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# Low PAPR Waveform Synthesis with Application to Wideband MIMO Radar

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***Abstract*—This paper considers the problem of waveform synthesis given a desired power spectrum. The properties of the designed waveforms are such that the overall system performance is increased. The metric used to evaluate the optimality of the synthesized time domain signals is the peak-to-average power ratio (PAPR). We discuss how to synthesize waveforms using the technique of partial transmit sequence (PTS). The key point is that the gradient can explicitly be derived from the objective function. Furthermore, the result is extended by allowing the power spectrum to deviate from its original shape, yielding a further reduction in the PAPR. The method is applied to derived power spectra for wideband multiple-input-multiple-output (MIMO) radar. It is shown that the proposed technique can achieve optimal or near optimal performance with PAPR below 0.5 dB.**

## I. Introduction

The design of time domain signal sequences that achieve certain system requirements has been an important research area for decades. The result can be used in several applications, and here the focus is to design waveforms for wideband multiple-input-multiple-output (MIMO) radar (for an overview of MIMO radar, see e.g. [1], [2], and [3]).

The radar system performance is directly linked to the time domain properties of the signal. As we are free to utilize arbitrary transmit signals, we seek waveforms that coincide with specific system requirements, for instance low peak-to-average power ratio (PAPR) or even constant modulus. The PAPR measures the largest power of a signal sample compared to the average power. Signals with large PAPR require higher dynamic range on the analog-to-digital converters as well as power amplifiers with a large linear range. Thus, the overall cost of the system is increased. The problem to synthesize a waveform from a periodic signal with a given power spectrum is studied already in [4]. Formulas to adjust the phase angles of periodic signals that yield a low PAPR are presented, and closed form solutions derived (for specific power spectra). Continuing, the problem to construct multitone signals with a low PAPR is addressed in [5], [6] and [7]. Furthermore, in [8], four different partial transmit sequence (PTS) algorithms are discussed, and an extended version of the time-frequency swapping algorithm [7] is proposed as the preferred method. On the other hand, if the covariance matrix of the waveforms is known, a cyclic algorithm that uses the results from [9] gives the possibility to synthesize waveforms with low PAPR or constant modulus, such that the covariance matrix is approximately accomplished [10]. In [11] an algorithm to generate waveforms that match a desired beampattern is proposed. Furthermore, algorithms that use the expression of the signal-to-noise ratio (SNR) to design waveforms that achieve low PAPR under the constraint of good Doppler resolution are discussed in [12].

Herein, we seek to synthesize waveforms that have an already derived desired power spectrum. The waveforms are designed to achieve a low PAPR. Obviously, PAPR is not the only concern in radar waveform design. In addition, the waveform must have a good ambiguity function. Note that the range resolution, i.e. the autocorrelation of the time domain waveform is directly given by the power spectrum. Thus, it is not of concern in this work. The Doppler resolution is of interest for moving targets. However, it is not considered in this paper.

A recently developed algorithm for MIMO communication [13] is exploited and reformulated to coincide with our problem formulation. The key point is to formulate an analytic expression of the objective function and to derive its gradient. The result is compared with the preferred approach in [8], where small modifications are necessary to fit our problem statement. However, this is only a minor difference in the application of the same principle. To further reduce the PAPR, we introduce an extended version of the proposed algorithm, where we allow the power spectrum to deviate from its original shape. Finally, we investigate the trade off between the reduction in PAPR and the loss in signal-to-noise-and-interference ratio (SNIR) when the optimal power spectrum is distorted.

*Notation*: In the sequel, time domain samples are denoted as lowercase letters with indices $n$, vectors by boldface lowercase letters and matrices with boldface uppercase letters. In comparison, frequency domain samples are denoted as uppercase letters with indices $k$. The real and imaginary part of a complex valued vector or matrix is denoted $\Re(\cdot)$ and $\Im(\cdot)$, respectively.

## II. Problem Formulation

Consider a discrete time signal sequence $y[n]$, $n = 0 \dots N-1$, that has an already derived optimal power spectrum. Thus, we seek to synthesize a waveform $\mathbf{y} = [y[0] \dots y[N-1]]^T$ whose frequency behavior is agreeable with $|Y[k]|^2 = P_k$, where $k = 0 \dots N-1$ and $Y[k]$ is the discrete Fourier transform (DFT) of $y[n]$ at frequency index $k$. Moreover, the designed signal sequence should possess a low PAPR yielding both enhancement of the system performance and an overall lower system cost. The PAPR is defined as

$$\text{PAPR} = \frac{\max_n |y[n]|^2}{\frac{1}{N}\sum_{n=0}^{N-1} |y[n]|^2}. \tag{1}$$

In the following section, we describe the waveform synthesis while maintaining the power spectrum. The result is then extended by allowing a small deviation of the desired power spectrum.

