On the Equivalence of TCM Encoders

On the Equivalence of TCM Encoders

Alex Alvarado[§], Alexandre Graell i Amat[‡], Fredrik Brännström[‡], and Erik Agrell[‡]
[§]Department of Engineering, University of Cambridge, UK
[‡]Department of Signals and Systems, Chalmers University of Technology, Gothenburg, Sweden
alex.alvarado@ieee.org, {alexandre.graell,fredrik.brannstrom,agrell}@chalmers.se

***Abstract*—Optimal trellis-coded modulation (TCM) schemes are obtained by jointly designing the convolutional encoder and the binary labeling of the constellation. Unfortunately this approach is infeasible for large encoder memories or constellation sizes. Traditional TCM designs circumvent this problem by using a labeling that follows the set-partitioning principle and by performing an exhaustive search over the encoders. Therefore, traditional TCM schemes are not necessarily optimal. In this paper, we study binary labelings for TCM and show how they can be grouped into classes, which considerably reduces the search space in a joint design. For the particular case of 8-ary modulation the search space for the labelings is reduced from $8!$ to 240. Using this classification, we formally prove that for any channel it is always possible to design a TCM system based on the binary-reflected Gray code with identical performance to the one proposed by Ungerboeck in 1982. Moreover, the classification is used to tabulate asymptotically optimal TCM schemes.**

I. Introduction

Trellis-coded modulation (TCM) systems are commonly constructed by coupling a convolutional encoder and a constellation labeled using the set-partitioning (SP) principle. TCM was introduced in [1], quickly adopted in the modem standards in the early 90s, and it is a well studied topic, cf. [2], [3, Ch. 18]. As an alternative to TCM, bit-interleaved coded modulation (BICM) [4], [5] was introduced in 1992. BICM is usually referred to as a pragmatic approach for coded modulation and is well suited for fading channels.

In this paper we study binary labelings for TCM. Of particular interest are the binary reflected Gray code (BRGC) [6] and the natural binary code (NBC) [7]. The BRGC is often used in BICM designs because it maximizes the BICM mutual information for medium and high signal-to-noise ratios [5, Sec. III] and the NBC is often used in TCM designs because it follows the SP principle when it is used with constellations having certain symmetries, cf. [1, Fig. 4], [8, Fig. 3].

The performance of BICM for the additive white Gaussian noise (AWGN) channel can be improved if the interleaver is removed [9], a configuration that was later called "BICM with trivial interleavers" (BICM-T) in [10]. BICM-T was recognized as a TCM transmitter used with a BICM receiver and it was shown to perform asymptotically as well as TCM if the convolutional encoder is properly selected [10, Table III] and the BRGC is used. The transmitters in [1, Table I] and [10, Table III] for the 8-state (memory $\nu = 3$) convolutional encoder[1] are shown in Fig. 1 (a) and Fig. 1 (c), respectively.

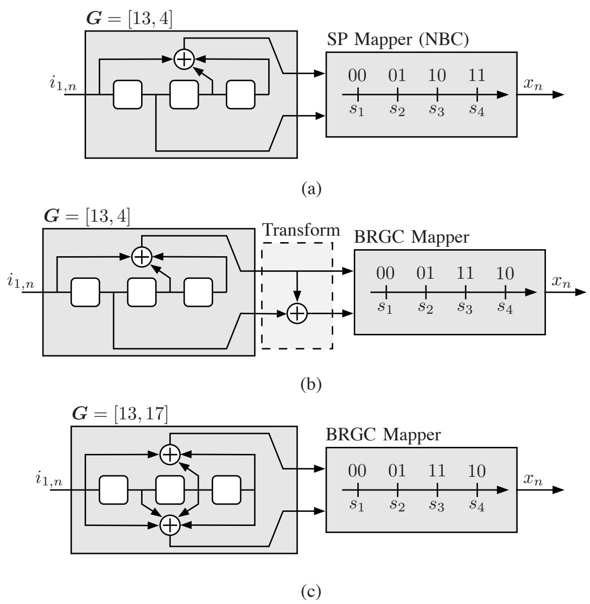

Fig. 1. Three equivalent TCM transmitters: (a) convolutional encoder with generators $\boldsymbol{G} = [13, 4]$ and an SP mapper [1]; (c) convolutional encoder with generators $\boldsymbol{G} = [13, 17]$ and a BRGC mapper [10]. The system in (b) shows how a transformation (binary addition) can be included in the mapper (to go from (b) to (a)) or in the code (to go from (b) to (c)).

The authors in [10] failed to note that in fact the optimal BICM-T configuration is *equivalent* to the one proposed by Ungerboeck 30 years earlier. For a 4-ary pulse amplitude modulation (PAM) constellation (shown in Fig. 1), Ungerboeck's SP mapper (which is in this case equivalent to the NBC) can be generated using the BRGC mapper plus one binary addition (*transform*) applied to its inputs, as shown in Fig. 1(b). If the transform is included in the mapper, the configuration in Fig. 1(a) is obtained, while if it is included in the code, the configuration in Fig. 1(c) is obtained. This equivalence also applies to convolutional encoders with larger number of memories[2] and simply reveals that a TCM transmitter

Research supported by The British Academy and The Royal Society via the Newton International Fellowship scheme, UK, by the European Community's Seventh's Framework Programme (FP7/2007-2013) under grant agreement No. 271986, by the Swedish Research Council (VR) under grants #621-2006-4872 and #621-2011-5950, and by the Swedish Agency for Innovation Systems (VINNOVA) under the P36604-1 MAGIC project.

