that $(\tilde{U}_1, \tilde{U}_2)$ break both $\mathrm{SU}(2)$ factors. Furthermore, the holonomies satisfy the almost commutation relation (3.20) with $m = (-\mathbb{1}, -\mathbb{1})$ which is identified with the identity through the division by K. To summarize, we have obtained two commuting elements $(\tilde{U}_1, \tilde{U}_2)$ in G_2 that break all the generators of the subgroup S as desired.

To complete the triple we must supply a third element in the commutant of $(\tilde{U}_1, \tilde{U}_2)$ which must be of the form $\tilde{U}_3 = (c_1, c_2)$ with $c_i \in C_{\mathrm{SU}(2)}$. Furthermore, $\tilde{U}_3$ must break G_2 to S which leads us to consider the adjoint representation of G_2. Under the subalgebra $\mathfrak{s}$ it decomposes according to

$$\mathbf{14} = (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{4}) \tag{3.55}$$

where $\mathbf{4}$ is the totally symmetric product of three fundamental $\mathbf{2}$ representations and the last summand contains the generators of G_2 not in S. Hence, $\tilde{U}_3$ is required to act non-trivially on $(\mathbf{2}, \mathbf{4})$ which implies that there are two possibilities provided by $(\mathbb{1}, -\mathbb{1})$ and $(-\mathbb{1}, \mathbb{1})$. However, these are identified as elements of S and consequently there is a single rank zero commuting triple in G_2 in agreement with the structure of the moduli space in (3.23).

In order to compute the spectrum of eigenvalues (z_1, z_2, z_3) we consider the first two summands of (3.55) whose contributions are obtained directly from the SU(n) result as $\{(\xi_1, \xi_2, 1) | \xi_i^2 = 1\}^\dagger$. Since there are two identical factors the eigenvalues are doubly degenerated. Finally, since the triple $(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3)$ is commuting the complete spectrum is obtained by requiring $\mathrm{SL}(3, \mathbb{Z})$-invariance as

$$(z_1, z_2, z_3) \in \{(\xi_1, \xi_2, \xi_3) | \xi_i^2 = 1\}^\dagger . \tag{3.56}$$

with two-fold degeneracy. Once again, we verify that the eigenvalues are complex roots of unity and that $2(2^3 - 1) = 14$ agrees with the dimension of G_2.

4

BPS partition functions in Yang-Mills theory on T^4

In supersymmetric theories there is an exceptionally important class of states characterized by the property that they are invariant under a fraction of the supersymmetries. They therefore furnish short representations of the supersymmetry algebra including fewer states than a generic representation. We have already encountered such a representation in the previous chapter when we considered the massless $\mathcal{N} = 4$ vector multiplet in four dimensions. Generically, however, we consider single particle states of some non-vanishing mass M in which case we obtain long multiplets where all supercharges Q act non-trivially on the states. The central charge Z_{ij} discussed in the previous chapter can then be non-vanishing. In fact, the central charge is a topological charge of the theory, measuring the topology of extended field configurations [47]. Classically, the mass is constrained by the BPS bound [48–50]

$$M^2 \leq |Z_i|^2 \,, \tag{4.1}$$

where Z_i are the independent components of the central charge matrix Z_{ij}. States for which this bound is saturated fall into short supermultiplets as mentioned above. In particular, this implies that in the quantum theory their mass cannot receive any corrections, since the bound (4.1) would then no longer be saturated and consequently the number of particle states in the multiplet would be changed [47, 51]. States which saturate the bound (4.1) and are protected from quantum corrections are called BPS states. A special case, which is the one we will concerned with below, is massless states which saturates the bound for $Z = 0$ and are $\frac{1}{2}$-BPS states.

The importance of the BPS-states lie in the protection offered by the saturation of the BPS bound. Since the number of states in a multiplet cannot change when the parameters of the theory are continuously varied, the BPS states can be considered at arbitrary values of the coupling strength of the theory. In particular, their mass spectrum can therefore be computed in perturbation theory in the weak coupling limit and reliably extrapolated to strong coupling. It is fair to say that much of the progress in understanding the strong coupling properties of supersymmetric gauge theories has been obtained through the use of BPS states.

The purpose of the present chapter is to provide an introduction to PAPER III, which concerns certain terms in the partition functions of $N = 4$ Yang-Mills theory on a Euclidean torus T^4 that only receive contributions from BPS states, and summarize the results obtained.

4.1 Partition functions in $\mathcal{N} = 4$

In this section we will consider the partition function of the $\mathcal{N} = 4$ Yang-Mills theory defined on an Euclidean four-torus T^4. After discussing certain aspects of the theory and its topological sectors we will consider certain terms in the expansion of the partition function around a background R-symmetry connection that preserves one of the supersymmetries. We find that they are independent of the coupling of the theory and can therefore be reliably computed at weak coupling.

4.1.1 The path integral

The first order of business in the present chapter is a quick review of the concept of path integrals and in particular the partition function in quantum field theory. This is an exceptionally useful tool because it contains essentially all information about the quantum theory and can be used to compute physical quantities like transition amplitudes and derive the Feynman rules of the theory. For a more exhaustive treatment than the working definitions offered here we recommend e.g. [52,53] which we will follow below. The starting point will be the Lorentzian path integral but we will also discuss the connection to its Euclidean counterpart through Wick rotation.

The canonical way of introducing path integrals is to first consider a single degree of freedom corresponding to a (generalized) coordinate q and the transition amplitude[1]

$$\langle q_f, T|q_i, 0\rangle = \langle q_f| \exp(-i\hat{H}T)|q_i\rangle \tag{4.2}$$

for evolution between two eigenstates $|q_i\rangle$ and $|q_f\rangle$ of the position operator $\hat{q}$ during a time T. By inserting a complete set of eigenstates of $\hat{q}$ at some intermediate time we get

$$\langle q_f, T|q_i, 0\rangle = \int dq\, \langle q_f, T|q, t\rangle\langle q, t|q_i, 0\rangle\, . \tag{4.3}$$

Repeating the division of the time interval into steps of equal length δ and taking the limit where $\delta \to 0$ we obtain (after some technical manipulations) the path integral

[1] The different notations refer to the Heisenberg and Schrödinger pictures.

representation of the transition amplitude

$$\langle q_f, T | q_i, 0 \rangle = \int [dq]_{q_i,0}^{q_f,T} \exp\left(i \int_0^T dt L(q, \dot{q}) \right) = \int [dq]_{q_i,0}^{q_f,T} \exp(iS) \,. \tag{4.4}$$

where S is the action functional. The measure $[dq]$ represents all paths from $|q_i\rangle$ at time $t = 0$ to $|q_f\rangle$ at time $t = T$, hence the name path integral.

The oscillating properties of the factor $\exp(iS)$ makes the Lorentzian path integral difficult to work with. An improvement of its properties can be achieved by performing an analytical continuation to imaginary time; a so called Wick rotation $t \to -iu$. The weight of the path integral is then $\exp(-S_E)$, where S_E is the Euclidean action, which improves the convergence properties. It is therefore often convenient to compute path integrals in Euclidean signature and analytically continue the result back to Lorentzian signature.

The main focus of the present chapter is the partition function Z which is perhaps most familiarly defined in Euclidean space (where the analogy with statistical mechanics is most transparent). It is defined as the trace

$$Z = \mathrm{Tr}_{\mathcal{H}} \exp(-U\hat{H}) \tag{4.5}$$

over the Hilbert space $\mathcal{H}$, which gives the sum over states weighted by their energy eigenvalue, and U is the equivalent of T in Euclidean signature[2]. The partition function can be represented in terms of the path integral by expressing the trace as an integral

$$Z = \int d\tilde{q} \, \langle \tilde{q} | \exp(-U\hat{H}) | \tilde{q} \rangle = \int d\tilde{q} \int [dq]_{\tilde{q},0}^{\tilde{q},U} \exp(-S_E) = \int [dq]_{T^2} \exp(-S_E) \,, \tag{4.6}$$

where $[dq]_{T^2}$ represents integration over all paths on the interval $[0, U]$ with $\tilde{q}(0) = \tilde{q}(U)$, which is equivalent to the integration over all closed paths on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. In Lorentzian signature the corresponding partition function is given by the Wick rotated expression

$$Z = \mathrm{Tr}_{\mathcal{H}} \exp(-iT\hat{H}) = \int [dq]_{T^2} \exp(iS) \,. \tag{4.7}$$

It is possible to modify the partition function by adding to the trace the operators corresponding to further conserved quantities in addition to the energy. Consider the partition function

$$Z = \mathrm{Tr}_{\mathcal{H}} \exp(-iT\hat{H} + iX\hat{P}) = \int d\tilde{q} \, \langle \tilde{q} + X, T | \tilde{q}, 0 \rangle = \int d\tilde{q} \int [dq]_{\tilde{q},0}^{\tilde{q}+X,T} \exp(iS) \,. \tag{4.8}$$

[2] In statistical physics the coefficient multiplying the Hamiltonian is conventionally denoted β.

Just like the parameter T appeared in the integral of the Lagrangian in (4.4) there is a connection between the parameter X and the manifold over which we integrate to obtain the action S. In particular, the partition function (4.8) can be expressed as the path integral

$$Z = \mathrm{Tr}_{\mathcal{H}} \exp(-iT\hat{H} + iX\hat{P}) = \int [dq]_{T^2} \exp(iS) \tag{4.9}$$

over all paths defined on the T^2 space-time defined by the the pair (T, X). The generalization to the corresponding partition function of field theory (with an infinite number of degrees of freedom) in four dimensions is then

$$Z = \mathrm{Tr}_{\mathcal{H}} \exp(-iT\hat{H} + i\mathbf{X}\cdot\hat{P}) = \int [D\Phi]_{T^4} \exp(iS) \tag{4.10}$$

where all fields Φ of the theory are defined on the torus T^4 defined by the vector $(T, \mathbf{X})$. Conversely, the inclusion of the momentum operator in the trace can be viewed as the restriction of the path integral to field configurations on the appropriate space-time T^4. We will use a further refinement of the partition function below, when we consider BPS states and their contribution to Z.

4.1.2 The $\mathcal{N} = 4$ theory on T^4

We will now consider the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on a manifold M which is a flat Euclidean torus. The geometry can be described as the quotient

$$M = T^4 = \mathbb{R}^4/\Gamma\,, \tag{4.11}$$

where Γ is a rank four lattice in $\mathbb{R}^4$ defining the torus. The gauge field A is the local connection one-form of a principal bundle P with structure (gauge) group G. In order to allow for the most general non-trivial bundle topology we take G of the adjoint form

$$G = \mathrm{SU}(n)/C \tag{4.12}$$

where C is the center subgroup of $\mathrm{SU}(n)$ which is (isomorphic to) $\mathbb{Z}/n\mathbb{Z}$. In order to simplify the analysis below we restrict n to be prime, which implies that C is a field. From section 2.2.3 we recall that since M is now four-dimensional we have not one but two characteristic classes describing the topological class of the bundle P; the 't Hooft flux $v \in H^2(M, C)$ and the instanton number $k \in H^4(M, \mathbb{Q})$ which are related through (2.33). The product of 't Hooft fluxes, appearing in the quantity[3]

$$\frac{1}{2} v \cdot v \in H^4(M, \mathbb{R}/\mathbb{Z}) \tag{4.13}$$

[3]Recall that for M a spin manifold, which is certainly the case for $M = T^4$, the product $v \cdot v$ is divisible by two in a canonical way.

(known as the Pfaffian of v) in the relation (2.33), is obtained by a composition of the cup product in cohomology and the pairing of center elements, given by identifying $C = \mathbb{Z}/n\mathbb{Z}$ and then computing the product of the lift to $\mathbb{Z}$ and reducing modulo n. For n prime the Pfaffian, which is invariant under the $\mathrm{SL}(4,\mathbb{Z})$ mapping class group of T^4, completely determines the equivalence class, i.e. the $\mathrm{SL}(4,\mathbb{Z})$ orbit, of the 't Hooft flux when $\frac{1}{2}v \cdot v \neq 0$. There are in addition two orbits with $\frac{1}{2}v \cdot v = 0$; one consisting of the trivial element $v = 0$ and the other being generated by an element of the form $v = e^1 \cup e^2$, where $\{e^\mu\}_{\mu=1}^4$ denotes a basis of $H^1(M,\mathbb{Z})$.

The field content of the Yang-Mills theory is the one described in the previous chapter, but for the purpose of studying the partition function of the theory it is convenient to remain in the four-dimensional description where the gauge field strength F is the curvature of a principal G-bundle, and the scalar and spinor fields Φ and Ψ are described as sections (i.e. zero-forms on M) of associated adjoint bundles. In addition, the fields transform respectively in the **6** and **4** representations of the $\mathrm{SU}(4) \cong \mathrm{SO}(6)$ R-symmetry. Furthermore, we recall that the supersymmetry generators Q transform in the same **4** representation as the spinor Ψ and are unbroken by the flat T^4.

Contrary to the case in the previous chapter, however, we will now consider gauging the R-symmetry by considering a connection B on a principal $\mathrm{SU}(4)$-bundle over M. The connection is non-dynamical and for simplicity we take the bundle to be topologically trivial and B to be flat. The kinetic part of the action of the Yang-Mills theory minimally coupled to the connection B is then given by

$$S_{\text{kin}} = \frac{\operatorname{Im}\tau}{4\pi}\int_M \operatorname{Tr}\left(F \wedge *F + (D+iB)\Phi \wedge *(D+iB)\Phi + \mathrm{Vol}_M \overline{\Psi}(\not{D} + i\not{B})\Psi\right) , \tag{4.14}$$

where D is the covariant derivative with respect to the connection A and the complex coupling constant is

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} , \tag{4.15}$$

where the gauge coupling constant is given by g. The action also contains the topological term

$$S_{\text{top}} = \frac{i\operatorname{Re}\tau}{4\pi}\int_M \operatorname{Tr}(F \wedge F) . \tag{4.16}$$

Neither the interaction terms of the full Yang-Mills action, nor the topological term will contribute in the weak coupling analysis which is the subject of the next section, and they have therefore been excluded from (4.14).