## III. Waveform Synthesis

Herein, we discuss the actual design of the waveforms. First, we make use of Parseval's theorem to write the energy of the signal sample as

$$\sum_{n=0}^{N-1} |y[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |Y[k]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} P_k. \tag{2}$$

Further, as we are interested in waveforms that coincide with a desired power spectrum, the use of the inverse discrete Fourier transform (IDFT) yields

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} \sqrt{P_k} \cdot e^{j\phi_k} e^{j2\pi\frac{kn}{N}}, \tag{3}$$

where the phases, $\phi_k \in [0\ 2\pi)$, do not change the given power spectrum. However, the PAPR changes dramatically with the choice of the phases [4]. Hence, by tuning the phases $\phi_k$, we can synthesize a signal with desirable properties while maintaining the power spectrum. Note that the PAPR is reduced by minimizing the maximal $|y[n]|^2$, since the energy is fixed. Thus, we seek to find the phase angles by solving

$$\hat{\boldsymbol{\phi}} = \arg\min_{\phi_k}\max_{n} |y[n]|^2. \tag{4}$$

Define the objective functions as $f_n(\boldsymbol{\phi}) = |y[n]|^2$, where $\boldsymbol{\phi} = [\phi_0 \dots \phi_{N-1}]^T$. Herein, we make use of a sequential quadratic programming (SQP) technique to solve the optimization problem (see [13] for pseudo code and complexity analysis). The optimization problem is formulated as

$$\begin{aligned} \min_{\boldsymbol{\phi}} \quad & \kappa \\ \text{subject to} \quad & f_n(\boldsymbol{\phi}) \leq \kappa,\ n = 0 \dots N-1 \end{aligned} \tag{5}$$

In (5), $\kappa$ is the maximum of $f_n(\boldsymbol{\phi})$. To avoid increasing the power of any other signal sample, the functions $f_n(\boldsymbol{\phi}) \leq \kappa$ are set as constraints to the minimization problem. The cost function has several local minima, which have slightly different levels [13]. Thus, by reaching a local minimum, we satisfy the PAPR reduction aim. Moreover, this implies that the algorithm is insensitive to the initialization vector. In order to derive the gradients of $f_n(\boldsymbol{\phi})$, rewrite (3) as

$$y[n] = \frac{1}{N}\sqrt{\mathbf{p}}^T \mathbf{W}_n \mathbf{g}(\boldsymbol{\phi}), \tag{6}$$

where $\sqrt{\mathbf{p}} = [\sqrt{P_0} \dots \sqrt{P_{N-1}}]^T$, $\mathbf{W}_n$ is a diagonal matrix with the inverse discrete Fourier transform (IDFT) coefficients, $(1, \dots, e^{j2\pi\frac{(N-1)n}{N}})$, on the diagonal and $\mathbf{g}(\boldsymbol{\phi}) = [e^{j\phi_0}, \dots e^{j\phi_{N-1}}]^T$. Rewriting, $e^{j\boldsymbol{\phi}} = \cos(\boldsymbol{\phi}) + j\sin(\boldsymbol{\phi})$ yields

$$\begin{aligned} f_n(\boldsymbol{\phi}) &= \frac{1}{N^2}|\sqrt{\mathbf{p}}^T \mathbf{W}_n \mathbf{g}(\boldsymbol{\phi})|^2 = \\ &= \frac{1}{N^2}\sqrt{\mathbf{p}}^T \underbrace{(\Re(\mathbf{W}_n)\cos(\boldsymbol{\phi}) - \Im(\mathbf{W}_n)\sin(\boldsymbol{\phi}))^2}_{A}\sqrt{\mathbf{p}} + \\ &+ \frac{1}{N^2}\sqrt{\mathbf{p}}^T \underbrace{(\Re(\mathbf{W}_n)\sin(\boldsymbol{\phi}) + \Im(\mathbf{W}_n)\cos(\boldsymbol{\phi}))^2}_{B}\sqrt{\mathbf{p}}. \end{aligned} \tag{7}$$