[1]Throughout this paper, all polynomial generators are given in octal form.

[2]For some particular values of ν this equivalence is not seen in the tables, because [10, Table III] lists the convolutional encoders in lexicographic order, and for some values of ν, there are multiple encoders with identical performance.

based on a BRGC mapper will have identical performance to Ungerboeck's TCM if the encoder is properly modified.

The previous discussion raises the question about the use of non-SP labelings for TCM. This problem has indeed been studied in the literature, see for example [11, Sec. 13.2.1, Problem 13–11], [3, Example 18.2] or the so-called pragmatic TCM [12, Ch. 8], [13]. In [9], a binary labeling for BICM-T was heuristically proposed for MPAM constellations for $M = 4, 8, 16$. Traditional TCM designs either optimize the convolutional encoder for a constellation using an SP labeling, cf. [1], [8], or simply connect a convolutional encoder designed for binary transmission with an ad-hoc binary labeling (Gray in [14] and non-Gray in [13]). TCM designs based on SP are considered heuristic [15, pp. 525, 531], and thus, they do not necessarily lead to an optimal design [11, p. 680]. Indeed, Ungerboeck's TCM design is based on heuristic rules that aim to increase the Euclidean distance (ED) when compared with uncoded transmission with the same spectral efficiency. This discussion raises another question, namely, the optimal joint design of the convolutional encoder and the labeling.

To the best of our knowledge there are no works formally addressing the *joint* design of the convolutional encoder and the labeling of a TCM system. In this paper we study this problem and formally prove that for any channel, binary labelings can be grouped into different classes that will result in equivalent TCM transmitters. The classes are closely related to the *Hadamard classes* introduced in [16] in the context of vector quantization. The proposed classification allows us to formally prove that in any TCM system the NBC labeling can be replaced by many other labelings (including the BRGC), provided that the convolutional encoder is properly modified.

II. Preliminaries

A. Notation Convention

Throughout this paper, scalars are denoted by italic letters x, row vectors are denoted by boldface letters $\boldsymbol{x} = [x_1, \dots, x_N]$, and matrices by capital boldface letters $\boldsymbol{X}$. Sets are denoted using calligraphic letters $\mathcal{C}$ and the binary set is defined as $\mathcal{B} \triangleq \{0, 1\}$. Binary addition is denoted by $a \oplus b$. We use $\mathcal{R}_m$ to denote the set of all reduced column echelon binary matrices of size $M \times m$ (see Section III) and $\mathcal{T}_m$ to denote the set of all invertible $m \times m$ binary matrices.

B. System Model

We consider the TCM encoder shown in Fig. 2 where a feedforward convolutional encoder of rate $R = k/m$ is serially connected to a mapper $\Phi_{\boldsymbol{L}}$ where the index $\boldsymbol{L}$ emphasizes the dependency of the mapper on the labeling (defined later). At each discrete time instant n, the information bits $i_{1,n}, \dots, i_{k,n}$ are fed to the convolutional encoder, which is fully determined by k shift registers and the way the input sequences are connected (through the registers) to its outputs. We denote the length of the pth shift register by ν_p, with $p = 1, \dots, k$, the *overall constraint length* (or *memory* of the convolutional encoder) by $\nu = \sum_{p=1}^{k} \nu_p$, and the *number of states* by 2^ν.

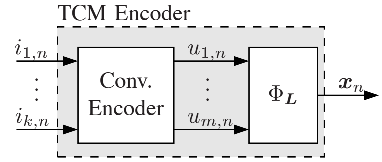

Fig. 2. Generic TCM encoder under consideration: A feedforward convolutional encoder of rate $R = k/m$ with 2^ν states serially concatenated with a memoryless mapper $\Phi_{\boldsymbol{L}}$.

The connection between the input and output bits is defined by the binary representation of the *convolutional encoder matrix*

$$\boldsymbol{G} \triangleq \begin{bmatrix} \boldsymbol{g}_1^{(1)} & \boldsymbol{g}_1^{(2)} & \cdots & \boldsymbol{g}_1^{(m)} \\ \boldsymbol{g}_2^{(1)} & \boldsymbol{g}_2^{(2)} & \cdots & \boldsymbol{g}_2^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{g}_k^{(1)} & \boldsymbol{g}_k^{(2)} & \cdots & \boldsymbol{g}_k^{(m)} \end{bmatrix} \quad (1)$$

where $\boldsymbol{g}_p^{(l)} \triangleq [g_{p,1}^{(l)}, \dots, g_{p,\nu_p+1}^{(l)}]^\mathsf{T} \in \mathcal{B}^{\nu_p+1}$ is a column vector representing the connection between the pth input sequence and the lth output sequence with $l = 1, \dots, m$. The coefficients $g_{p,1}^{(l)}, \dots, g_{p,\nu_p+1}^{(l)}$ are associated to the input bits $i_{p,n}, \dots, i_{p,n-\nu_p}$, respectively, and $\boldsymbol{G} \in \mathcal{B}^{(\nu+k)\times m}$. Throughout this paper we will show the vectors $\boldsymbol{g}_p^{(l)}$ defining $\boldsymbol{G}$ either in binary or octal notation. When shown in octal notation, $g_{p,1}^{(l)}$ will always represent the most significant bit (cf. Fig. 1).