The partition function of the $\mathcal{N} = 4$ theory is defined according to the previous section and depends, in addition to the coupling constant τ, on the flat metric on T^4 and the background R-symmetry connection B:

$$Z(\tau|\Gamma, B) = \int \mathcal{D}A\mathcal{D}\Phi\mathcal{D}\Psi \exp(-S) . \tag{4.17}$$

However, the integral over all gauge inequivalent connections A decomposes into a sum of contributions from P belonging to distinct topological classes of principal bundles over T^4. In particular, we will find it useful to consider an organization of the contributions according to the 't Hooft flux v and introduce the quantities

$$Z_v(\tau|\Gamma, B) = \sum_k \int \mathcal{D}A\mathcal{D}\Phi\mathcal{D}\Psi \exp(-S)\,, \qquad (4.18)$$

where the sum is over all instanton numbers satisfying (2.33), or equivalently over all isomorphism classes of bundles with 't Hooft flux v.

4.1.3 S-duality

One of the many remarkable properties of the $\mathcal{N} = 4$ Yang-Mills theory is a (conjectured) $\mathrm{SL}(2,\mathbb{Z})$ symmetry, called S-duality, acting on the coupling parameter τ. It generalizes the strong-weak $\mathbb{Z}_2$ duality of Montonen and Olive [54, 55] taking $\tau \to -1/\tau$ and exchanging the electric and magnetic charges of the theory (corresponding in the present case to exchanging $G = \mathrm{SU}(n)/C$ for its dual group $G^\vee = \mathrm{SU}(n)$). Using supersymmetry it was shown in [47] that the dyonic mass formula of $\mathcal{N} = 2$ Yang-Mills theory is invariant under the exchange of electric and magnetic charges. Furthermore, it saturates the BPS bound (4.1) implying that the masses receive no quantum corrections as discussed in the introduction to this chapter. In the $\mathcal{N} = 4$ theory the spectra of electrically charged elementary particles and magnetically charged topological monopole excitations are also identical [51], as required by S-duality.

In addition to direct evidence in field theory [13, 56] it is possible to argue for the S-duality of the $\mathcal{N} = 4$ theory through its appearance as a low energy effective theory in string theory [57]. Equivalently, as we will discuss in more detail in the next chapter, the $\mathrm{SL}(2,\mathbb{Z})$ duality is made manifest if we consider $\mathcal{N} = 4$ theory as the dimensional reduction of $(2,0)$ theory on T^2, in which case it corresponds to the mapping class group of the torus.

In the present chapter we will consider the transformation properties of the partition function of $\mathcal{N} = 4$ Yang-Mills theory which, as we saw above, decomposes according to the 't Hooft flux v. The S-duality conjecture can then be extended to linear relations between the components $Z_v(\tau|\Gamma, B)$ under $\mathrm{SL}(2,\mathbb{Z})$ transformations of τ. For $M = T^4$ the group of transformations is generated [13] by the S-transformation[4]

$$Z_v(-1/\tau|\Gamma, B) = |C|^{-3} \sum_{v'} \exp\left(2\pi i \int_M v \cdot v'\right) Z_{v'}(\tau|\Gamma, B)\,, \qquad (4.19)$$

[4] This transformation can be thought of as exchanging the electric and magnetic 't Hooft fluxes in a Hamiltonian formulation (see section 3.2.1) which are related by a Fourier transform [32, 33]. A more detailed discussion for arbitrary M is given in [13].

where $|C|$ is the order of the finite center subgroup, and the T-transformation which simply shifts $\tau \to \tau + 1$ and acts on the $Z_v(\tau|\Gamma, B)$ by a multiplicative factor corresponding to the change in the topological term in the Yang-Mills action.

4.1.4 Flat connections on T^4

Below we will be particularly interested in flat connections on principal bundles. In analogy with flat connections on principal bundles over T^3, which was the topic of chapter 3, flat connections on principal bundles over $M = T^4$ can be described by their holonomies around generators of $\pi_1(T^4)$ yielding a quadruple of commuting group elements whose lift to the universal covering group satisfy almost commutation relations encoded by $v \in H^2(M, C)$.

Since the gauge group G is not simply connected the 't Hooft flux v can be non-trivial and the lift of the holonomies of A can be almost commuting. The moduli space $\mathcal{M}$ consists of disjoint subspaces $\mathcal{M}(v)$ corresponding to topologically distinct bundles P just as in three dimensions. However, the instanton number introduces significant differences. As we saw in section 2.3.2 the instanton number k is expressed in terms of the curvature as

$$k = \frac{1}{8\pi^2}\int_{T^4} \mathrm{Tr}(F \wedge F)\,. \tag{4.20}$$

Consequently, only bundles with $k = 0$ admit flat connection. In particular, due to the relation (2.33) this implies that only the moduli spaces $\mathcal{M}(v)$ with $\frac{1}{2}v \cdot v = 0$ are non-empty. As a consequence the remarkable relation (3.22) is only applicable in three dimensions and does not generalize to connections on bundles over T^4.

For n prime we can then investigate the moduli space of flat connections by considering the distinct 't Hooft fluxes which allows for an instanton number $k = 0$. For $v = 0$ the quadruple of holonomies is commuting after the lift to $\mathrm{SU}(n)$ and can therefore be simultaneously conjugated to a maximal torus of $\mathrm{SU}(n)$. Consequently, the corresponding moduli space contains a single maximal rank component of dimensions $4(n-1)$. As mentioned above, the 't Hooft flux $v \neq 0\,,\, \frac{1}{2}v \cdot v = 0$ can be put on the form $v = e^1 \cup e^2$, in analogy with the the case of almost commuting triples, using $\mathrm{SL}(4, \mathbb{Z})$ transformation. Since any element is a generator of C for n prime we can construct n rank zero triples according to the prescription in section 3.3.2. Since the commutant of the almost commuting pair $(\tilde{U}_1, \tilde{U}_2)$ is the center C any quadruple of holonomies must have $\tilde{U}_3, \tilde{U}_4 \in C$ yielding n^2 rank zero quadruples (showing again that (3.22) is violated in the case of T^4). However, since these are all identified in $G = \mathrm{SU}(n)/C$ they define a unique flat connection on P.

Using its holonomies we can also describe the R-symmetry connection B by a quadruple of commuting elements of $\mathrm{SU}(4)$. (The elements are commuting by virtue of the R-symmetry group being simply connected implying that the $\mathrm{SU}(4)$ bundle necessarily has vanishing 't Hooft flux.) Since, as we argued in the previous paragraph, such elements can be simultaneously conjugated to a maximal torus $\mathcal{T}$ of SU(4) we can identify

$$B \in H^1(M, \mathcal{T})\,. \tag{4.21}$$

We thus need to describe the action of $\mathcal{T}$ on the modules of the **4** and **6** representations relevant to the Yang-Mills multiplet. In fact, it is sufficient to consider the fundamental **4** representation since the action in an arbitrary representation is then determined by its tensor product construction. Denoting the weights of **4** by w_i , $i = 1, 2, 3, 4$ we can identify an element in $\mathcal{T}$ by the phase by which it acts on the corresponding weight spaces. The weights furthermore satisfy $w_1+w_2+w_3+w_4 = 0$ which implies that the connection can be described by a quartet of elements $B = (B^1, B^2, B^3, B^4)$ in $H^1(M, \mathbb{R}/\mathbb{Z})$ subject to the relation

$$B^1 + B^2 + B^3 + B^4 = 0\,. \tag{4.22}$$

In the manipulations involved in the computations described in the remainder of this chapter we will need to lift the elements B^i to $H^1(M, \mathbb{R})$. The lift preserves the property (4.22) and is conveniently denoted by the same symbols.

Finally, we will need to describe the action of B as it appears in (4.14), which implies that we must consider it at the level of the Lie algebra. The relation to the above description is given by the exponential map and consequently the action of B on an element $v_i \in V_{w_i}$ in the w_i weight space is given by

$$Bv_i = 2\pi B^i v_i\,, \tag{4.23}$$

where there is no summation of indices. Note that this expression is applicable to the weight spaces of arbitrary representations and not just that of the fundamental **4** representation.

4.2 The BPS terms

After introducing all of the necessary machinery we can now begin to make contact with results of Paper III by considering an expansion of the R-symmetry connection around some $B = (B^1, B^2, B^3, B^4)$ which leaves one of the supersymmetries unbroken. From (4.23) it follows that a generic B acts non-trivially in all weight spaces of **4** and consequently breaks all supersymmetries. In order to preserve one of the supersymmetries we must thus take (say) $B^4 = 0$ which leaves the Q of weight w_4 unbroken.

We can then expand the R-symmetry connection around B by adding a perturbation $\delta B = (0, 0, 0, \delta B^4)$. Since the B^1, B^2, B^3 are arbitrary we need only consider a non-vanishing perturbation in the fourth component. Again we denote by δB also the lift of the perturbation to $H^1(M, \mathbb{R})$, which we require to be minimal to ensure that perturbation theory in δB makes sense. The partition function can then be expressed as power series in δB according to

$$Z_v(\tau|\Gamma, B + \delta B) = Z_v^{(0)}(\Gamma, B) + Z_v^{(1)}(\Gamma, B|\delta B) + \mathcal{O}((\delta B)^2) \,. \tag{4.24}$$

That the components $Z_v^{(0)}(\Gamma, B)$ and $Z_v^{(1)}(\Gamma, B|\delta B)$ of order zero and one are independent of τ, as indicated by the notation, can be shown in the following way: Consider a Wick rotated Hamiltonian formulation with the decomposition of the lattice Γ into temporal and spatial parts induced by δB. As we discussed above, the partition function takes the form

$$Z_v(\tau|\Gamma, B) = \mathrm{Tr}_{\mathcal{H}_v} \left((-1)^F \exp(-it\hat{H} + i\mathbf{x} \cdot \hat{P} + iB_0) \right) \tag{4.25}$$

where $(t, \mathbf{x})$ defines the torus T^4 and we have refined the expression by inserting the fermion number operator and the time component B_0 of the R-symmetry connection. Since the connection $B = (B^1, B^2, B^3, 0)$ breaks three of the supersymmetries, only the two pairs of creation and annihilation operators associated to the weight w_4 act non-trivially on $\mathcal{H}_v$. Therefore, all states in the Hilbert space must fall into $\mathcal{N} = 1$ multiplets. In particular, the contribution of long (non-BPS) multiplets with $E^2 > \mathbf{p}^2$ to the partition function is proportional to

$$1 - 2\exp(2\pi i \delta B_0^4) + \exp(4\pi i \delta B_0^4) = \mathcal{O}((\delta B)^2) \,, \tag{4.26}$$

which shows that only short (BPS) multiplets contribute to the terms $Z_v^{(0)}(\Gamma, B)$ and $Z_v^{(1)}(\Gamma, B|\delta B)$. The mass of the short multiplet (or equivalently the relation $E^2 = \mathbf{p}^2$) is protected under the continuous deformations of the theory and the spatial momenta $\mathbf{p}$ are quantized and therefore independent of τ. We can thus indeed conclude that the $Z_v^{(0)}(\Gamma, B)$ and $Z_v^{(1)}(\Gamma, B|\delta B)$ are independent of the coupling constant τ and are consequently referred to as the BPS terms of the partition function (expanded around the loci where one of the supersymmetries is unbroken).

Having identified quantities that are independent of the coupling constant τ we must now proceed to investigate if there is any value of τ where we can in fact compute their value in order to capitalize on this feature. The limit of weak gauge coupling $\tau \to i\infty$ (almost) allows us to perform such a computation. To see how this is accomplished we use the fact that

$$\int_M \mathrm{Tr}(F \wedge *F) \pm \int_M \mathrm{Tr}(F \wedge F) \geq 0 \,. \tag{4.27}$$

Since the real part of the action contains only positive definite terms we have

$$\mathrm{Re}\, S \geq \frac{1}{g^2} \int_M \mathrm{Tr}(F \wedge *F) \geq \frac{8\pi^2}{g^2} \left| \int_M k \right| \tag{4.28}$$

where we have used the definition (4.20) of the instanton number in terms of the curvature. Consequently, the contributions to the partition function from connections on bundles P with non-vanishing k are exponentially suppresses as we take $g \to 0$.

Using the result of the previous section that bundles with 't Hooft fluxes in the $\frac{1}{2}v \cdot v \neq 0$ necessarily have $k \neq 0$ we can immediately conclude that the partition function is identically vanishing in this limit, which in particular implies that we have

$$Z_v^{(0)}(\Gamma, B) = 0 \quad , \quad Z_v^{(1)}(\Gamma, B|\delta B) = 0 \tag{4.29}$$

for these orbits.

4.2.1 Non-trivial 't Hooft flux

We now proceed to consider bundles with 't Hooft fluxes in the $\frac{1}{2}v \cdot v = 0\,,\, v \neq 0$ $\mathrm{SL}(4,\mathbb{Z})$ orbit, in which case there is an isomorphism class with $k = 0$ which gives a non-vanishing contribution to $Z_v(\tau|\Gamma, B)$ in the weak coupling limit. According to the results discussed in section 4.1.4 this bundle admits a unique flat connection $\mathcal{A}$. In analogy with the previous chapter we then expand the fields of the $\mathcal{N} = 4$ theory around the corresponding isolated vacuum configuration according to

$$A = \mathcal{A} + ga \;\;,\;\; \Phi = g\phi \;\;,\;\; \Psi = g\psi\,, \tag{4.30}$$

and denote by $\mathcal{D}_\mu$ the covariant derivative with respect to the flat background connection by $\mathcal{A}$.