The gradients of (7) are now calculated as

$$\begin{aligned} \frac{\partial f_n(\boldsymbol{\phi})}{\partial \phi_k} &= -\frac{2}{N^2}\sqrt{\mathbf{p}}^T A\Bigg(\cos(2\pi\frac{nk}{N})\sin(\phi_k) + \\ &+ \sin(2\pi\frac{nk}{N})\cos(\phi_k)\Bigg)\sqrt{\mathbf{p}} + \frac{2}{N^2}\sqrt{\mathbf{p}}^T B \cdot \\ &\cdot \Bigg(\cos(2\pi\frac{nk}{N})\cos(\phi_k) - \sin(2\pi\frac{nk}{N})\sin(\phi_k)\Bigg)\sqrt{\mathbf{p}}. \end{aligned} \tag{8}$$

Next, by allowing the power spectrum to deviate from its original shape with a constant, $\epsilon_k$, i.e. $\sqrt{\tilde{P}_k} = \sqrt{P_k} + \epsilon_k$, we investigate if further reduction of the PAPR is possible. Hence, the optimization problem is reformulated as

$$\begin{aligned} \min_{\epsilon_k, \phi_k}\max_{n} \quad & |\frac{1}{N}(\sqrt{\mathbf{p}} + \boldsymbol{\epsilon})^T \mathbf{W}_n \mathbf{g}(\boldsymbol{\phi})|^2 + \lambda||\boldsymbol{\epsilon}||^2 \\ \text{subject to} \quad & \sqrt{P_k} + \epsilon_k \geq 0 \text{ and } ||\sqrt{\mathbf{p}} + \boldsymbol{\epsilon}||^2 \leq P^{\text{tot}}, \end{aligned} \tag{9}$$

Here, $\lambda$ is a design parameter that controls the total amount of allowed distortion for all entries, $\boldsymbol{\epsilon} = [\epsilon_0 \dots \epsilon_{N-1}]^T$ and $P^{\text{tot}} = \sum_{k=0}^{N-1} P_k$. To solve the complete optimization problem, an alternating approach is used. Thus, first the optimal phase angles, $\boldsymbol{\phi}$, are found for a fixed $\boldsymbol{\epsilon}$ by solving (5). Second, (9) is minimized with respect to $\boldsymbol{\epsilon}$ with fixed phase angles. The alternation continues until the reduction in the PAPR is below a pre-determined convergence criterion. The initial values of $\epsilon_k$ are chosen as zero.

## IV. Relation to Previous Work

To evaluate the performance of the proposed waveform synthesis, a comparison with equivalent algorithms is performed.

Herein, we compare our method to other PTS based techniques, e.g. [4], [5], [6] and [8]. In particular, [8] discusses four different algorithms to reduce the PAPR, and a preferred time-frequency swapping algorithm is presented in detail. Thus, this algorithm is compared with our approach. For the preferred method to coincide with our proposed approach, the stated complex multitone signal of [8] with $N$ tones

$$\underline{m}(t) = \sum_{n=0}^{N-1} e^{j\phi_n} e^{jn\Delta\omega t} \tag{10}$$

is reformulated into

$$\underline{m}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \sqrt{P_k} e^{j\phi_k} e^{j2\pi \frac{kn}{N}}. \tag{11}$$

However, the same proposed algorithm can be used also in this more general setting.

## V. Experimental Validation

To evaluate the proposed approach, the derived frequency behavior of the signal is obtained from [14], where the optimal waveform designs for wideband MIMO radar are investigated. However, any system with prior knowledge of the power spectrum can be used as a reference. In the sequel, we present the case where three waveforms are synthesized, each associated with a specific power spectrum. The power spectra are visualized in Fig. 1.

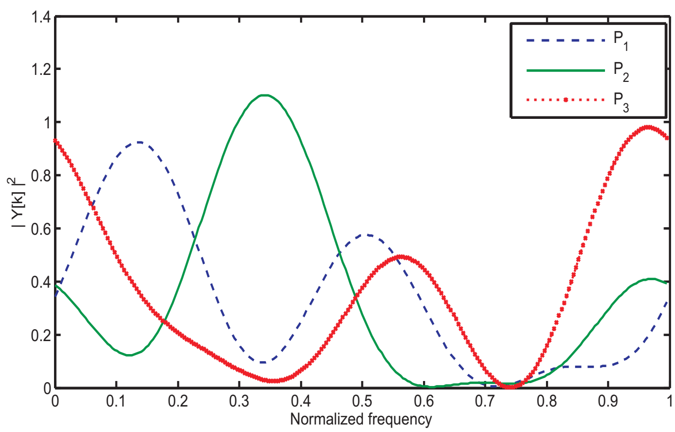


Fig. 1. Desired power spectra for the three waveforms.