The convolutional encoder matrix (1) allows us to express the output of the convolutional encoder at time n, which we denote by $\boldsymbol{u}_n = [u_{1,n}, \dots, u_{m,n}]$, as a function of $(\nu + k)$ information bits, i.e.,

$$\boldsymbol{u}_n = \boldsymbol{j}_n \boldsymbol{G} \quad (2)$$

where $\boldsymbol{j}_n \triangleq [\boldsymbol{i}_n^{(1)}, \dots, \boldsymbol{i}_n^{(k)}]$ with $\boldsymbol{i}_n^{(p)} \triangleq [i_{p,n}, \dots, i_{p,n-\nu_p}]$ are the information bits, and the matrix multiplication is in GF(2).

The coded bits $\boldsymbol{u}_n$ are mapped to N-dimensional real constellation symbols using the mapper $\Phi_{\boldsymbol{L}} : \mathcal{B}^m \to \mathcal{X}$ where $\mathcal{X} \subset \mathbb{R}^N$ is the constellation used for transmission, with $|\mathcal{X}| = M = 2^m$. We use $\boldsymbol{x}_n \in \mathcal{X}$ to denote the transmitted symbols at time n and we use the matrix $\boldsymbol{S} = [\boldsymbol{s}_1^\mathsf{T}, \dots, \boldsymbol{s}_M^\mathsf{T}]^\mathsf{T}$ with $\boldsymbol{s}_q \in \mathbb{R}^N$ and $q = 1, \dots, M$ to denote the ordered constellation points.

The binary labeling of the qth symbol in $\boldsymbol{S}$ is denoted by $\boldsymbol{c}_q = [c_{q,1}, \dots, c_{q,m}] \in \mathcal{B}^m$, where $c_{q,l}$ is the bit associated to the lth input of the mapper in Fig. 2. The labeling matrix is defined as $\boldsymbol{L} = [\boldsymbol{c}_1^\mathsf{T}, \dots, \boldsymbol{c}_M^\mathsf{T}]^\mathsf{T}$, where $\boldsymbol{c}_q$ in $\boldsymbol{L}$ corresponds to the binary labeling of the symbol $\boldsymbol{s}_q$ in $\boldsymbol{S}$. The resulting spectral efficiency of the system is k [bit/symbol].

C. Binary Labelings for TCM

The NBC of order m is defined as $\boldsymbol{N}_m \triangleq [\boldsymbol{n}_1^\mathsf{T}, \dots, \boldsymbol{n}_M^\mathsf{T}]^\mathsf{T}$, where $\boldsymbol{n}_q = [n_{q,1}, \dots, n_{q,m}] \in \mathcal{B}^m$ is the base-2 representation of the integer $q-1$, where $n_{q,m}$ is the least significant bit. The BRGC of order m is defined as $\boldsymbol{B}_m \triangleq [\boldsymbol{b}_1^\mathsf{T}, \dots, \boldsymbol{b}_M^\mathsf{T}]^\mathsf{T}$, where $\boldsymbol{b}_q = [b_{q,1}, \dots, b_{q,m}] \in \mathcal{B}^m$. The bits of the BRGC can be generated from the NBC as $b_{q,1} = n_{q,1}$ and $b_{q,l} = n_{q,l-1} \oplus n_{q,l}$ for $l = 2, \dots, m$.

Example 1: The NBC and BRGC of order $m=3$ are

$$\boldsymbol{N}_3=\begin{bmatrix}00001111\\00110011\\01010101\end{bmatrix}^{\mathsf{T}},\quad \boldsymbol{B}_3=\begin{bmatrix}00001111\\00111100\\01100110\end{bmatrix}^{\mathsf{T}}. \quad (3)$$

Alternatively, we have that $n_{q,l}=b_{q,1}\oplus\ldots\oplus b_{q,l-1}\oplus b_{q,l}$ for $l=1,\ldots,m$, or, in matrix notation, $\boldsymbol{B}_m=\boldsymbol{N}_m\boldsymbol{T}$ and $\boldsymbol{N}_m=\boldsymbol{B}_m\boldsymbol{T}^{-1}$, where

$$\boldsymbol{T}=\begin{bmatrix}110\ldots00\\011\ldots00\\001\ldots00\\\vdots\quad\ddots\quad\vdots\\000\ldots11\\000\ldots01\end{bmatrix},\quad \boldsymbol{T}^{-1}=\begin{bmatrix}111\ldots11\\011\ldots11\\001\ldots11\\\vdots\quad\ddots\quad\vdots\\000\ldots11\\000\ldots01\end{bmatrix}. \quad (4)$$

D. System Optimization and Search Problems

For a given constellation $\boldsymbol{S}$ and a given ν, a *TCM encoder* is fully defined by the convolutional encoder matrix $\boldsymbol{G}$ and the labeling of the constellation $\boldsymbol{L}$. In this paper, a TCM encoder is defined by the pair $\Theta=[\boldsymbol{G},\boldsymbol{L}]$. For given integers k, m, and ν, we define the *convolutional encoder universe* as the set $\mathcal{G}_{k,m,\nu}$ of all $(\nu+k)\times m$ binary matrices $\boldsymbol{G}$.[3] In [1], [8] Ungerboeck optimized TCM *codes* in terms of the minimum ED over all possible convolutional codes for well-structured one- and two-dimensional constellations and a labeling $\boldsymbol{L}$ that follows the SP principle, e.g., the NBC. On the other hand, in [10, Sec. IV-C] BICM-T systems over all $\boldsymbol{G}\in\mathcal{G}_{1,2,\nu}$ ($R=1/2$ and 4PAM) and $\boldsymbol{L}=\boldsymbol{B}_2$ were optimized. We are also interested in the *labeling universe,* defined for a given integer m as the set $\mathcal{L}_m$ of all $M\times m$ binary matrices whose M rows are all distinct.