In order to evaluate the partition function we must also account for gauge fixing. We use the conventional generalized Feynman gauge (with gauge fixing functional $\mathcal{D}_\mu a^\mu$ and real parameter ξ) which modifies the effective Lagrangian of the gauge field and introduces a complex fermionic ghost field which we denote ω. The action to be inserted into the path integral is then given by

$$\begin{aligned} S \;=\; & \int_M \mathrm{Tr}\,\big(*a \wedge \big(*\mathcal{D} * \mathcal{D} + \xi^{-1}\mathcal{D} * \mathcal{D}*\big)\, a + \mathrm{Vol}_M \overline{\psi}(\not{\mathcal{D}} + i\not{B})\psi \\ & - *\phi \wedge *(\mathcal{D} + iB) * (\mathcal{D} + iB)\phi + *\bar{\omega} \wedge *\mathcal{D} * \mathcal{D}\omega\big) + \mathcal{O}(g) \end{aligned} \tag{4.31}$$

so that in the weak coupling limit the partition function is given by the one-loop contribution

$$Z_v^{\mathrm{one-loop}}(\Gamma, B) = \frac{\det\left(*\mathcal{D} * \mathcal{D}\right) \det\left(\not{\mathcal{D}} + i\not{B}\right)}{\det^{1/2}\left(*\mathcal{D} * \mathcal{D} + \xi^{-1}\mathcal{D} * \mathcal{D}*\right) \det^{1/2}\left(*(\mathcal{D} + iB) * (\mathcal{D} + iB)\right)}\,. \tag{4.32}$$

Computing the determinants in the one-loop partition functions requires knowledge of the spectra of the various differential operators appearing in the weak coupling effective action. Since the action of B on the various fields is determined through their respective representations the only information missing is the spectrum of the covariant derivative with respect to the flat connection $\mathcal{A}$. In complete analogy with the construction described for the case of a three-torus in the previous chapter we can construct simultaneous eigenfunctions u_p, satisfying

$$i\mathcal{D}_\mu u_p = 2\pi p_\mu u_p \tag{4.33}$$

and spanning the space of sections of the ad(P) bundle. (Here the index μ refers to the basis $\{e^\mu\}$ of $H^1(M,\mathbb{Z})$ introduced in the previous section.) In order to describe the (real) eigenvalues p, constituting the allowed momenta on M for a bundle of 't Hooft flux v, in a covariant manner we introduce the set

$$S_v = \big\{\rho \in H^1(M,C)\,|\,\rho \neq 0\,,\ \rho\cdot v = 0\big\} \tag{4.34}$$

where $\rho \cdot v \in H^3(M,\mathbb{R}/\mathbb{Z})$ using the same pairing as above. (Note that the set S_v is well-defined for arbitrary 't Hooft flux v.) The allowed momentum eigenvalues for a 't Hooft flux v in the orbit $\frac{1}{2}v\cdot v = 0\,,\ v \neq 0$ presently under consideration are then given by the sets

$$P_\rho = \left\{p \in H^1\left(M, \frac{1}{n}\mathbb{Z}\right)\,\Big|\,[np] = \rho\right\} \tag{4.35}$$

for $\rho \in S_v$, where $[np]$ denotes reduction of np modulo n.

Before proceeding it is illuminating to make contact with the formalism of chapter 3 by considering the particular example of the 't Hooft flux $v = e^1 \cup e^2$, in which case the allowed ρ are on the form

$$\rho = (\rho_1, \rho_2, 0, 0) \quad , \quad (\rho_1,\rho_2) \neq (0,0) \tag{4.36}$$

where we have identified $C = \mathbb{Z}/n\mathbb{Z}$ so that $\rho_i \in \mathbb{Z}/n\mathbb{Z}$. These are the (logarithms of) eigenvalues of the adjoint action of the holonomies on the Lie algebra $\mathfrak{su}(n)$ of G. The allowed momenta then take the form

$$p = \left(\frac{\rho_1}{n} + \mathbb{Z}, \frac{\rho_2}{n} + \mathbb{Z}, \mathbb{Z}, \mathbb{Z}\right) \tag{4.37}$$

in analogy with the three-dimensional formula (3.28).

It is possible to show that the one-loop partition function is free of ultraviolet divergences and can be used to compute the BPS terms for the $\frac{1}{2}v\cdot v = 0\,,\ v \neq 0$ orbit. The detailed arguments and computations are given in PAPER III and the result is most conveniently presented as

$$Z_v^{(0)}(\Gamma, B) = 1 \quad , \quad Z_v^{(1)}(\Gamma, B|\delta B) = \sum_{\rho\in S_v} \Xi(\rho, B|\delta B)\,, \tag{4.38}$$

where we have introduced a shorthand notation for the rather unwieldy quantity

$$\begin{aligned}\Xi(\rho, B|\delta B) &= \sum_{p\in P_\rho}\left(\frac{(p+B^1+B^2)\cdot\delta B^4}{(p+B^1+B^2)^2}+\frac{(-p+B^1+B^2)\cdot\delta B^4}{(-p+B^1+B^2)^2}\right.\\ &+ \frac{(p+B^1+B^3)\cdot\delta B^4}{(p+B^1+B^3)^2}+\frac{(-p+B^1+B^3)\cdot\delta B^4}{(-p+B^1+B^3)^2}\\ &+ \left.\frac{(p+B^2+B^3)\cdot\delta B^4}{(p+B^2+B^3)^2}+\frac{(-p+B^2+B^3)\cdot\delta B^4}{(-p+B^2+B^3)^2}\right). \end{aligned} \tag{4.39}$$

The reader might justly ask at this point if the omission of this expression would not have served this introductory part of the thesis well. Unfortunately, it turns out that the final result obtained below is succinctly presented in terms of the quantity $\Xi(\rho, B|\delta B)$, and its definition is therefore included for completeness.

4.2.2 Trivial 't Hooft flux from S-duality

At this point, it remains only to compute the BPS terms for the single 't Hooft flux $v = 0$. Just as in the previous section there exist an isomorphism class of such bundles with instanton number $k = 0$ which can give non-vanishing contributions to the partition function in the limit of weak gauge coupling. As we saw above, however, this bundle has a $4(n-1)$ dimensional moduli space of flat connections corresponding to vacuum states of the theory. Just as was the case when considering flat connections over T^3, the unbroken generators of G correspond to zero modes which make the perturbative approach used for the $\frac{1}{2}v\cdot v = 0\,,\ v \neq 0$ orbit intractable. Consequently, it is not possible to directly compute the partition function for the trivial orbit $v = 0$ even at weak coupling.

At this point we recall that S-duality of the $\mathcal{N} = 4$ (or more specifically the generating S-transformation introduced in the previous section) relates the partition functions $Z_v(\tau|\Gamma, B)$ of distinct 't Hooft fluxes, albeit for couplings related by $\tau \to -1/\tau$. However, since the BPS terms are independent of the coupling they can still be computed using (4.19). The result for arbitrary 't Hooft flux v, obtained in Paper III, can then be summarized in terms of the set S_v as

$$Z_v^{(0)}(\Gamma, B) = \frac{|S_v|}{n^2-1} \quad , \quad Z_v^{(1)}(\Gamma, B|\delta B) = \sum_{\rho\in S_v}\Xi(\rho, B|\delta B)\,, \tag{4.40}$$

where $|S_v|$ denotes the cardinality of S_v. As mentioned above, S_v can be defined for any $v \in H^2(M, C)$ and in particular is empty for the non-trivial orbits $\frac{1}{2}v\cdot v \neq 0$ in agreement with the result of the previous subsection. Finally, it remains to verify that the result is self-consistent, i.e. compatible with (4.19) which is an overdetermined system of constraints on the $Z_v(\tau|\Gamma, B)$. In Paper III it is shown that (4.40) is in fact the unique solution which reproduces the weak coupling partition functions for non-trivial v.

5

$(2,0)$ theory on circle fibrations

In this final chapter of the introduction we will consider the second example of a maximally supersymmetric theory, namely the superconformal $(2,0)$ theory in six dimensions. In contrast to the previous two chapters, where the space-time geometry was that of a torus, we will now consider more general geometries, in particular manifolds with non-vanishing curvature. Therefore, we will begin with a brief recollection of the basic geometrical concepts of Riemannian manifolds and, because of the application to $(2,0)$ theories we have in mind, their conformal structure. Throughout this chapter, we will implicitly assume all manifolds to have Lorentzian signature.

Subsequently, we give an overview of the $(2,0)$ theories and their role in string theory and M-theory before proceeding to a summary of the work presented in PAPER IV-V concerning $(2,0)$ theory on (spatial) circle fibrations and the low energy effective theory obtained upon reduction on the circle.

5.1 Riemannian geometry

In this first section we will discuss some of the concepts from Riemannian geometry appearing in the study of $(2,0)$ theory on circle fibrations. We will review the definition of conformal equivalence classes of metrics and the introduction of Lorentz spinors in curved space-time of arbitrary dimension before restricting our attention to the special case of six-dimensional circle fibrations considered in the following sections.

5.1.1 Conformal structure

We consider a manifold M parametrized by local coordinates y^M, with $M, N = 0, 1, \ldots, d-1$ and equip M with a symmetric and non-degenerate (Lorentzian signature) metric tensor field G_{MN}, providing an inner product of vectors $X, Y \in \chi(M)$ through[1]

$$(X, Y) = G_{MN} X^M Y^N \,. \tag{5.1}$$

We define the covariant derivative[2] $\hat{\nabla}_M$ using the unique symmetric and metric compatible Levi-Civita connection

$$\hat{\Gamma}^P_{MN} = \frac{1}{2} G^{PQ} \left(\partial_M G_{NQ} + \partial_N G_{MQ} - \partial_Q G_{MN} \right) , \tag{5.2}$$

where G^{MN} denotes the inverse metric. The inner product is then preserved by parallel transport using the covariant derivative. The Riemann curvature tensor

$$\hat{R}^P{}_{QMN} = \partial_M \hat{\Gamma}^P_{NQ} - \partial_N \hat{\Gamma}^P_{MQ} + \hat{\Gamma}^P_{MS} \hat{\Gamma}^S_{NQ} - \hat{\Gamma}^P_{NS} \hat{\Gamma}^S_{MQ} \tag{5.3}$$

measures the non-commutativity of covariant derivatives, or equivalently the change in a vector parallel transported around a closed loop[3] in M.

A diffeomorphism $f : M \to M$ that preserves the metric,

$$f^* G_{f(p)} = G_p \,, \tag{5.4}$$

and thus the pair (M, G) constituting a Riemannian manifold, is called an isometry. (In particular, the Riemann curvature is therefore invariant under isometries.) The generators of isometries are Killing vector fields X satisfying $\mathcal{L}_X g = 0$, which in local coordinates takes the form $\hat{\nabla}_{(M} X_{N)} = 0$, or in other words; the flow generated by a Killing vector field preserves the metric.

As we saw in chapter 2 the metric G_{MN} allows the definition of an isomorphism between the spaces of differential forms of different degrees through the Hodge dual operation $*_G : \Omega^r(M) \to \Omega^{d-r}(M)$. Given an r-form ω we define its dual as

$$*_G\, \omega = \frac{-1}{r!(d-r)!} \frac{1}{\sqrt{-G}}\, \omega_{M_1 \ldots M_r} \epsilon^{M_1 \ldots M_r}{}_{M_{r+1} \ldots M_d} dy^{M_{r+1}} \wedge \ldots \wedge dy^{M_d} \tag{5.5}$$

where G is the determinant of the metric and $\epsilon^{M_1 \ldots M_d}$ is the totally antisymmetric tensor density with $\epsilon^{0 \ldots d-1} = 1$. In particular, in six dimensions this operator squares to unity in the middle dimension (that is acting on three-forms we have $*_G^2 = \mathbb{1}$), inducing a decomposition of $\Omega^3(M)$ into self-dual and anti-self-dual parts.

[1] Throughout the present chapter we will always work in the tensor calculus formalism of Riemannian geometry. For an introduction to the differential form description of Cartan we refer to e.g. [5, 6].

[2] Anticipating the dimensional reduction in subsequent parts of this chapter we denote by hatted symbols six-dimensional quantities whose reduced counterparts cannot be conveniently indicated by replacing upper by lower case symbols.

[3] The connection and curvature of Riemannian geometry can be alternatively be described in terms of fibre bundles using the machinery developed in previous chapters. We will not pursue such a description here.

We will be particularly interested below in the description of field theories with conformal symmetry, i.e. that are invariant under diffeomorphisms $f : M \to M$ that preserve the metric up to scale transformations

$$f^* G_{f(p)} = e^{-2\sigma} G_p \,, \tag{5.6}$$

where $\sigma \in \mathcal{F}(M)$ is an arbitrary function. Note that this is not a coordinate transformation; a point $p \in M$ is taken to a different point $f(p) \in M$. Conformal transformations are generated by conformal Killing vectors satisfying the generalization of the Killing vector equation to $\hat{\nabla}_{(M} X_{M)} - \frac{1}{d} G_{MN} \hat{\nabla}_P X^P = 0$. A transformation (5.6) furthermore induces transformations of the various fields of a theory defined on M. Following [58] we will use a convenient formulation of conformal invariance of a field theory: A theory is invariant under conformal transformations if it is coordinate invariant and invariant under simultaneous rescalings $G_{MN} \to e^{-2\sigma} G_{MN}$ (called Weyl rescaling) of the metric and of the fields $\Phi \to e^{w\sigma}\Phi$, where w is called the conformal weight of Φ. Correspondingly, the equations describing the dynamics and symmetries of the theory are required to transform covariantly.