First, the waveforms are synthesized to maintain their desired power spectra. It is assumed that convergence of the algorithms are achieved when the reduction in PAPR $\leq 10^{-4}$ per iteration. Figure 2 illustrates the synthesized waveform without the use of a PAPR reduction technique, when the phases, $\phi$, are drawn from a uniform distribution on the interval $\phi \in [0\ 2\pi)$. In comparison, Fig. 3 visualizes the corresponding waveform, designed to achieve a low PAPR. As seen, the maximum level of the signal is reduced.

The PAPR for the optimized signal sequences is compared with the initial PAPR and the time-frequency swapping method [8], see Table I. Unless stated otherwise, the results are average over 200 Monte Carlo trials. As seen,

TABLE I
THE PAPR BEFORE ($\text{PAPR}_0$) AND AFTER ($\text{PAPR}_1$) OPTIMIZATION COMPARED WITH THE TIME–FREQUENCY SWAPPING ALGORITHM [8].

| | $\text{PAPR}_0$ [dB] | $\text{PAPR}_1$ [dB] | PAPR [8] [dB] |
|---|---|---|---|
| Waveform 1 | 7.6 | 0.32 | 1.2 |
| Waveform 2 | 6.6 | 0.23 | 1.0 |
| Waveform 3 | 7.3 | 0.31 | 1.1 |

the proposed method performs significantly better than the

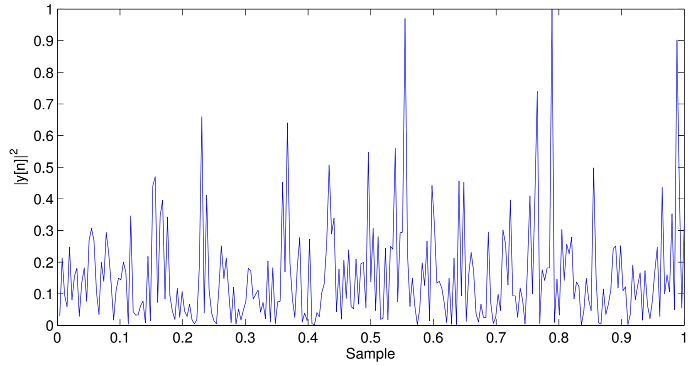


Fig. 2. Synthesized time domain waveform associated with $P_1$ without using a PAPR reduction technique.

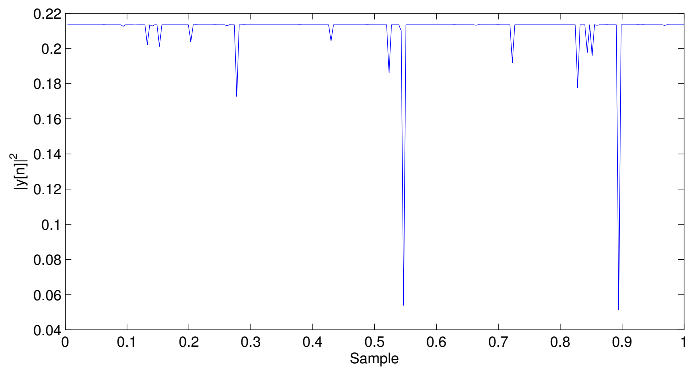


Fig. 3. Synthesized time domain waveform associated with $P_1$ optimized to achieve a low PAPR.

comparable approach. However, the time–frequency algorithm is less complex than that of the proposed approach, and is therefore less time consuming. In the sequel, the initial guess for the proposed algorithm is the output from the time-frequency swapping algorithm. Thus, faster convergence is ensured as the initial guess is closer to a minimum.

Continuing, we investigate the optimal solution to the minimization problem and to the time-frequency swapping algorithm. To identify the variation of the achieved PAPR, 200 test trials are performed. The cumulative distribution function (CDF) for the PAPR, illustrated in Fig. 4, shows that $93\%$ of the obtained PAPR is contained in the interval PAPR $\in [0.2\ 0.4]$ dB for the proposed approach.