To the best of our knowledge, there are no works addressing the problem of designing a TCM encoder by *exhaustively* searching over the labeling universe and the convolutional encoder universe. In this paper, we show how a joint optimization over all $\boldsymbol{G}\in\mathcal{G}_{k,m,\nu}$ and $\boldsymbol{L}\in\mathcal{L}_m$ can be restricted, without loss of generality, to a joint optimization over all $\boldsymbol{G}\in\mathcal{G}_{k,m,\nu}$ and over a subset of $\mathcal{L}_m$.

III. Equivalent Labelings for TCM Encoders

The output of a given TCM encoder $\Theta=[\boldsymbol{G},\boldsymbol{L}]$ at time n depends on $(\nu+k)$ information bits. Using (2), the transmitted symbol at time n can then be expressed as $\boldsymbol{x}_n=\Phi_{\boldsymbol{L}}(\boldsymbol{u}_n)=\Phi_{\boldsymbol{L}}(\boldsymbol{j}_n\boldsymbol{G})$.

Definition 1: Two TCM encoders $\Theta=[\boldsymbol{G},\boldsymbol{L}]$ and $\tilde{\Theta}=[\tilde{\boldsymbol{G}},\tilde{\boldsymbol{L}}]$ are said to be *equivalent* if they give the same output symbol for the same information bit sequence, i.e., if they fulfill $\Phi_{\boldsymbol{L}}(\boldsymbol{j}\boldsymbol{G})=\Phi_{\tilde{\boldsymbol{L}}}(\boldsymbol{j}\tilde{\boldsymbol{G}})$ for any $\boldsymbol{j}\in\mathcal{B}^{\nu+k}$.

Remark 1: For any channel, equivalent TCM encoders have the same bit error rate and frame error rate (FER).

Lemma 1: $\Phi_{\boldsymbol{L}}(\boldsymbol{c})=\Phi_{\tilde{\boldsymbol{L}}}(\boldsymbol{c}\boldsymbol{T})$ where $\tilde{\boldsymbol{L}}=\boldsymbol{L}\boldsymbol{T}$, for any two mappers $\Phi_{\boldsymbol{L}}$ and $\Phi_{\tilde{\boldsymbol{L}}}$ that use the same constellation $\boldsymbol{S}$, any $\boldsymbol{T}\in\mathcal{T}_m$, and any $\boldsymbol{c}\in\mathcal{B}^m$.

Proof: Let $\boldsymbol{v}_q\triangleq[0,\ldots,0,1,0,\ldots,0]$ be a vector of length M, where the one is in position q. Thus $\boldsymbol{c}_q=\boldsymbol{v}_q\boldsymbol{L}$ for $q=1,\ldots,M$. The mapping $\Phi_{\boldsymbol{L}}$ satisfies by definition $\Phi_{\boldsymbol{L}}(\boldsymbol{c}_q)=\boldsymbol{s}_q$ or, making the dependency on $\boldsymbol{L}$ explicit,

$$\Phi_{\boldsymbol{L}}(\boldsymbol{c})=\boldsymbol{s}_q,\quad \text{if } \boldsymbol{c}=\boldsymbol{v}_q\boldsymbol{L} \quad (5)$$

for any $\boldsymbol{c}\in\mathcal{B}^m$. Similarly, for any $\boldsymbol{c}\in\mathcal{B}^m$,

$$\begin{aligned}\Phi_{\tilde{\boldsymbol{L}}}(\boldsymbol{c}\boldsymbol{T})&=\boldsymbol{s}_q,\quad \text{if } \boldsymbol{c}\boldsymbol{T}=\boldsymbol{v}_q\tilde{\boldsymbol{L}}\\&=\boldsymbol{s}_q,\quad \text{if } \boldsymbol{c}=\boldsymbol{v}_q\boldsymbol{L}\end{aligned} \quad (6)$$

where the last step follows because $\boldsymbol{L}=\tilde{\boldsymbol{L}}\boldsymbol{T}^{-1}$. Since the right-hand sides of (5) and (6) are equal, $\Phi_{\tilde{\boldsymbol{L}}}(\boldsymbol{c}\boldsymbol{T})=\Phi_{\boldsymbol{L}}(\boldsymbol{c})$ for all $\boldsymbol{c}\in\mathcal{B}^m$. □

Theorem 1: For any $\boldsymbol{G}\in\mathcal{G}_{k,m,\nu}$, $\boldsymbol{L}\in\mathcal{L}_m$, and $\boldsymbol{T}\in\mathcal{T}_m$, the two TCM encoders $\Theta=[\boldsymbol{G},\boldsymbol{L}]$ and $\tilde{\Theta}=[\tilde{\boldsymbol{G}},\tilde{\boldsymbol{L}}]$ are equivalent, where $\tilde{\boldsymbol{L}}=\boldsymbol{L}\boldsymbol{T}$ and $\tilde{\boldsymbol{G}}=\boldsymbol{G}\boldsymbol{T}$.