We are thus led to consider manifolds M equipped not with a metric structure but with a conformal structure, which is an equivalence class of metrics defined by the relation

$$G_{MN} \sim e^{-2\sigma} G_{MN} \,. \tag{5.7}$$

The Weyl transformations thus relate different representatives of a conformal class of metrics. Furthermore, the Riemann curvature tensor (5.3) is not invariant under conformal rescalings of G_{MN} but it is possible to extract the part of the curvature that is: It is given by the traceless part of of $\hat{R}^P{}_{QMN}$ called the Weyl tensor

$$\begin{aligned} \hat{C}_{MNPQ} &= \hat{R}_{MNPQ} - \frac{1}{d-2}\left(\hat{R}_{MP}G_{NQ} - \hat{R}_{NP}G_{MQ} + \hat{R}_{NQ}G_{MP} - \hat{R}_{MQ}G_{NP}\right) \\ &\quad + \frac{\hat{R}}{(d-1)(d-2)}\left(G_{MP}G_{NQ} - G_{NP}G_{MQ}\right) . \end{aligned} \tag{5.8}$$

Here, we have introduced the Ricci tensor and the scalar curvature as the usual contractions of the Riemann tensor

$$\hat{R}_{MN} = \hat{R}^P{}_{MPN} \quad , \quad \hat{R} = G^{MN}\hat{R}_{MN} \,. \tag{5.9}$$

5.1.2 Spinors on curved manifolds

Since the focus of this thesis is supersymmetric theories we must also be able to describe spinors on curved manifolds. The obstacle in doing so is that for a general manifold M coordinate transformations are not encoded in SO$(d-1,1)$ but in GL$(d-1,1)$ matrices. Fields of the theory are usually required to transform in irreducible representations of this group[4], which has no spinor representations.

[4]In particular the covariant derivative $\hat{\nabla}_M$ introduced above acts on fields in such representations to produce objects with definite transformation properties.

The way to introduce spinors is instead to introduce an orthonormal frame of the tangent space at each point, called the vielbein E_M^A. Here, the indices $A, B = 0, 1, \ldots, d-1$ enumerate the basis vectors and are raised and lowered using the Minkowski metric η_{AB}. The vector index M is raised and lowered using the metric G_{MN}. Orthonormality of the E_M^A then implies the two equivalent relations

$$E_A^M E_{MB} = \eta_{AB} \quad , \quad E_M^A E_{NA} = G_{MN} \,. \tag{5.10}$$

The key observation is now that the choice of basis is arbitrary up to local Lorentz transformations acting on the (thusly named) Lorentz index A. In other words, there is a local $\mathrm{SO}(d-1,1)$ invariance in the vielbein formalism. We can therefore introduce the gauge field $\Omega_M{}^A{}_B$, called the spin connection, corresponding to this symmetry as the connection of the principal orthogonal frame bundle of M. The spin connection can then be used to define a derivative that in is covariant with respect to both local Lorentz transformations and general coordinate transformations.

Let us now return to the motivating example for the introduction of the spin connection, namely that of a field $\Psi(x)$ on M transforming in the spinor representation of $\mathrm{SO}(d-1,1)$. The generators on this representation are $\Sigma_{AB} = \frac{1}{2}\Gamma_{AB}$, which is the antisymmetric product of two Γ_A Minkowski Γ-matrices, and the covariant derivative is given as

$$\nabla_M \Psi = \partial_M \Psi + \frac{1}{4}\Omega_M^{AB}\Gamma_{AB}\Psi \,. \tag{5.11}$$

The quantity transforms as a covariant vector under general coordinate transformations and as a spinor under local Lorentz transformations as required[5]. Given a vector on M we can also use the vielbein to obtain a Lorentz vector or vice versa. In particular, this practice can be applied to the ordinary matrices Γ_A generating the spinor representation, yielding the curved space gamma matrices $\Gamma_M = E_M^A \Gamma_A$, satisfying $\{\Gamma_M, \Gamma_N\} = 2G_{MN}$, which are needed to describe the dynamics of the spinor field as we will see below.

We must impose the consistency condition that the vielbeins are covariantly constant, i.e. that $\nabla_M E_N^A = 0$, in order for the transition between flat and curved vector indices A and M to be well-defined. From this relation it is possible to solve for Ω_M^{AB} in terms of E_M^A

$$\Omega_M^{AB} = E^{NA}\partial_{[M}E_{N]}^B - E^{NB}\partial_{[M}E_{N]}^A - E^{PA}E^{QB}\partial_{[P}E_{Q]C}E_M^C \,, \tag{5.12}$$

and obtain the usual form (5.2) of the Levi-Civita connection. Finally, we note that the field strength of the spin connection is related to the curvature of M through the vielbein. Useful references for more detailed accounts of the material reviewed above are [5, 59, 60].

[5] This construction can of course be generalized to arbitrary representations of $\mathrm{SO}(d-1,1)$.

5.1.3 Metrics on circle fibrations

So far, the discussion of Riemannian geometry has been very general. We will now specialize to the geometry of interest in the remainder of the present chapter and in Paper IV-V. Following [61] we therefore consider a manifold M_6 of dimension $d = 6$ which is a fibration of a circle S^1 over some five-dimensional base manifold M_5. This construction is equivalent to the existence of a free action of U(1) on M_6 which preserves the metric G_{MN}. The base manifold is then reconstructed as the quotient $M_5 = M_6/\mathrm{U}(1)$, obtained by identifying all points in the orbits of U(1), or equivalently by the projection operator $\pi : M_6 \to M_5$ in the bundle description.

Since the U(1) action preserves the metric there exists an isometry along the S^1 fibre direction. Using this isometry we can obtain the general form of a metric on a circle fibration. To this end we split the curved vector index according to $M = (\mu, \varphi)$, with $\mu = 0, 1, \ldots, 4$, and denote the local coordinates on M_5 by x^μ while S^1 is parametrized by a coordinate φ of periodicity 2π. In terms of local coordinates we can write the invariant length squared on the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + r^2 \left(d\varphi + \theta_\mu dx^\mu\right)^2 \,. \tag{5.13}$$

The isometry along the S^1 implies that the $g_{\mu\nu}$, r and θ_μ are all independent of the coordinate φ. Thus, $g_{\mu\nu}(x)$ can be interpreted as the metric on M_5, $r(x)$ as the radius of the S^1 fibre and the vector $\theta_\mu(x)$ as an obliqueness parameter. The vielbein on M_5 is denoted e_μ^a where the flat vector index is split according to $A = (a, 5)$ with $a = 0, \ldots, 4$ in analogy with the curved index M. In Paper IV explicit expressions are presented for the metric, vielbein, Levi-Civita connection and spin connection on M_6 in terms of the corresponding quantities on M_5.

From the expression (5.13) it is clear that a reparametrization $\varphi \to \varphi + \lambda(x)$ corresponds to a transformation

$$\theta_\mu \to \theta_\mu + \partial_\mu \lambda \,, \tag{5.14}$$

implying that we can interpret θ_μ as the gauge field of the U(1) action on the fibre of $M_6 \to M_5$. The gauge field θ_μ is non-dynamical in the sense that it is determined by the geometry of M_6 and not governed by equations of motion derived from any action functional. However, reparametrization invariance in six dimensions implies that all physical quantities on M_5 must be invariant under θ_μ gauge transformations (5.14). Generically, this means that they can only include a dependence on θ_μ through its gauge invariant field strength

$$\mathcal{F}_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu \,, \tag{5.15}$$

which is the curvature of the U(1) bundle over M_5.

5.2 The $(2,0)$ theory

Even though the stated purpose of the present chapter is the study of the $(2,0)$ theories we have yet to introduce them. In order to remedy this we devote the present section to describing the $(2,0)$ supersymmetry algebra, which give these theories their name, and its simplest representation that will feature prominently in the rest of this chapter. We also give a basic overview of the appearance (and importance) of $(2,0)$ theory in string theory and M-theory, even though the work described below makes little use of this connection.

In the first part of this section we will consider the space-time to be flat Minkowski space, contrary to what is suggested by the content of the present chapter so far. The reason is that on a generic curved manifold all supersymmetries will be broken by the geometry, rendering the name somewhat misleading. It is therefore appropriate to be more specific: $(2,0)$ theory on an arbitrary curved manifold is taken to mean the conformally invariant theory that in flat space is invariant under the $(2,0)$ supersymmetry algebra. We will return to the question of which manifolds admit unbroken supersymmetries below.

5.2.1 The supersymmetry algebra

In six flat dimensions the $(2,0)$ supersymmetry algebra is the maximally extended Poincaré superalgebra which can be considered without the inclusion of gravity[6]. (In fact, there exist no supersymmetry algebra which can accommodate conformal symmetry in higher dimensions [29].) The bosonic part of the algebra is isomorphic to $\mathfrak{so}(5,1) \oplus \mathfrak{so}(5)$, where the two factors are respectively the algebras of the $\mathrm{SO}(5,1)$ Lorentz group and the $\mathrm{USp}(4) \cong SO(5)$ R-symmetry group. As we saw in section 3.1 it follows from the general form of the supersymmetry algebra that the fermionic generators furnish a representation of the bosonic subalgebra, and in particular a spinorial representation of the Lorentz algebra. In the case of the $(2,0)$ algebra the supercharges Q^α transform as $(\mathbf{4};\mathbf{4})$ under the bosonic subalgebra [29, 30], where the two $\mathbf{4}$'s refer to the chiral spinor representation of $\mathrm{SO}(5,1)$ and the fundamental spinor representation of $\mathrm{USp}(4)$ respectively. The index $\alpha = 1, \ldots, 4$ refers to the latter, and can be used to impose a symplectic reality condition

$$(Q^\alpha)^* = M_{\alpha\beta} B_{(6)} Q^\beta \,. \tag{5.16}$$

Here, $M_{\alpha\beta}$ is the $\mathrm{USp}(4)$-invariant symplectic metric (with inverse $T^{\alpha\beta}$) used to lower (and raise) spinor indices $\alpha, \beta, \ldots$ in the fundamental $\mathbf{4}$ representation of $\mathrm{USp}(4)$ and $B_{(6)}$ is related to the charge conjugation matrix in six dimensions. Various spinor conventions (we will always suppress spinor indices below) and conventions regarding the symplectic structure related to the R-symmetry are given in the appendix to Paper IV.

[6] C.f. the $\mathcal{N} = 4$ algebra in four dimensions considered in the two previous chapters.

From the transformation properties of Q^α under the bosonic algebra $\mathfrak{so}(5,1) \oplus \mathfrak{so}(5)$ we can derive the most general form of the anticommutator $\{Q^\alpha, Q^\beta\}$ by considering the symmetric tensor product representation

$$[(\mathbf{4};\mathbf{4}) \otimes (\mathbf{4};\mathbf{4})]_s = (\mathbf{6};\mathbf{1}) \oplus (\mathbf{6};\mathbf{5}) \oplus (\mathbf{10}_+;\mathbf{10})\,, \tag{5.17}$$

where $\mathbf{6}$ is a Lorentz vector, $\mathbf{10}$ and $\mathbf{5}$ are respectively symmetric and traceless antisymmetric USp(4) bispinors, and $\mathbf{10}_+$ is a self-dual three-form tensor. As a result, the anticommutation relation is given by [62]

$$\{Q^\alpha, Q^\beta\} = -2iM^{\alpha\beta}\Gamma^M P_M + \Gamma^M Z_M^{\alpha\beta} + \Gamma^{MNP} Z_{MNP}^{\alpha\beta}\,, \tag{5.18}$$

where P_M is the energy-momentum vector[7] while $Z_M^{\alpha\beta}$ and $Z_{MNP}^{\alpha\beta}$ are central charges transforming according to the above tensor product decomposition. In this chapter we will predominantly be concerned with the tensor multiplet representation for which the central charges vanish, but we will find the general form of the algebra useful for understanding $(2,0)$ theory in the context of string theory and M-theory below.

Before proceeding we note that according to [29] the supersymmetry algebra described above can be extended to the superconformal algebra whose bosonic subalgebra is $\mathfrak{so}(6,2) \oplus \mathfrak{so}(5)$. Even though we will not consider this algebra in detail here, its existence is what allows us to consider the $(2,0)$ theories, which in addition to invariance under the super-Poincaŕe group exhibit conformal symmetry.