Second, the waveforms are synthesized to approximately coincide with their desired power spectra. The reduction in the PAPR for the alternating algorithm with $\lambda = 10^{-2}$ is depicted in Fig. 5. On the x-axis, one iteration corresponds to one step in the alternation method. As illustrated, the algorithm converges fast and the PAPR is decreased in each iteration. To investigate the degradation in performance, the SNIR defined as

$$\text{SNIR} = \frac{P_{\text{target}}}{P_{\text{interference}} + P_{\text{noise}}}, \tag{12}$$

where $P_{\text{target}}$, $P_{\text{interference}}$ and $P_{\text{noise}}$ are the power contribution from the target, interference and receiver noise, respectively, is evaluated for the distorted power spectra. The original

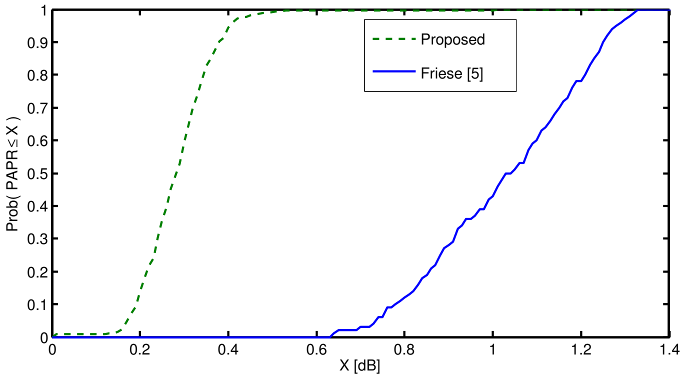


Fig. 4. Prob(PAPR $\leq$ X) for 200 Monte Carlo trials

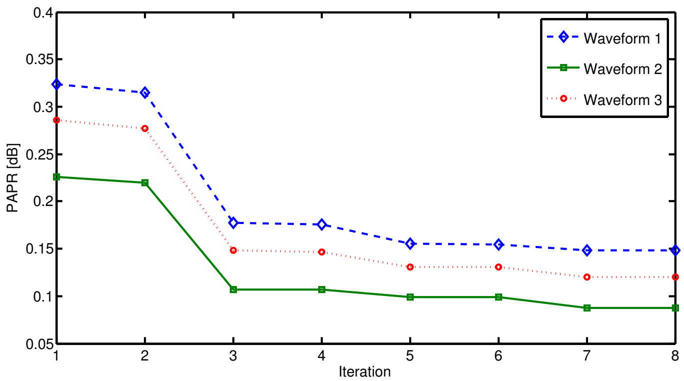


Fig. 5. The average reduction of the PAPR versus iteration count for the alternating algorithm.

power spectra are optimal for a specific radar scenario, consisting of: a target, deceptive jamming and receiver noise (for further information see [14]). The results are presented in Fig. 6, where the SNIR versus the PAPR for a varying $\lambda$ is depicted. Here, 2 complete iterations of the alternating algorithm is performed and $\lambda = [1, 10^{-1}, 50^{-2}, 10^{-2}, 50^{-3}, 10^{-3}, 50^{-4}, 10^{-4}, 50^{-5}, 10^{-5}]$. Note that the illustrated PAPR is the average of the combined waveforms, and that the design parameter, $\lambda$, decreases from left to right. As seen, small variations of the power spectra do not drastically affect the SNIR.

## VI. Conclusion

In this paper, a waveform synthesis algorithm that reduces the PAPR is presented. The designed time domain expression has a specific frequency behavior, that is agreeable with a desired power spectrum. The proposed algorithm is compared with the time-frequency swapping method described in [8], and shows a larger reduction in the PAPR. However, a drawback is that the proposed method is more time consuming. The algorithm is extended by allowing a small deviation of the power spectrum. Results show that further reduction of the PAPR is possible and that the system performance, measured as the SNIR, is not drastically degraded for a small distortion.

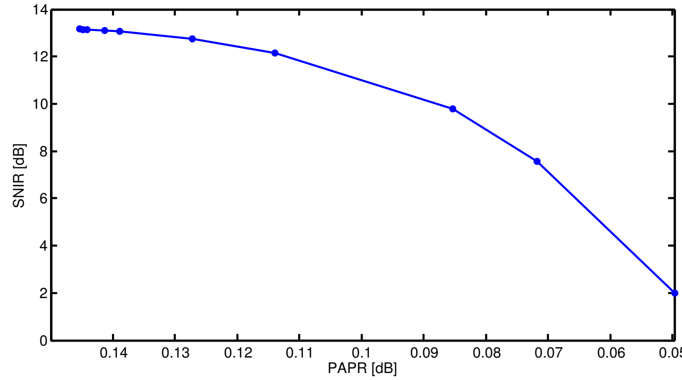


Fig. 6. The degradation in SNIR versus PAPR obtained for different values of the design parameter $\lambda$.

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