Proof: For any $\boldsymbol{j}\in\mathcal{B}^{\nu+k}$, $\Phi_{\tilde{\boldsymbol{L}}}(\boldsymbol{j}\tilde{\boldsymbol{G}})=\Phi_{\tilde{\boldsymbol{L}}}(\boldsymbol{j}\boldsymbol{G}\boldsymbol{T})=\Phi_{\boldsymbol{L}}(\boldsymbol{j}\boldsymbol{G})$, where the last equality follows by Lemma 1. The theorem now follows using Definition 1. □

Theorem 1 tells us that an exhaustive search over $\mathcal{G}_{k,m,\nu}$ and $\mathcal{L}_m$ will include many pairs of equivalent TCM encoders. Therefore, an optimal TCM encoder with given parameters can be found by searching over a subset of $\mathcal{G}_{k,m,\nu}$ and the whole set $\mathcal{L}_m$ or vice versa. In this paper, we choose the latter approach, searching over a subset of $\mathcal{L}_m$.

A *reduced column echelon matrix*[4] is, in the context of binary labelings, defined as a binary labeling matrix in which (i) the first "1" in any column is in a row where all other elements are "0" and (ii) the number of leading zeros decreases in every column. The matrix $\boldsymbol{N}_3$ in Example 1 (or more generally $\boldsymbol{N}_m$) is an example of a reduced column echelon matrix. On the other hand, $\boldsymbol{B}_m$ is not a reduced column echelon matrix. The following theorem, adapted from [17, p. 187, Corollary 1] to the concept of reduced column echelon matrices, shows an important matrix factorization which will be used in Example 2 and Theorem 3.

Theorem 2: Any binary labeling $\boldsymbol{L}\in\mathcal{L}_m$ can be uniquely factorized as

$$\boldsymbol{L}=\boldsymbol{L}_{\mathrm{R}}\boldsymbol{T} \quad (7)$$

where $\boldsymbol{T}\in\mathcal{T}_m$ and $\boldsymbol{L}_{\mathrm{R}}\in\mathcal{R}_m$.

Theorem 2 shows that all binary matrices $\boldsymbol{L}$ can be uniquely generated by finding all the invertible matrices $\boldsymbol{T}$ (the set $\mathcal{T}_m$) and all the different reduced column echelon matrices $\boldsymbol{L}_{\mathrm{R}}$ (the set $\mathcal{R}_m$). In particular, we have [16, eq. (18)] $M_{\mathrm{T}}\triangleq|\mathcal{T}_m|=\prod_{k=1}^{m}(M-2^{k-1})$ and $|\mathcal{L}_m|=M!=M_{\mathrm{R}}M_{\mathrm{T}}$, where $M_{\mathrm{R}}\triangleq|\mathcal{R}_m|$. In Table I, the values for M_{R} and M_{T} for $1\leq m\leq 6$ are shown. In view of Theorem 1 and Table I, for a joint design and 8-ary constellations ($m=3$), the total number of different binary labelings that must be tested is reduced from $8!=40320$ to 240.

A *modified Hadamard class* is defined as the set of matrices $\boldsymbol{L}$ that can be generated via (7) using the same reduced column

[3]Note that whenever $\boldsymbol{G}$ is given in its binary form, $\nu_1,\ldots,\nu_k$ are also needed to interpret $\boldsymbol{G}$ correctly according to (1).

[4]The only difference between a reduced column echelon matrix and the commonly used reduced row echelon matrix [17, pp. 183–184] is a transpose.

TABLE I
NUMBER OF CLASSES ($M_{\mathrm{R}} = |\mathcal{R}_m|$) AND THEIR CARDINALITY ($M_{\mathrm{T}} = |\mathcal{T}_m|$) FOR DIFFERENT VALUES OF m.

m	1	2	3	4	5	6
M_{R}	2	4	240	$1.0378 \cdot 10^9$	$2.6315 \cdot 10^{28}$	$6.2943 \cdot 10^{78}$
M_{T}	1	6	168	20160	$9.9994 \cdot 10^6$	$2.0159 \cdot 10^{10}$
$M!$	2	24	40320	$2.0923 \cdot 10^{13}$	$2.6313 \cdot 10^{35}$	$1.2689 \cdot 10^{89}$

echelon matrix $\boldsymbol{L}_{\mathrm{R}}$. Note that these modified Hadamard classes are narrower than the regular Hadamard classes defined in [16], each including M reduced column echelon matrices. There are thus M_{R} modified Hadamard classes, each with cardinality M_{T}.

The problem of finding the set $\mathcal{R}_m$ of reduced column echelon matrices for a given m can be solved by using a modified version of the full linear search algorithm introduced in [16, Sec. VIII]. Such an algorithm would generate one member of each modified Hadamard class, the one that corresponds to a reduced column echelon matrix $\boldsymbol{L}_{\mathrm{R}}$.

Example 2: For $m = 2$, where $M_{\mathrm{R}} = 4$, we have

$$\mathcal{R}_2 = \left\{ \begin{bmatrix} 0\,0\,1\,1 \\ 0\,1\,0\,1 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} 0\,0\,1\,1 \\ 1\,0\,0\,1 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} 0\,1\,0\,1 \\ 1\,0\,0\,1 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} 0\,1\,1\,0 \\ 1\,0\,1\,0 \end{bmatrix}^{\mathsf{T}} \right\} \quad (8)$$

where the first element in $\mathcal{R}_2$ is the NBC (cf. Section II-C). The $M_{\mathrm{T}} = 6$ binary invertible matrices for $m = 2$ are

$$\mathcal{T}_2 = \left\{ \begin{bmatrix} 0\,1 \\ 1\,0 \end{bmatrix}, \begin{bmatrix} 0\,1 \\ 1\,1 \end{bmatrix}, \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}, \begin{bmatrix} 1\,0 \\ 1\,1 \end{bmatrix}, \begin{bmatrix} 1\,1 \\ 0\,1 \end{bmatrix}, \begin{bmatrix} 1\,1 \\ 1\,0 \end{bmatrix} \right\}. \quad (9)$$

Using Theorem 2, all the 24 binary labelings in $\mathcal{L}_2$ (cf. Table I) can be generated by multiplying the matrices in $\mathcal{R}_2$ and in $\mathcal{T}_2$.