5.2.2 The tensor multiplet

We will now review the construction [30] of the free tensor multiplet representation of the $(2,0)$ supersymmetry algebra. With vanishing central charges we once again use Wigner's method of induced massless representations. The little group is in this case $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{USp}(4)$, with respect to which the representation of the supercharge Q^α is decomposed according to

$$(\mathbf{4};\mathbf{4}) = (\mathbf{2},\mathbf{1};\mathbf{4}) \oplus (\mathbf{1},\mathbf{2};\mathbf{4})\,. \tag{5.19}$$

The first term corresponds to the active supercharges $Q_{1/2}$ while the second term corresponds to the unbroken generators annihilating physical states. In order to determine a set of creation and annihilation of the Clifford algebra generated by $Q_{1/2}$ we consider a further decomposition relative to the SO(2) subgroup of SU(2) yielding

$$(\mathbf{2};\mathbf{4}) = \mathbf{4}_{1/2} \oplus \mathbf{4}_{-1/2}\,, \tag{5.20}$$

where we have suppressed the the SU(2) singlet and denote the SO(2) weight with a subscript. In this way we achieve the separation of $Q_{1/2}$ into creation and annihilation operators related by hermitian conjugation. Acting repeatedly on a vacuum

[7] The factor of $-2i$ in the P_M term is introduced to conform with conventions in PAPER IV-V.

state $|0\rangle$ in the $\mathbf{1}_{-1}$ representation with the creation operators in $\mathbf{4}_{1/2}$ we obtain the tensor multiplet representation as

$$(\mathbf{1}_{-1} \oplus \mathbf{1}_0 \oplus \mathbf{1}_1) \oplus \mathbf{5}_0 \oplus \left(\mathbf{4}_{-1/2} \oplus \mathbf{4}_{1/2}\right) \tag{5.21}$$

or expressed in terms of representations of the SO(4) × USp(4) little group

$$(\mathbf{3}, \mathbf{1}; \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \mathbf{5}) \oplus (\mathbf{2}, \mathbf{1}; \mathbf{4}) \,. \tag{5.22}$$

Having established the tensor multiplet representation content we can proceed to identify the corresponding fields realizing the representation. The first term is perhaps the least obvious; it corresponds to a two-form gauge field B_{MN} with self-dual three-form field strength $H = dB$, transforming trivially under the USp(4) R-symmetry. The second term describes five Lorentz scalars transforming in the vector representation **5** of USp(4) ≅ SO(5) which we describe as an antisymmetric bispinor $\Phi^{\alpha\beta}$ satisfying the tracelessness condition $M_{\alpha\beta}\Phi^{\alpha\beta} = 0$. Furthermore, we impose the symplectic reality condition

$$(\Phi^{\alpha\beta})^* = \Phi_{\alpha\beta} = M_{\alpha\gamma} M_{\beta\delta} \Phi^{\gamma\delta} \,. \tag{5.23}$$

Finally, the last term in (5.22) corresponds to the fermionic part of the tensor multiplet consisting of a positive chirality Lorentz spinor Ψ^α transforming in the **4** spinor representation of USp(4) and satisfying the same symplectic Majorana condition as the supercharges

$$(\Psi^\alpha)^* = M_{\alpha\beta} B_{(6)} \Psi^\beta \,. \tag{5.24}$$

The above realization of the tensor multiplet was first constructed in [63] using a superfield formalism.

We are now ready to consider the dynamics of the free tensor multiplet. In doing so we will transition to the general case of an arbitrary manifold M_6. As mentioned above, we thus refer to the free theory of the fields described above as the tensor multiplet of $(2,0)$ theory. In order to ensure that the theory depends only on the conformal structure of M_6 we must, according to the discussion in the previous subsection, impose invariance under simultaneous Weyl rescalings of the metric and corresponding rescalings of the fields.

Throughout, we will consider a description using only the three form tensor field H which is then taken to satisfy the equations of motion

$$dH = 0 \quad , \quad H = *_G H \,, \tag{5.25}$$

which are manifestly conformally covariant. The self-duality of H makes a Lagrangian description of the three-form complicated since the candidate for Lagrangian density $H \wedge *_G H$ is identically vanishing. We will therefore only consider the three-form at the level of equations of motion[8].

The massless scalar field satisfies the conformal Klein-Gordon equation

$$G^{MN}\hat{\nabla}_M\hat{\nabla}_N\Phi^{\alpha\beta} + c\hat{R}\Phi^{\alpha\beta} = 0\,, \tag{5.26}$$

which is covariant in $d = 6$ provided the conformal transformation $\Phi^{\alpha\beta} \to e^{2\sigma}\Phi^{\alpha\beta}$ and $c = -\frac{1}{5}$. Finally, the fermionic spinor Ψ^α satisfies the ordinary Dirac equation

$$\Gamma^M\hat{\nabla}_M\Psi^\alpha = 0\,, \tag{5.27}$$

which is conformally covariant for spinors scaling with conformal weight $\Psi^\alpha \to e^{\frac{5}{2}\sigma}\Psi^\alpha$. Unlike the three-form H the remaining fields of the tensor multiplet admit Lagrangian descriptions: Both (5.26) and (5.27) follow from conformally invariant action functionals.

We can now return to the question of supersymmetry in curved geometries. Since we consider the theory at the level of equations of motion, supersymmetry amounts to the closure of the set of solutions to (5.25), (5.26) and (5.27) under the supersymmetry transformations

$$\begin{aligned}
\delta H_{MNP} &= 3\hat{\nabla}_{[M}\left(\overline{\Psi}_\alpha\Gamma_{NP]}\mathcal{E}^\alpha\right) && (5.28)\\
\delta\Phi^{\alpha\beta} &= 2\overline{\Psi}^{[\alpha}\mathcal{E}^{\beta]} - \frac{1}{2}T^{\alpha\beta}\overline{\Psi}_\gamma\mathcal{E}^\gamma && (5.29)\\
\delta\Psi^\alpha &= \frac{i}{12}H_{MNP}\Gamma^{MNP}\mathcal{E}^\alpha + 2iM_{\beta\gamma}\hat{\nabla}_M\Phi^{\alpha\beta}\Gamma^M\mathcal{E}^\gamma && (5.30)\\
&\quad + \frac{4i}{3}M_{\beta\gamma}\Phi^{\alpha\beta}\Gamma^M\hat{\nabla}_M\mathcal{E}^\gamma\,, && (5.31)
\end{aligned}$$

where the fermionic parameter $\mathcal{E}^\alpha$ is a symplectic Majorana spinor of negative chirality[9] in the **4** of USp(4). The variations satisfy the same equations of motion as the original fields up to terms proportional to the differential operator $\hat{\nabla}_M - \frac{1}{d}\Gamma_M\Gamma^N\hat{\nabla}_N$ acting on $\mathcal{E}^\alpha$. Thus, supersymmetry imposes a condition on the geometry of M_6: It must admit non-trivial solutions to the conformal Killing spinor equation

$$\hat{\nabla}_M\mathcal{E}^\alpha - \frac{1}{d}\Gamma_M\Gamma^N\hat{\nabla}_N\mathcal{E}^\alpha = 0\,. \tag{5.32}$$

A special case is manifolds M_6 admitting covariantly constant spinors $\hat{\nabla}_M\mathcal{E}^\alpha = 0$. Finally, we note that the supersymmetry variations satisfy the correct conformal rescaling properties and that (5.32) is covariant provided the rescaling of the supersymmetry parameter according to $\mathcal{E}^\alpha \to e^{-\frac{1}{2}\sigma}\mathcal{E}^\alpha$, ensuring consistent superconformal symmetry.

[8]There exist alternative constructions introducing an auxiliary scalar field which allows a Lagrangian description reproducing the correct equations of motion for H [64, 65], or considering an action which does not imply but allow self-duality [66]. We will not further consider either of these approaches here.

[9]The negative chirality is also required for the bosonic generator $\overline{\mathcal{E}}_\alpha Q^\alpha$ to be non-vanishing.

5.2.3 Origin in String/M-theory

We will now proceed to make good on our promise to provide an overview of the $(2,0)$ theory in the context of string theory and M-theory. Since this is somewhat out of the main line of the thesis we will keep the exposition very brief and not attempt to give a complete account of the developments in the field or the corresponding bibliographic references. However, the higher dimensional perspective provides an important motivation for the study of $(2,0)$ theory for readers who fail to be sufficiently fascinated by the prospect of studying the highest-dimensional superconformal theory that exists [29].

Even though the $(2,0)$ algebra and the tensor multiplet representation were both previously known, the the discovery of $(2,0)$ theory (in the sense it is most commonly used) and its appearance in string theory was made in [67], based on results by Witten presented at the Strings '95 conference in Los Angeles. The argument is based on the T-duality of type IIA and type IIB string theory in a space-time of the form $\mathbb{R}^{5,1} \times K3$, where $K3$ is a particular compact four-dimensional hyper-Kähler manifold. The moduli space of the $K3$ contains certain singular points, obeying an ADE classification, where N of its linearly independent two-cycles collapse to a singularity. At these points the type IIA theory develops an enhanced gauge symmetry indicated by the appearance of additional massless vector particles (W bosons) [67–69]. (The symmetry enhancement can be understood in terms of another string theory duality between the type IIA string on $\mathbb{R}^{5,1} \times K3$ and heterotic string on $\mathbb{R}^{5,1} \times T^4$ [68].) Witten's realization was that T-duality of type IIA and type IIB further compactified on a circle predicted the existence of self-dual strings in type IIB theory compactified on $K3$, which become tensionless at the singular points of the $K3$ moduli space corresponding to the symmetry enhancement.

We will now review the compactification of type IIB theory on $K3$ and the emergence of the A_N series of $(2,0)$ theory, which in particular includes the self-dual six-dimensional strings featuring in the previous paragraph. The first step is to employ the description of $K3$ close to the A_N singularity as a $N+1$ multi-centered Taub-NUT space[10] TN_{N+1} [70–72] with metric

$$ds^2 = U\delta_{ij}dx^i dx^j + U^{-1}(d\varphi + \theta_i dx^i)^2 \tag{5.33}$$

where

$$U = \frac{1}{\lambda^2} + \sum_{i=1}^{N+1} \frac{1}{|\vec{r} - \vec{r}_i|} \tag{5.34}$$

and $dU = *_\delta d\theta$. This space has a compact direction φ whose radius vanishes at the centers located at the points $\vec{r}_i$ in the three remaining coordinate directions $\vec{r} = (x^1, x^2, x^3)$ parametrizing $\mathbb{R}^3$.

[10] We will return to discuss the single-centered Taub-NUT space in more detail below in a different application.

We can construct N linearly independent two-cycles as the surfaces $I_{i,i+1} \times S^1$, where $I_{i,i+1}$ is the line connecting the center locations $\vec{r}_i$ and $\vec{r}_{i+1}$, which have the topology of a sphere due to the pinching of the cylindrical region at the centers. The intersection matrix of the two-cycles is given by the Cartan matrix of A_N [73], which implies that they can be associated with the simple roots of the Lie algebra. The singularity where the two-cycles collapse is then obtained when all centers coincide and the approach to the singular point of the $K3$ is described by $4N$ parameters. Furthermore, for each center in (5.33) there exists a unique anti-self-dual two-form Ω_i localized near $\vec{r}_i$, which can be viewed as the connections of certain complex line bundles over TN_{N+1} [74] generalizing the construction for the single-centered case previously described in e.g. [72].

Having a convenient description of the two-cycles in terms of the Taub-NUT geometry we can construct the A_N $(2,0)$ theory by considering the four-form potential C_4 of type IIB string theory sourced by the D3-brane. The coupling is self-dual in the sense that C_4 has self-dual five-form field strength and couples both electrically and magnetically to the D3-brane. In the compactification on $K3$ the D3-brane may wrap any of the N two-cycles, which from the uncompactified dimensions $\mathbb{R}^{5,1}$ is perceived as a string propagating in six dimensions. Furthermore, the corresponding C_4 gauge field is decomposed according to

$$C_4 = \sum_i B_i \wedge \Omega_{i,i+1} \,, \tag{5.35}$$

where $\Omega_{i,i+1}$ is the linear combination of two-forms supported near the two-sphere connecting $\vec{r}_i$ and $\vec{r}_{i+1}$. The two-forms B_i appearing in the decomposition has self-dual three-form field strength $H = dB$ and couples to the string in $\mathbb{R}^{5,1}$ both electrically and magnetically. Thus, the string is indeed self-dual, as indicated above, in the sense that its electric and magnetic charges under B are equal. Finally, the RR two-form gauge potential gives rise to N moduli, making a total of $5N$ real moduli of the $(2,0)$ theory which correspond to the vacuum expectation values of $5N$ real scalars. Together with the B gauge fields they fall into N massless tensor multiplet representations of the $(2,0)$ algebra.

The tension of the strings are proportional to the area of the two-cycles in $K3$ wrapped by the D3-brane, which implies that close to the singularity the string, and consequently the $(2,0)$ theory, decouples from gravity. To make contact with [67] we consider a further compactification to $\mathbb{R}^{4,1} \times S^1 \times K3$, in which case the tensor multiplets describe N massless vector bosons (a result which is obtained for the generalized case of a circle fibration in Paper IV and summarized below). The self-dual strings wrapping the S^1 on the other hand describe vector particles with mass proportional to the string tension (see also [75]). The five-dimensional theory is thus $\mathrm{SU}(N+1)$ supersymmetric Yang-Mills theory spontaneously broken by the moduli to $\mathrm{U}(1)^N$. The tensor multiplets consequently correspond to the Cartan subalgebra of $\mathrm{SU}(N+1)$ while the strings related to the linearly independent two-cycles correspond to the simple roots. (Strings not wrapping the S^1 describe the magnetic strings dual to the gauge particles.) At the origin of moduli space all strings become tensionless and the $\mathrm{SU}(N+1)$ symmetry is restored, corresponding to the gauge symmetry enhancement of the type IIA theory. The above argument can be generalized to the D- and E- series of singularities and $(2,0)$ theory.

Next, we will briefly describe the construction of the A_N series of $(2,0)$ theory in M-theory (which when compactified on a circle gives the type IIA construction T-dual to the type IIB construction described above) containing extended M2- and M5-branes. Considering first a single M5-brane in $\mathbb{R}^{10,1}$ one finds that its presence breaks the Lorentz group $\mathrm{SO}(10,1) \to \mathrm{SO}(5,1) \times \mathrm{SO}(5)$, half of the 32 supersymmetry generators and certain gauge symmetries of the M-theory three-form. The world-volume theory of the M5-brane, describing the dynamics of the Goldstone modes corresponding to the broken symmetries, is described by the $(2,0)$ tensor multiplet [76, 77] which is invariant under the 16 unbroken supersymmetries.