As a consequence of Theorems 1 and 2, the two TCM encoders $[\boldsymbol{G}, \boldsymbol{L}]$ and $[\boldsymbol{G}\boldsymbol{T}^{-1}, \boldsymbol{L}_{\mathrm{R}}]$ are equivalent for any $\boldsymbol{G} \in \mathcal{G}_{k,m,\nu}$ and $\boldsymbol{L} \in \mathcal{L}_m$, where $\boldsymbol{L}_{\mathrm{R}}$ and $\boldsymbol{T}$ are given by the factorization (7). In other words, all nonequivalent TCM encoders can be generated using one member of each modified Hadamard class only, and thus, a joint optimization over all $\boldsymbol{G} \in \mathcal{G}_{k,m,\nu}$ and $\boldsymbol{L} \in \mathcal{L}_m$ can be reduced to an optimization over all $\boldsymbol{G} \in \mathcal{G}_{k,m,\nu}$ and $\boldsymbol{L} \in \mathcal{R}_m$ with no loss in performance. This means that the search space is reduced by a factor of $M_{\mathrm{T}} = M!/M_{\mathrm{R}}$.

A. NBC and BRGC

Another way of interpreting the result in Theorem 1 is that for any TCM encoder $\tilde{\Theta} = [\tilde{\boldsymbol{G}}, \tilde{\boldsymbol{L}}]$, a new equivalent TCM encoder can be generated using a convolutional encoder $\boldsymbol{G} = \tilde{\boldsymbol{G}}\boldsymbol{T}^{-1}$ and a labeling $\boldsymbol{L} = \tilde{\boldsymbol{L}}\boldsymbol{T}^{-1}$ that belongs to the same modified Hadamard class as the original labeling $\tilde{\boldsymbol{L}}$. One direct consequence of this result is that any TCM encoder using the NBC labeling can be constructed using the BRGC and an appropriately selected encoder.

Example 3: For the two TCM encoders in Fig. 1, the NBC and BRGC labelings are related via $\boldsymbol{B}_2 = \boldsymbol{N}_2\boldsymbol{T}$, i.e.,

$$\begin{bmatrix} 0\,0\,1\,1 \\ 0\,1\,1\,0 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 0\,0\,1\,1 \\ 0\,1\,0\,1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 1\,1 \\ 0\,1 \end{bmatrix}. \quad (10)$$

Thus, the BRGC and the NBC of order $m = 2$ belong to the same modified Hadamard class, and convolutional encoders can be chosen to make the two resulting TCM encoders equivalent. This was illustrated in Fig. 1, where the transform block corresponds to the transform matrix $\boldsymbol{T} = [[1,1]^{\mathsf{T}}, [0,1]^{\mathsf{T}}]^{\mathsf{T}} = \boldsymbol{T}^{-1}$. Since $\boldsymbol{N}_2 = \boldsymbol{B}_2\boldsymbol{T}^{-1}$, the TCM encoders $[\boldsymbol{G}_{[13,17]}, \boldsymbol{B}_2]$ and $[\boldsymbol{G}_{[13,4]}, \boldsymbol{N}_2]$ are equivalent, where

$$\boldsymbol{G}_{[13,4]} = \begin{bmatrix} 1\,0\,1\,1 \\ 0\,1\,0\,0 \end{bmatrix}^{\mathsf{T}} = \boldsymbol{G}_{[13,17]}\boldsymbol{T}^{-1} = \begin{bmatrix} 1\,0\,1\,1 \\ 1\,1\,1\,1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 1\,1 \\ 0\,1 \end{bmatrix}.$$

The above relation between the NBC and the BRGC is generalized to an arbitrary order m in the following theorem.

Theorem 3: The BRGC and the NBC of any order m belong to the same modified Hadamard class.

Proof: The BRGC and NBC are related via $\boldsymbol{B}_m = \boldsymbol{N}_m\boldsymbol{T}$, with $\boldsymbol{T}$ given by (4). The theorem now follows from Theorem 2 and the definition of a modified Hadamard class. □

Theorem 3 can be understood as follows. Any TCM encoder using the NBC $\boldsymbol{N}_m$ and a convolutional encoder $\boldsymbol{G}$ is equivalent to a TCM encoder using the BRGC $\boldsymbol{B}_m$ and a convolutional encoder $\boldsymbol{G}\boldsymbol{T}$ with $\boldsymbol{T}$ given by (4).