To obtain the A_N type $(2,0)$ theory we consider instead a configuration $N+1$ parallel M5-branes, and M2-branes stretching between them [78]. The dynamics of each of the M5-branes is described by a tensor multiplet, but the sum (associated to the center of mass motion) decouples from the theory giving N tensor multiplets corresponding to the Cartan generators of A_N. From the world-volume perspective the M2-branes stretching between two adjacent M5-branes appear as self-dual strings coupling to the two-forms of the corresponding tensor multiplets. The moduli of the theory are the vacuum expectation values of the $5N$ scalar fields parametrizing the (relative) position in the transverse directions of the M5-branes. The tension of the strings is proportional to the distance between the two branes, implying that close to the origin of moduli space, where the branes coincide, the strings decouple from the gravity bulk modes resulting in a purely six-dimensional theory without gravity on the world-volume of the stack. The compactification to five dimensions is analogous to the one described above in the type IIB case, justifying the identification of the N self-dual strings with the simple root generators of A_N and reproducing the symmetry enhancement to $\mathrm{SU}(N+1)$ gauge theory when the M5-branes coincide[11].

Before closing the present subsection we need to make a few remarks concerning the $(2,0)$ theory constructed in string and M-theory. Above we have associated the self-dual strings with the N simple roots of the A_N algebra. The remaining positive roots are obtained by linear combinations corresponding in the type IIB case to wrapping the D3-brane on two-spheres connecting any two centers of the TN_{N+1} space and in the M-theory construction to M2-branes stretching between any two M5-branes. (Negative roots are obtained by reversing the string orientation.) In the compactification to five dimensions this produces the remaining massive vector multiplets of spontaneously broken $\mathrm{SU}(N)$ Yang-Mills theory. The A_N construction in M-theory can be extended to the D_N series of $(2,0)$ theory by introducing an

[11] The center of mass tensor multiplet corresponds to the Cartan generator of the abelian factor in $\mathrm{U}(N+1) \cong \mathrm{SU}(N+1) \times \mathrm{U}(1)$.

orientifold plane [73] in the constructions above, while the $(2,0)$ theory corresponding to the three exceptional simply laced algebras have no known simple geometric M-theory interpretation.

Finally, the self-dual string appearing above deserves a few further comments. In particular, we can now reconnect to the general form of the $(2,0)$ supersymmetry algebra in (5.18). While we saw that the tensor multiplet had vanishing central charges, the presence of the central charge $Z_M^{\alpha\beta}$ allows the construction of representations including an extended one-dimensional object [79, 80]. This agrees with the general result [81] that a p-dimensional extended object gives rise to a p-form central charge in the supersymmetry algebra[12].

5.2.4 Dimensional reduction to $d = 5$ and $d = 4$

As we saw in the previous subsection the compactification of the six-dimensional model on $\mathbb{R}^{4,1} \times S^1$ provides crucial insight into the $(2,0)$ theory. In fact, it is fair to say that the full six-dimensional formulation of the interacting $(2,0)$ theories, which in particular possesses no weak coupling limit due to the self-dual coupling of the string, remains elusive. (The exception is the free tensor multiplet described above.) Much of the progress towards an understanding of the theory was therefore initially obtained by considering the compactification to the maximally supersymmetric Yang-Mills theory in five dimensions. The perturbative low-energy theory described by the vector multiplets obtained from the tensor multiplets and strings wrapping the S^1 is complemented by the unwrapped strings and Kaluza-Klein modes of the massless tensor multiplets and strings. This suggests that $(2,0)$ theories should be interpreted as the UV completion of the (power-counting non-renormalizable) $\mathcal{N} = 4$ Yang-Mills theory in five dimensions including both perturbative and solitonic (non-perturbative) degrees of freedom.

Recent work suggests that the connection between maximally supersymmetric Yang-Mills theory and $(2,0)$ theory is in fact even stronger: In [82] a non-abelian generalization of the tensor multiplet is constructed and its dynamics is found to be essentially five-dimensional. The reduction to five dimensions is further investigated in [83] where the Kaluza-Klein modes of the self-dual string (whether it is wrapping the S^1 or not) are found to correspond to non-perturbative states of the $d = 5$ Yang-Mills theory. This indicates that $(2,0)$ theory and Yang-Mills theory are equivalent (see also [84]), rather than the former being the UV completion of the latter, since no additional high energy modes need to be introduced into the SYM theory (which in turn suggests that it is a well-defined quantum theory despite its being power-counting non-renormalizable).

Finally, we can now elucidate the connection between the $(2,0)$ theory and $\mathcal{N} = 4$ Yang-Mills theory in $d = 4$ by considering a further compactification to $\mathbb{R}^{3,1} \times T^2$ [67,

[12] Consequently, from (5.18) we also expect the appearance of a three-dimensional extended object in $(2,0)$ theory. Such an object can indeed be constructed [62] but will not be consider here.

75, 85]. The low-energy gauge theory on $\mathbb{R}^{3,1}$ obtained has maximal supersymmetry and depends on the complex coupling constant τ, which is the modular parameter of the torus T^2, through the Yang-Mills action discussed in previous chapters. Thus, S-duality of the $\mathcal{N}=4$ theory is manifest in the description as dimensionally reduced $(2,0)$ theory due to the $\mathrm{SL}(2,\mathbb{Z})$ reparametrization invariance of T^2. In particular, electrically and magnetically charged particles correspond to self-dual strings in $d=6$ wrapping the two distinct one-cycles of T^2, and the $\mathbb{Z}_2$ symmetry exchanging the cycles corresponds to electric-magnetic duality.

5.3 Dimensional reduction on circle fibrations

In this section we will proceed to review the topic of PAPER IV: The dimensional reduction of $(2,0)$ theory on the spatial S^1 fibre of a manifold M_6 which is a $\mathrm{U}(1)$-fibration over some arbitrary base manifold M_5 (which is in general a curved space-time). The geometry of such a situation was described in section 5.1.3. Full details on the decomposition of the connections needed to define a derivative ∇_μ, which is covariant with respect to both general coordinate transformations and local Lorentz transformations on M_5 are computed from the metric (5.13) and given in PAPER IV. Furthermore, the Riemann tensor is decomposed in the reduction implying in particular that the curvature scalar $\hat{R}$ appearing in the equation of motion for $\Phi^{\alpha\beta}$ becomes

$$\hat{R} = R - \frac{1}{4}r^2\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - 2\frac{1}{r}\nabla_\mu\nabla^\mu r \tag{5.36}$$

where R is the curvature scalar of the metric $g_{\mu\nu}$ on M_5.

In order to derive the low-energy effective theory on M_5 obtained in the reduction explicitly, we first restrict our attention to the free tensor multiplet which possesses a classical description in terms of equations of motion reviewed in the previous section. We will then generalize the abelian gauge theory obtained to a non-abelian Yang-Mills theory in order to produce a candidate for the reduction of interacting $(2,0)$ theory. In particular, we will verify that the correct symmetries expected from superconformal invariance in six dimensions are reproduced.

5.3.1 The tensor multiplet

The tensor multiplet and its dynamics were discussed in section 5.2.2 for an arbitrary manifold M_6. For $M_6 = \mathbb{R}^{4,1}\times S^1$ with a product metric it is well-known [67, 75] that reduction on the S^1 produces (maximally) supersymmetric Maxwell theory in five dimensions, whose R-symmetry group is $\mathrm{USp}(4)$ just as for the original theory. The fields of the dimensionally reduced theory furnish the massless $\mathcal{N}=4$ vector multiplet consisting of a gauge field A_μ with field strength $F_{\mu\nu}$, five scalars $\phi^{\alpha\beta}$ satisfying $\phi^{(\alpha\beta)}=0$, $M_{\alpha\beta}\phi^{\alpha\beta}=0$ and $(\phi^{\alpha\beta})^*=\phi_{\alpha\beta}$, and four fermionic spinors ψ^α transforming in the $\mathbf{4}$ of $\mathrm{USp}(4)$ and satisfying the five-dimensional analogue of the symplectic Majorana condition (5.24). The gauge coupling, furthermore, is given by $\tilde{g} = \sqrt{r}$ and is a constant on $M_5 = \mathbb{R}^{4,1}$ due to the direct product structure of the metric.

The generalization to an arbitrary circle fibration M_6 implies that the Maxwell theory on M_5 receives certain modifications. The dimensional reduction in the general case is described in detail in PAPER IV and will not be repeated in its entirety here[13]. Instead, in this subsection and the next, we aim at providing a summary of the resulting theory, which in particular has the same (dynamical) field content $(A_\mu, \phi^{\alpha\beta}, \psi^\alpha)$ as the ordinary Maxwell theory.

In the most general metric (5.13) compatible with a circle fibration, the radius $r(x)$ is generically a function on M_5 and the non-dynamical U(1) gauge field θ_μ (corresponding to the connection of the bundle M_6 as explained in section 5.1.3) is non-vanishing. The dimensional reduction of the scalar and spinor fields of the tensor multiplet is relatively straightforward and produces the corresponding fields $\phi^{\alpha\beta}$ and ψ^α on M_5. With an appropriate rescaling to obtain canonically normalized fields in five dimensions their dynamics are governed by the action functionals

$$S_\phi = \int d^5x\sqrt{-g}\left(-\frac{1}{r}\nabla_\mu\phi_{\alpha\beta}\nabla^\mu\phi^{\alpha\beta} - \frac{1}{5}\frac{1}{r}R\phi_{\alpha\beta}\phi^{\alpha\beta} + K(g,r,\theta)\phi_{\alpha\beta}\phi^{\alpha\beta}\right) \tag{5.37}$$

and

$$S_\psi = \int d^5x\sqrt{-g}\left(\frac{1}{r}i\overline{\psi}_\alpha\gamma^\mu\nabla_\mu\psi^\alpha - \frac{1}{8}\mathcal{F}_{\mu\nu}\overline{\psi}_\alpha\gamma^{\mu\nu}\psi^\alpha\right). \tag{5.38}$$

Here, we have introduced the quantity

$$K(g,r,\theta) = \frac{1}{r^3}\nabla_\mu r\nabla^\mu r - \frac{3}{5}\frac{1}{r^2}\nabla_\mu\nabla^\mu r + \frac{1}{20}r\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}, \tag{5.39}$$

containing information about the geometry of the manifold M_6.

The dimensional reduction of the self-dual three-form H requires a slightly more detailed treatment, especially in view of the developments considered in the last section of this chapter. It can be decomposed according to

$$H = E + F \wedge d\varphi \tag{5.40}$$

where (in the low energy effective theory) E and F are respectively a three-form and a two-form on M_5. The self-duality of H implies that E can be eliminated and the dimensionally reduced theory on M_5 expressed in terms of F. From the tensor field dynamics it follows that F satisfies $dF = 0$, allowing the (local) interpretation as the field strength of a gauge field $F = dA$, and the equation of motion obtained as the stationary point of the action

$$S_F = \int\left(-\frac{1}{r}F\wedge *_g F + \theta\wedge F\wedge F\right). \tag{5.41}$$

[13]In particular, the decomposition of Lorentz spinors and the truncation of the Kaluza-Klein modes obtained in the reduction are discussed.

Summarizing the implications of the above results we find that for a generic fibration the coupling strength (which is still given by the square root of the fibre radius) is no longer a constant but a function on M_5. Furthermore, the gradient of $r(x)$ appears in the quadratic potential terms in (5.37) and (5.38) together with a coupling to the gauge field θ_μ through the gauge invariant curvature $\mathcal{F}$ of the U(1) bundle M_6. The gauge field action is modified by the appearance of a topological term $\int \theta \wedge F \wedge F$ (generalizing the familiar θ-term of SYM theory in four dimensions discussed in previous chapters). The consequences of the presence of this term is discussed in [61] (see also [86]) and will be considered in more detail below.

5.3.2 Superconformal symmetry

Before proceeding to consider a non-abelian generalization of the results obtained in the previous subsection, we will review the manifestation of the superconformal symmetry of the $(2,0)$ theory in the dimensionally reduced abelian gauge theory on M_5. The fact that the theory of the $(2,0)$ tensor multiplet depends only on the conformal structure on M_6 gives rise to a generalized conformal symmetry in five dimensions as follows[14]: According to (5.13) a Weyl rescaling of the metric G_{MN} induces rescalings not only of the metric $g_{\mu\nu}$ but also of the fibre radius $r(x)$

$$g_{\mu\nu} \to e^{-2\sigma} g_{\mu\nu} \quad , \quad r \to e^{-\sigma} r \,. \tag{5.42}$$

The simultaneous rescaling of the tensor multiplet fields gives the corresponding transformations of the fields on M_5 (taking the rescaling mentioned above into account)

$$\phi^{\alpha\beta} \to e^{\sigma} \phi^{\alpha\beta} \quad , \quad \psi^\alpha \to e^{\frac{3}{2}\sigma} \psi^\alpha \,. \tag{5.43}$$

It is easy to verify that the action functionals (5.37) and (5.38) are invariant under such a generalized conformal transformation[15], which thus constitutes a symmetry of the theory on M_5. We note that in particular, this symmetry implies a rescaling of the coupling strength parameter of the Maxwell theory; a rather peculiar feature.

Since it is a consequence of the six-dimensional theory being defined only in terms of the conformal class of the metric G_{MN}, the generalized conformal symmetry persists for arbitrary circle fibrations $M_6 \to M_5$. However, the same is not true for supersymmetry. More specifically, we saw in section 5.2.2 that supersymmetry required the existence of non-trivial conformal Killing spinors on M_6 satisfying the condition (5.32). For the circle fibration reduction, this condition can be formulated as a condition on the geometry of M_5 in terms of the supersymmetry parameter ε^α

[14] Ordinary (supersymmetric) Maxwell theory in five dimensional Minkowski space is not conformally invariant due to the dimensionality of the coupling constant.

[15] The Maxwell action (5.41) is of course manifestly invariant.

on M_5, by requiring the existence of non-trivial solutions to

$$\nabla_\mu \varepsilon^\alpha = \frac{1}{2}\frac{1}{r}\nabla^\nu r \gamma_\mu \gamma_\nu \varepsilon^\alpha + \frac{i}{8} r \mathcal{F}^{\rho\sigma} \gamma_\mu \gamma_{\rho\sigma} \varepsilon^\alpha + \frac{i}{4} r \mathcal{F}_\mu{}^\nu \gamma_\nu \varepsilon^\alpha \,. \tag{5.44}$$

The supersymmetry transformation of the dynamical fields of the Maxwell theory on M_5 are then obtained by dimensional reduction of the corresponding transformations (5.28), (5.29) and (5.30) of the tensor multiplet on M_6:

$$\begin{aligned}
\delta\phi^{\alpha\beta} &= 2\overline{\psi}^{[\alpha}\varepsilon^{\beta]} - \frac{1}{2}T^{\alpha\beta}\bar{\psi}_\gamma \varepsilon^\gamma & (5.45)\\
\delta F_{\mu\nu} &= -2i\nabla_{[\mu}\overline{\psi}_\alpha \gamma_{\nu]}\varepsilon^\alpha + i\frac{1}{r}\nabla^\rho r \overline{\psi}_\alpha \gamma_{\mu\nu\rho}\varepsilon^\alpha - 2i\frac{1}{r}\nabla_{[\mu} r\overline{\psi}_\alpha \gamma_{\nu]}\varepsilon^\alpha \\
&\quad + r\mathcal{F}_{\mu\nu}\overline{\psi}_\alpha \varepsilon^\alpha + \frac{3}{2} r\mathcal{F}_{[\mu}{}^\rho \overline{\psi}_\alpha \gamma_{\nu]\rho}\varepsilon^\alpha - \frac{1}{4} r\mathcal{F}^{\rho\sigma}\overline{\psi}_\alpha \gamma_{\mu\nu\rho\sigma}\varepsilon^\alpha & (5.46)\\
\delta\psi^\alpha &= \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\varepsilon^\alpha + 2iM_{\beta\gamma}\nabla_\mu \phi^{\alpha\beta}\gamma^\mu \varepsilon^\gamma \\
&\quad + 2i\frac{1}{r}M_{\beta\gamma}\phi^{\alpha\beta}\nabla_\mu r \gamma^\mu \varepsilon^\gamma - rM_{\beta\gamma}\phi^{\alpha\beta}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^\gamma \,. & (5.47)
\end{aligned}$$

Indeed, the complete Maxwell action is invariant under this supersymmetry transformation provided that ε^α satisfies (5.44). Finally, we note that for the case of a direct product metric we have $\partial_\mu r = 0$ and $\theta_\mu = 0$, implying the vanishing of all the geometric terms in both the action and supersymmetry variations as required to reproduce the ordinary Maxwell theory for $M_6 = \mathbb{R}^{4,1} \times S^1$.

5.3.3 Non-abelian generalization

The $(2,0)$ theory for the simply laced groups of the ADE classification cannot be described in terms of classical field theory, rendering a direct generalization of the procedure outlined above for the free tensor multiplet theory inaccessible. However, it is known that $(2,0)$ theory associated to a group G upon reduction on the S^1 fibre of $M_6 \to M_5$ should be described by gauge fields with gauge group G on M_5 [61]. We expect the non-abelian theory to couple to the background U(1) gauge field θ_μ (through the field strength $\mathcal{F}_{\mu\nu}$) and to exhibit the same generalized conformal invariance that was found for the free tensor multiplet above. Finally, when M_6 admits conformal Killing spinors (or equivalently M_5 admits non-trivial solutions to (5.44)) the theory should be supersymmetric. In fact, in the case of $\mathbb{R}^{5,1} \times S^1$ the long-distance physics on M_5 is described by $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [67, 75].

In order to arrive at a candidate for the reduction of interacting $(2,0)$ theory, starting from the Maxwell theory on M_5 described above, we must first generalize the fields of the vector multiplet to $(A^a_\mu, \phi^{\alpha\beta}_a, \psi^\alpha_a)$ transforming in the adjoint representation of the Lie algebra $\mathfrak{g}$ of the gauge group G. In other words, we promote A to the connection of a principal G-bundle over M_5 (of which F is the curvature) and the remaining fields to sections of associated adjoint bundles. We also introduce the gauge covariant derivative D_μ, acting on a field Ξ^a in the adjoint representation in the usual way

$$D_\mu \Xi^a = \nabla_\mu \Xi^a + f^a{}_{bc} A^b_\mu \Xi^c \,. \tag{5.48}$$

Replacing all derivatives ∇_μ with D_μ and taking the trace in the action functionals we achieve gauge covariance on M_5. If we are to reproduce the ordinary Yang-Mills theory we must also include the Yukawa and ϕ^4 interaction terms

$$\begin{aligned} S_{\rm YM} &= \ldots + \int d^5x \sqrt{-g} \left(2\frac{1}{r} f^{abc} M_{\alpha\gamma} M_{\beta\delta} \phi^{\alpha\beta}_a \overline{\psi}^\gamma_b \psi^\delta_c \right. \\ &\left. + \frac{1}{r} f^{ab}{}_e f^{cde} M_{\sigma\alpha} M_{\beta\gamma} M_{\delta\lambda} M_{\tau\rho} \phi^{\alpha\beta}_a \phi^{\gamma\delta}_b \phi^{\lambda\tau}_c \phi^{\rho\sigma}_d \right) \,, \end{aligned} \tag{5.49}$$

and modify the supersymmetry variation of the fermionic field with the usual non-linear term

$$(\delta\psi^\alpha_a)_{\rm YM} = \ldots + 2 f_a{}^{bc} M_{\beta\gamma} M_{\delta\lambda} \phi^{\alpha\beta}_b \phi^{\gamma\delta}_c \varepsilon^\lambda \,, \tag{5.50}$$

where the ellipsis denotes the gauge covariantized quantities derived for the tensor multiplet.

Since the additional terms contain no derivatives the theory trivially exhibits the same generalized conformal invariance as in the abelian case. Supersymmetry of the model is less trivial, but by a straightforward calculation it is possible to verify. It should be emphasized that the theory described by (5.49) and (5.50) is not obtained by direct derivation; it is simply the minimal generalization required to produce all the known properties of interacting $(2,0)$ theory reduced on a circle fibration. However, it appears difficult to construct a competing candidate due to the strong restrictions imposed by the generalized conformal symmetry and supersymmetry (given the existence of non-trivial parameters ε^α).

5.4 Singular fibrations

Having considered the case of a general circle fibration in the previous section it is interesting to extend the scope to include manifolds M_6 whose description as an S^1 fibred over M_5 becomes singular over some hypersurface W in M_5. The appropriate way to understand this situation is to consider an action of U(1) on M_6. When the action is free the manifold M_6 can be described as a circle fibration over the base manifold $M_5 = M_6/\mathrm{U}(1)$, but generically this description breaks down on the hypersurfaces W in M_5 where the action is non-free. We will consider the special case of a codimension four hypersurface W defined as the fixed point locus of the U(1) action. More specifically, we will consider the free $(2,0)$ tensor multiplet on $M_6 = \mathbb{R}^{1,1} \times TN$, where TN is the four-dimensional (single center) Taub-NUT space introduced above. In this case, as we shall see explicitly below, we have

$W = \mathbb{R}^{1,1} \times \{0\}$ and the quotient M_5 is in fact a smooth manifold[16], even though the description of M_6 as a U(1) fibration is only valid over $M_5 \backslash W$.

5.4.1 WZW model on the singularity

In [61] Witten considers (among other things) the low energy effective theory on M_5 for the case when the free U(1) action on the space normal to W can be described by the natural action on $\mathbb{C}^2$. (In particular, this is the case for $M_6 = \mathbb{R}^{1,1} \times TN$.) The obstruction to extending the bundle over all of M_5 can be measured by non-triviality of the first Chern class of the bundle over $M_5 \backslash W$, which can in turn be viewed as a singularity along W of the curvature $\mathcal{F}$ of the bundle. Such a singularity is described by[17] $d\mathcal{F} = c\delta_W$ where δ_W, defined by

$$\int_{M_5} \delta_W \wedge \omega = \int_W \omega \tag{5.51}$$

for an arbitrary test-form $\omega \in \Omega^2(M_5)$, is the Poincaré dual of W.

In this situation the singularity induces an anomalous transformation of the topological term in the gauge theory on M_5. This is most easily seen by rewriting the topological term as $\int \mathcal{F} \wedge \mathrm{CS(A)}$ and noting that CS(A) is gauge invariant only up to an exact form. The equivalence of the two forms can be understood by considering a manifold M bounded by M_5. The variation of $\int_M \mathcal{F} \wedge F \wedge F$ localizes on the boundary and agrees with the one obtained from the variation of the two five-dimensional expressions, neither of which are strictly speaking well-defined.

Consequently, a modification of the low energy theory is required in order to cancel the anomaly. Since the description obtained in the previous section remains valid away from the singular locus the modification must be localized on W. In [61] it is argued that the correct modification is obtained by introducing additional degrees of freedom along W, more specifically a holomorphic WZW model [87, 88].

We can elucidate the appearance of the WZW model by considering the equation of motion for the gauge field $F_{\mu\nu}$ given for $M_5 \backslash W$ by

$$-\,d\left(\frac{1}{r} *_g F\right) + \mathcal{F} \wedge F = 0 \tag{5.52}$$

obtained from the variation of (5.41). The fact that $d\mathcal{F}$ is non-trivial along W prevents the extension of these equations to all of M_5. In fact, from the relationship $d\mathcal{F} = c\delta_W$ we can deduce the modification required to extend the gauge theory over all of M_5, namely

$$-\,d\left(\frac{1}{r} *_g F\right) + \mathcal{F} \wedge F = \delta_W \wedge J\,. \tag{5.53}$$

[16] The argument for smoothness can be found in [61].

[17] The proportionality constant c depends on the precise definition of Chern classes employed.

The current one-form J (inducing a current on W through the pull-back with the inclusion map) is thus given by

$$dJ = -cF_{|W} \,. \tag{5.54}$$

Thus, J indeed represent additional degrees of freedom (since it is only its derivative that is determined in terms of the gauge field strength) and anomaly cancellation requires it to the be the current of a gauged (holomorphic) WZW model. The relation (5.54) provides the WZW equations of motion for J [88].

5.4.2 The Taub-NUT example

We will now conclude this section, and the introductory part of the thesis, by examining an example of a singular fibration in more detail. We recall from 5.2.3 the definition of the Taub-NUT spaces and restrict considerations to the single center case

$$ds^2 = U\delta_{ij}dx^i dx^j + U^{-1}(d\varphi + \theta_i dx^i)^2 \tag{5.55}$$

with

$$U = \frac{1}{\lambda^2} + \frac{1}{|\vec{x}|} \,, \tag{5.56}$$

$dU = *_\delta \mathcal{F}$ and notational conventions according to section 5.2.3. At the origin of $\mathbb{R}^3$ the radius, which is given by $r = U^{-1/2}$ (c.f. the general expression in (5.13)), of the circle parametrized by φ vanishes[18], indicating the breakdown of the description of the space TN as an S^1 fibred over $\mathbb{R}^3$. Alternatively, to make contact with the introduction to the present section, the situation can be described as the U(1) action on TN having a fix-point at the origin $\vec{x} = 0$ of $\mathbb{R}^3$.

We then consider $M_6 = \mathbb{R}^{1,1} \times TN$ and take the U(1) to act trivially on the first factor so that $W = \mathbb{R}^{1,1} \times \{0\}$. The coordinates on M_5 are $x^\mu = (\sigma^a, x^i)$, where σ^a are light-cone coordinates on $\mathbb{R}^{1,1}$. Away from W we identify the U(1) connection θ_μ with the TN connection. Using the explicit form of U we compute the obstruction to extending the bundle over the singularity to be

$$d\mathcal{F} = -4\pi\delta_W \,. \tag{5.57}$$

Furthermore, the decomposition (5.40) of the three-form of the tensor multiplet breaks down when the fibre radius vanishes, eliminating the coordinate φ. This can be viewed as the explanation for the appearance of the current term in the equations of motion.

Finally, we will illustrate the appearance of the WZW model by considering a particular solution to the six-dimensional equations of motion for the TN geometry.

[18] With the form of the Taub-NUT metric we currently employ the periodicity of the S^1 coordinate is 4π in contrast to the previous parts of the present chapter.

This solution is obtained from the unique anti-self-dual harmonic two-form (mentioned above in the context of the multi-center Taub-NUT spaces) Ω, on TN which takes the form [74] $\Omega = d\Lambda$ with

$$\Lambda = \frac{1}{\lambda^2} U^{-1}(d\varphi + \theta_i dx^i)\,. \tag{5.58}$$

From Ω we get a closed self-dual three-form field strength $H = f(\sigma^+) d\sigma^+ \wedge \Omega$ on M_6 for some holomorphic function $f(\sigma^+)$. Note that this solution is well-defined on M_6 since Ω is regular at the singular point of TN; in fact it is localized near the origin. Away from W the decomposition (5.40) is still valid and can be used to extract the field strength, whose only non-vanishing component is

$$F_{i+} = -f(\sigma^+) \frac{\lambda^2}{|\vec{x}|(|\vec{x}| + \lambda^2)^2} x_i\,. \tag{5.59}$$

Inserting this particular solution and the explicit form of $\mathcal{F}$ into (5.53) we obtain the current

$$J = -4\pi f(\sigma^+) d\sigma^+\,. \tag{5.60}$$

Indeed, J contains only left-moving modes as required if it is to correspond to a WZW model and cancel the anomalous transformation of the topological term in the action. Further details on this construction are presented in Paper V where we also consider the possibility of finding a generalization of the solution (5.59) to the gauge field equations of motion.

Bibliography

[1] C. N. Yang and R. L. Mills, *Conservation of isotopic spin and isotopic gauge invariance*, Phys. Rev. **96** (1954) 191.

[2] S. Coleman and J. Mandula, *All possible symmetries of the S-matrix*, Phys. Rev. **159** (1967) 1251.