IV. APPLICATION: ASYMPTOTICALLY OPTIMAL TCM

In this section we show how the classification introduced in this paper can be used to find asymptotically optimal TCM encoders in terms of FER. We use a union bound on the FER that is a straightforward generalization of the bound presented in [18] for convolutional codes. For the AWGN channel, and a block length of K symbols, we obtain

$$\mathrm{FER} \le \sum_{d \in \mathcal{D}} K A_d Q\left(\sqrt{d^2 \frac{E_{\mathrm{s}}}{2N_0}}\right). \quad (11)$$

In (11), E_{s} is the average symbol energy, $N_0/2$ is the variance of the noise, A_d is the distance multiplicity of the TCM system, which gives the average number of pairs of sequences at ED d [19, eq. (6.9)] and $\mathcal{D}$ is the set of all EDs $\{d_1, d_2, d_3, \ldots\}$ ($d_i < d_{i+1}$) between *any* two sequences[5] of the TCM system, where d_1 is the minimum ED.

We call the infinite set of pairs (d, A_d) the distance spectrum (DS) of a given TCM encoder $\Theta = [\boldsymbol{G}, \boldsymbol{L}]$, where $d \in \mathcal{D}$. An optimal DS TCM (ODSTCM) encoder is defined in the same way optimum distance spectrum convolutional encoders are defined in [20]. This means that to minimize the FER, the DS (d, A_d) must be sequentially optimized, i.e., first d_1 is maximized, then A_{d_1} is minimized, then d_2 maximized, etc.

We performed a search over $\boldsymbol{G} \in \mathcal{G}_{k,m,\nu}$ and $\boldsymbol{L} \in \mathcal{L}_m$ for $k = 1$ and 4PAM ($m = 2$ and $s_q = (2q-5)/\sqrt{5}$). The results are shown in Table II, where the ODSTCM encoders are shown as $[\cdot,\cdot]^*$. For 4PAM, the possible squared EDs can be expressed as $d_l^2 = d_1^2 + 0.8(l-1)$ for $l = 2, 3, \ldots$ In this table, we also list the encoders proposed by Ungerboeck

[5] Since TCM systems are in general not linear, A_d should be calculated without making the assumption that the all-zero sequence was transmitted [2, p. 101] [11, Problem 13–11].

TABLE II
UNGERBOECK AND ODSTCM ENCODERS: 4PAM, NBC, AND $k = 1$.

ν	d_1^2	$\boldsymbol{G}$	$A_{d_1}, A_{d_2}, A_{d_3}, A_{d_4}, A_{d_5}$
1	4.00	–	–
		$[3, 1]^*$	$0.50, 0.50, 0.50, 0.50, 0.50$
2	7.20	$[5, 2]^{\mathrm{U}}$	$1.00, 1.25, 1.75, 2.56, 3.81$
		$[7, 2]^*$	$0.50, 1.25, 1.63, 2.56, 3.78$
3	8.00	$[13, 4]^{\mathrm{U}}$	$0.25, 1.00, 1.56, 2.75, 3.14$
		$[13, 4]^*$	$0.25, 1.00, 1.56, 2.75, 3.14$
4	8.80	$[23, 4]^{\mathrm{U}}$	$0.63, 0.50, 2.00, 2.02, 2.03$
		$[23, 10]^*$	$0.13, 0.50, 1.88, 2.39, 3.72$
5	10.40	$[45, 10]^{\mathrm{U}}$	$1.13, 1.52, 2.59, 3.58, 5.29$
		$[55, 4]^*$	$0.75, 2.13, 2.14, 4.47, 5.45$
6	11.20	$[103, 24]^{\mathrm{U}}$	$2.34, 0.00, 2.82, 0.00, 7.60$
		$[107, 32]^*$	$0.13, 1.44, 1.41, 1.73, 4.58$
7	12.80	$[235, 126]^{\mathrm{U}}$	$2.19, 0.00, 3.05, 0.00, 10.09$
		$[313, 126]^*$	$1.46, 0.00, 4.77, 0.00, 15.42$
8	13.60	$[515, 362]^{\mathrm{U}}$	$0.53, 1.89, 1.66, 3.81, 6.03$
		$[677, 362]^*$	$0.36, 1.06, 1.47, 3.44, 5.25$

in [8, Table I] (shown as $[\cdot,\cdot]^{\mathrm{U}}$). The search was performed numerically considering 5 terms in the spectrum[6]. Although no gains in terms of minimum ED were obtained, the DS of the ODSTCM encoders is better than those in [8, Table I]. Also, the NBC was among the optimal labelings found for all values of ν and is therefore the chosen labeling (first one in lexicographical order) in Table II. This is however not the case for other combinations of m, k, and ν, which are not shown here due to space limitations. Fig. 3 shows that the new TCM schemes have better FER performance not only asymptotically but also for realistic signal-to-noise ratios.

V. CONCLUSIONS

We analyzed the problem of jointly designing the convolutional encoder and the labeling of a TCM scheme, by grouping the labelings into classes. Theoretically, this contributes to a better understanding of the interplay between code and labeling in TCM systems. Practically, it enables more powerful optimization methods for TCM schemes. As a proof of concept, TCM schemes were found that improve on Ungerboeck's celebrated designs by up to 0.3 dB.

VI. ACKNOWLEDGMENT

The authors would like to thank Prof. Dr.-Ing. Robert Fischer for pointing out the equivalence between TCM systems with convolutional encoders optimized for the BRGC and the NBC, and showing how the convolutional encoders in [10] and [1] are related. These observations inspired this paper.

[6]Note that in some cases there are multiple TCM encoders yielding the same 5-term spectrum. In this case, the first pair of labeling and convolutional encoder (tested in a lexicographic order) is chosen.