[3] R. Haag, J. Lopuszanski and M. Sohnius, *All possible generators of supersymmetries of the S-matrix*, Nucl. Phys. **B88** (1975) 257.

[4] E. Witten, *Conformal field theory in four and six dimensions*, in *Topology, Geometry and Quantum Field Theory*, U. Tillman, ed., p. 405. Cambridge University Press, Cambridge, 2004. `arXiv:0712.0157 [math.RT]`.

[5] T. Eguchi, P. B. Gikey and A. J. Hanson, *Gravitation, gauge theories and differential geometry*, Phys. Rept. **66** (1980) 213.

[6] M. Nakahara, *Geometry, Topology and Physics.* Taylor & Francis, Boca Raton, 2003.

[7] S. Helgason, *Differential Geometry and Symmetric Spaces.* Academic Press, New York, 1962.

[8] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory.* Springer, New York, 1972.

[9] H. Georgi, *Lie Algebras in Partical Physics.* The Benjamin/Cummings Publishing Company, Inc., Reading, 1982.

[10] J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations.* Cambridge University Press, Cambridge, 2003.

[11] J. Morgan, *An introduction to gauge theory*, in *Gauge theory and the topology of four-manifolds*, R. Friedman and J. Morgan, eds., pp. 51–144. IAS/Park City Mathematics Institute, 1994.

[12] J. J. Duistermaat, *Principal Fiber Bundles.* Lecture Notes, Utrecht Spring School, 2004.

[13] C. Vafa and E. Witten, *A strong coupling test of S duality*, Nucl. Phys. **B431** (1994) 3, `arXiv:hep-th/9408074`.

[14] P. A. M. Dirac, *Quantised singularities in the electromagnetic field*, Proc. Roy. Soc. London **A133** (1931) 60.

[15] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, Phys. Lett. **B59** (1975) 85.

[16] V. I. Arnold, *Mathematical Methods of Classical Mechanics.* Springer, New York, 1989.

[17] J. J. Duistermaat, *Symplectic Geometry.* Lecture Notes, Utrecht Spring School, 2004.

[18] H. Goldstein, C. Poole and J. Safko, *Classical Mechanics.* Addison Wesley, San Fransisco, 2002.

[19] P. A. M. Dirac, *Generalized Hamiltonian dynamics*, Canad. J. Math. **2** (1950).

[20] P. A. M. Dirac, *Lectures on Quantum Mechanics.* Yeshiva University, New York, 1964.

[21] J. Figueroa-O'Farrill, *BRST Cohomology and its Application to Two Dimensional Conformal Field Theory.* PhD thesis, State University of New York at Stony Brook, 1989.

[22] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems.* Princeton University Press, Princeton, 1992.

[23] S. Weinberg, *The Quantum Theory of Fields Volume I: Foundations.* Cambridge University Press, Cambridge, 1995.

[24] M. F. Sohnius, *Introducing supersymmetry*, Phys. Rept. **128** (1985) 39.

[25] J. Wess and J. Bagger, *Supersymmetry and supergravity.* Princeton University Press, Princeton, 1992.

[26] S. Weinberg, *The Quantum Theory of Fields Volume III: Supersymmetry.* Cambridge University Press, Cambridge, 2000.

[27] J. Scherk, *Extended supersymmetry and extended supergravity thoeries*, in *Recent Developments in Gravitation, Cargèse 1978*, M. Lévy and S. Deser, eds., pp. 479–517. Plenum Press, New York, 1979.

[28] T. Kugo and P. Townsend, *Supersymmetry and the division algebras*, Nucl. Phys. **B221** (1983) 357.

[29] W. Nahm, *Supersymmetries and their representations*, Nucl. Phys. **B135** (1978) 149.

[30] J. Strathdee, *Extended Poincaré supersymmetry*, Int. J. Mod. Phys. **A2** (2087) 273.

[31] L. Brink, J. Schwarz and J. Scherk, *Supersymmetric Yang-Mills theories*, Nucl. Phys. **B121** (1977) 77.

[32] G. 't Hooft, *On the phase transition towards permanent quark confinement*, Nucl. Phys. **B138** (1978) 1.

[33] G. 't Hooft, *A property of electric and magnetic flux in non-abelian gauge theories*, Nucl. Phys. **B153** (1979) 141.

[34] E. Witten, *Supersymmetric index in four-dimensional gauge theories*, Adv. Theor. Math. Phys. **5** (2002) 841, `arXiv:hep-th/0006010`.

[35] M. Henningson and N. Wyllard, *Zero-energy states of $\mathcal{N} = 4$ SYM on T^3: S-duality and the mapping class group*, JHEP **04** (2008) 066, `arXiv:0802.0660 [hep-th]`.

[36] E. Witten, *Toroidal compactification without vector structure*, JHEP **02** (1998) 006, `arXiv:hep-th/9712028`.

[37] A. Borel, R. Friedman and J. Morgan, *Almost commuting elements in compact Lie groups*, `arXiv:math/9907007 [math.GR]`.

[38] E. Witten, *Constraints on supersymmetry breaking*, Nucl. Phys. **B202** (1982) 253.

[39] M. Henningson and N. Wyllard, *Low-energy spectrum of $\mathcal{N} = 4$ super-Yang-Mills on T^3: flat connections, bound states at threshold and S-duality*, JHEP **06** (2007) 084, `arXiv:hep-th/0703172`.

[40] M. Henningson and N. Wyllard, *Bound states in $\mathcal{N} = 4$ SYM on T^3:* Spin($2n$) *and the exceptional groups*, JHEP **07** (2007) 001, `arXiv:0706.2803 [hep-th]`.

[41] A. Borel and J.-P. Serre, *Sur certain sous-groupes des groupes de Lie compacts*, Comment. Math. Helv. **27** (1953) 128.

[42] A. Borel, *Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes*, Tohoku Math. J. **13** (1961) 216.

[43] A. Keurentjes, A. Rosly and A. Smilga, *Isolated vacua in supersymmetric Yang-Mills theories*, Phys. Rev. **D58** (1998) 081701, `arXiv:hep-th/9805183`.

[44] V. G. Kac and A. V. Smilga, *Vacuum structure in supersymmetric Yang-Mills theories with any gauge group*, in *The many faces of the superworld*, M. A. Shifman, ed., pp. 185–234. World Scientific Publishing Co. Pte. Ltd., Singapore, 2000. `arXiv:hep-th/9902029`.

[45] A. Keurentjes, *Non-trivial flat connections on the 3-torus I: G_2 and the orthogonal groups*, JHEP **05** (1999) 001, `arXiv:hep-th/9901154`.

[46] A. Keurentjes, *Non-trivial flat connections on the 3-torus II: The exceptional groups F_4 and $E_{6,7,8}$*, JHEP **05** (1999) 014, `arXiv:hep-th/9902186`.

[47] E. Witten and D. Olive, *Supersymmetry algebras that include topological charges*, Phys. Lett. **78B** (1978) 97.

[48] M. K. Prasad and C. M. Sommerfield, *Exact classical solution for the 't Hooft monopole and the Julia-Zee dyon*, Phys. Rev. Lett. **35** (1975) 760.

[49] E. B. Bogomolny, *Stability of classical solutions*, Sov. J. Nucl. Phys. **24** (1976) 449.

[50] S. Coleman, S. Parke, A. Neveu and C. M. Sommerfield, *Can one dent a dyon?*, Phys. Rev. **D15** (1977) 544.

[51] H. Osborn, *Topological charges for $N = 4$ supersymmetric gauge theories and monopoles of spin* 1, Phys. Lett. **B83** (1979) 321.

[52] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*. Westview Press, Boulder, 1995.

[53] J. Polchinski, *String Theory Volume I: An Introduction to the Bosonic String*. Cambridge University Press, Cambridge, 1998.

[54] C. Montonen and D. Olive, *Magnetic monopoles as gauge particles?*, Phys. Lett. **B72** (1977) 117.

[55] P. Goddard, J. Nuyts and D. Olive, *Gauge theories and magnetic charge*, Nucl. Phys. **B125** (1977) 1.

[56] A. Sen, *Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space and* $\mathrm{SL}(2,\mathbb{Z})$ *invariance in string theory*, Phys. Lett. **B329** (1994) 217, `arXiv:hep-th/9402032`.

[57] J. Schwarz and A. Sen, *Duality symmetries of 4D heterotic strings*, Phys. Lett. **B312** (1993) 105, `arXiv:hep-th/9305185`.

[58] T. Fulton, F. Rohrlich and L. Witten, *Conformal invariance in physics*, Rev. Mod. Phys. **34** (1962) 442.

[59] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory, Volume 2*. Cambridge University Press, Cambridge, 1987.

[60] D. Z. Freedman and A. Van Proeyen, *Supergravity*. Cambridge University Press, Cambridge, 2012.

[61] E. Witten, *Geometric Langlands from six dimensions*, `arXiv:0905.2720 [hep-th]`.

[62] P. S. Howe, N. D. Lambert and P. C. West, *The three-brane soliton of the M-fivebrane*, Phys. Lett. **B419** (1998) 79, `arXiv:hep-th/9710033`.

[63] P. S. Howe, G. Sierra and P. K. Townsend, *Supersymmetry in six dimensions*, Nucl. Phys. **B221** (1983) 331.

[64] P. Pasti, D. P. Sorokin and M. Tonin, *Covariant action for a D=11 five-brane with the chiral field*, Phys. Lett. **B398** (1997) 41, `arXiv:hep-th/9701037`.

[65] I. A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin and M. Tonin, *Covariant action for the super-five-brane of M-theory*, Phys. Rev. Lett. **78** (1997) 4332, `arXiv:hep-th/9701149`.

[66] M. Cederwall, B. E. W. Nilsson and P. Sundell, *An action for the super-5-brane in $D = 11$ supergravity*, JHEP **04** (1998) 007, `arXiv:hep-th/9712059`.

[67] E. Witten, *Some comments on string dynamics*, in *Future Perspectives in String Theory*, I. Bars, P. Bouwknegt, J. Minahan, D. Nemeschansky, H. Pilch, K. Saleur and N. Warner, eds., pp. 501–523. World Scientific Publishing Co. Pte. Ltd., Singapore, 1996. `arXiv:hep-th/9507121`.

[68] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. **B443** (1995) 85, `arXiv:hep-th/9503124`.

[69] P. S. Aspinwall, *K3 Surfaces and String Duality*, in *Differential Geometry Inspired by String Theory*, S. T. Yau, ed., pp. 1–95. International Press of Boston, Somerville, 1999. `arXiv:hep-th/9611137`.

[70] S. W. Hawking, *Gravitational instantons*, Phys. Lett. **60A** (1977) 81.

[71] G. W. Gibbons and S. W. Hawking, *Classification of gravitational instanton symmetries*, Commun. Math. Phys. **66** (1979) 291.

[72] T. Eguchi and A. J. Hanson, *Self-dual solutions to euclidean gravity*, Ann. Phys. **120** (1979) 82.

[73] A. Sen, *A note on enhanced gauge symmetries in M- and string theory*, JHEP **09** (1997) 001, arXiv:hep-th/9707123.

[74] E. Witten, *Branes, instantons, and Taub-NUT spaces*, JHEP **09** (2009) 067, arXiv:0902.0948 [hep-th].

[75] N. Seiberg, *Notes on theories with* 16 *supercharges*, Nucl. Phys. Proc. Suppl. **67** (1998) 158, arXiv:hep-th/9705117.

[76] D. M. Kaplan and J. Michelson, *Zero modes for the* $D = 11$ *membrane and five-brane*, Phys. Rev. **D53** (1996) 3474, arXiv:hep-th/9510053.

[77] T. Adawi, M. Cederwall, U. Gran and B. E. W. Nilsson, *Goldstone tensor modes*, JHEP **02** (1999) 001, arXiv:hep-th/9811145.

[78] A. Strominger, *Open p-branes*, Phys. Lett. **B383** (1996) 44, arXiv:hep-th/9512059.

[79] P. S. Howe, N. D. Lambert and P. C. West, *The self-dual string soliton*, Nucl. Phys. **B515** (1998) 203, arXiv:hep-th/9709014.

[80] A. Gustavsson and M. Henningson, *A short representation of the six-dimensional* $(2,0)$ *algebra*, JHEP **06** (2001) 054, arXiv:hep-th//0104172.

[81] J. A. de Azcárraga, J. M. Gauntlett, J. M. Izquierdo and P. K. Townsend, *Topological extensions of the supersymmetry algebra for extended objects*, Phys. Rev. Lett. **63** (1989) 2443.

[82] N. Lambert and C. Papageorgakis, *Nonabelian* $(2,0)$ *tensor multiplets and* 3*-algebras*, JHEP **08** (2010) 083, arXiv:1007.2982 [hep-th].

[83] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, *M*5*-branes, D*4*-branes and quantum* 5*D super-Yang-Mills*, JHEP **01** (2011) 083, arXiv:1007.2982 [hep-th].

[84] M. R. Douglas, *On* $D = 5$ *super Yang-Mills theory and* $(2,0)$ *theory*, JHEP **02** (2011) 011, arXiv:1012.2880 [hep-th].

[85] E. Verlinde, *Global aspects of electric-magnetic duality*, Nucl. Phys. **B455** (1995) 211, arXiv:hep-th/9506011.

[86] D. Gaiotto and E. Witten, *Janus configurations, Chern-Simons couplings, and the* θ*-angle in* $\mathcal{N} = 4$ *super Yang-Mills theory*, JHEP **06** (2010) 097, arXiv:0804.2907 [hep-th].

[87] E. Witten, *Non-abelian bosonization in two dimensions*, Commun. Math. Phys. **92** (1984) 455.

[88] E. Witten, *On holomorphic factorization of WZW and coset models*, Commun. Math. Phys. **144** (1992) 189.