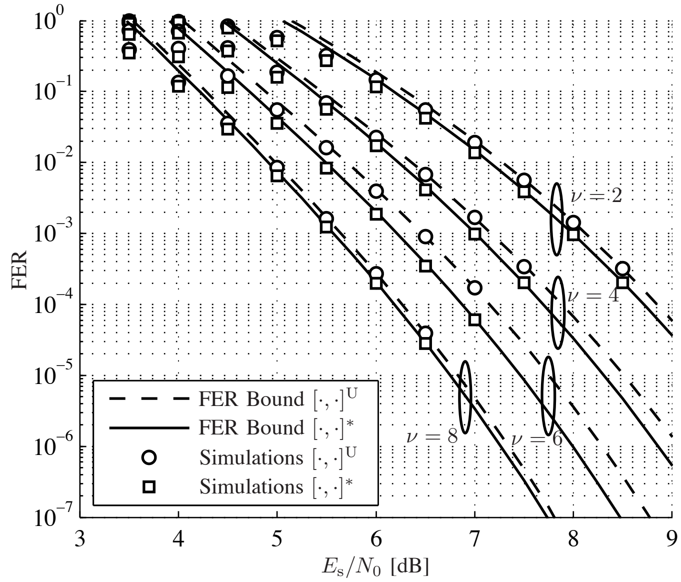

Fig. 3. FER bound in (11) using the 20 first terms of the DS and simulations for Ungerboeck's encoders ($[\cdot,\cdot]^{\mathrm{U}}$) and the ODSTCM encoders ($[\cdot,\cdot]^*$) from Table II together with NBC labeling for $K = 1000$.

REFERENCES

[1] G. Ungerboeck, "Channel coding with multilevel/phase signals," *IEEE Trans. Inf. Theory*, vol. 28, no. 1, pp. 55–67, Jan. 1982.

[2] E. Biglieri, D. Divsalar, P. J. McLane, and M. K. Simon, *Introduction to Trellis-Coded Modulation with Applications*. Prentice Hall, 1992.

[3] S. Lin and D. J. Costello, Jr., *Error Control Coding*, 2nd ed. Englewood Cliffs, NJ: Prentice Hall, 2004.

[4] E. Zehavi, "8-PSK trellis codes for a Rayleigh channel," *IEEE Trans. Commun.*, vol. 40, no. 3, pp. 873–884, May 1992.

[5] G. Caire, G. Taricco, and E. Biglieri, "Bit-interleaved coded modulation," *IEEE Trans. Inf. Theory*, vol. 44, no. 3, pp. 927–946, May 1998.

[6] E. Agrell, J. Lassing, E. G. Ström, and T. Ottosson, "On the optimality of the binary reflected Gray code," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3170–3182, Dec. 2004.

[7] E. Agrell and A. Alvarado, "Optimal alphabets and binary labelings for BICM at low SNR," *IEEE Trans. Inf. Theory*, vol. 57, no. 10, pp. 6650–6672, Oct. 2011.

[8] G. Ungerboeck, "Trellis-coded modulation with redundant signal sets Part II: State of the art," *IEEE Commun. Mag.*, vol. 25, no. 2, pp. 12–21, Feb. 1987.

[9] C. Stierstorfer, R. F. H. Fischer, and J. B. Huber, "Optimizing BICM with convolutional codes for transmission over the AWGN channel," in *International Zurich Seminar on Communications*, Zurich, Switzerland, Mar. 2010.

[10] A. Alvarado, L. Szczecinski, and E. Agrell, "On BICM receivers for TCM transmission," *IEEE Trans. Commun.*, vol. 59, no. 10, pp. 2692–2702, Oct. 2011.

[11] J. B. Barry, E. A. Lee, and D. G. Messerschmitt, *Digital Communication*, 3rd ed. Springer, 2004.

[12] G. C. Clark, Jr. and J. B. Cain, *Error-correction coding for digital communications*, 2nd ed. Plenum Press, 1981.

[13] A. J. Viterbi, J. K. Wolf, E. Zehavi, and R. Padovani, "A pragmatic approach to trellis-coded modulation," *IEEE Commun. Mag.*, vol. 27, no. 7, pp. 11–19, July 1989.

[14] J. P. Odenwalder, A. J. Viterbi, I. M. Jacobs, and J. A. Heller, "Study of information transfer optimization for communication satellites," Linkabit Corporation, San Diego, CA, Tech. Rep. NASA-CR-114561, Jan. 1973, available at http://hdl.handle.net/2060/19730012162.

[15] J. G. Proakis, *Digital Communications*, 4th ed. McGraw-Hill, 2000.

[16] P. Knagenhjelm and E. Agrell, "The Hadamard transform—a tool for index assignment," *IEEE Trans. Inf. Theory*, vol. 42, no. 4, pp. 1139–1151, July 1996.

[17] G. Birkhoff and S. Mac Lane, *A Survey of Modern Algebra*, 4th ed. New York: Macmillan, 1977.

[18] G. Caire and E. Viterbo, "Upper bound on the frame error probability of terminated trellis codes," *IEEE Commun. Lett.*, vol. 2, no. 1, pp. 2–4, Jan. 1998.

[19] C. B. Schlegel and L. C. Perez, *Trellis and Turbo Coding*, 1st ed. John Wiley & Sons, 2004.

[20] P. Frenger, P. Orten, and T. Ottosson, "Convolutional codes with optimum distance spectrum," *IEEE Trans. Commun.*, vol. 3, no. 11, pp. 317–319, Nov. 1999.